TYPE I DEGENERATION OF KUNEV SURFACES

Sampei USUI

Dedicated to Professor Friedrich Hirzebruch
on the occasion of his sixtieth birthday

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Introduction.

In this article we determine completely the main
components of type I degenerations of Kunev surfaces, i.e.,
degenerations of Kunev surfaces with finite local monodromy.
The main results here were already announced in [Us.4] only
with some idea of proofs.

A *Kunev surface* X is defined as a canonical surface,
i.e., canonical model of a surface of general type, with
$\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 2$, ω_X : the dualizing sheaf, which

has an involution σ such that $Y' := X/\sigma$ is a K3 surface with rational double points (RDP, for short). It is well-known that X has only RDP hence ω_X is a line bundle. It is also known that the linear system $|\omega_X^{\otimes 2}|$ gives a finite Galois cover $f : X \longrightarrow \mathbb{P}^2$, factoring through Y' , with Galois group $(\mathbb{Z}/2)^{\oplus 2}$ whose branch locus consists of two cubics $\sum C_j$ and a line L satisfying the conditions:

(0.1) $\sum C_j$ has only simple singularity. $C_1 \cap C_2 \cap L = \phi$. The pull-back of L on the minimal model Y of Y' is reduced. Conversely, given two cubics $\sum C_j$ and a line L on $\mathbb{P}^2$ satisfying (0.1), we can reconstruct a Kunev surface X in the following way:

(i) Take a double cover Y' of $\mathbb{P}^2$ branched along $\sum C_j$. Then Y' is a K3 surface only with RDP.

(ii) Let Y be the minimal model of Y' . Set $\alpha_1 : Y \longrightarrow \mathbb{P}^2$. Let E_i $(1 \leq i \leq 9)$ be the exceptional curves for α_1 whose multiplicity in $\alpha_1^* C_j$ is odd. These are called *distinguished (-2)-curves*.

(iii) Take a double cover X' of Y branched along $\alpha_1^* L + \sum E_i$. Then the canonical model X of X' becomes a Kunev surface.

By the structure of Kunev surfaces described above, we can construct their coarse moduli space $\mathfrak{M}$ in two ways; by the geometric invariant theory applied for the branch loci $\sum C_j + L$ and by the period map for K3 surfaces Y . In order to see it more precisely, set

$$G := \{\sum C_j \in \operatorname{Sym}^2 |\mathcal{O}_{\mathbb{P}^2}(3)| \mid \sum C_j \text{ has only simple singularity}\}$$

$$H* := G \times |O_{\mathbb{P}^2}(1)|$$

$$H := \{\textstyle\sum C_j + L \in H* \,|\, \textstyle\sum C_j + L \text{ satisfies } (0.1)\}$$

Recall the fact that a plane sexetic curve is properly stable with respect to the natural action of $SL_3(\mathbb{C})$ if and only if it has only simple singularity (cf. [H.2] , [Sh]). Hence we can see in the first method that

$$N := G/SL_3(\mathbb{C})$$

is the coarse moduli space of triples $(Y', \alpha_1^* O_{\mathbb{P}^2}(1), \sum_1^9 E_i)$, which are called $K3$ *surfaces of Kunev type,* and that the coarse moduli space of Kunev surfaces is

$$\mathbb{m} = H/SL_3(\mathbb{C}) \ .$$

On the other hand, by the second method, the projection

$$\Phi_2 : \mathbb{m} \longrightarrow N$$

can be seen as a period map of the second cohomology for Kunev surfaces. This is proved by suitable versions of the Torelli theorem and surjectivity of the period map for $K3$ surfaces of Kunev type and the lattice theory of Nikulin in [T.2] and [Mo.2] (there are some ambiguous points in the former; the latter is rigorous) (cf. (2.8)). This together with the Kulikov list of degenerations of $K3$ surfaces ([Ku], [PP]) implies that

$$\mathfrak{m}* := H*/SL_3(\mathbb{C})$$

is a partial compactification of $\mathfrak{m}$ obtained by adding those points which correspond to type I degenerations of Kunev surfaces.

Now we define two functions on $H*$ by

$$m(\textstyle\sum C_j, L) := \sum_{P \in \mathbb{P}^2} \min\{I(P, L \cap C_j) \mid j = 1, 2\}$$

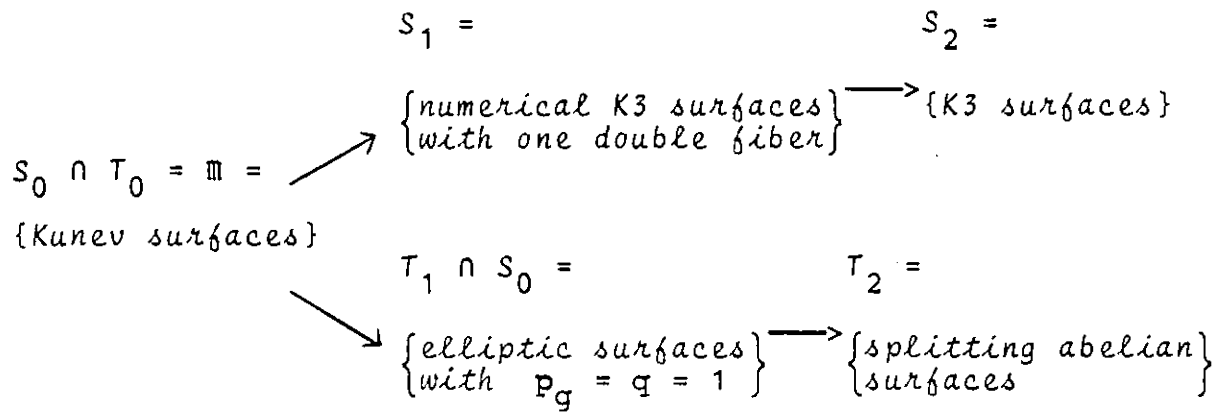$$n(\textstyle\sum C_j, L) := \#\{\text{triple points of } C_j \text{ on } L, j = 1, 2\}$$

where $I(P, L \cap C_j)$ is the intersection multiplicity of L and C_j at $P \in \mathbb{P}^2$. It is easy to see that the value of $m(\ )$ (resp. $n(\ )$) is $0, 1, 2$ or 3 (resp. $0, 1$ or 2). These functions induce ones on $\mathfrak{m}*$ and they define two stratifications:

$$\mathfrak{m}* = S_0 \amalg S_1 \amalg S_2 \quad \text{where} \quad S_m = \{s \in \mathfrak{m}* \mid m = \min\{2, m(s)\}\}$$

$$\mathfrak{m}* = T_0 \amalg T_1 \amalg T_2 \quad \text{where} \quad T_n = \{s \in \mathfrak{m}* \mid n = n(s)\} .$$

The main result in this paper is stated as

(0.2) *In the above notation, the partial compactification* $\mathfrak{m}*$ *is divided into five parts by the above stratifications and they correspond to two series of degenerations:*

$$S_1 = \qquad\qquad\qquad S_2 =$$

$$S_0 \cap T_0 = \mathfrak{m} = \qquad \left\{\begin{matrix} numerical\ K3\ surfaces \\ with\ one\ double\ fiber \end{matrix}\right\} \longrightarrow \{K3\ surfaces\}$$

$$\{Kunev\ surfaces\}$$

$$T_1 \cap S_0 = \qquad\qquad T_2 =$$

$$\left\{\begin{matrix} elliptic\ surfaces \\ with\ p_g = q = 1 \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} splitting\ abelian \\ surfaces \end{matrix}\right\}$$

The period map of the second cohomology for respective surfaces is essentially equal to the restriction of the projection

$$\Phi_2 : \mathfrak{m}^* \longrightarrow N$$

(by the Mayer-Vietoris sequence and the Clemens-Schmid sequence) and $\Phi_2|_{S_m}$ (resp. $\Phi_2|_{T_n}$, $\Phi_2|_{S_m \cap T_n}$) has pure relative dimension 2-m (resp. 2-n, 2-(m+n)) (Theorem (2.6), Corollaries (2.10), (2.13)).

Here we use the terminology a *numerical K3 surface* with one double fiber, which means a minimal elliptic surface with $p_g = 1$, $q = 0$ and $c_1^2 = 0$ and with one double fiber. For $\sum c_j + L \in H^*$, the minimal model $\hat{X}$ of the corresponding surface can be obtained in an analoguous way as the reconstruction (i) - (iii) above of Kunev surfaces, i.e., we can construct a diagram:

(0.3)

$$X' \longleftarrow X^* \longrightarrow \hat{X}$$

$$Y' \longleftarrow Y \longleftarrow Y^*$$

$$\mathbb{P}^2 \longleftarrow \mathbb{P}^*$$

with α_1

where Y' is the double cover of $\mathbb{P}^2$ branched along $\sum C_j$, Y is the canonical resolution of $Y' \longrightarrow \mathbb{P}^2$, X' is the double cover of Y branched along $\alpha_1^* L + \sum_1^9 E_i$, X^* is the canonical resolution of $X' \longrightarrow Y$, and $\hat{X}$ is the minimal model of X^* . Diagram (0.3) suggests an idea of a proof of (0.2). The essential part is the computation of the branch locus $\alpha_1^* L + \sum E_i$ on the minimal K3 surface Y .

Historically the phenomenon of appearance of positive dimensional fibers of a period map is first observed for Kunev surfaces in [T.1], [Us.1] and [Us.2] (for Todorov surfaces in [T.2]) then for elliptic surfaces with $p_g = q = 1$ in [Sa.M]. It is new for numerical K3 surfaces with one double fiber. The present result (0.2) explains uniformly these phenomena by degeneration (Corollary (2.13)).

We explain here the background of Kunev surfaces. The minimal model of a Kunev surface is a simply connected surface with $p_g = c_1^2 = 1$. Let $\widetilde{\mathbb{M}}$ be the coarse moduli space of surfaces with $p_g = c_1^2 = 1$, then $\widetilde{\mathbb{M}}$ is irreducible, rational and with $\dim \widetilde{\mathbb{M}} = 18$ which contains Kunev locus $\mathbb{M}$ with codimension 6 ([Ca.1], [Ca.2]). On the Hodge theoretic viewpoint, these surfaces are interesting materials. After Kunev constructed an example of a Kunev surface as a counterexample to the infinitesimal Torelli theorem, the following results are known:

(0.4) *The generic infinitesimal Torelli theorem holds for surfaces in* $\widetilde{\mathbb{M}}$ ([Ca.1]).

(0.5) *The period map* Φ_2 *of surfaces in* $\widetilde{\mathfrak{m}}$ *has some positive dimensional fibers* ([T.1], [Us.1], [Us.2]; [T.1] treats only Kunev surfaces).

(0.6) $\mathfrak{m}$ *in* $\widetilde{\mathfrak{m}}$ *is characterized by* $\dim \Phi_2^{-1}\Phi_2([X]) = 2$, *which is the maximal dimension of the fibers of* Φ_2 ([Us.1]).

(0.7) *The infinitesimal mixed Torelli theorem holds for pairs* (X,C) *of surfaces* X *in* $\widetilde{\mathfrak{m}}$ *and their smooth canonical curves* C ([Us.3]).

(0.8) *The generic mixed Torelli theorem holds for Kunev surfaces* ([L], [SSU]; there is a point about monodromy which is not clear in [L]).

(0.9) *There exists a Zariski open subset* U *of* $\mathfrak{m}$ *such that* $\Phi^{-1}\Phi(U) = U$, *where* $\Phi : \widetilde{\mathfrak{m}} \longrightarrow \Gamma \backslash D$ *is the mixed period map* ([SSU]).

Hence, in order to solve the mixed Torelli problem for surfaces in $\widetilde{\mathfrak{m}}$ via Kunev locus $\mathfrak{m}$, it is necessary to study the following:

(0.10) *A compactification of the mixed period map*
$\Phi : \widetilde{\mathfrak{m}} \longrightarrow \Gamma \backslash D$.

(0.11) *The monodromy* Γ *in* (0.9), *where we used a geometric one*.

(For a general reference of the above as well as for the terminology such as *mixed* period map, *mixed* Torelli etc., see [SSU].) Problem (0.10) is one of the motivations of the present work. Our result here is not its answer but a by-product.

Section 1 is preliminaries. We shall recall the canonical resolution of a double cover and related results, the Clemens-Schmid sequence and monodromy criteria, the canonical bundle formula for elliptic surfaces and definitions of Kunev surfaces

and numerical K3 surfaces and some of their properties for our latter use.

In Section 2, we shall construct an integral family of surfaces $f : X \longrightarrow \mathbb{P}^2$ over a fixed K3 surface of Kunev type, which is a degeneration of Kunev surfaces. We shall state the main theorem (2.6) and explain this result perspectively in the framework of a type I partial compactification $\mathbb{M}^*$ of the coarse moduli space $\mathbb{M}$ of Kunev surfaces (Corollary (6.10)). We shall also explain uniformly the phenomenon of appearance of positive dimensional fibers of the period map for Kunev surfaces, numerical K3 surfaces with one double fibers and elliptic surfaces with $p_g = q = 1$. The main part of the proof of Theorem (2.6) will be postponed to Sections 4 and 5.

In Section 3, we shall study locally over the singular points P of $\sum C_j + L \subset \mathbb{P}^2$ for $\sum C_j + L \in H*$ and give tables of configurations of $\sum C_j + L$, the branch loci $B_Y(P)$ on minimal K3 surfaces Y and the canonical divisors $K_{\hat{X}_1}(P)$ of type I degenerations of Kunev surfaces corresponding to $\sum C_j + L$. All of these will be described locally over the critical points P in this section. The result here plays the key role in the proofs of Theorem (2.6.3).

Section 4 contains tables of global configurations of $\sum C_j + L \in H*$, the branch loci B_Y on Y and the canonical divisors of the minimal model $\hat{X}$ of type I degenerations of Kunev surfaces corresponding to $\sum C_j + L$. These tables give a proof of Theorem (2.6.3), which is clumsy but elementary and fruitful.

Section 5 contains another proof of Theorem (2.6.3). We shall use the local classification in Section 3 as well as an elliptic fibration on the minimal model $\hat{X}$ induced by the pencil of lines on $\mathbb{P}^2$ through a critical point P of $\sum C_j + L$ for $\sum C_j + L \in S_m \cup T_n$ $(m > 0, n > 0)$.

We use the following terminology:

(-1)-$curve$: an irreducible exceptional curve of the first kind on a smooth surface.

(-2)-$curve$: an irreducible rational curve with self-intersection -2 on a smooth surface, i.e., a nodal curve.

(n)-$(bi)section$: a (bi)section of a fibration on a smooth surface with self-intersection n.
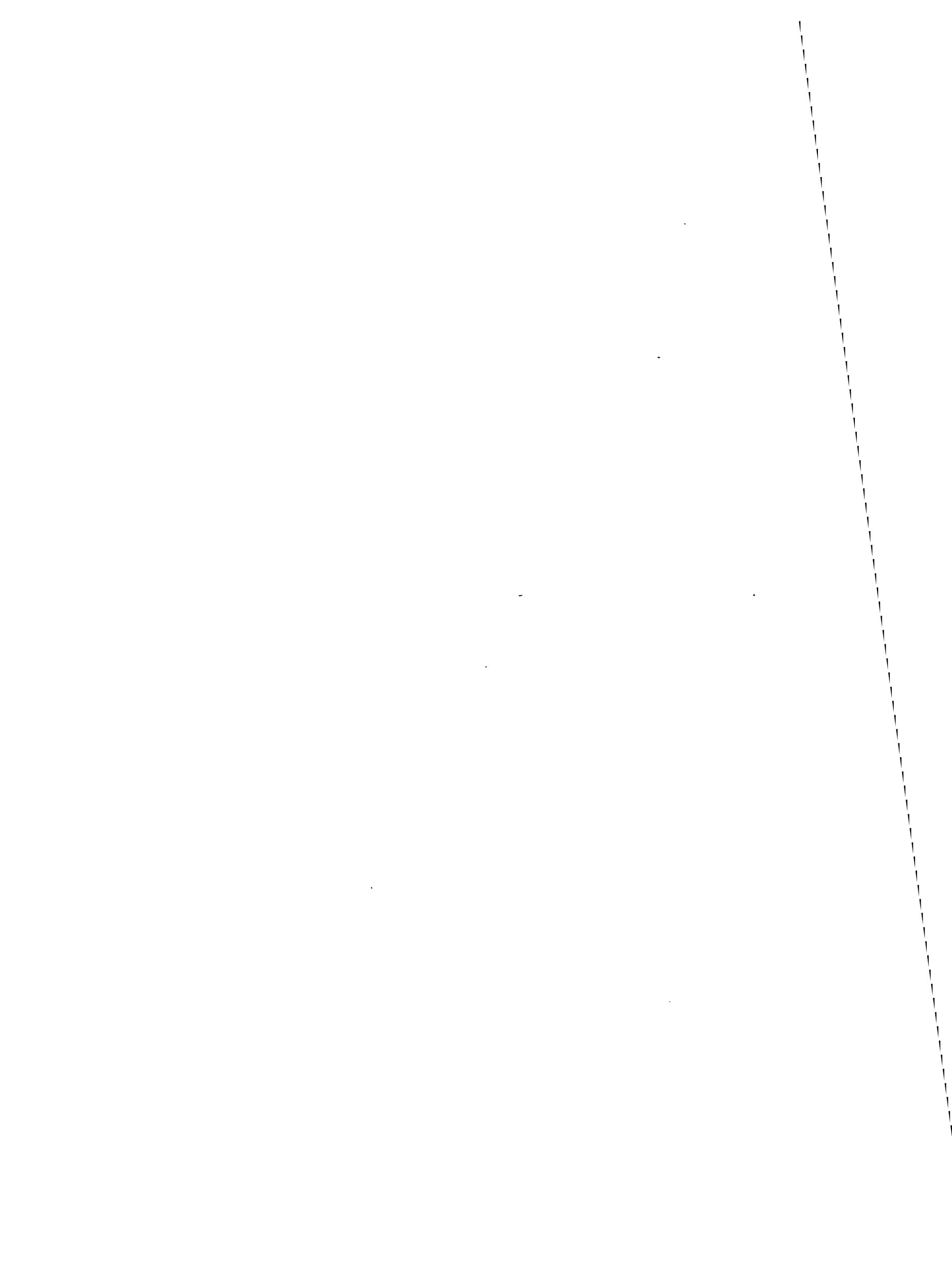
1. Preliminaries.

(1.1) *Canonical resolution.* In this subsection, we shall sumarize the process of a canonical resolution and related results in [H.1] in a slightly general form for our later use.

Let Y be a smooth surface, $B = \Sigma b_i D_i$ an effective divisor on Y and $\mathcal{F}$ a line bundle on Y such that $\mathcal{O}_Y(B) = \mathcal{F}^{\otimes 2}$. Then we can associate the double cover $X = \text{Spec}(\mathcal{O}_Y \oplus \mathcal{F}^{-1}) \longrightarrow Y$ branched along B, where $\mathcal{O}_Y \oplus \mathcal{F}^{-1}$ is endowed an $\mathcal{O}_Y$-algebra structure by $s : \mathcal{F}^{\otimes(-2)} \longrightarrow \mathcal{O}_Y$ for $s \in H^0(Y, \mathcal{F}^{\otimes 2})$ with $(s = 0) = B$. If B is non-reduced (resp. reduced but singular), X is non-normal (resp. has isolated singularity).

The process to obtain the canonical resolution X^* of X is as follows:

0) Set $Y_0 = Y$, $B_0 = B_{\text{odd,red}} := B - 2\Sigma[b_i/2]D_i$ and $\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{O}_Y(-\Sigma[b_i/2]D_i)$, and take the double cover $X_0 = \text{Spec}(\mathcal{O}_Y \oplus \mathcal{F}_0^{-1})$ branched along B_0. Let $p_0 : X_0 \longrightarrow X$ be the birational morphism induced by $\mathcal{F}_0 \hookrightarrow \mathcal{F}$.

i) Let $q_1 : Y_1 \longrightarrow Y$ be a blowing-up with center at a singular point P_1 of B_0. Let e_1 be the multiplicity of $P_1 \in B_0$ and $E_1 = q_1^{-1}(P_1)$ the exceptional divisor. Set $B_1 = q_1^* B_0 - 2[e_1/2]E_1$ and $\mathcal{F}_1 = q_1^* \mathcal{F}_0 \otimes \mathcal{O}_{Y_1}(-[e_1/2]E_1)$ and take the double cover $X_1 = \text{Spec}(\mathcal{O}_{Y_1} \oplus \mathcal{F}_1^{-1})$. Let $p_1 : X_1 \longrightarrow X_0$ be the birational morphism induced by $\mathcal{F}_1 \hookrightarrow q_1^* \mathcal{F}_0$.

After a finite number, say n, of repetition of the process i), we get a non-singular model $X^* := X_n$ which is called the *canonical*

resolution of X. The whole procedure is given by the diagram:

$$X \xleftarrow{\ p_0\ } X_0 \xleftarrow{\ p_1\ } X_1 \xleftarrow{\ p_2\ } \ldots \xleftarrow{\ p_n\ } X_n = X^*$$

(1.1.1)

$$Y = Y_0 \xleftarrow{\ q_1\ } Y_1 \xleftarrow{\ q_2\ } \ldots \xleftarrow{\ q_n\ } Y_n =: Y^*$$

$$B_i = q_i^* B_{i-1} - 2[e_i/2]E_i$$

$$\mathcal{F}_i = q_i^* \mathcal{F}_{i-1} \otimes \mathcal{O}_Y(-[e_i/2]E_i) \qquad (1 \leq i \leq n)$$

A singularity at a point on a reduced curve is called *simple* if its multiplicity is not greater than three and if it is not an infinitely near triple point. Note that, in the procedure of the canonical resolution (1.1.1), the curve B_0 has at most simple singularity if and only if $[e_i/2] = 1$ for all i.

(1.1.2) **Lemma.** In the above notation, if B_0 has at most simple singularity, then the canonical resolution of X coincides with the minimal resolution of X and we have

$$K_{X^*} = \varphi^*(K_Y \otimes \mathcal{F}_0), \qquad \text{where } \varphi : X^* \longrightarrow Y.$$

If, moreover, Y is a minimal K3 surface and $p_g(X^*) = 1$, then

$$q(X^*) = -((B_0)^2/8 + 2).$$

Proof. The first assertion follows from $K_{Y_i} \otimes \mathcal{F}_i = q_i^*(K_{Y_{i-1}} \otimes \mathcal{F}_{i-1})$ and this follows from the remark just before this lemma (cf. [H.1]). We shall prove the second assertion. By construction,

$$h^i(\mathcal{O}_{X^*}) = h^i(\mathcal{O}_{Y_n}) + h^i(\mathcal{F}_n^{-1}) = h^i(\mathcal{O}_{Y_n}) + h^{2-i}(K_{Y_n} \otimes \mathcal{F}_n)$$

$$= h^i(\mathcal{O}_Y) + h^{2-i}(K_Y \otimes \mathcal{F}_0) = h^i(\mathcal{O}_Y) + h^{2-i}(\mathcal{F}_0)$$

Since $h^i(\mathcal{O}_{X^*}) = h^i(\mathcal{O}_Y) = 1$ for $i = 0, 2$, we see $h^i(\mathcal{F}_0) = 0$ for $i = 0, 2$. Hence, by the Riemann-Roch theorem on Y, we have

$$q(X^*) = h^1(\mathcal{F}_0) = -\chi(\mathcal{F}_0) = -((\mathcal{F}_0)^2/2 + \chi(\mathcal{O}_Y)) = -((B_0)^2/8 + 2).$$

QED.

(1.2) *Clemens-Schmid sequence and monodromy criteria.* We shall summarize the Clemens-Schmid sequence and related results ([Cl.1], [Sc]) in the form for our later use. There are good expositions on this topic in [P] and [Mo.1].

Let U be the unit disk. Let $f : \mathfrak{X} \longrightarrow U$ be a semi-stable degeneration of surfaces, i.e., f is a proper flat holomorphic map, $\mathfrak{X}$ is a Kähler manifold, $X_t := f^{-1}(t)$ is smooth for $t \neq 0$ and $X_0 := f^{-1}(0) = \Sigma V_i$ is a reduced divisor with simple normal crossings. In this situation, we have the Clemens-Schmid exact sequence

(1.2.1) $\qquad 0 \longrightarrow H^0_{lim} \overset{\beta}{\longrightarrow} H_4 \overset{\alpha}{\longrightarrow} H^2 \overset{\iota}{\longrightarrow} H^2_{lim} \overset{N}{\longrightarrow} H^2_{lim} \overset{\beta}{\longrightarrow} H_2$

where

$H^i_{lim} = H^i(X_t, \mathbb{Q})$ endowed with the limiting mixed Hodge structure,

$H^i = H^i(X_0, \mathbb{Q})$ endowed with the functorial mixed Hodge structure of Deligne [D],

$H_i = H_i(X_0, \mathbb{Q})$ endowed with the dual mixed Hodge structure,

$N = \mathrm{id} - T$ for the local monodromy T acting on H^i_{lim},

β, α, ι and N are morphisms of mixed Hodge structure of type $(-2,-2)$, $(3,3)$, $(0,0)$ and $(-1,-1)$ respectively.

As a corollary of the Clemens-Schmid sequence, we have:

(1.2.2) **Lemma-Definition.** In the above notation, $N^3 = 0$ and $p_g(X_t) \geq \Sigma p_g(V_i)$ always hold. $N^2 = 0$ if and only if $H^2(\Gamma) = 0$ for the dual graph Γ of $X_0 = \Sigma V_i$. $N = 0$ if and only if $p_g(X_t)$

$= \Sigma \, p_g(V_i)$.   The semi-stable degeneration $f : \mathfrak{X} \longrightarrow U$ is called type I (resp. II, III) if $N = 0$ (resp. $N \neq 0$ and $N^2 = 0$, $N^2 \neq 0$ and $N^3 = 0$).

(1.3) *Some results for elliptic fibrations*.   We include here the canonical bundle formula [Ko.2, Theorem 12] and the positivity of the direct image of relative dualizing sheaf [Ue, Remark in Appendix] for elliptic fibrations for our later use.

Let X be a non-singular compact complex surface and let f : X $\longrightarrow$ C be a *relatively minimal* elliptic fibration, i.e., a general fiber of f is a non-singular elliptic curve and no fiber of f contains (-1)-curves.

(1.3.1) *Canonical bundle formula*.   The canonical bundle K_X of an elliptic surface X has the form

$$K_X = f^*(K_C \otimes (R^1 f_* \mathcal{O}_X)^{-1}) \otimes \mathcal{O}_X(\Sigma(m_i - 1)F_i)$$

where $m_i F_i$, i = 1, 2, ..., n, are all multiple singular fibers. The line bundle $R^1 f_* \mathcal{O}_X$ is dual to $f_* \omega_{X/C}$ where $\omega_{X/C} = K_X \otimes (f^* K_C)^{-1}$, and

$$\deg R^1 f_* \mathcal{O}_X = -\chi(\mathcal{O}_X).$$

A simpler proof of the above formula can be found in [Ue, Appendix].

For the degree of the line bundle $R^1 f_* \mathcal{O}_X$, or equivalently of $f_* \omega_{X/C}$, we can see more:

(1.3.2) *Positivity of* $f_* \omega_{X/C}$.   We have

$$\deg f_* \omega_{X/C} \geq 0.$$

- 4 -

The equality holds if and only if the elliptic fibration $f : X \longrightarrow C$ has constant J-invariant and has only multiple singular fibers of type ${}_mI_0$ (for the notation, see [Ko.1]).

There is a full proof of (1.3.2) in [BPV, p.110] by reducing the assertion to the case of a semi-stable fibration.

By the definition of the Kodaira dimension, the following assertion can be obtained as an exercise of intersection theory (for a proof, see, e.g., [BPV, p.194]).

(1.3.3) If a non-singular compact complex surface X has Kodaira dimension $\kappa(X) = 1$, then X is an elliptic surface.

(1.4) *Some surfaces and their properties*. We include here the definitions of somewhat unfamiliar surfaces and their properties which will appear later.

(1.4.1) Definition. A *Kunev surface* X is a canonical surface with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$, ω_X the dualizing sheaf, which has an involution σ such that X/σ is a K3 surface with at most rational double points (R.D.P. for short).

Let $\hat{X}$ be the minimal model of a Kunev surface X. The following properties are known ([Ca.1]):

(1.4.2) $\hat{X}$ is simply connected. $p_g(\hat{X}) = 1$. $c_1^2(\hat{X}) = 1$.

(1.4.3) The canonical model X can be represented as a weighted complete intersection of type (6,6) in $P(1,2,2,3,3)$ with

at most R.D.P., whose partially normalized equations are

$$\begin{cases} f = z_3^2 + f^{(3)}(x_0^2, y_1, y_2) \\ g = z_4^2 + g^{(3)}(x_0^2, y_1, y_2) \end{cases}$$

where $\deg x_0 = 1$, $\deg y_i = 2$ ($i = 1, 2$), $\deg z_i = 3$ ($i = 3, 4$), and $f^{(3)}$ and $g^{(3)}$ are cubics in $y_0 := x_0^2$, y_1, y_2.

(1.4.4) **Definition.** A minimal surface X is called a *numerical K3 surface* if $p_g = 1$, $q = 0$ and $c_1^2 = 0$.

The following are known:

(1.4.5) Every simply connected numerical K3 surface X belongs to one of two oriented homotopy types according to its Whitney class, i.e., $c_1(X)$ mod 2 ([Mi]).

(1.4.6) A simply connected numerical K3 surface is characterized as either a K3 surface or an elliptic surface with $p_g = 1$ and $q = 0$ which has at most two multiple fibers and, in the case that there are two, their multiplicities m_1 and m_2 are mutually prime ([Ko.3, Proposition 1, Lemma 6]).

(1.4.7) **Remark.** Kodaira [Ko.3] called a simply connected surface with the same oriented homotopy type as K3 surface a *homotopy K3 surface*. By definition, a homotopy K3 surface (resp. K3 surface) is equivalent to a simply connected numerical K3 surface with $c_1(X) \equiv 0$ mod 2 (resp. $c_1(X) = 0$). While we shall come across numerical K3 surfaces with one double fiber later.

2. Construction of families of surfaces and statements of the main results.

(2.0) In this section, we shall construct families of surfaces which are degenerations of Kunev surfaces over a fixed K3 surface and state the main resuls. We postpone the proof of Theorem (2.6.3) in Section 4 and Section 5, where we shall give two different proofs after a preparation in Section 3.

(2.1) Let X be a Kunev surface defined in (1.4.1). Then by (1.4.3) the bicanonical bundle $\omega_X^{\otimes 2}$ gives a Galois cover $X \longrightarrow P^2$ with Galois group $(Z/2Z)^{\oplus 2}$. The branch locus consists of two cubics $C_1 = \{f^{(3)}(y_0, y_1, y_2) = 0\}$ and $C_2 = \{g^{(3)}(y_0, y_1, y_2) = 0\}$ and a line $L = \{y_0 = 0\}$. The K3 surface $Y' := X/\sigma$ in (1.4.1) can be seen as a weighted hypersurface of type (6) in $P(1,1,1,3)$ defined by an equation $h = u_3^2 + f^{(3)}(y_0, y_1, y_2) \, g^{(3)}(y_0, y_1, y_2)$ with $\deg y_i = 1$ $(0 \leq i \leq 2)$ and $\deg u_3 = 3$. By construction, the K3 surface Y' with R.D.P. is the double cover of P^2 branched along the two cubics ΣC_j, hence ΣC_j on P^2 has only simple singularity.

(2.2) For sexetic curves on P^2, curves with at most simple singularity coincide with properly stable curves with respect to the action of $SL_3(C)$ ([H.2], [Sh]). Set
$$\mathcal{G} = \{\Sigma C_j \in Sym^2 | \mathcal{O}_{P^2}(3) | \ | \ \Sigma C_j \text{ has only simple singularity}\}$$
$$\mathfrak{N} = \mathcal{G}/SL_3(C)$$
Then, as a consequence of Theorem (2.6.3) below, $\mathfrak{N}$ can be seen as the coarse moduli space of the polarized K3 surfaces with R.D.P. which are quotients of Kunev surfaces X by their involution σ

plus the data of the distinguished (-2)-curves defined in (2.4.2)

below (cf. (2.7), (2.8) below).   We call the K3 surfaces equipped

with these data *K3 surfaces of Kunev type*.   We have a projection

$p : \mathfrak{M} \longrightarrow \mathfrak{N}$, $[X] \longmapsto [X/\sigma]$.

(2.3) For any fixed $\Sigma C_j \in \mathcal{G}$, we define functions in $t \in \check{P}^2$

by

$$m(t) = \Sigma_{P \in P^2} \min\{I(P, L_t \cap C_j) \mid j = 1, 2\}$$

$$n(t) = \#\{\text{triple points of } C_j \text{ on } L_t, \quad j = 1, 2\}$$

Notice that if C_j has a triple point then C_j consists of three

distinct lines with a common point.   These functions define two

stratifications of $\check{P}^2$:

$\check{P}^2 = S_0 \amalg S_1 \amalg S_2$ where $S_m = \{t \in \check{P}^2 \mid m = \min\{2, m(t)\}\}$

$\check{P}^2 = T_0 \amalg T_1 \amalg T_2$ where $T_n = \{t \in \check{P}^2 \mid n = n(t)\}$

Notice that $\operatorname{codim} S_m = m$, $\operatorname{codim} T_0 = 0$, and $\operatorname{codim} T_n = n$ if T_n

is non-empty (n = 1, 2).

(2.4) For $\Sigma C_j \in \mathcal{G}$, we denote by Y the minimal K3 surface

which is obtained as the minimal resolution of the double cover of

P^2 branched along ΣC_j.   Let $\alpha_1 : Y \longrightarrow P^2$ be the projection

and E_i be the exceptional curves for α_1, i.e., (-2)-curves.

Then we have the following lemma whose proof is easy and we omit it.

(2.4.1) Lemma.   The sets $\{E_i \mid$ the multiplicity of E_i in the

total transform of C_j is odd} (j = 1, 2) coincide and the number

of their elements is nine.

(2.4.2) Remark.   The nine (-2)-curves in the above lemma is an

equivalent datum to the one of the *distinguished partial*

desingularization of a K3 surface of Kunev (more generally, Todorov) type in [Mo.2]. We call the former the *distinguished (-2)-curves*. They appeared in A.D.E. configuration of exceptional curves over R.D.P. as in Table (3.2.2) in Section 3.

(2.5) We reorder the numbering so that E_i ($1 \le i \le 9$) are the nine distinguished (-2)-curves on Y, and set $\delta_i = \check{P}^2 \times E_i$ ($1 \le i \le 9$). Denote by $\mathscr{L} \subset \check{P}^2 \times P^2$ the total space of the universal family of lines on P^2. We can construct families of surfaces $\check{f}' : \mathscr{X}' \longrightarrow \check{P}^2$, $\check{f} : \mathscr{X} \longrightarrow \check{P}^2$ and $f : \mathscr{X} \longrightarrow \check{P}^2$ in the following way: (0) Set $\alpha = 1 \times \alpha_1 : \check{P}^2 \times Y \longrightarrow \check{P}^2 \times P^2$. (i) Let $\beta : \mathscr{Y} \longrightarrow \check{P}^2 \times Y$ be the blowing-up along $\alpha^{-1}\mathscr{L} \cap (\Sigma_1^9 \delta_i)$. Denote by $\mathscr{W}_i$ ($1 \le i \le 9$) the exceptional divisors. (ii) Take the double cover $\gamma : \mathscr{X}' \longrightarrow \mathscr{Y}$ branched along $(\alpha\beta)^{-1}\mathscr{L} + \beta^{-1}(\Sigma \delta_i)$. (iii) Let $\delta : \mathscr{X}' \longrightarrow \mathscr{X}$ be the contraction of $(\beta\gamma)^{-1}(\Sigma \delta_i)$. (iv) Let $\varepsilon : \mathscr{X} \longrightarrow \mathscr{X}$ be the contraction of $\delta\gamma^{-1}(\Sigma \mathscr{W}_i)$. (In the notation above, we used $\alpha^{-1}\mathscr{L}$ etc. as the proper transforms.)

Set $\mathscr{L}_{\mathscr{X}} = (\delta(\alpha\beta\gamma)^{-1}\mathscr{L}$ with reduced structure) and $\mathscr{W}_{\mathscr{X}, i} = \delta\gamma^{-1}\mathscr{W}_i$.

(2.6) Theorem. In the above notation, $f : \mathscr{X} \longrightarrow \check{P}^2$ is an integral family of degenerations of Kunev surfaces over the fixed $\Sigma C_j \in \mathscr{G}$. This family has the following properties:

(1) The singularity of the total space $\mathscr{X}$ consists of disjoint nine compounds Veronese cone over $S_1 \amalg S_2 = (C_1 \cap C_2)^\vee$, i.e., analytically isomorphic to the product of a line and the cone over the Veronese embedding of $P^2 \subset P^5$ by $|\mathscr{O}_{P^2}(2)|$. $\varepsilon : \mathscr{X} \longrightarrow \mathscr{X}$ is

- 9 -

a desingularization and the exceptional divisor $W_{\mathcal{X},i}$ is a family of P^2 over a line in $(C_1 \cap C_2)^\vee$ $(1 \le i \le 9)$. $K_{\mathcal{X}} - \tilde{f}^* K_{\check{P}^2} = \mathcal{L}_{\mathcal{X}} + \Sigma W_{\mathcal{X},i}$.

(2) The fiber $\tilde{X}_t := \tilde{f}^{-1}(t) = V_t + \Sigma W_{i,t}$, where V_t is the main component, i.e., the component with $p_g = 1$, and $W_{i,t} := W_{\mathcal{X},i}|\tilde{X}_t$. Hence the dualizing sheaf of V_t coincides with $\mathcal{O}(\mathcal{L}_{\mathcal{X}}|V_t)$.

(3) V_t is a (singular) Kunev surface, numerical K3 surface with one double fiber, K3 surface, elliptic surface with $p_g = q = 1$, or splitting abelian surface according to $t \in S_0 \cap T_0$, S_1, S_2, $T_1 \cap S_0$ or T_2.

Proof of (1) and (2). In the notation in (2.5), notice that $\mathcal{L}$ and $\check{P}^2 \times (\Sigma C_j)$ intersect transversally on $\check{P}^2 \times P^2$ hence so do $\alpha^{-1}\mathcal{L}$ and $(\Sigma_1^9 \delta_i)$ on $\check{P}^2 \times Y$. This implies that $\alpha^{-1}\mathcal{L} \cap (\Sigma \delta_i)$ consists of nine disjoint P^1-bundles over the lines $\check{P}_k$ on $\check{P}^2$, $P_k \in C_1 \cap C_2$. Therefore the branch locus $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma \delta_i)$ on $\mathcal{Y}$ is a smooth divisor. It follows that $\mathcal{X}'$ is smooth.

Since $\delta_i = \check{P}^2 \times E_i$ and E_i is a (-2)-curve on Y, we see $N_{\delta_i/\check{P}^2 \times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}(-2)$. Hence $\beta^{-1}\delta_i$ on $\mathcal{Y}$ is a P^1-bundle whose normal bundle restricted to any fiber is isomorphic to $\mathcal{O}_{P^1}(-2)$. This implies that $(\beta\gamma)^{-1}\delta_i$ on $\mathcal{X}'$ is a P^1-bundle over $\check{P}^2$ whose normal bundle restricted to any fiber is isomorphic to $\mathcal{O}_{P^1}(-1)$. Thus we get a smooth variety $\mathcal{X}$ in Step (iii).

The P^1-bundle $\alpha^{-1}\mathcal{L} \cap \delta_i$ over the line $\check{P}_k$ has $N_{\alpha^{-1}\mathcal{L} \cap \delta_i/\check{P}^2 \times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2)$, because $N_{E_i/\check{P}^2 \times Y} \simeq \mathcal{O}_{P^1}^{\oplus 2} \oplus \mathcal{O}_{P^1}(-2)$ and $\alpha^{-1}\mathcal{L} \cap \delta_i \cap (H \times Y) = E_i$ transversally, for any line H on $\check{P}^2$

other than $\check{P}_k$, and $N_{H\times Y/\check{P}^2\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}$. Hence $\mathscr{W}_i$ is a

Σ_2-bundle over $\check{P}_k$, where $\Sigma_2 := \text{Proj}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2))$. This implies

that $\gamma^{-1}\mathscr{W}_i$ is a Σ_1-bundle over $\check{P}_k$ intersecting with $(\beta\gamma)^{-1}\delta_i$

along the P^1-bundle over $\check{P}_k$ whose fiber is the (-1)-section on Σ_1.

Thus we get a P^2-bundle $\delta\gamma^{-1}\mathscr{W}_i$ over the line $\check{P}_k$ on $\mathfrak{X}$ in

Step(iii).

Since $N_{H\times Y/\check{P}^2\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}$ as above, $\beta^{-1}(H \times Y)$ intersects

with $\mathscr{W}_i$ along a (2)-section on Σ_2 over the point $H \cap \check{P}_k$, hence

$(\beta\gamma)^{-1}(H \times Y)$ intersects with $\gamma^{-1}\mathscr{W}_i$ along a (4)-bisection on Σ_1.

Therefore $\delta(\beta\gamma)^{-1}(H \times Y)$ intersects with $\delta\gamma^{-1}\mathscr{W}_i$ along a conic on

P^2 over the point $H \cap \check{P}_k$. Thus we see that $\delta\gamma^{-1}\mathscr{W}_i$ contracts to

a compound Veronese cone over the line $\check{P}_k$.

Now the other assertions in (1) and (2) follows easily by the

adjunction formula.   QED.

    (2.7) Set

$\mathscr{K}^* := \{\Sigma C_j + L \in \text{Sym}^2|\mathcal{O}_{P^2}(3)| \times |\mathcal{O}_{P^2}(1)| \mid \Sigma C_j \in \mathscr{G}\}$

(for the notation $\mathscr{G}$, see (2.2)).

Now we consider the functions $m(t)$ and $n(t)$ in (2.3) as functions

$m(\Sigma C_j, L)$ and $n(\Sigma C_j, L)$ on $\mathscr{K}^*$ and define

$\mathscr{K} := \{\Sigma C_j + L \in \mathscr{K}^* \mid m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0\}$

$\mathscr{K}^\circ := \{\Sigma C_j + L \in \mathscr{K} \mid$ both cubics C_j are smooth and

                             they intersect transversally.

                       L and ΣC_j intersect transversally.$\}$

By the natural action of $SL_3(\mathbb{C})$, we can take their quotients (cf.

(2.2)):

    $\mathfrak{M}^* := \mathscr{K}^*/SL_3(\mathbb{C})$, $\mathfrak{M} = \mathscr{K}/SL_3(\mathbb{C})$, $\mathfrak{M}^\circ := \mathscr{K}^\circ/SL_3(\mathbb{C})$,

where the midle equality is a consequence of Theorem (2.6.3).

(2.8) In order to explain Teorem (2.6.3) more perspectively, we recall here briefly the construction of the coarse moduli space of Kunev (more generally, Todorov) surfaces by the period map. These were first constructed by Todorov [T.2] by using the Torelli theorem and surjectivity of the period map for K3 surfaces, but it contains some ambiguous points in lattice theoretic part. Morrison [Mo.2] then gave a rigorous proof based on Nikulin's works.

For the economy of pages, we take a reference point $0 = (\Sigma\ C_j,\ L) \in \mathcal{H}^\circ$ and construct the following diagram in a similar way as (0)-(iv) in (2.5):

(2.8.1)

$$
\begin{array}{ccc}
X_0 & \xleftarrow{\ \pi\ } & X_0^* \\
\downarrow & & \downarrow \varphi \\
Y_0' & \longleftarrow & Y_0 \\
\downarrow \quad \alpha_1 \nearrow & & \downarrow \\
P^2 & \longleftarrow & P^*
\end{array}
$$

where Y_0' is the double cover of P^2 branched along $\Sigma\ C_{j,0}$,

$\quad Y_0$ is the minimal resolution of Y_0' on which sit the nine distinguished (-2)-curves $\Sigma\ E_{i,0}$ coming from the nine ordinary double points on Y_0',

$\quad X_0^*$ is the double cover of the minimal K3 surface Y_0 branched along $B_{Y_0} := \alpha_1^* L_0 + \Sigma\ E_{i,0}$, and

$\quad X_0$ is the contraction of the nine (-1)-curves on X_0^* lying over $\Sigma\ E_{i,0}$ on Y_0.

Set

$\quad \Lambda := H^2(Y_0,\ Z)$

$$\lambda := \text{class}(\alpha_1^* \mathcal{O}_{\mathbf{P}2}(1)) \in \Lambda$$

$$N := \{\xi \in \Lambda \mid \xi \cdot \lambda = \xi \cdot E_i = 0 \quad (1 \leq i \leq 9)\}$$

Notice that

$$\{\omega \in P(N \otimes \mathbf{C}) \mid \omega \cdot \omega = 0, \quad \omega \cdot \bar{\omega} > 0\}$$

has two connected components, interchanging by complex conjugation. Choose the component D containing $H^{2,0}(Y_0)$, a *period domain*. This choice is called the *sign structure*.

Now let Y' be any K3 surface with R.D.P., $\mu : Y \longrightarrow Y'$ the minmal resolution and $\{D_k\}$ the set of exceptional (-2)-curves for μ. Set

$$I^2(Y') := \{\xi \in H^2(Y, \mathbf{Z}) \mid \xi \cdot D_k = 0 \quad \text{for all} \quad k\}$$

A *marking* of Y' is an embedding of lattice

$$\varphi_0 : I^2(Y') \lhook\joinrel\longrightarrow \Lambda$$

for which there exists an isometry $\varphi : H^2(Y, \mathbf{Z}) \overset{\sim}{\longrightarrow} \Lambda$ such that $\varphi|_{I^2(Y')} = \varphi_0$.

Glueing together local deformations by virtue of a suitable version of the Torelli theorem and surjectivity of the period map for K3 surfaces with R.D.P., we can construct the unversal family $g : \mathcal{Y}' \longrightarrow D$ of marked K3 surfaces of Kunev type and a relatively ample line bundle $L_{\mathcal{Y}'}$ on $\mathcal{Y}'$ whose first Chern class on each fiber is mapped to λ by the marking. Here the markings of the fibers are required to have images in the span of λ and N, and to send the holomorphic 2-forms on the minimal model of each fiber into D (cf. [Mo.2, §7]). This yields a $\mathbf{P}^2$-bundle

$$P(g_* L_{\mathcal{Y}'}) \longrightarrow D.$$

Let $\mathcal{V}$ be the Zariski open set of $P(g_* L_{\mathcal{Y}'})$ consisting of those points (ω, L_ω) which satisfies the condition: the pull-back $\mu^* L_\omega$

of the divisor L_ω on the minimal model $\mu : Y_\omega \longrightarrow Y'_\omega$ of the K3 surface has at most simple singularity and it is disjoint from the distinguished (-2)-curves on Y_ω.

We denote by $\tilde{\Gamma}$ the subgroup of orthogonal group $O(\Lambda)$ of the K3 lattice Λ consisting of those elements which preserves the polarization λ, the (unordered) set of distinguished (-2)-curves $\{E_1, \ldots, E_9\}$ and the sign structure. By definition there is the natural homomorphism $\tilde{\Gamma} \longrightarrow O(N)/\{\pm 1\}$, where $O(N)$ is the orthogonal group of the lattice N. We denote its image by Γ. Then we can see that the action of Γ on D lifts to the P^2-bundle $P(g_* L_{\vartheta'}) \longrightarrow D$ which preserves the open set $\mathcal{V}$ and that the quotients $\mathcal{V}/\Gamma \longrightarrow D/\Gamma$ are the coarse moduli spaces of Kunev surfaces and K3 surfaces of Kunev type, which are irreducible (cf. [Mo.2, (7.3), (7.5), (7.8)]).

(2.9) Thus we get the coarse moduli spaces in two ways, via geometric invariant theory and via period map:

$$\mathfrak{M} = \mathcal{K}/SL_3(\mathbb{C}) \simeq \mathcal{V}/\Gamma$$
$$\cap \qquad\qquad \cap$$
$$\mathfrak{M}^* = \mathcal{K}^*/SL_3(\mathbb{C}) \simeq P(g_* L_{\vartheta'})/\Gamma$$
$$\downarrow p \qquad\qquad \downarrow$$
$$\mathfrak{N} = \mathcal{G}/SL_3(\mathbb{C}) \simeq D/\Gamma$$

As a consequence, we see in particular that the partial compactification $\mathfrak{M}^*$ of $\mathfrak{M}$ consists of all the points whose period is an interior point of $D/\Gamma = \mathfrak{N}$, i.e., type I degeneration.

By construction, the functions $m(\Sigma C_j, L)$ and $n(\Sigma C_j, L)$ on $\mathcal{K}^*$ defined in (2.7) and (2.3) induce ones on $P(g_* L_{\vartheta'})$ and on $\mathfrak{M}^*$, and these functions on $\mathfrak{M}^*$ define two stratifications of $\mathfrak{M}^*$ as in

(2.3):

$$\mathfrak{M}^* = \mathcal{G}_0 \amalg \mathcal{G}_1 \amalg \mathcal{G}_2 \quad \text{where} \quad \mathcal{G}_m = \{s \in \mathfrak{M}^* \mid m = \min(2, m(s))\}$$

$$\mathfrak{M}^* = \mathcal{T}_0 \amalg \mathcal{T}_1 \amalg \mathcal{T}_2 \quad \text{where} \quad \mathcal{T}_n = \{s \in \mathfrak{M}^* \mid n = n(s)\}$$

Theorem(2.6.3) implies:

(2.10) Corollary. The partial compactification $\mathfrak{M}^*$ of the coarse moduli space $\mathfrak{M}$ of Kunev surfaces consists of all the points of type I degenerations and $\mathfrak{M}^*$ is divided into five parts

$$\mathcal{G}_0 \cap \mathcal{T}_0 = \mathfrak{M}, \quad \mathcal{G}_1, \quad \mathcal{G}_2, \quad \mathcal{T}_1 \cap \mathcal{G}_0, \quad \mathcal{T}_2$$

whose points correspond to Kunev surfaces, numerical K3 surfaces with one double fiber, K3 surfaces, elliptic surfaces with $p_g = q = 1$, and splitting abelian surfaces respectively. $\mathfrak{M}^{\circ}$ is a Zariski open subset of $\mathfrak{M}$ consisting of those points which correspond smooth Kunev surfaces, i.e., the canonical model is smooth.

(2.11) In the remaining part of this section, we shall explain uniformly by Theorem (2.6.3) the appearance of positive dimensional fibers of the period map for the second cohomology of Kunev surfaces, numerical K3 surfaces with one double fiber and elliptic surfaces with $p_g = q = 1$. This phenomenon was observed separately before in [T.1], [Us.1], [Us.2] for the first surfaces and in [Sa.M] for the third. It is new for the second surfaces.

Let $f : \mathfrak{X} \longrightarrow \check{P}^2$ and $\check{f} : \mathfrak{X} \longrightarrow \check{P}^2$ be the families of degenerations of Kunev surfaces constructed in (2.5) for a fixed $\Sigma C_j \in \mathcal{G}$. Starting from these, we can construct semi-stable degenerations as follows (cf. [Us.5]):

(2.11.1) *Base* $t \in S_1$: We may assume that ΣC_j are smooth

cubics intersecting transversally because other cases are limit of this. For a general point $t_0 \in S_1$, say $t_0 \in \check{P} \subset (C_1 \cap C_2)^\vee$, let U be a small polydisk neighberhood with center $(0,0) = t_0 \in \check{P}^2$. Then the restriction over U of the family $\tilde{f} : \tilde{X} \longrightarrow \check{P}^2$ gives a 2-parameter family of semi-stable degeneration of Kunev surfaces whose singular fibers lie over the line $\check{P} \cap U = \{(t,0) \mid |t| < 1\}$. For $(t,0) \in \check{P} \cap U$, the fiber $\tilde{X}_{t,0} := \tilde{f}^{-1}(t,0) = V_{t,0} + W_{t,0}$ where $V_{t,0}$ is a minimal numerical K3 surface with one double fiber and $W_{t,0} \simeq P^2$. The double locus $V_{t,0} \cap W_{t,0}$ is a smooth bisection with self-intersection -4 on $V_{t,0}$ and a smooth conic on $W_{t,0}$.

(2.11.2) *Base* $t \in T_1 \cap S_0$: For a general point $t_0 \in T_1$, say $t_0 \in \check{Q}$ for the triple point Q of C_j, take a small polydisk neighborhood U with center $(0,0) = t_0 \in \check{P}^2$. Then the restriction over U of the family $f : X \longrightarrow \check{P}^2$ (equivallently, $\tilde{f} : \tilde{X} \longrightarrow \check{P}^2$) gives a 2-parameter family of degeneration of Kunev surfaces whose singular fibers are non-normal and lie over the line $\check{Q} \cap U$. Extending the base to the double cover $\pi : U_2 \longrightarrow U$ branched along the line $\check{Q} \cap U$, we can construct a semi-stable family $\hat{f} : \hat{X} \longrightarrow U_2$ whose singular fibers lie over the line $\pi^{-1}(\check{Q} \cap U) = \{(s,0) \mid |s| < 1\}$. For $(s,0) \in \pi^{-1}(\check{Q} \cap U)$, the fiber $\hat{X}_{s,0} = \hat{f}^{-1}(s,0) = \hat{V}_{s,0} + \hat{W}_{s,0}$ where $\hat{V}_{s,0}$ is a minimal elliptic surface with $p_g = q = 1$ and with a section which is a smooth elliptic curve with self-intersection -1 and $\hat{W}_{s,0}$ is a rational surface constructed, for example, from P^2 by blowing-up twice at each of the four 2-torsion points on a smooth cubic endowed with a well-known abelian group structure. The double locus $\hat{V}_{s,0} \cap \hat{W}_{s,0}$ is the section

mentioned above on $\hat{V}_{s,0}$ and the proper transform of the above cubic on $\hat{W}_{s,0}$.

(2.12) Recall the spectral sequence for a reduced simple normal crossing variety $Z = \Sigma\, Z_k$:

$$E_1^{p,q} = H^q(Z^{[p]},\ \mathbb{Q}) \longrightarrow E^{p+q} = H^{p+q}(Z,\ \mathbb{Q})$$

where $Z^{[p]} = \displaystyle\coprod_{k_0 \leq \ldots \leq k_p} X_{k_0} \cap \ldots \cap X_{k_p}$.

It is known that it degenerates at $E_2 = E_\infty$ (cf. [D], [GS]). Applying this to $Z = \tilde{X}_{t,0}$ or $\hat{X}_{s,0}$, the singular fibers of the semi-stable degenerations in (2.11), we can observe easily in both cases that $E_2^{2,0} = E_2^{1,1} = E_2^{1,2} = 0$ hence we have an exact sequence

(2.12.1) $\quad 0 \longrightarrow H^2(Z) \xrightarrow{\ \nu\ } H^2(Z_1) \oplus H^2(Z_2) \longrightarrow H^2(Z_1 \cap Z_2) \longrightarrow 0.$

On the other hand, since the local monodromies of the semi-stable families obtained in (2.11) are trivial, the Clemens-Schmid sequence (1.2.1) becomes in both cases

(2.12.2) $\quad 0 \longrightarrow H^0_{\lim} \longrightarrow H_4 \longrightarrow H^2 \xrightarrow{\ \iota\ } H^2_{\lim} \longrightarrow 0.$

The morphism of Hodge structure (H.S. for short) ν in (2.12.1) relates the variation of Hodge structure (V.H.S. for short) associated to the smooth family $\{V_{t,0}\}_{|t|<1}$ of numerical K3 surfaces with one double fiber (resp. $\{\hat{V}_{s,0}\}_{|s|<1}$ of elliptic surfaces with $p_g = q = 1$) with the V.H.S. associated to the flat family $\{\tilde{X}_{t,0}\}_{|t|<1}$ (resp. $\{\hat{X}_{s,0}\}_{|s|<1}$) and they coincide essentially because $W_{t,0}$ (resp. $\hat{W}_{s,0}$) is a rational surface hence its associated V.H.S. is trivial. While the morphism ι in (2.12.2) relates the V.H.S. associated to the flat family $\{\tilde{X}_{t,0}\}$ (resp. $\{\hat{X}_{s,0}\}$) with the variation of limiting H.S. associated to the 2-parameter family of sem-stable degeneration of Kunev surfaces

$(\tilde{X}_{t,t'})$ (resp. $(\hat{X}_{s,s'})$), taking limit as $t' \rightarrow 0$ (resp. $s' \rightarrow 0$), and they coincide essentially because H_4 in (2.12.2) carries a trivial H.S. in both cases.

Thus we get:

(2.13) Corollary.   In the above notation, the following assertions hold and they are related by degeneration as above:

(1)   The 2-parameter smooth families $(\tilde{X}_{t,t'})_{t' \neq 0}$ and $(\hat{X}_{s,s'})_{s' \neq 0}$ of minimal Kunev surfaces have 2-dimensional moduli and the associated V.H.S. are trivial.

(2)   The 1-parameter smooth family $(V_{t,0})$ of minimal numerical K3 surfaces with one double fiber has 1-dimensional moduli and the associated V.H.S. is trivial.

(3)   The 1-parameter smooth family $(\hat{V}_{s,0})$ of minimal elliptic surfaces with $p_g = q = 1$ has 1-dimensional moduli and the associated V.H.S. is trivial.

Proof.   The assertion on the V.H.S. has already proved before the corollary.   As for the assertion on the moduli, the case (1) is obvious by construction (cf. (2.2)).   The case (2) follows from an observation that the moduli of the double fiber of $V_{t,0}$ varies (cf. Proposition (5.3) and its proof).   The case (3) follows from an observation that the moduli of the section of $\hat{V}_{s,0}$ varies (cf. the proof of Proposition (5.3)).   QED.

3. Local study over critical points.

(3.0) Let $\Sigma_1^2 \, C_j$ be two cubics on P^2 with at most simple singularity, i.e., $\Sigma \, C_j \in \mathscr{G}$ in the notation of (2.2), and let L be a line. In this section we study locally over the singular points of $\Sigma \, C_j + L$ on P^2. The tables obtained in this section will play the key role in the proof of Theorem (2.6.3).

(3.1) For the cubics $\Sigma \, C_j$, we constructed the minimal K3 surface Y and the families of surfaces $\widetilde{f}' : \mathfrak{X}' \longrightarrow \check{P}^2$, $\widetilde{f} : \mathfrak{X} \longrightarrow \check{P}^2$ and $f : \mathfrak{X} \longrightarrow \check{P}^2$ in (2.5). Let V', V and X be the main components of the fibers $\widetilde{f}'^{-1}(t)$, $\widetilde{f}^{-1}(t)$ and $f^{-1}(t)$ over the point $t \in \check{P}^2$, $L_t = L$, respectively. Then the morphisms γ, δ and ε in Steps (ii), (iii) and (iv) in (2.5) induce the morphisms (abuse of the notation):

$$(3.1.1) \qquad Y \overset{\gamma}{\longleftarrow} V' \overset{\delta}{\longrightarrow} V \overset{\varepsilon}{\longrightarrow} X.$$

By construction, we see that δ and ε in (3.1.1) are birational morphisms and that γ is the finite double cover branched along B_Y := $\alpha_1^* L + \Sigma_1^9 \, E_i - 2\Sigma_i \, E_i$, where in the last term the index i runs over the set $\{i \mid 1 \le i \le 9, \; E_i \subset \alpha_1^* L\}$. Here we use the notation $\alpha_1 : Y \longrightarrow P^2$, the canonical resolution of the double cover Y' of P^2 branched along $\Sigma \, C_j$, and E_i $(1 \le i \le 9)$, the nine distinguished (-2)-curves, in (2.4).

The minimal model $\hat{X}$ of X is obtained by the succesive contraction of (-1)-curves, starting from the canonical resolution X^* of the double cover γ in (3.1.1). This procedure is indicated by the diagram:

$$\begin{array}{ccccccc}
V^{\cdot} & \longleftarrow & X^* & \overset{\pi_1}{\longrightarrow} & \hat{X}_1 & \overset{\pi_2}{\longrightarrow} & \hat{X} \\
\downarrow & {}^{\varphi}\!\!\nearrow & \downarrow & & & & \\
\end{array}$$

(3.1.2)

$$\begin{array}{ccccc}
Y^{\cdot} & \longleftarrow & Y & \longleftarrow & Y^* \\
\downarrow & {}^{\alpha_1}\!\!\nearrow & \downarrow & & \\
P^2 & \longleftarrow & P^* & &
\end{array}$$

where $\pi_1 : X^* \longrightarrow \hat{X}_1$ is the succesive contraction of the

(-1)-curves each of which is mapped to a singular points of $\Sigma\, C_j + L$

on P^2 and $\pi_2 : \hat{X}_1 \longrightarrow \hat{X}$ is the succesive contraction of the

(-1)-curves each of which is mapped onto the line L on P^2.

We use the notation:

(3.1.3) $B_Y := (\alpha_1^* L + \Sigma_1^9\, E_i)_{odd,red} = (B_Y^{\cdot})_{odd,red}.$

Here, for an effective divisor D, $(D)_{odd,red}$ means the reduced

divisor whose support consists of those components with odd

multiplicity in D.

(3.2) Notice that, in Diagram (3.1.2), all the processes but

$\pi_2 : \hat{X}_1 \longrightarrow \hat{X}$ are local over a singular point of $\Sigma\, C_j + L$ on P^2.

For a singular point $P \in \mathrm{Sing}(\Sigma\, C_j + L)$, we denote by $\alpha_1^* L(P)$

(resp. $B_Y(P)$, $K_{\hat{X}_1}(P)$) the pull-back of the line $\alpha_1^* L$ on Y (resp.

the divisor B_Y on Y in (3.1.3), the canonical divisor $K_{\hat{X}_1}$ of

$\hat{X}_1$) restricted over an open neighborhood of the point $P \in P^2$. We

can classify the singular points $P \in \mathrm{Sing}(\Sigma\, C_j + L)$, where $\Sigma\, C_j$

has at most simple singularity, and compute the divisors $\alpha_1^* L(P)$,

$B_Y(P)$ and $K_{\hat{X}_1}(P) = \pi_1\varphi^* B_Y(P)/2$ locally over the point P. Note

that the last equality follows from the observation that $B_Y(P)$ has

at most simple singularity, which is a consequence of the computation.

All of these classification and computations are elementary,

- 20 -

hence we give here the tables. For the computation of $B_Y(P)$, we use Table (3.2.2) below of the distinguished (-2)-curves.

In order to devide the cases, we define functions $m_P(\Sigma C_j, L)$ and $n_P(\Sigma C_j, L)$ in $P \in P^2$ and $(\Sigma C_j, L) \in \mathcal{R}^*$ by

$$m_P(\Sigma C_j, L) = \min\{I(P, L \cap C_j) \mid j = 1, 2\}$$

$$n_P(\Sigma C_j, L) = \begin{cases} 1 & \text{if } \text{mult}_P C_j = 3 \text{ for } j = 1 \text{ or } 2 \text{ and if } P \in L \\ 0 & \text{otherwise} \end{cases}$$

Hence the summations of these functions over $P \in P^2$ give

$$m(\Sigma C_j, L) = \Sigma_{P \in P^2} m_P(\Sigma C_j, L)$$
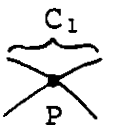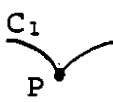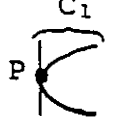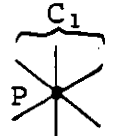
$$n(\Sigma C_j, L) = \Sigma_{P \in P^2} n_P(\Sigma C_j, L)$$

(see (2.7)). We also use the following notation:
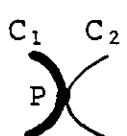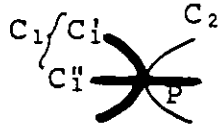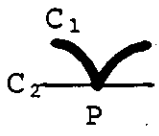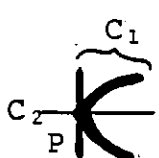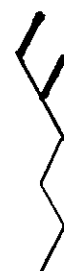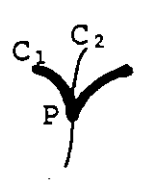
$L_Y := \alpha_1^{-1} L$ the proper transform of L on Y

$L_{\hat{X}_1} := \pi_1(\alpha_1\varphi)^{-1} L$ the proper transform of L on $\hat{X}_1$

(3.2.1) Case $m_P(\Sigma C_j, L) = n_P(\Sigma C_j, L) = 0$, $P \in \text{Sing } C_1 - (C_2 + L)$:

on P^2	exceptional curves on Y		$B_Y(P)$	exceptional curves on $\hat{X}_1$	$K_{\hat{X}_1}(P)$
		A_1	0	$2A_1$	0
		A_2	0	$2A_2$	0
		A_3	0	$2A_3$	0
		D_4	0	$2D_4$	0

(3.2.2) Case $m_P(\Sigma C_j, L) = n_P(\Sigma C_j, L) = 0$, $P \in C_1 \cap C_2 - L$

(distinguished (-2)-curves):

on $\mathbf{P}^2$	exceptional curves on Y, $B_Y(P)$: bold lines	exceptional curves on $\hat{X}_1$	$K_{\hat{X}_1}(P)$
C_1 C_2 P $I(P, C_1 \cap C_2)$ $= a$		A_{2a-1}	A_{a-1}	0
C_1 / C_1' C_2 C_1'' P $I(P, C_1' \cap C_2)$ $= a$		D_{2a+2}	D_{a+2}	0
C_1 C_2 P		D_5	A_5	0
C_1 C_2 P		D_6	A_7	0
C_1 C_2 P		E_7	E_6	0

(3.2.3) Case $m_P(\Sigma\, C_j,\, L) = n_P(\Sigma\, C_j,\, L) = 0,\quad P \in L:$

on $\mathbf{P}^2$	$\alpha_1^* L$ with multiplicity, $B_Y(P)$: bold curves	$\pi_1(\alpha_1\varphi)^* L$ on $\hat{X}_1$, $K_{\hat{X}_1}(P)$: bold curves with multiplicity

$$L = C_1 \quad\text{with}\quad P$$

L_Y $\quad$ $\dfrac{1}{L_Y}$ $\quad$ $\dfrac{1}{L_{\hat{X}_1}}$

A_1

A_2

$L = C_1$ $\quad$ $\dfrac{2}{L_Y}$ $\quad$ 0

A_1 $\quad$ A_3

A_1 $\quad$ D_4

$L_{\hat{X}_1}$

C_1

L P

L_Y 2 $\left.\right\}A_1$ 1

$L_{\widehat{X}1}$ 0

C_1

L P

L_Y 1   1   1 $\left.\right\}A_2$

$L_{\widehat{X}1}$ 1   1   1   1   1 $\left.\right\}A_5$

C_1

L P

L_Y 1   1   1 $\left.\right\}A_2$

$L_{\widehat{X}1}$ 1   2   2   3   1   2   1 $\left.\right\}E_6$

C_1

L P

L_Y 1   1   1   1   1 $\left.\right\}A_3$

$L_{\widehat{X}1}$ 1   1   1   1   1   1   1 $\left.\right\}A_7$

C_1

L P

L_Y 2   1   2   1 $\left.\right\}A_3$

$L_{\widehat{X}1}$ $\left.\right\}A_1$ 0

(3.2.4) Case $n_P(\Sigma C_j, L) = 1$:

on $\mathbf{P}^2$

α_1^*L with multiplicity,

$B_Y(P)$: bold curves

$\pi_1(\alpha_1\varphi)^*L$ on $\hat{X}_1$,

$K_{\hat{X}_1}(P)$: bold curves with multiplicity,

( ) : self-intersection

(3.2.5) Case $m_P(\Sigma C_j, L) = 1$:

on $\mathbf{P}^2$

α_1^*L with multiplicity,

$B_Y(P)$: bold curves

$\pi_1(\alpha_1\varphi)^*L$ on X ,

$K_{\hat{X}_1}(P)$: bold curves with multiplicity,

( ) : self-intersection

C_1 C_2

L

P

C_1 C_2

L

P

$L=C_1$ C_2

P

C_1 C_1' C_2

C_1''

L P

$I(P, C_1' \cap C_2)$

$= a$

1 1

L_Y $\Big\}A_1$

A_1

L_Y $\Big\{$ 1 1 1

1 A_1

2

L_Y

1

1

2

2

2

2

1 2 1

L_Y

D_{2a+2}

$L_{\hat{X}_1}$ 1 (-3)

(-3)

(-3)

1 1

$L_{\hat{X}_1}$ 1 A_1

(-3)

$2A_1$

$L_{\hat{X}_1}$ 0

Case
a = 1:

$4A_1$

(-3)

$L_{\hat{X}_1}$ 1

Case
a ≥ 2:

D_{a+1}

$2A_1$

(-3)

$L_{\hat{X}_1}$ 1

$C_1 \diagdown C_1'$

C_1'' C_2

L P

$I(P, C_1' \cap C_2)$

$= a$

1

1

2

2

2

D_{2a+2}

2

2

2

L_Y 1

1

$2A_1$

Case
a = 1:

(-3)

1

1 1

1

1

A_1

$L_{\hat{X}_1}$

Case
a ≥ 2:

D_{a+1}

(-3)

1

1 1

1

1

A_1

$L_{\hat{X}_1}$

$C_1 \diagdown C_1'$

$C_1''=L$ C_2

P

$I(P, C_1' \cap C_2)$

$= a$

1

1

2

2

2

2

2

D_{2a+2}

L_Y 2

D_{a+2}

$L_{\hat{X}_1}$
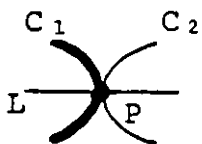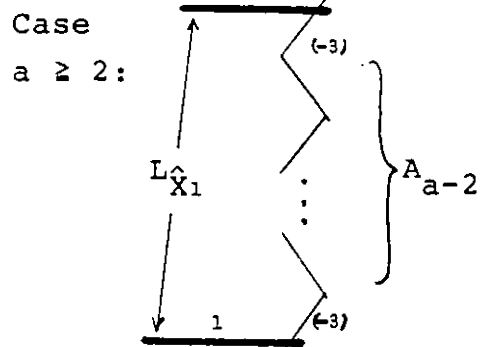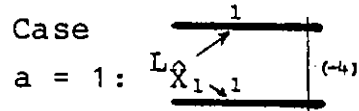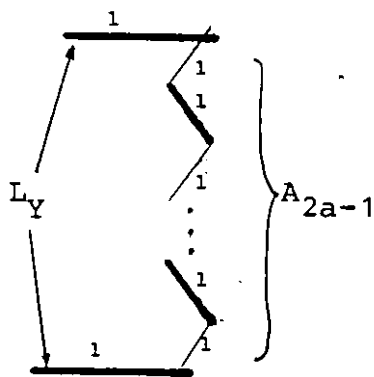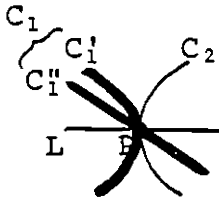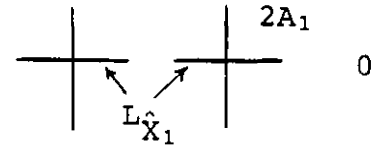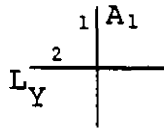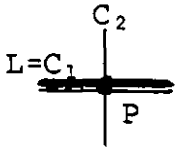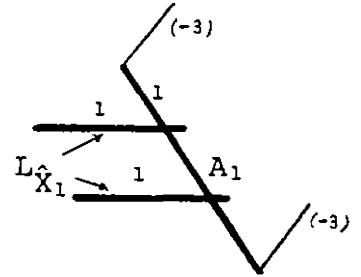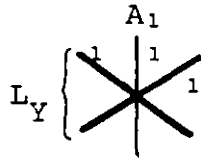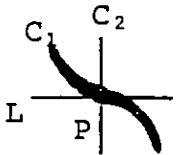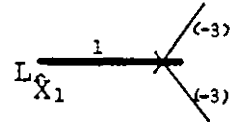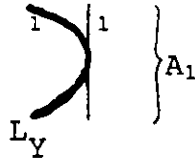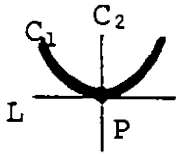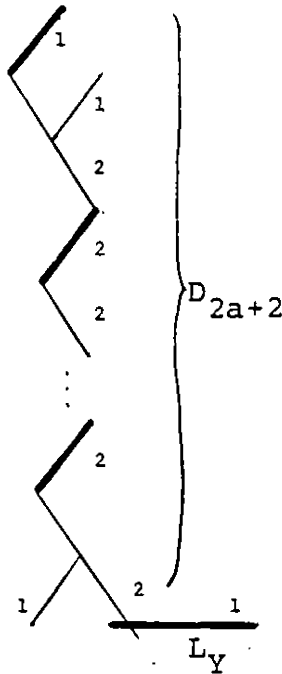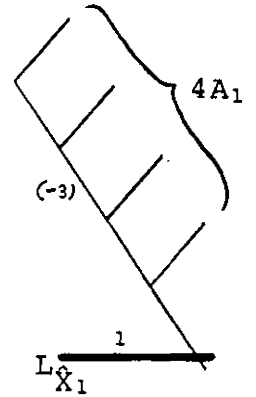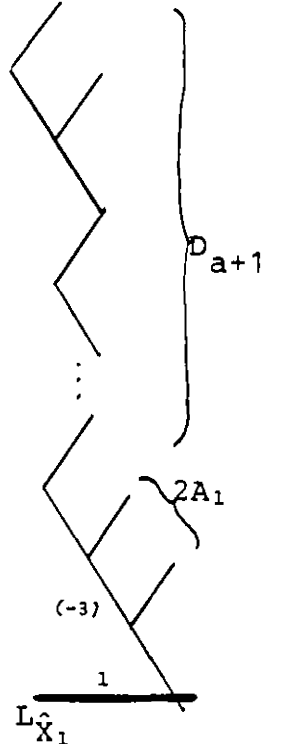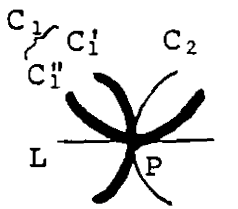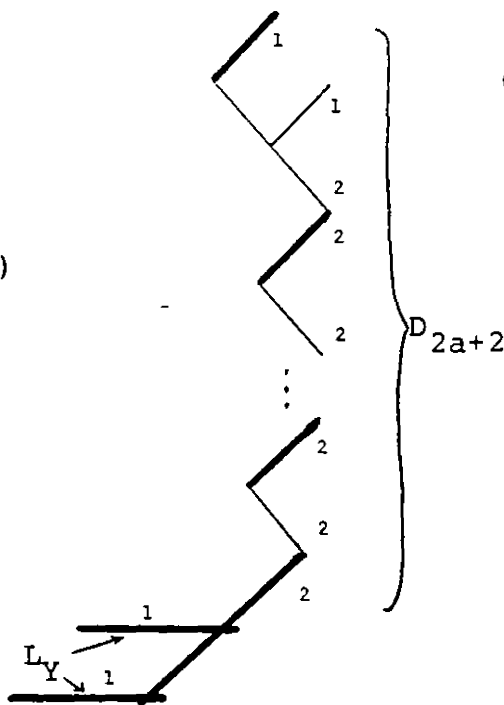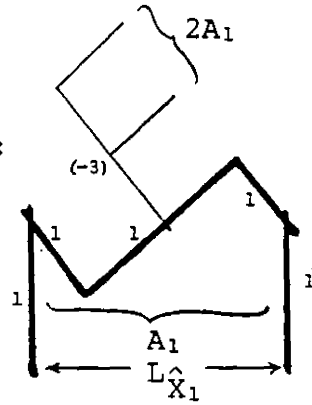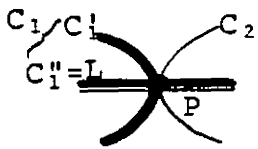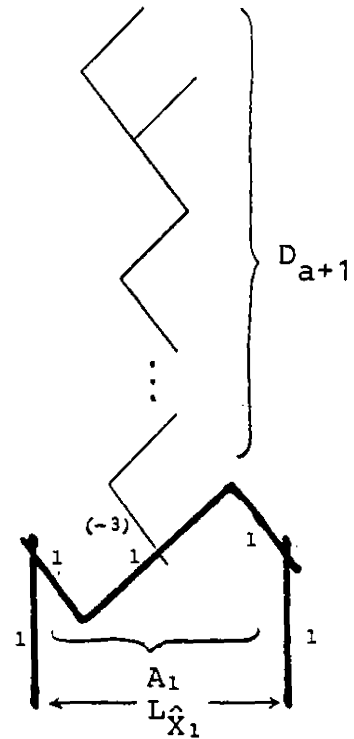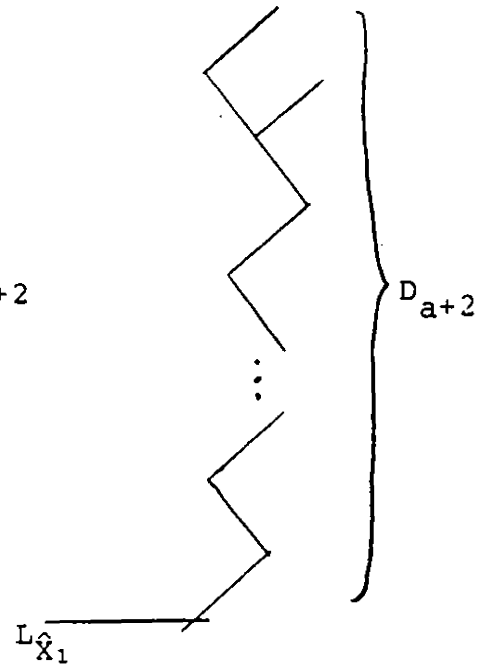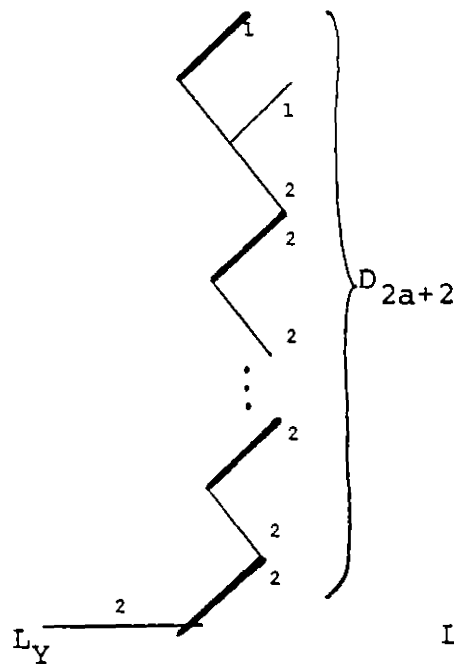
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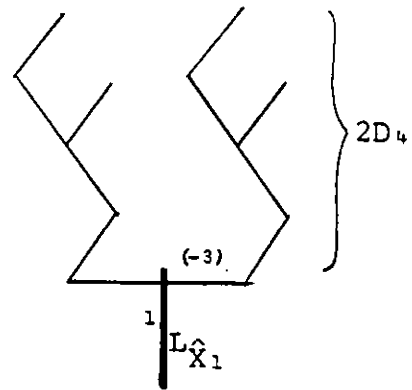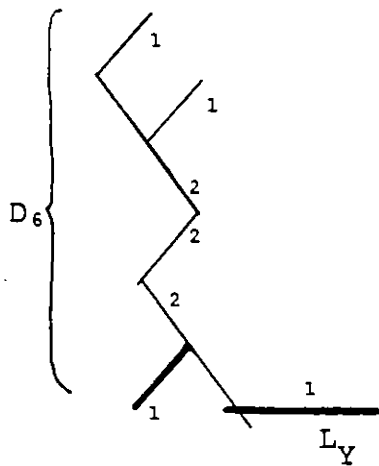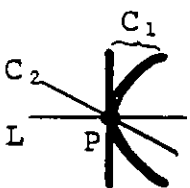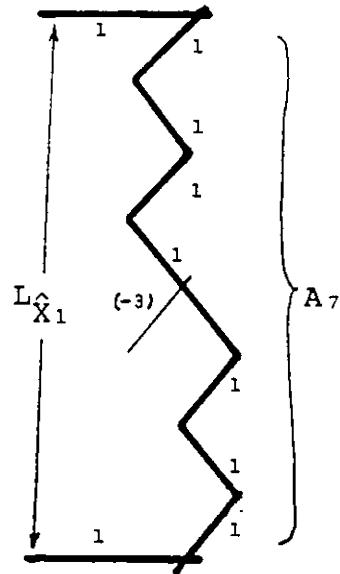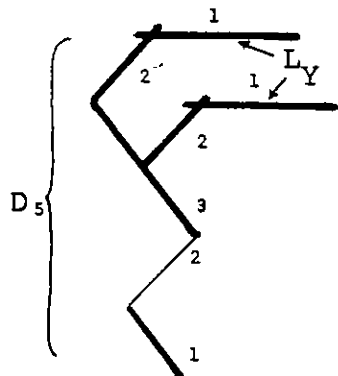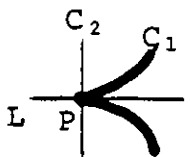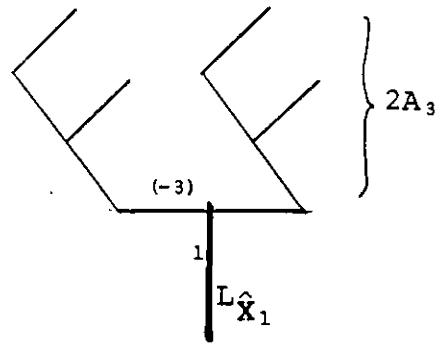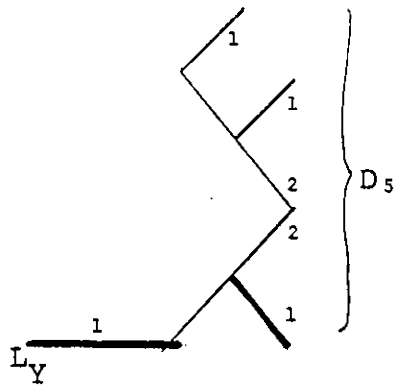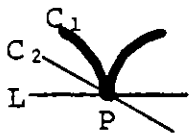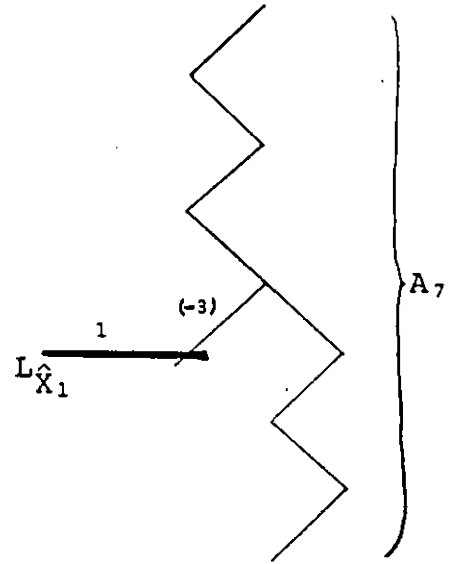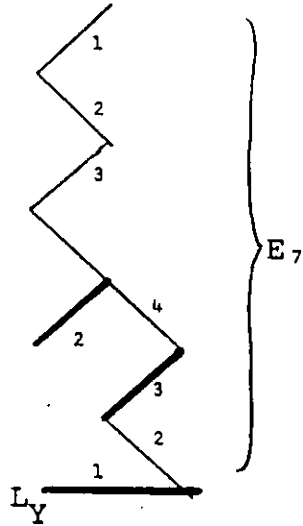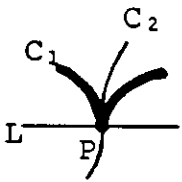
(3.2.6) Case $m_P(\Sigma C_j, L) \geq 2$:

on $\mathbf{P}^2$ Set $a := I(P, C_1 \cap C_2)$

$b := I(P, L \cap C_1)$

$c := I(P, L \cap C_2)$

and assume $b \geq c$, then $3 \geq b \geq c \geq 2$.
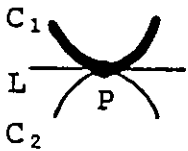
$\alpha_1^* L$ with $\pi_1(\alpha_1\varphi)^* L$ on $\hat{X}_1$,

         multiplicity, $K_{\hat{X}_1}(P)$: bold curves

on $\mathbf{P}^2$ $B_Y(P)$: bold curves   with multiplicity,

                              $(\ )$: self-intersection

$(a;b,c)$

$= (2;3,2)$

C_1

L P

C_2

Case   (a;b,c)

    = (2;2,2):

1

1   2

L_Y 1

$\Big\}\, A_3$

1

$\Big\}\, 2A_1$

1

$L_{\hat{X}_1}$ (-4)

1

$\Big\}\, 2A_1$

Case   (a;b,c)

    = (a;2,2)

with   a ≥ 3:

1

1

2

2

2

L_Y

⋮

2

1

2

1

$\Big\}\, A_{2a-1}$

$\Big\}\, 2A_1$

(-3)

1

$L_{\hat{X}_1}$

⋮

1

$\Big\}\, A_{a-3}$

(-3)

$\Big\}\, 2A_1$

C_2

$L=C_1$ P

1

2

L_Y 2

1

$\Big\}\, A_3$

$L_{\hat{X}_1}$

$\Big\}\, 2A_3$ 0

Case
(a;b,c)
= (3;3,3):

C_1 C_2

L P

L_Y

1 2 3 1 1 2 1

A_5

$L_{\hat{X}_1}$ 1 1 (-4)
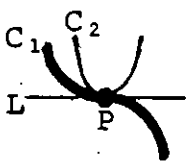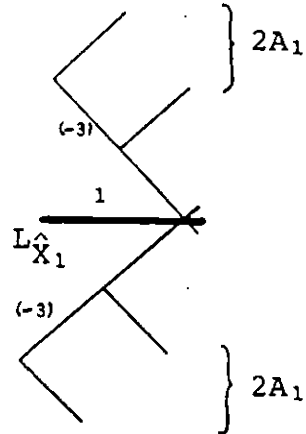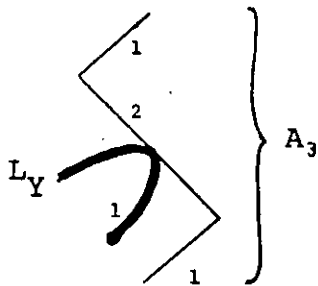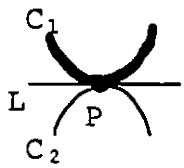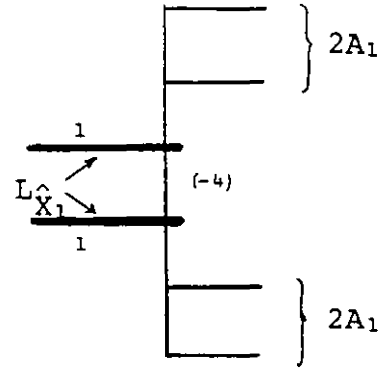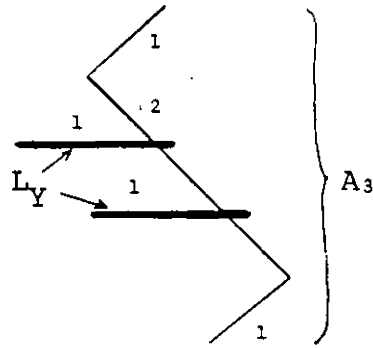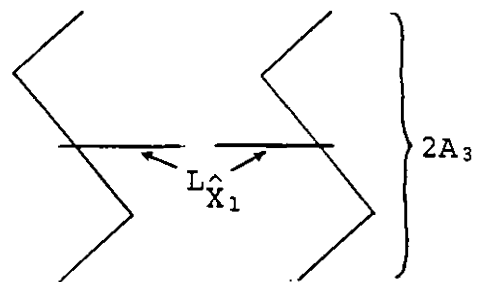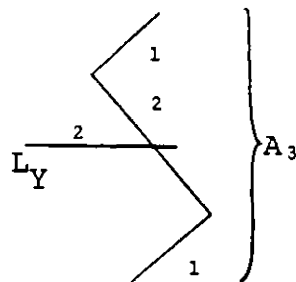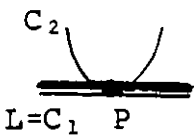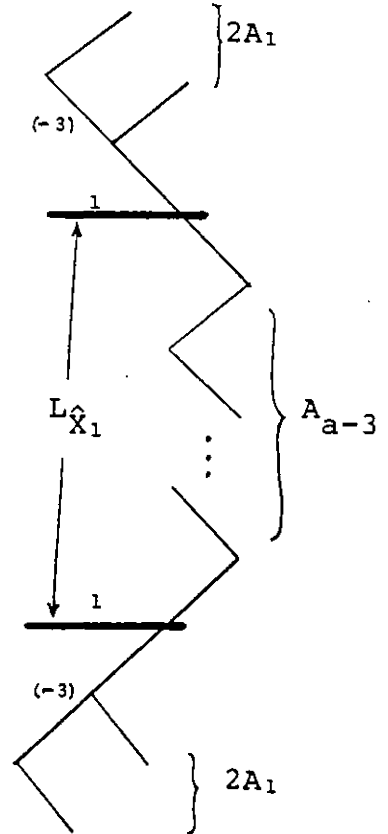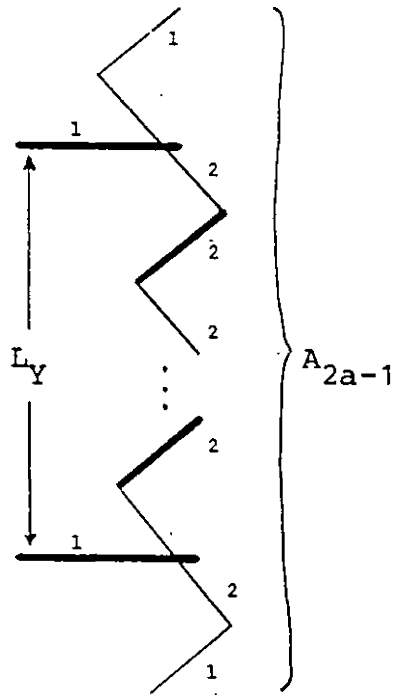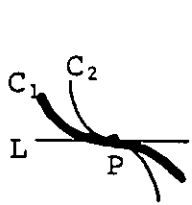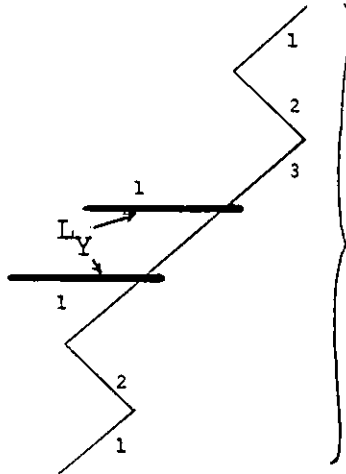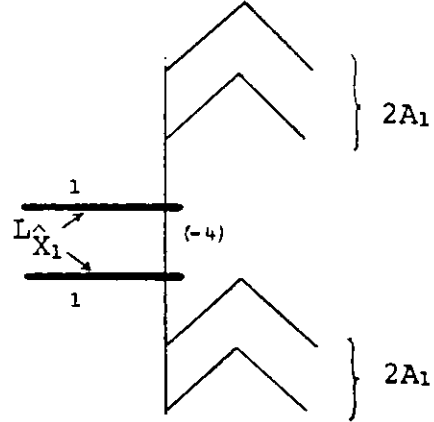
$2A_1$

$2A_1$

Case
(a;b,c)
= (a;3,3)
with a ≥ 4:

1 2 3 1 3 3 3 3 1 2 1

L_Y

A_{2a-1}

$L_{\hat{X}_1}$

(-3) 1 1 (-3)

$2A_1$

A_{a-4}

$2A_1$

C_2

$L = C_1$ P

L_Y

1 2 2 3 2 1

A_5

$L_{\hat{X}_1}$

$2A_5$

0

C_1

$L = C_2$ P

L_Y D_4

$2D_4$ $L_{\hat{X}_1}$

(*)

C_1
C_1' C_1''
L P
C_2

$a = I(P,\ C_1' \cap C_2)$
$\geq I(P,\ L \cap C_2)$
$= 2$

D_{2a+2}

D_a

L_Y

$L_{\hat{X}_1}$

$L_{\hat{X}_1}$ (-3) $2A_3$

(*)

C_1
C_2
L P

L_Y D_6

$L_{\hat{X}_1}$ (-4) $2A_5$

(*)

C_2

P

$C_1' = L$

C_1''

C_1

L_Y 2 3 2 4 3 2 1 D_6

$L_{\hat{X}_1}$ A_7 0

(*)

C_2

P

$C_1' = L$

C_1''

C_1

L_Y 2 4 3 6 5 4 3 2 1 D_8

$L_{\hat{X}_1}$ A_{11} 0

(*)

C_1

$L = C_2$ P

L_Y 2 2 3 4 3 2 1 D_6

$L_{\hat{X}_1}$ $2D_6$ 0

C_1

C_2

P

L

1

1

2

2

2

2

1

L_Y

D_6

(-3) (-3)

1 $L_{\hat{X}_1}$

$2D_5$

C_1

$L=C_2$

P

1

1

2

2

2

2

2

L_Y

D_6

$L_{\hat{X}_1}$

$2D_6$ 0

$(*)$

C_2

C_1''

$C_1 \{$

$C_1'=L$ P

L_Y 2

3

2

4

3

2

1

D_6

$L_{\hat{X}_1}$

$D_4 \{$
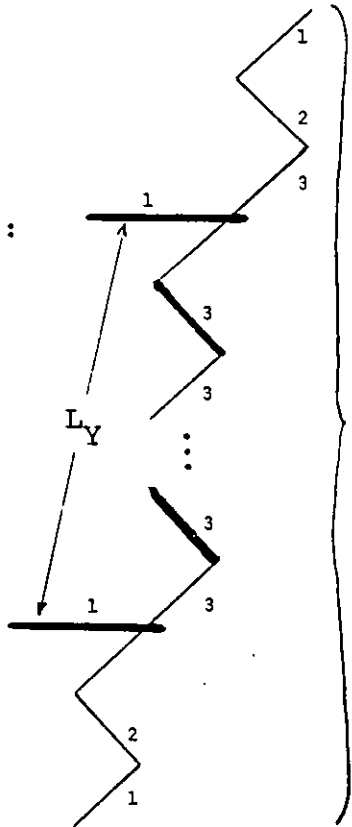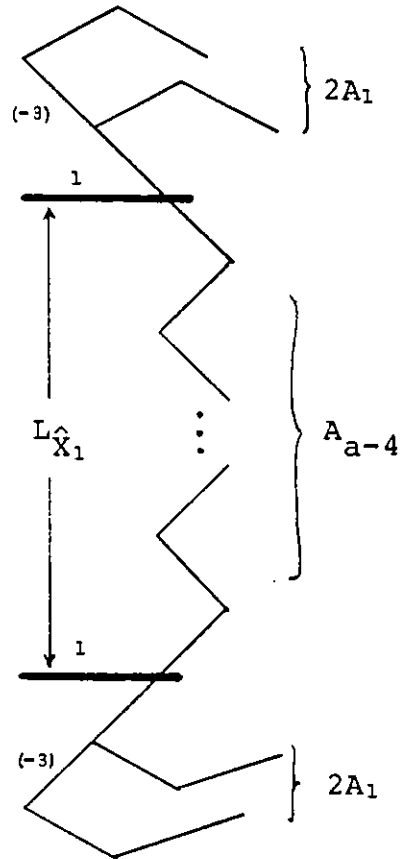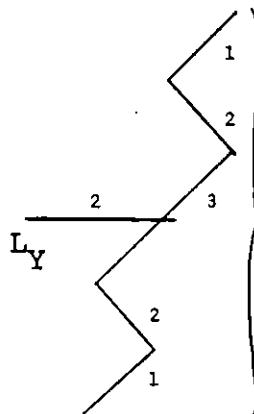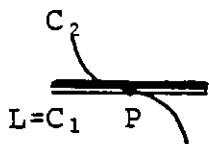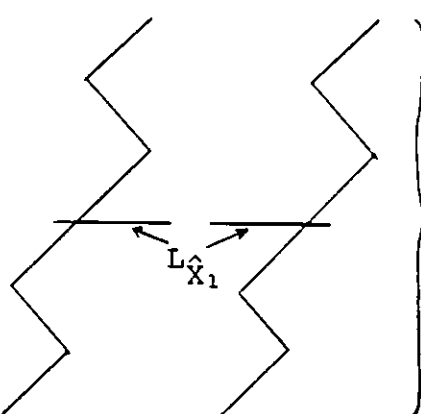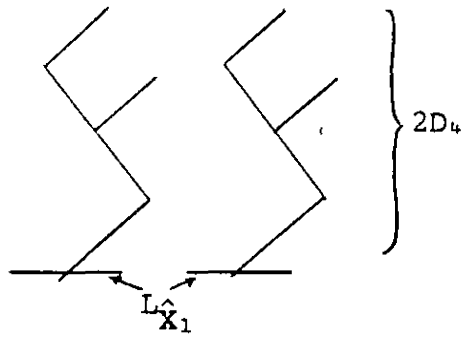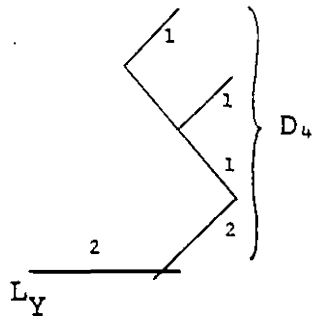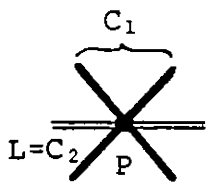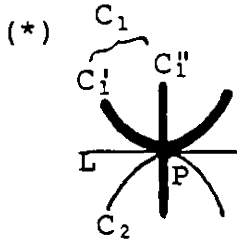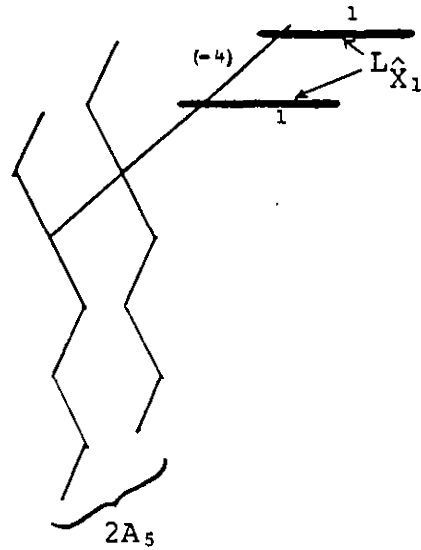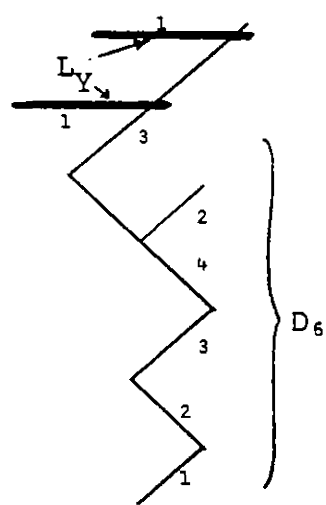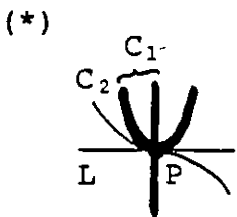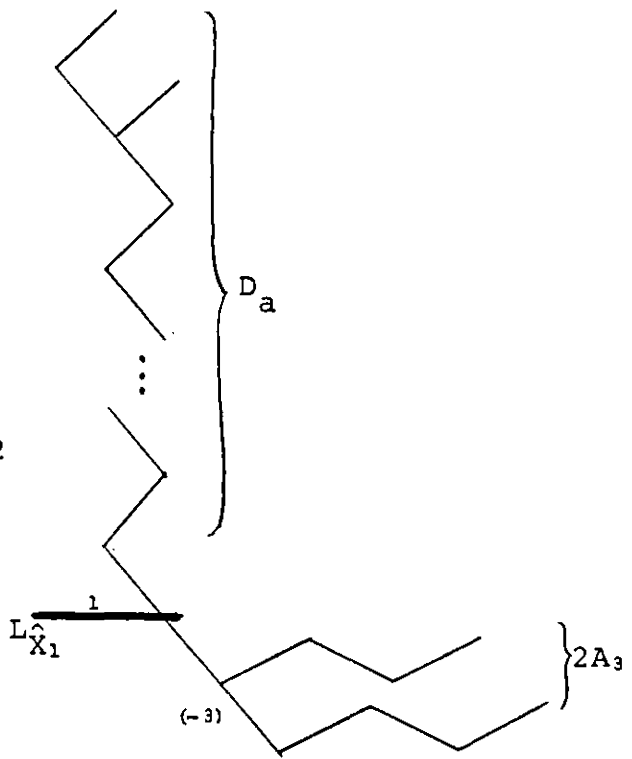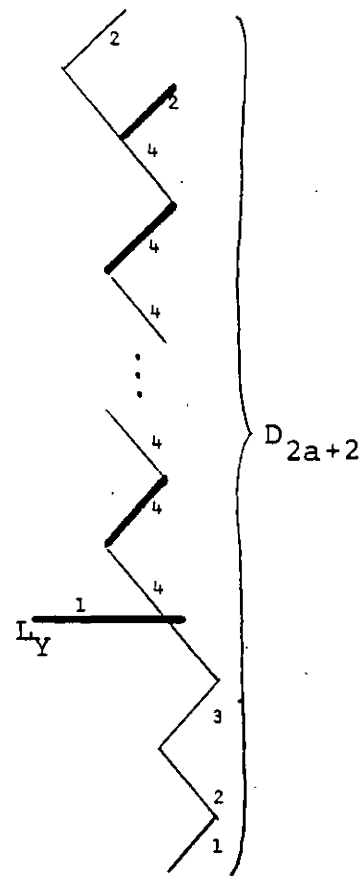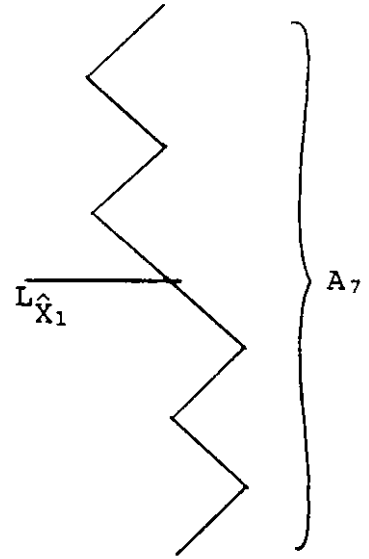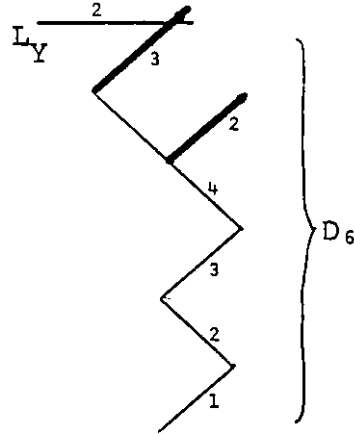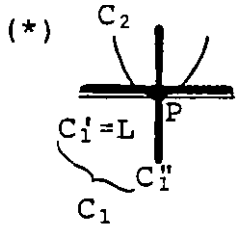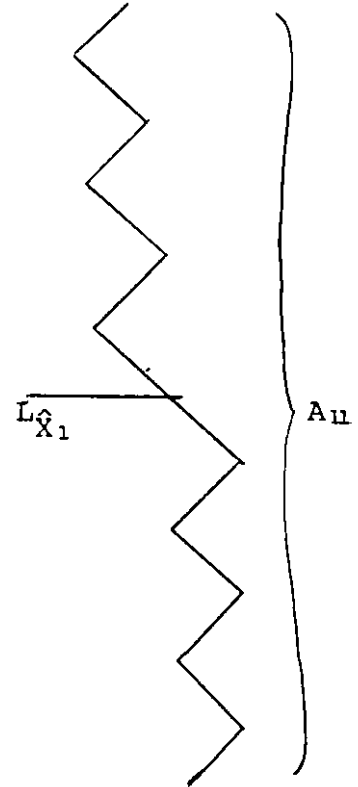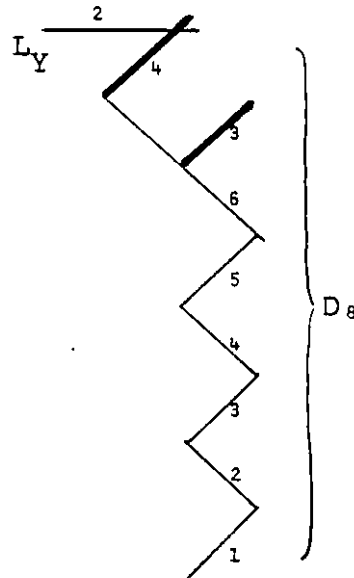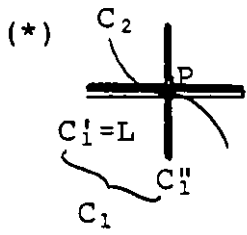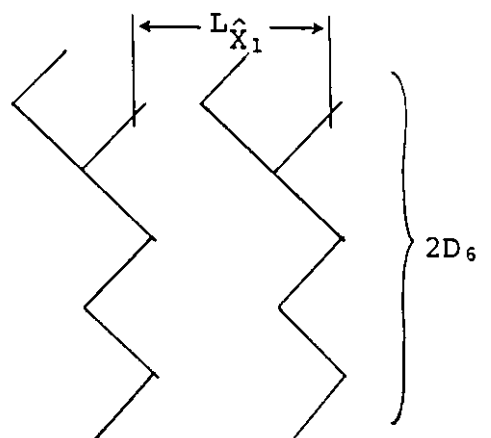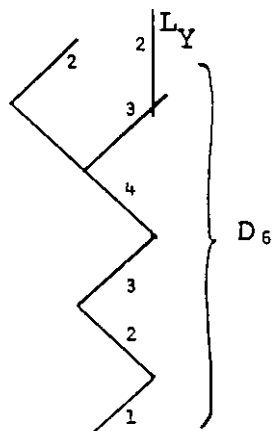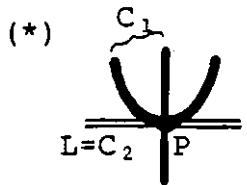
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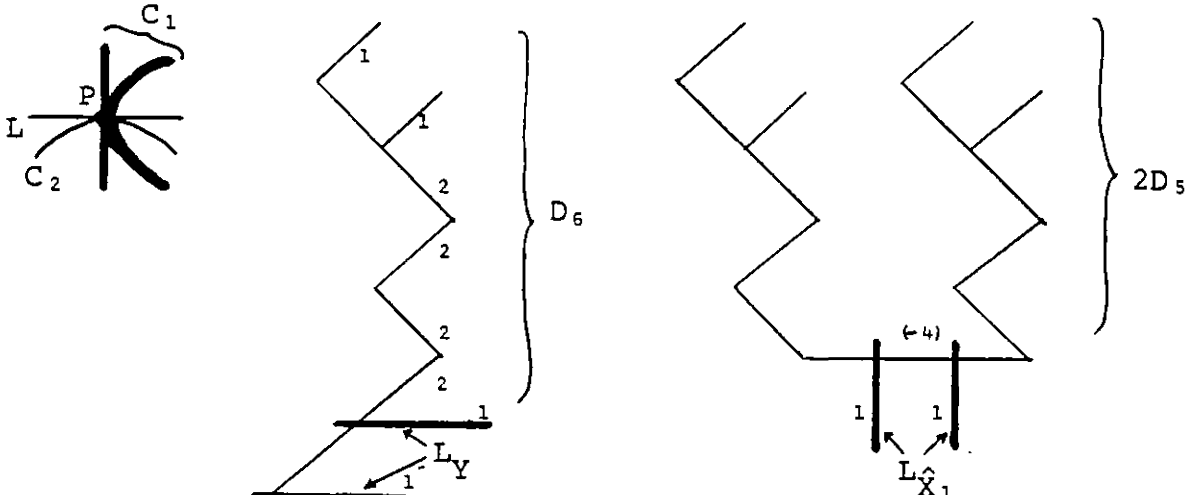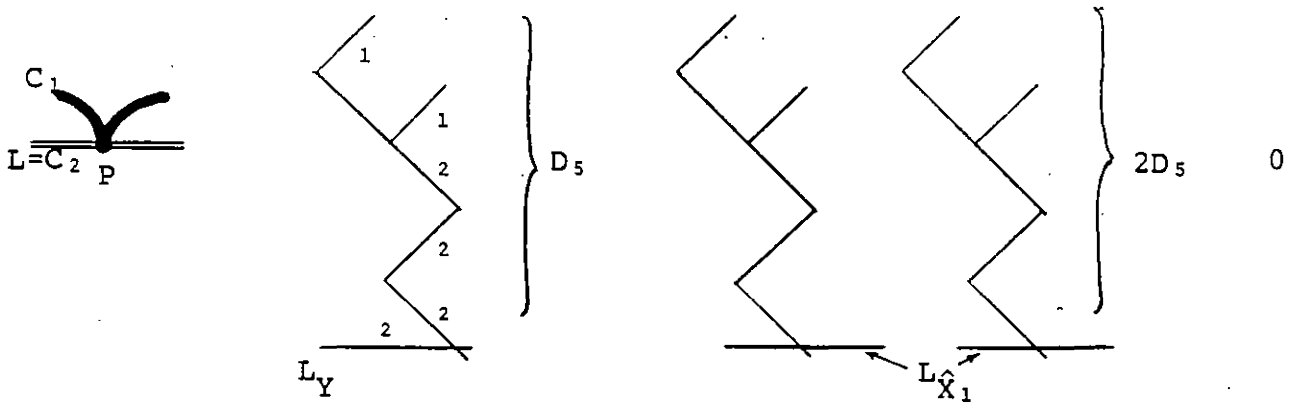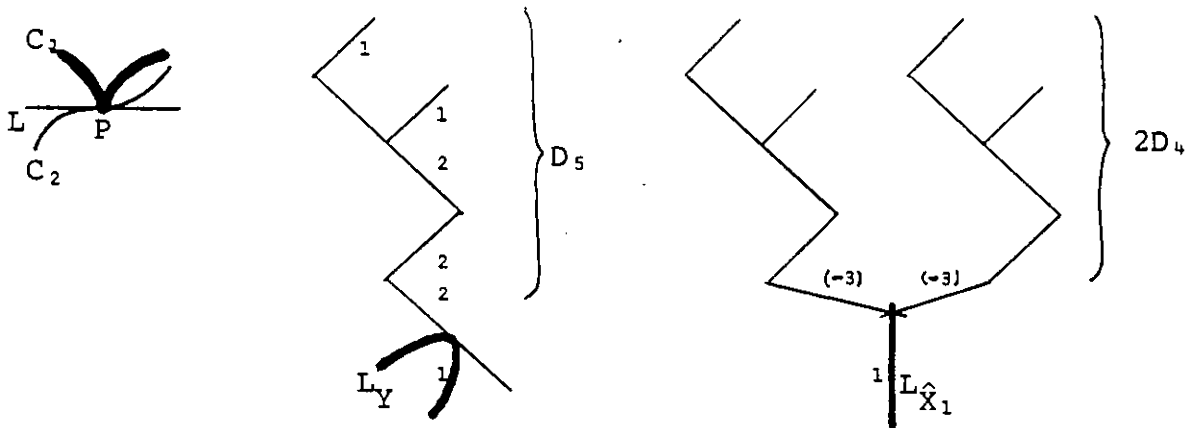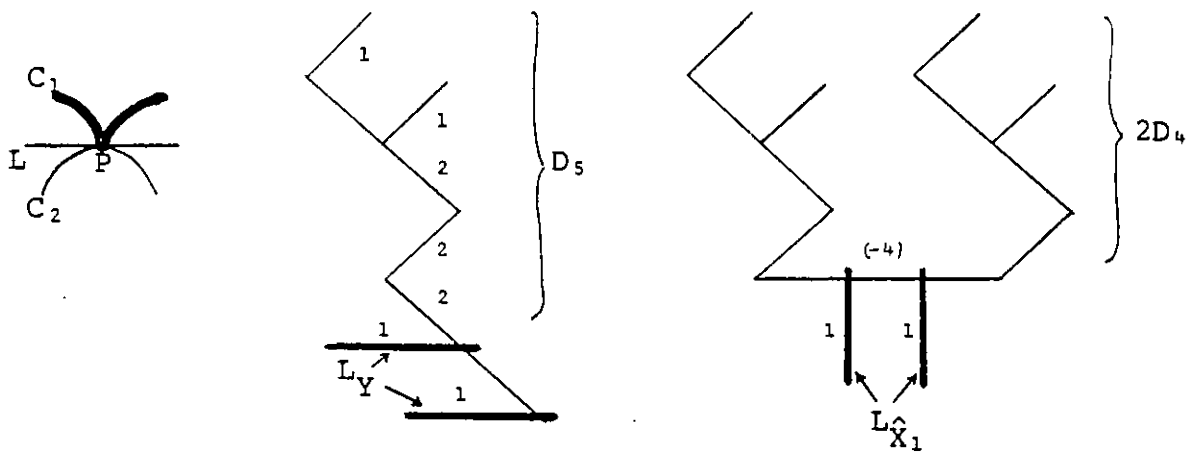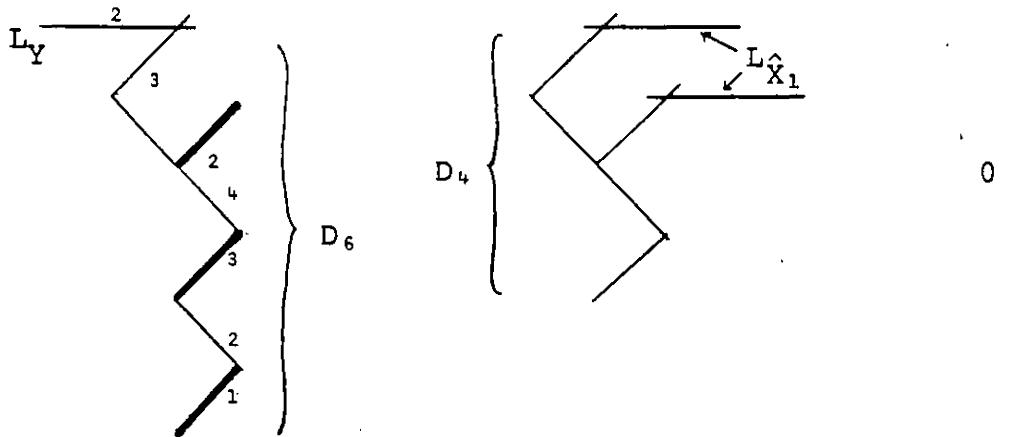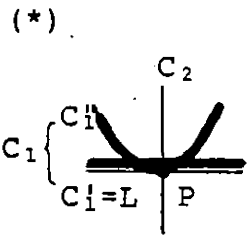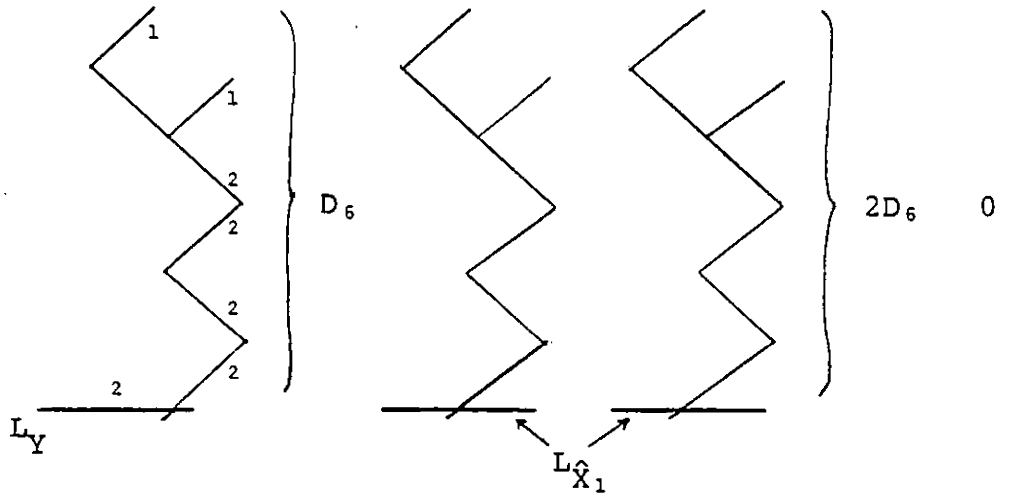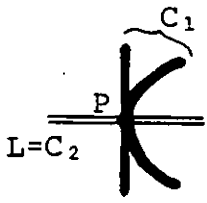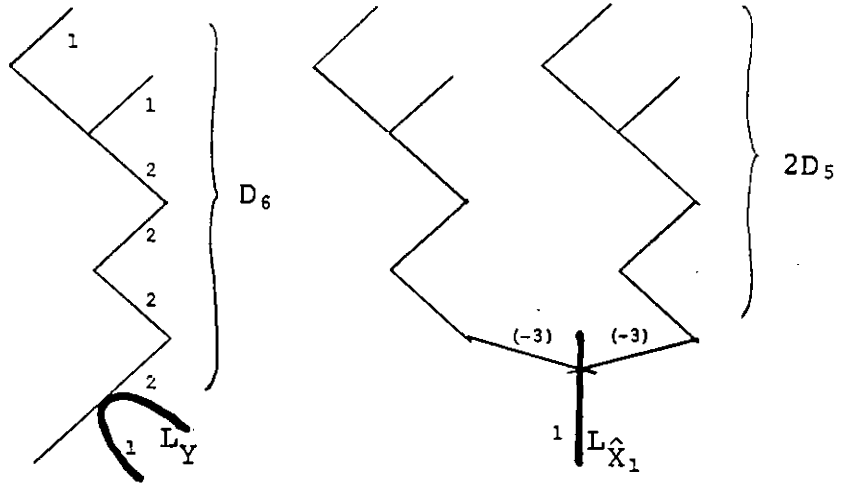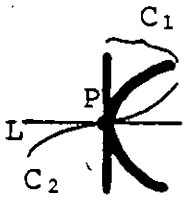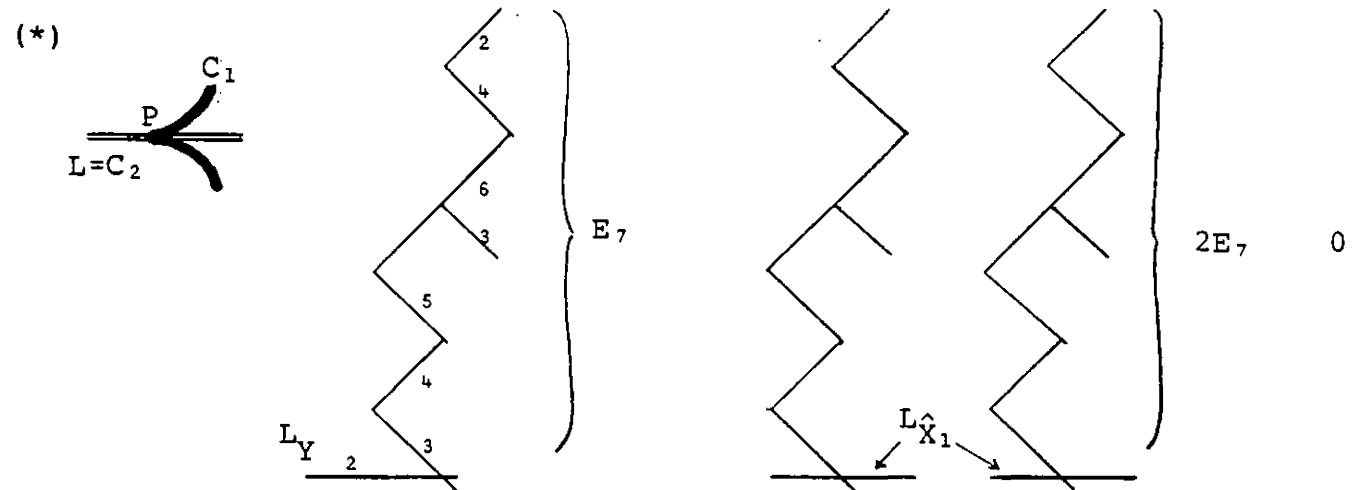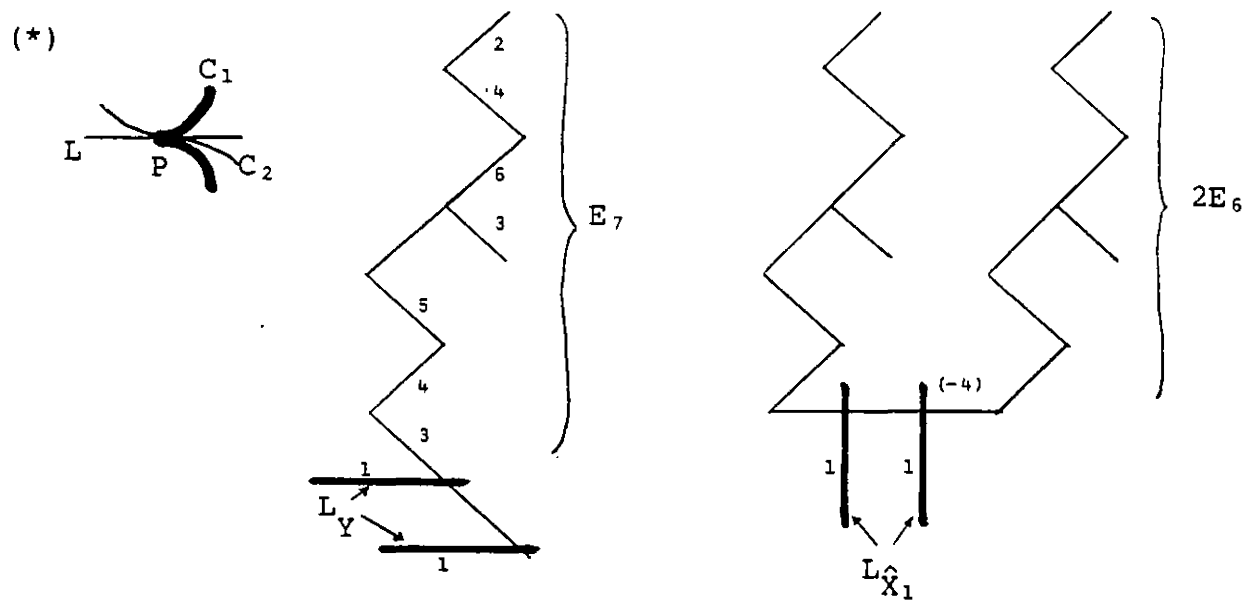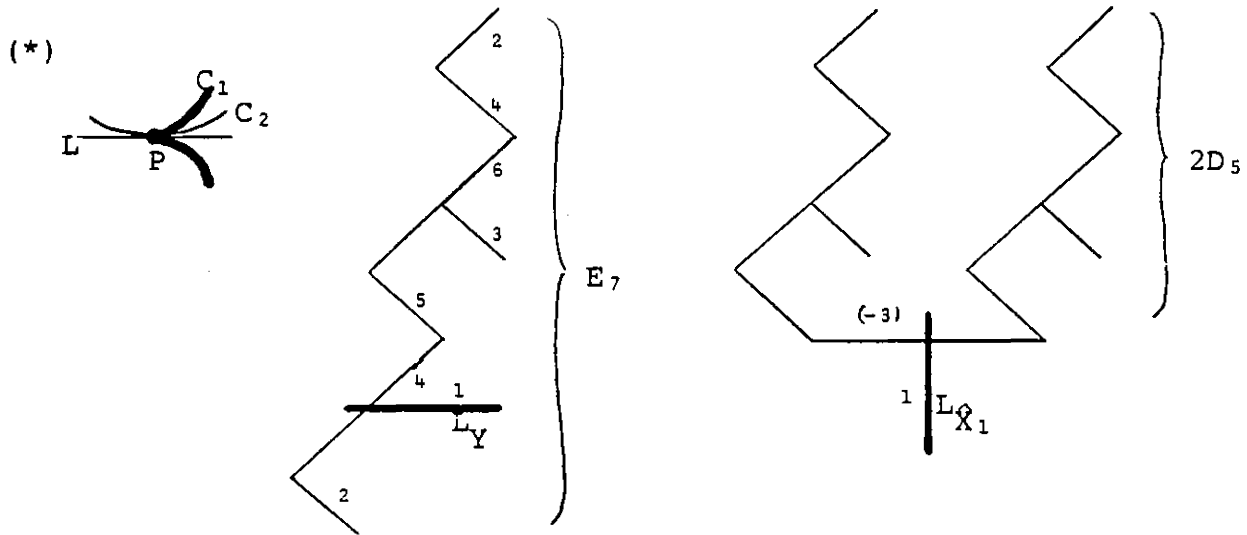
(Curves with (*) are unstable as plane curves of degree 7, cf. [Se].)

(3.3) Observation. We employ the above notation. By Tables in (3.2), we can observe the following:

(1) In all cases, the divisor B_Y on the minimal K3 surface Y has only simple singularity and the canonical divisor $K_{\hat{X}}$ of the minimal model $\hat{X}$ is connected and not multiple.

(2) In the case $m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0$, $\alpha_1^* L$ has only simple singularity on the minimal K3 surface Y and the morphism $\pi : X^* \longrightarrow \hat{X}$ in Diagram (3.1.2) contracts only the nine (-1)-curves coming from the nine distinguished (-2)-curves on Y.

(3.4) Proposition. In the above notation, if $m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0$, the corresponding $\hat{X}$ is the minimal model of Kunev surface.

Proof. We use the notation in Diagram (3.1.2). Denote by $\mathcal{F}$ the line bundle on Y such that $\mathcal{O}_Y(B_Y) = \mathcal{F}^{\otimes 2}$. Then, since $V' = \text{Spec}(\mathcal{O}_Y \oplus \mathcal{F}^{-1})$, we have

$$\chi(\mathcal{O}_{\hat{X}}) = \chi(\mathcal{O}_{V'}) = \chi(\mathcal{O}_Y) + \chi(\mathcal{F}^{-1}) = 2 + \chi(\mathcal{F}).$$

By the Riemann-Roch theorem on Y,

$$\chi(\mathcal{F}) = (\mathcal{F})^2/2 + \chi(\mathcal{O}_Y) = (B_Y)^2/8 + 2 = (2 + 9(-2))/8 + 2 = 0.$$

On the other hand, since $K_{X^*} = \varphi^* \mathcal{F}$ by Observation (3.3.2) and Lemma (1.1.2), we see

$$c_1^2(\hat{X}) = c_1^2(X^*) + 9 = (\varphi^* \mathcal{F})^2 + 9 = 2(\mathcal{F})^2 + 9 = (B_Y)^2/2 + 9$$
$$= (2 + 9(-2))/2 + 9 = 1.$$

Let X be the canonical model of $\hat{X}$. Then, by construction and Observation (3,3,2), the bicanonical map f of $\hat{X}$ is a morphism which factors as

$$f : \hat{X} \xrightarrow{\;f_1\;} X \xrightarrow{\;f_2\;} Y' \xrightarrow{\;f_3\;} P^2$$

where f_1 is a birational morphism and f_2 and f_3 are finite double covers. Hence X is a Kunev surface with an involution σ which is the covering transformation of $f_2 : X \longrightarrow Y'$. QED.

As a corollary, we have the following result, which will be used in Sections 4 and 5:

(3.4.1) Corollary. We use the above notation and the notation in (2.7). For any $(\Sigma C_j, L) \in \mathscr{K}^*$, the corresponding minimal model $\hat{X}$ has $p_g(\hat{X}) = 1$.

Proof. We use the flat family of surfaces $f : \mathfrak{X} \longrightarrow \check{P}^2$ constructed in (2.5). Take a small disk U in $\check{P}^2$ with center $0 = (\Sigma C_j, L) \in \check{P}^2$ such that $(\Sigma C_j) \cap L$ are six nodes for all $t \in U - \{0\}$, and denote by $f_U : \mathfrak{X}_U \longrightarrow U$ the restriction of the family f over U. Then, by construction and Proposition (3.4), the fibers of f_U over all $t \in U - \{0\}$ are desingularizations of Kunev surfaces. Let $\hat{f} : \hat{\mathfrak{X}} \longrightarrow U_r$ be a semi-stable reduction with base extension $U_r \longrightarrow U$ of $f_U : \mathfrak{X}_U \longrightarrow U$ (cf. [Mu]) and let $\hat{f}^{-1}(0) = \Sigma \hat{V}_k$ be the decomposition of the central fiber. Then we see

$$1 \leq p_g(\hat{X}) \leq \Sigma \, p_g(\hat{V}_k) \leq p_g(\hat{f}^{-1}(\hat{t})) = 1 \qquad \text{for } \hat{t} \neq 0.$$

For the first inequality, we use the fact that the minimal model $\hat{X}$ carries a holomorphic 2-form coming from one on a K3 surface Y. The second inequality follows from the fact that there is a component $\hat{V}_k$ dominating X, and the third follows from (1.2.2). QED.

4. Global computation of the branch locus B_Y on Y and the canonical bundle $K_{\hat{X}}$ on $\hat{X}$.

(4.0) We use the notation in (3.0) and (3.1) throughout this section. In this section we shall compute globally the branch locus B_Y on the minimal K3 surface Y and the canonical divisor $K_{\hat{X}}$ of the minimal model $\hat{X}$ of the fiber $X = X_t = f^{-1}(t)$, $t \in \check{P}^2$.

(4.1) We divide the distinguished (-2)-curves E_i $(1 \le i \le 9)$ on Y (cf. Table (3.2.2)) into two types:

Type I. $\alpha_1(E_i) \notin L$.

Type II. $\alpha_1(E_i) \in L$.

Then the branch locus B_Y on Y is divided into

(4.1.1) $B_Y = B_Y(I) + B_Y(L)$

where $B_Y(I)$ is the reduced divisor consisting of the mutually disjoint (-2)-curves of Type I and $B_Y(L)$ is the reduced divisor consisting of the components of B_Y which are mapped to L by α_1. Notice that $B_Y(I)$ is disjont from $B_Y(L)$ and become mutually disjoint (-1)-curves on the canonical resolution X^* and contract to points on $\hat{X}_1$.

(4.2) As for $B_Y(L)$, we can compute it following the procedure of Diagram (3.1.2). Each process is elementary. We give here the tables of the configurations of the two cubics and the line $\Sigma\, C_j + L$ near L on P^2, the divisor $B_Y(L)$ and the number of the components $\#B_Y(I)$ of the divisor $B_Y(I)$ on the minimal K3 surface Y (see (4.1.1)), and the canonical divisor $K_{\hat{X}}$ of the minimal model $\hat{X}$ of X, which is always not multiple by Observation (3.3.1).

(4.2.1) Case $m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0$ (36 types):

on $\mathbf{P}^2$,	$B_Y(L)$,	$\#B_Y(I)$	K_X'
C_1: bold curves	( ): self-intersection		( ): self-intersection

C_1 C_2, L ... (2)		9	(1) genus 2
... (2)		9	(-1) elliptic A_1
... (2)		9	(-1) elliptic A_2
... (0) A_1		9	(-1) A_3 elliptic
... (0) A_1		9	D_4 (-1) elliptic
... (0) A_2		9	(-1) A_5 elliptic
... (0) A_2		9	(-1) elliptic E_6
... (0) A_3		9	(-1) A_7 elliptic

(2)   9   (−3) A_1 A_1

(2)   9 A_1 (−3) A_2

(0) A_1 9   (−3) A_1 A_3

(0) A_1 9 D_4 A_1 (−3)

(0) A_2 9   (−3) A_1 A_5

(0) A_2 9 A_1 (−3) E_6

(0) A_3 9   (−3) A_1 A_7

(2) A_2 (−3) A_2

- 41 -

(4.2.2) Case $m(\Sigma\, C_j,\, L) = 0$, $n(\Sigma\, C_j,\, L) > 0$ (8 + 1 = 9 types):

on $\mathbf{P}^2$, $B_Y(L)$: bold curves, $\#B_Y(I)$ $K_{\hat{X}}$,

C_1: bold curves   ( ): self-intersection   ( ): self-intersection

	(0)	9	elliptic (0)
	(0)	9	(-2)
	(0)	9	(-2)
	(0)	9	(-2)
	(-2)	9	(-2)
	(-2)	9	(-2)
	(-2)	9	(-2)
	(-2)	9	(-2)
	(-2)	9	0

- 45 -

on $\mathbf{P}^2$, $B_Y(L)$: bold curves, $\#B_Y(I)$ $K_{\hat{X}'}$

C_1: bold curves   ( ): self-intersection   ( ): self-intersection

$\overbrace{\quad}^{C_1} \overbrace{\quad}^{C_2}$

L P

$a = I(P, C_1 \cap C_2)$

(0) A_{2a-1} 9 - a   (0)   elliptic

(0)   9 - a   (-2)

(-2)   9 - a   (-2)

(-2)   9 - a   (-2)

(-2)   9 - a   (-2)

(-2)   (-2)   9 - a   (-2)   (-2)

(-2)   (-2)   9 - a   (-2)   (-2)

9 - a

9 - a

9 - a

9 - a

9 - a

9 - a

9 - a

9 - a

(0)

elliptic

(0)

(−2)

8

8

8

8

8

8

8

8

8

8

L $\quad$ C₁′ $\quad$ ... P

$a = I(P, C_1' \cap C_2)$

D_{2a+2}

(0)

8 − a

(0)

elliptic

8 − a

(−2)

(0)

(0)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

(−2)

8 - a

8 - a

8 - a

$C_1^!$ $C_2^!$

L

P

$a = I(P, C_1^! \cap C_2^!)$

D_{2a+2}

8 - a

8 - a

8 - a

8 - a

8 - a

7   elliptic

7

7

7

7

7

(0)

(-2)

(-2)

(-2)

6

6

6

6

(-2)

(-2)

(-2)

(-2)

(4.2.4) Case $m(\Sigma C_j, L) \geq 2$ (69 types):

In this case, all curves in $B_Y(L)$ are (-2)-curves hence $K_{\hat{X}}$ is 0.

on $\mathbf{P}^2$, C_1: bold curves

$B_Y(L)$: bold curves, ( ): $\#B_Y(I)$ self-inter-section

on $\mathbf{P}^2$, C_1: bold curves

$B_Y(L)$: bold curves, ( ): $\#B_Y(I)$ self-inter-section

L

P_1 P_2

$a_i = I(P_i, C_1 \cap C_2)$

(i = 1, 2)

$A_{2a_i - 1}$

$9 - a_1 - a_2$

$8 - a_1$

$7 - a_1$

$7 - a_1$

$6 - a_1$

7

C_2'

L

P

D_{2a+2}

$7 - a$

$a = I(P, C_1 \cap C_2')$

- 52 -

6

6

5

L P_1 P_2 P_3

$9-a_1-a_2-a_3$

$a_i = I(P_i, C_1 \cap C_2)$

$(i = 1, 2, 3)$ A_{2a_i-1}

C_1' C_2'

L

P_1 P_2

$7-a_1-a_2$

$a_1 = I(P_1, C_1' \cap C_2)$

$a_2 = I(P_2, C_1 \cap C_2')$

D_{2a_1+2}

$6-a_1$

$6-a_1$

$5-a_1$

5

5

4

5

4

3

L P

$I(P, C_1 \cap C_2)$

$= a \geq 2$ A_{2a-1}

$9-a$

L P Q

$b = I(Q, C_1 \cap C_2)$

$9-a-b$

A_{2b-1}

7

- 53 -

(*)

6

6

6

6

6

(*)

6

6

6

(*)

6

6

6

6

6

(*)

6

6

6

(*)

6

D_{2a+2}

$$a = I(P, C_1 \cap C_2)$$

Cl

L

P

6 - a

6 - a

$$6 - a$$

$$6 - a$$

$$6 - a$$

(*)

$$5$$

(*)

$$5$$

(*)

$$5$$

(*)

$$5$$

C_1' C_1'

L

P_1 P_2

$a_i = I(P_i, C_1' \cap C_2)$

$(i = 1, 2)$

$D_{2a_i + 2}$

$$6 - a_1 - a_2$$

(Curves with (*) are unstable as plane curves of degree 7, cf. [Se].)

(4.3) As a consequence of the above classification of the branch locus B_Y on Y and the canonical divisor $K_{\hat{X}}$ on $\hat{X}$, we get a proof of Theorem (2.6.3).

Proof of Theorem (2.6.3). We use the above notation and the notation in (2.3). The case $t \in S_0 \cap T_0$ is already settled in Proposition (3.4). In all cases, $p_g(\hat{X}) = 1$ by Corollary (3.4.1). By Lemma (1.1.2), $c_1^2(\hat{X})$ and $q(\hat{X})$ can be computed from the result of the above classification of B_Y. $K_{\hat{X}}$ is always connected and not multiple by Observation (3.3.1) or by the above result of classification. By construction, we see $\dim |2K_{\hat{X}}| = 2 - \max\{m, n\}$ for $t \in S_m \cap T_n$. This together with the value of $c_1^2(\hat{X})$ determines $\kappa(\hat{X})$. Thus we get:

Case	$\kappa(\hat{X})$	$c_1^2(\hat{X})$	$K_{\hat{X}}$	$p_g(\hat{X})$	$q(\hat{X})$	Type of $\hat{X}$
$t \in S_0 \cap T_0$	2	1	nef & big	1	0	Kunev
$t \in S_1$	1	0	connected & not multiple	1	0	numerical K3 with one double fiber
$t \in S_2$	0	0	0	1	0	K3
$t \in T_1 \cap S_0$	1	0	connected & not multiple	1	1	elliptic with $p_g = q = 1$
$t \in T_2$	0	0	0	1	2	abelian

Here we use (1.3.3) and the canonical bundle formula (1.3.1) for determination of the type of $\hat{X}$ in case $\kappa(\hat{X}) = 1$. In fact, for an elliptic fibration $f : \hat{X} \longrightarrow \Delta$ with multiple fibers $m_i F_i$, (1.3.1) says

$$K_{\hat{X}} = f^* Z + \Sigma (m_i - 1) F_i$$

for a divisor Z on the base curve Δ with

$$\deg Z = \chi(\mathcal{O}_{\hat{X}}) - 2\chi(\mathcal{O}_{\Delta}) = \begin{cases} 2g(\Delta) & \text{in case} \quad t \in S_1 \\ 2g(\Delta) - 1 & \text{in case} \quad t \in T_1 \cap S_0 \end{cases}$$

Since $K_{\hat{X}}$ is connected & not multiple, we have the only possibility for the type of the elliptic fibration:

In case $t \in S_1$, there exists unique double fiber and the base curve is rational.

In case $t \in T_1 \cap S_0$, $K_{\hat{X}}$ is a fiber and the base curve is elliptic.   QED.

(4.4) **Remark.** The global classification in (4.2) is clumsy but fruitful. Besides the elementary proof of Theorem (2.6.3) given in (4.3), we can observe, for example, series of degenerations of the canonical curves in each case by Tables in (4.2).

5. Elliptic fibrations in case $m(t) > 0$ or $n(t) > 0$.

(5.0) We continue to use the notation in the previous sections. Throughout this section we assume that $\Sigma C_j \in \mathcal{G}$, i.e., the sum of two cubics ΣC_j on P^2 has at most simple singularity. In the case that the functions $m(t) > 0$ or $n(t) > 0$ (see (2.3)), the pencils of lines through a critical point on P^2 induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$. We shall study these elliptic fibrations in this section. This together with Proposition (3.4) gives another proof of Theorem (2.6.3), which does not depend on the global classification in the previous section.

(5.1)   We first treat the case $n(t) > 0$ and $m(t) = 0$.
Recall that in case $n(t) = 1$ and $m(t) = 0$ one of the cubics on
P^2,   say C_1,   consists of three different lines passing through a
common point P_1 and the line $L = L_t$ also passes P_1 but L is
not a component of C_1 nor passes the triple point of C_2 if exists.
In case $n(t) = 2$ each cubic C_j on P^2 consists of three
different lines passing through a common point P_j ($j = 1, 2$), $P_1 \neq$
P_2, $L = L_t$ is the line joining these two points P_1 and P_2,   and
the seven lines $\Sigma\, C_j + L$ are different.

(5.2) Proposition.   In the notation in (5.1), if $n(t) = 1$
and $m(t) = 0$,   the pencil of lines through P_1 on P^2 induces an
elliptic fibration both on the minimal K3 surface Y and on the
minimal model $\hat{X}$ with section.   The section on Y is a (-2)-curve
and that on $\hat{X}$ is a smooth elliptic curve with self-intersection -1.
These elliptic fibrations have constant J-invariants if and only if
the other cubic C_2 has also a triple point.   In any case, $\hat{X}$ is
an elliptic surface with $\kappa(\hat{X}) = p_g(\hat{X}) = q(\hat{X}) = 1$.

Proof. $p_g(\hat{X}) = 1$ is already known in Corollary (3.4.1).
Let $\check{P}_1$ be the pencil of lines through the point P_1 on P^2.
Following the procedure of Diagram (3.1.2), we shall first prove
that $\check{P}_1$ induces elliptic fibrations on Y and on $\hat{X}$.   Let $\Sigma_0^3\, D_i'$
be the exceptional curves on P^* over the point P_1 on P^2 such
that $D_0' \cdot D_i' = 1$ ($i = 1, 2, 3$).   Then the branch locus B_{P^*} on P^*
becomes $B_{P^*} = D_0' + q^{-1}C_2 + D''$,   where $q^{-1}C_2$ is the proper
transform of C_2 by $q : P^* \longrightarrow P^2$ and D'' is the effective

divisor defined by the above equation. For a line $M \in \check{P}_1$, the
proper transform $q^{-1}M$ intersects with B_p* at four distinct points
provided that M is not contained in C_1 nor passes a singular
point of ΣC_j other than P_1 nor touches C_2. Hence these lines
$M \in \check{P}_1$ become smooth irreducible elliptic curves on Y. This shows
that the pencil of lines $\check{P}_1$ on P^2 induces an elliptic fibration
on Y. This fibration has a section D which is the component of
the ramification divisor on Y lying over D_0'. D is a (-2)-curve.

Since the branch locus on Y is $B_Y = (\alpha_1^*L + \Sigma_1^9 E_i)_{odd,red}$
(see (3.1.3)), the branch locus on Y^* is contained in a finite
number of fibers of the elliptic fibration on Y^*. Therefore the
elliptic fibration on Y^* induces one on X^*. The canonical
divisor K_X* of X^* is contained in $\varphi^*B_Y/2$ (actually they
coincide because B_Y has at most simple singularity, which is a
consequence of the local classification in Section 3). Hence the
exceptional divisor for $\pi : X^* \longrightarrow \hat{X}$ is contained in a finite
number of fibers on X^*. Thus we get an elliptic fibration on $\hat{X}$.

Next we shall prove that the elliptic fibration on $\hat{X}$ has a
section which is a smooth elliptic curve. For this purpose, note
that

$$\alpha_1^{-1}L + 2D + \Sigma_1^3 D_i + F = \alpha_1^*L$$

$$2(\alpha^{-1}C_1)_{red} + 6D + 2\Sigma_1^3 D_i + \Sigma_1^9 E_i + 2G = \alpha_1^*C_1$$

where $\alpha_1^{-1}(\ )$ means the proper transform, D_i is the pull-back of
D_i' on Y ($i = 1, 2, 3$) and F and G are the effective divisors
defined by the above equations. From this we get

(5.2.1) $\quad B_Y = (\alpha_1^*L + \Sigma_1^9 E_i)_{odd,red} = \alpha_1^{-1}L + \Sigma_1^3 D_i + F + \Sigma_1^9 E_i$

$$= \alpha_1^*(L + C_1) - 2(4D + (\alpha_1^{-1}C_1)_{red} + \Sigma_1^3 D_i + G).$$

This shows that B_Y is linearly equivalent to twice of a divisor whose support is contained in a finite number of fibers on Y. This property is preserved on Y^* and we see that $X^* = \mathrm{Spec}(\mathcal{O}_{Y^*} \oplus \mathcal{F}^{-1})$ for a line bundle $\mathcal{F}$ on Y^* whose restriction to a fiber on Y^* is trivial. This implies that the pull-back of the fibers on Y^*, appart from B_{Y^*}, divide into two disjoint copies on X^* hence $D^* := \varphi^* D$ is a section and so is $\hat{D} = \pi D^*$. $\hat{D}$ is isomorphic to D^* and D^* is a smooth elliptic curve with selfintersection -4 on Y^* because D is a (-2)-curve on Y whose neighborhood is isomrphic to one on Y^* and $D^* \longrightarrow D$ is a double cover branched four different points $D \cap (\alpha_1^{-1} L + \Sigma_1^9 D_i)$.

For the assertion on J-invariant on $\hat{X}$, it is enough to show it on Y because most of the fibers on Y devide into two copies on $\hat{X}$. We recall here an elementary fact that for a line $M \in \check{P}_1$ the cross-ratio of the branch points on M, i.e., the points $M \cap C_2$ and P_1, gives the J-invariant of the elliptic curve on Y induced by M up to ordering of the four points (cf., e.g., [C1.2]). It is easy to see that these cross-ratio upto ordering are constant if and only if C_2 is cocurrent three lines. Thus we get our assertion.

We shall now compute $q(\hat{X})$ by using a theorem of Ueno (1.3.2) and the Leray spectral sequence applying to the elliptic fibration $f : \hat{X} \longrightarrow \hat{D}$. In order to check the condition of the above theorem, the only thing we should do is that the elliptic fibration on $\hat{X}$ has singular fiber other than $_m I_0$ in the case that the two cubics C_1 and C_2 are pairs of cocurrent three lines. But in this case we can perform easily the procedure of Diagram (3.1.2) and we see that there are two singular fibers of type I_0^* on $\hat{X}$ coming from the

line joining two triple points P_1 and P_2 on P^2.

Finally we shall prove that the section $\hat{D}$ on $\hat{X}$ has self-intersection -1. By Observation (3.3.1), the canonical bundle $K_{\hat{X}}$ is connected & not multiple. Hence $K_{\hat{X}}$ consists of one fiber by the canonical bundle formula (1.3.1), because we have already known that the base curve, i.e., the section $\hat{D}$, is an elliptic curve and $p_g(\hat{X}) = q(\hat{X}) = 1$. Now $(\hat{D})^2 = -1$ follows from the adjunction formula $(K_{\hat{X}} + \hat{D}) \cdot \hat{D} = \deg K_{\hat{D}} = 0$. QED.

(5.3) Remark. A smooth elliptic curve with self-intersection -1 on a smooth surface is the exceptional divisor of the minimal resolution of a *simple elliptic singularity of type* $\tilde{E}_8$ in the sense of K. Saito (cf. [Sa.K]).

(5.4) Proposition. In the notation in (5.1), if $n(t) = 2$, the minimal model $\hat{X}$ is isomorphic to a product $\hat{D}_1 \times \hat{D}_2$ of two smooth elliptic curves $\hat{D}_j$ $(j = 1, 2)$, whose two trivial elliptic fibrations coincide with those induced by the pencils of lines through the point P_j $(j = 1, 2)$ on P^2.

Proof. In the present case, we can go on the same line as the proof of Proposition (5.2). Actually it is simpler than before because the configuration of the two cubics $\Sigma\, C_j$ and the line L is unique. We do not repeat it here. Consequently the two pencils of lines $\check{P}_j$ through the triple point P_j of C_j induce two elliptic fiber bundles with a section $\hat{D}_j$ coming from the first order infinitely near point of P_j, which becomes a fiber of the

other elliptic fiber bundle $(j = 1, 2)$. Hence the projections induce an isomorphism $\hat{X} \xrightarrow{\sim} \hat{D}_1 \times \hat{D}_2$.   QED.

(5.5) **Remark.** Proposition (5.4) shows that if the two cubics C_1 and C_2 consist of two pairs of cocurrent three lines and $\Sigma\, C_j$ has at most simple singularity then the minimal K3 surface Y is an elliptic Kummer surface associated to the splitting abelian surface $\hat{X} \simeq \hat{D}_1 \times \hat{D}_2$ obtained in that proposition.

(5.6) Next we deal with the case $m(t) > 0$. In this case the sexetic $\Sigma\, C_j$ has at most simple singularity and the line $L = L_t$ passes through common points P_i of two cubics C_1 and C_2. $m(t) = 1$ if and only if the number $\#\{P_i\} = 1$ and L is transversal to one of C_j $(j = 1, 2)$ at P_1.

(5.7) **Proposition.** In the notation in (5.6), if $m(t) = 1$, the pencil of lines through P_1 on P^2 induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$ over a rational curve with non-constant J-invariant. The latter has one double fiber. Hence $\hat{X}$ is a numerical K3 surface with one double fiber.

Proof. $p_g(\hat{X}) = 1$ is already known in Corollary (3.4.1). Let $\check{P}_1$ be the pencil of lines through the point P_1 on P^2. The argument in the present case is similar to that in the proof of Proposition (5.2) but there are some points essentially different, hence we shall write down a full proof.

As before, following the procedure of Diagram (3.1.2), we shall first prove that the pencil of lines $\check{P}_1$ induces elliptic fibrations both on Y and on $\hat{X}$. Let $\Sigma_1 \longrightarrow P^2$ be the blowing-up at the point P_1 and let D_1 be the exceptional curve. Then the pencil of lines $\check{P}_1$ induces the ruling of Σ_1. By construction we have a commutative diagram:
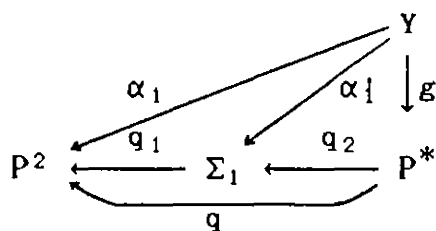
Because of the procedure of the canonical resolution, the proper transform $D' := q_2^{-1}D_1$ does not appear in the branch locus B_{P^*} on P^* if P_1 is a double point of ΣC_j on P^2, while D' remains as a component of B_{P^*} if P_1 is a triple point of ΣC_j. Set

(5.7.1) $B_{P^*} = \Sigma q^{-1}C_j + F'$ in double point case, and

$B_{P^*} = \Sigma q^{-1}C_j + D' + F''$ in triple point case.

Then we see in both cases that, for a fiber M on Σ_1, $q_2^{-1}M$ intersects with B_{P^*} at four distinct points provided that M does not touch $\Sigma q^{-1}C_j$ nor passes a singular point of $\Sigma q^{-1}C_j$. Hence, for these fibers M on Σ_1, α_1^*M are smooth irreducible curve on Y. This shows that the pencil of lines $\check{P}_1$ on P^2 induces an elliptic fibration $Y \xrightarrow{\alpha_1} \Sigma_1 \xrightarrow{pr} D_1$. By the local classification (3.2.5), we can observe that $D := (g^{-1}D')_{red} = \alpha_1^{-1}D_1$, which is a component in the exceptional divisor for α_1 meeting with $\alpha_1^{-1}L$, does not appear in the branch locus $B_Y = (\alpha_1^*L + \Sigma E_i)_{odd,red}$ on Y. Hence the support of B_Y is contained in a finite number of fibers. Therefore the elliptic fibration on Y induces one on X^* then on

$\hat{X}$ by the same argument in the proof of Proposition (5.2).

Next we shall find out the base curve of the elliptic fibration on $\hat{X}$. We observe again the local classification (3.2.5) or its process to get the following:

$$\alpha_1^* L = \alpha_1^{-1} L + g^* D' + E$$

(5.7.2) $\quad \alpha_1^* C_2 = 2(\alpha_1^{-1} C_2)_{red} + g^* D' + F$

$D \subset \Sigma_1^9 E_i$ and $g^* D' = D$ if P_1 is a double point of ΣC_j.

$D \not\subset \Sigma_1^9 E_i$ and $g^* D' = 2D$ if P_1 is a triple point.

Here E and F are effective divisors defined by the above equations. Notice that their supports are contained in a finite number of fibers on Y. From (5.7.2) we get:

$$\begin{aligned}
B_Y &= (\alpha_1^* L + \Sigma_1^9 E_i)_{odd,red} \\
&= \alpha_1^*(L + C_2) - 2((\alpha_1^{-1} C_2)_{red} + g^* D' + G) \\
&\equiv 2(2\alpha_1^* H - (\alpha_1^{-1} C_2)_{red} - g^* D' - G) \\
&\equiv 2(\alpha_1^* H - (\alpha_1^{-1} C_2)_{red} - G')
\end{aligned}$$

where H is a line on P^2 and G and G' are some divisors on Y whose supports are contained in a finite number of fibers. Set $\mathcal{F}' = \mathcal{O}_Y(\alpha_1^* H - (\alpha_1^{-1} C_2)_{red} - G')$. By (5.7.1), we see that the restriction of the line bundle $\mathcal{F}'$ to smooth fibers on Y is non-trivial 2-torsion. This property is preserved by the line bundle $\mathcal{F}$ with $\mathcal{F}^{\otimes 2} = \mathcal{O}_{Y^*}(B_Y*)$ on Y^*. This implies that the pull-back of the fibers on Y^* are still connected on $\hat{X}$. Thus we see that D_1 is the base curve of the elliptic fibration $\hat{X}$.

For the J-invariant, we use the argument of the cross-ratio as in the proof of Proposition (5.2). Note that a smooth fiber on $\hat{X}$ are isogeneous to the corresponding fiber on Y. Hence it is enough to show that the J-invariant on Y is non-constant. If this is

constant, then ΣC_j should contain cocurrent four lines. But this contradicts our assumption that ΣC_j has at most simple singularity.

Now we see $q(\hat{X}) = h^1(\mathcal{O}_{D_1}) = 0$ by the same reasoning in the proof of Proposition (5.2).

As for the multiple fibers on $\hat{X}$, we use the last part of the argument in (4.3).

Thus we get our assertion. QED.

(5.8) Proposition. In the notation in (5.6), if $m(t) \geq 2$, the pencils of lines through P_i on P^2 induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$ over a rational curve with non-constant J-invariant and without multiple fibers. The canonincal divisor $K_{\hat{X}} = 0$, hence $\hat{X}$ is a K3 surface.

$\mathcal{P}roof$. It is enough to show $K_{\hat{X}} = 0$. In fact, we can prove the assertions on elliptic fibrations in the same way as the proof of Proposition (5.7) and the assertion on multiple fibers follow from $K_{\hat{X}} = 0$ by the canonical bundle formula (1.3.1).

In order to see $K_{\hat{X}} = 0$, we devide the cases:

(a) L is a component of ΣC_j.

(b) $m(t) = 2$.

(c) $m(t) = 3$ and not the case (a).

In case (a), $K_{\hat{X}} = 0$ follows from the local classification in Section 3. By the local classification, we can observe that the proper transform $\alpha_i^{-1}L$ on Y is one (-2)-curve (resp. two (-2)-curves) in case (b) (resp. case (c)), and that in both cases B_Y

$= (\alpha_1^* L + \Sigma E_i)_{odd,red}$ consists of disjoint (-2)-curves. From these observations, we get $K_{\hat{X}} = 0$ in these cases. QED.

(5.9) **Remark.** A more sophisticated proof of Proposition (5.8) will be given by using Kulikov's list of degeneration of K3 surfaces ([Ku], [PP]), i.e., by virtue of this list it is enough to show that $\hat{X}$ is a K3 surface in generic case with $m(\Sigma C_j, L) = 2$ and in this case the verification is easy. We omit the details.

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