2000]Primary 52C35, 55P35; Secondary 20F14, 20F36, 20F40

LIE ALGEBRAS ASSOCIATED TO FIBER-TYPE ARRANGEMENTS

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Abstract. Given a hyperplane arrangement in a complex vector space of dimension ℓ, there is a natural associated arrangement of codimension k subspaces in a complex vector space of dimension $k\ell$. Topological invariants of the complement of this subspace arrangement are related to those of the complement of the original hyperplane arrangement. In particular, if the hyperplane arrangement is fiber-type, then, apart from grading, the Lie algebra obtained from the descending central series for the fundamental group of the complement of the hyperplane arrangement is isomorphic to the Lie algebra of primitive elements in the homology of the loop space for the complement of the associated subspace arrangement. Furthermore, this last Lie algebra is given by the homotopy groups modulo torsion of the loop space of the complement of the subspace arrangement. Looping further yields an associated Poisson algebra, and generalizations of the "universal infinitesimal Poisson braid relations."

1. Introduction

Two classical constructions of interest in group theory and topology are:

(i) The Lie algebra arising from the filtration quotients associated to the descending central series of a discrete group G; and

(ii) The Lie algebra of primitive elements in the singular homology of the loop space of a space X, for certain topological spaces X.

The purpose of this article is to illustrate that these two a priori unrelated Lie algebras are, in fact, isomorphic in certain natural cases. This work is motivated by recent results relating the Lie algebras of (i) and (ii) arising in the context of classical configuration spaces, and resolves a conjecture of the second two authors concerning the generalization of these results to spaces arising from certain hyperplane arrangements.

The main result here asserts that the Lie algebra associated to the fundamental group G of the complement of a fiber-type hyperplane arrangement is, up to regrading, isomorphic to the Lie algebra of primitive elements in the homology of the loop space of the complement of a higher dimensional analogue of the arrangement. The main theorem is, in fact, stronger. The Samelson product for the loop space gives rise to a graded Lie algebra given by the homotopy groups modulo torsion. This Lie algebra is, again up to regrading, also isomorphic to the Lie algebra associated to the descending central series quotients of G. In addition, after looping further, there are natural related Poisson algebras arising from the homology of associated iterated loop spaces.

Given a discrete group G, let G_n be the n-th stage of the descending central series, defined inductively by $G_1 = G$, and $G_{n+1} = [G_n, G]$ for $n \geq 1$, and let $E_0^n(G) = G_n/G_{n+1}$ be the n-th associated quotient. Let $E_0^*(G) = \bigoplus_{n\geq 1} E_0^n(G)$ be the Lie algebra obtained from the descending central series of G, with Lie algebra structure induced by the commutator map $G \times G \to G$, $(x,y) \mapsto xyx^{-1}y^{-1}$. For each positive

1991 *Mathematics Subject Classification.* [.

Key words and phrases. fiber-type arrangement, descending central series, loop space homology.

[♭]Partially supported by grants LEQSF(1996-99)-RD-A-04 and LEQSF(1999-2002)-RD-A-01 from the Louisiana Board of Regents, and by grant MDA904-00-1-0038 from the National Security Agency.

[♮]Partially supported by the National Science Foundation.

[♯]Partially supported by the CINVESTAV of the IPN and by the MPI für Mathematik.

integer k, use the ungraded Lie algebra $E_0^*(G)$ to define a related graded Lie algebra as follows.

Definition 1.1. For a group G, let $E_0^*(G)_k$ be the graded Lie algebra given by

$$E_0^q(G)_k = \begin{cases} E_0^n(G) & \text{if } q = 2nk, \\ 0 & \text{otherwise,} \end{cases}$$

with Lie bracket structure induced by that of the Lie algebra $E_0^*(G)$ obtained from the descending central series of G in the obvious manner.

A theorem relating the Lie algebras of (i) and (ii) above is described next. Let P_n be the Artin pure braid group, the fundamental group of the configuration space $F(\mathbb{C}, n)$. The results on configuration spaces alluded to above, due to Fadell and Husseini [7] and Cohen and Gitler [3], may be summarized as follows.

Theorem 1.2. *For $k \geq 1$, the homology of the loop space of the configuration space $F(\mathbb{C}^{k+1}, n)$ is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(P_n)_k$. Moreover,*

(a) *The image of the Hurewicz homomorphism*

$$\pi_*(\Omega F(\mathbb{C}^{k+1}, n)) \to H_*(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z})$$

is isomorphic to $E_0^*(P_n)_k$; and

(b) *The Hurewicz homomorphism induces isomorphisms of graded Lie algebras*

$$\pi_*(\Omega F(\mathbb{C}^{k+1}, n)) / \text{Torsion} \to \operatorname{Prim} H_*(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z}) \cong E_0^*(P_n)_k,$$

where Prim $\bullet$ *denotes the module of primitive elements, and the Lie algebra structure of the source is induced by the classical Samelson product.*

The Lie algebra arising in the above theorem is the "universal Yang-Baxter Lie algebra" $\mathcal{L}(n)$, the quotient of the free Lie algebra on a free abelian group of rank $\binom{n}{2}$ by relations recorded in (4.1) below, and known variously as the "infinitesimal pure braid relations" or the "horizontal four-term relations and framing independence." Furthermore, the homology of an iterated loop space of configuration space, $\Omega^q F(\mathbb{C}^{k+1}, n)$ for $q > 1$, admits the structure of a graded Poisson algebra, see [4]. The associated relations are called the "universal infinitesimal Poisson braid relations." This Poisson algebra structure on $H_*(\Omega^q F(\mathbb{C}^{k+1}, n))$ has recently been used in the context of algebraic groups by Lehrer and Segal [19].

The universal Yang-Baxter Lie algebra, and infinitesimal pure braid relations, arise in a number of contexts. These include the classification of pure braids by Vassiliev invariants, see Kohno [18], and the Knizhnik-Zamolodchikov differential equations from conformal field theory, where the relations appear as integrability conditions on the associated Gauss-Manin connection, see Varchenko [22]. Moreover, any finite dimensional representation of the Lie algebra $\mathcal{L}(n)$ induces a representation of the pure braid group P_n on the same vector space, see Kapovich and Millson [15].

An important ingredient in the proof of Theorem 1.2 is a classical result of Fadell and Neuwirth [8] which shows that configuration spaces admit iterated bundle structure. Similar results are known to hold for certain orbit configuration spaces [24, 2, 6], which admit analogous bundle structure, and are described in more detail below. All of these spaces fit in the following general framework.

For each natural number ℓ, let X_ℓ be a functor from Euclidean spaces, with morphisms restricted to endomorphisms, to topological spaces. For a Euclidean space $\mathbb{E}$, let $\mathcal{Q}_\ell(\mathbb{E})$ be a discrete subset of $\mathbb{E}$ of fixed (possible infinite) cardinality depending on ℓ. Assume that there are natural transformations $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ which satisfy the following conditions.

1. The space $X_1(\mathbb{E}) = \mathbb{E} \setminus \mathcal{Q}_1(\mathbb{E})$ is the complement of a discrete subset of $\mathbb{E}$.
2. The map $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ is a fiber bundle projection, with fiber $\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})$.
3. Each bundle $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ admits a cross-section.
4. If $\mathbb{E} \cong \mathbb{C}$, the fundamental group of $X_{\ell-1}(\mathbb{E})$ acts trivially on the homology of the fiber $\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})$.

The prototypical examples are given by the configuration spaces $X_\ell(\mathbb{E}) = F(\mathbb{E}, \ell)$, where $\mathbb{E} = \mathbb{C}^k$. Further examples are given below.

It seems likely that for many choices of X_ℓ, the Lie algebras associated to $X_\ell(\mathbb{E})$ as $\mathbb{E}$ varies are related in a manner analogous to those arising in Theorem 1.2. If $\mathbb{E} \cong \mathbb{C}$, conditions (1) and (2) imply that $X_\ell(\mathbb{E})$ is a $K(G,1)$ space, where $G = \pi_1(X_\ell(\mathbb{E}))$ is the fundamental group of $X_\ell(\mathbb{E})$, as is readily seen from the homotopy sequence of a bundle. In this case, condition (3) further implies that the group G admits the structure of an iterated semidirect product of free groups, and condition (4) restricts the type of free group automorphisms arising in this structure. These conditions determine the additive structure of the Lie algebra $E_0^*(G)$, see [9] and Section 4. For higher dimensional $\mathbb{E}$, conditions (1)–(3) imply that the homology of the loop space of $X_\ell(\mathbb{E})$ is isomorphic to the universal enveloping algebra of the Lie algebra $\pi_*(\Omega X_\ell(\mathbb{E}))/\,\mathrm{Torsion}$, and determine the additive structure of $\mathrm{Prim}\, H_*(\Omega X_\ell(\mathbb{E}); \mathbb{Z})$, see [6] and Section 5. For higher dimensional $\mathbb{E}$, these conditions have analogous implications for the homology of an iterated loop space $\Omega^q X_\ell(\mathbb{E})$ with $q > 1$, and the Poisson algebra structure admitted by this homology, see [4] and Section 6.

A brief indication how one may analyze and compare the Lie algebras arising for various choices of $\mathbb{E}$ is given next. First, there is a variant of the classical Freudenthal suspension, relating reduced suspensions and loop spaces as indicated below, where the maps are induced by (homology) suspensions.

$$\begin{tikzcd}
H_{2k-2}(\Omega X_\ell(\mathbb{C}^k)) & H_{2k-1}(X_\ell(\mathbb{C}^k)) \arrow[l] \arrow[d] & \\
& H_{2k}(\Sigma X_\ell(\mathbb{C}^k)) \arrow[r] & H_{2k}(\Omega X_\ell(\mathbb{C}^{k+1}))
\end{tikzcd}$$

If $k \geq 2$, conditions (1)–(3) above imply that these maps are all (additive) isomorphisms. In the case $k = 1$, these maps yield an additive isomorphism $E_0^1(G) = H_1(X_\ell(\mathbb{C})) \cong H_2(\Omega X_\ell(\mathbb{C}^2))$ where $G = \pi_1(X_\ell(\mathbb{C}))$. While this comparison does not in general preserve the structures of these Lie algebras, it does provide a geometric way to compare indecomposable elements in these Lie algebras.

To determine the Lie algebra structure, let S be a sphere of appropriate dimension and $A : S \to X_\ell(\mathbb{E})$ a map representing a (reduced) homology generator in minimal degree. Consider the pullback $\xi(\mathbb{E})$ of the bundle $X_{\ell+1}(\mathbb{E}) \to X_\ell(\mathbb{E})$ along the map A:

$$\begin{tikzcd}
\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E}) \arrow[r] \arrow[d, equal] & \xi(\mathbb{E}) \arrow[r] \arrow[d] & S \arrow[d, "A"] \\
\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E}) \arrow[r] & X_{\ell+1}(\mathbb{E}) \arrow[r] & X_\ell(\mathbb{E})
\end{tikzcd}$$

These bundles admit compatible cross-sections by condition (3). There is consequently a morphism of extensions of Lie algebras

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{L}(\xi(\mathbb{E})) \arrow[r] \arrow[d] & \mathcal{L}(S) \arrow[r] \arrow[d, "A_*"] & 0 \\
0 \arrow[r] & \mathcal{L}(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})) \arrow[r] & \mathcal{L}(X_{\ell+1}(\mathbb{E})) \arrow[r] & \mathcal{L}(X_\ell(\mathbb{E})) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{L}(\bullet)$ denotes the Lie algebra obtained from the descending central series of the fundamental group if $\mathbb{E} \cong \mathbb{C}$, and the graded Lie algebra of primitive elements in the homology of the loop space for higher dimensional $\mathbb{E}$. Knowledge of the extension $0 \to \mathcal{L}(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})) \to \mathcal{L}(\xi(\mathbb{E})) \to \mathcal{L}(S) \to 0$ and the map $A_* : \mathcal{L}(S) \to \mathcal{L}(X_\ell(\mathbb{E}))$ for all homology generators completely determines the structure of the Lie algebra $\mathcal{L}(X_{\ell+1}(\mathbb{E}))$. In favorable situations, one can show that the extensions of Lie algebras which arise as $\mathbb{E}$ varies are, apart from grading, isomorphic by carefully combining these considerations with the aforementioned comparison of indecomposables.

Several natural families of examples which fit in the framework described above are given next. These examples either may be or have been studied using (variants of) the techniques sketched above. Let M be a manifold, and Γ a group which acts properly discontinuously on M. The orbit configuration space $F_\Gamma(M, \ell)$ consists of all ℓ-tuples of points in M, no two of which lie in the same Γ-orbit.

First, consider orbit configuration spaces of the form $F_\Gamma(\mathbb{E}\times \mathbb{C}^n, \ell)$, where Γ operates diagonally of $\mathbb{E} \times \mathbb{C}^n$, and trivially on $\mathbb{C}^n$. Relevant examples include the following.

(a) A parameterized lattice Γ acting on $\mathbb{E} = \mathbb{C}$, so that the orbit space is an elliptic curve. The orbit configuration spaces associated to the action of the standard integral lattice were the subject of [6], where it is shown that the analogue of Theorem 1.2 holds for these spaces.

(b) A discrete group Γ acting properly discontinuously on the upper half-plane $\mathbb{E} = \mathbb{H}$, so that the orbit space is a complex curve.

(c) A torsion free subgroup of $\Gamma < Sp(2g, \mathbb{Z})$ acting properly discontinuously on Siegel upper half-space $\mathbb{E} = \mathbb{H}^g$.

(d) A torsion free subgroup Γ of the mapping class group for genus g surfaces, acting on Teichmuller space $\mathbb{E}$.

Second, let $M = \mathbb{C}^k \setminus \{0\}$ and let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ act freely on M by rotations. The orbit configuration spaces $F_\Gamma(M, \ell)$ were the subject of [24] and [2], the results of which combine to show that the analogue of Theorem 1.2 also holds for these spaces.

In the instances where conditions (1)–(4) hold, one obtains generalizations of the universal Yang-Baxter Lie algebra, parameterized by the group Γ. This is the case for the family $F_\Gamma(M, \ell)$ of orbit configuration spaces above, where $M = \mathbb{C}^k \setminus \{0\}$ and $\Gamma = \mathbb{Z}/p\mathbb{Z}$. As noted by D. Matei, the resulting generalized Yang-Baxter Lie algebra with cyclic symmetry is of use in constructing Vassiliev invariants of links in the lens space $L(p, 1)$. The Lie algebras arising from other families of orbit configuration spaces may be of similar use for other three-manifolds, among other potential applications. The orbit configuration spaces $F_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{C}^k \setminus \{0\}, \ell)$, and the classical configuration spaces $F(\mathbb{C}^k, \ell)$, may be realized as complements of finite hyperplane or subspace arrangements. This led to speculation in [6] that similar results may hold for fiber-type arrangements whose complements, like configuration spaces, admit iterated bundle structure.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, a finite collection of codimension one affine subspaces, with complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. See Orlik and Terao [20] as a general reference on arrangements. Given a hyperplane $H \subset \mathbb{C}^\ell$, let H^k be the codimension k affine subspace of $\mathbb{C}^{k\ell} = (\mathbb{C}^\ell)^k$ consisting of all k-tuples of points in $\mathbb{C}^\ell$, each of which lies in H. For each positive integer k, the elements of the hyperplane arrangement $\mathcal{A}$ may be used in this way to obtain an arrangement $\mathcal{A}^k$ of complex codimension k subspaces in $\mathbb{C}^{k\ell}$, with complement $M(\mathcal{A}^k) = \mathbb{C}^{k\ell} \setminus \bigcup_{H\in\mathcal{A}} H^k$.

When $\mathcal{A}$ is a fiber-type hyperplane arrangement, the behavior of the family of spaces $\{X_\ell(\mathbb{C}^k) = M(\mathcal{A}^k), k \geq 1\}$ is reminiscent of that of the family $\{F(\mathbb{C}^k, n), k \geq 1\}$ of configuration spaces. Let $G = \pi_1(M(\mathcal{A}))$ be the fundamental group of the complement of the fiber-type arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, and let $E_0^*(G)$ be the Lie algebra obtained from the descending central series of G. The main result of this article is as follows.

Theorem 1.3. *For $k \geq 1$, the homology of $\Omega M(\mathcal{A}^{k+1})$, the loop space of the complement of the subspace arrangement $\mathcal{A}^{k+1}$ in $\mathbb{C}^{(k+1)\ell}$, is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(G)_k$. Moreover,*

(a) *The image of the Hurewicz homomorphism*

$$\pi_*(\Omega M(\mathcal{A}^{k+1})) \to H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z})$$

is isomorphic to $E_0^*(G)_k$; and

(b) *The Hurewicz homomorphism induces isomorphisms of graded Lie algebras*

$$\pi_*(\Omega M(\mathcal{A}^{k+1}))/\,\text{Torsion} \to \operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z}) \cong E_0^*(G)_k,$$

where the Lie algebra structure of the source is induced by the Samelson product.

The paper is organized as follows.

§2. Given a hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^\ell$, there is an associated arrangement of codimension k subspaces $\mathcal{A}^k \subset \mathbb{C}^{k\ell}$. The combinatorics and topology of the subspace arrangement $\mathcal{A}^k$ are studied in this section.

§3. The topology of the subspace arrangement $\mathcal{A}^k$, in the instance where the underlying hyperplane arrangement $\mathcal{A}$ is fiber-type, is further studied in this section.

§4. The (known) structure of the Lie algebra $E_0^*(G)$ associated to the descending central series of the fundamental group $G = \pi_1(M(\mathcal{A}))$ of the complement of a fiber-type hyperplane arrangement $\mathcal{A}$ is analyzed in this section.

§5. The structure of the Lie algebra of primitive elements in the homology of the loop space of the complement of the subspace arrangement $\mathcal{A}^k$ is analyzed in this section, and the isomorphisms of graded Lie algebras asserted in Theorem 1.3 are established.

§6. The Poisson algebra structure on the homology of an iterated loop space of the complement of the subspace arrangement $\mathcal{A}^k$ is briefly analyzed in this section.

2. Redundant Arrangements

Let H be an affine hyperplane in $\mathbb{C}^\ell$, an affine subspace of codimension one. For each positive integer k, there is an affine subspace H^k of codimension k in $\mathbb{C}^{k\ell}$ obtained from H in the following manner. Choose coordinates $\mathbf{x} = (x_1, \dots, x_\ell)$ on $\mathbb{C}^\ell$, and $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ on $\mathbb{C}^{k\ell} = \mathbb{C}^\ell \times \cdots \times \mathbb{C}^\ell$, where for each i, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,\ell}) \in \mathbb{C}^\ell$. Then, if the hyperplane H in $\mathbb{C}^\ell$ is given by $H = \{\mathbf{x} \in \mathbb{C}^\ell \mid \sum_{j=1}^{\ell} a_j x_j = b\}$, define a codimension k affine subspace H^k in $\mathbb{C}^{k\ell}$ by $H^k = \{(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k\ell} \mid \sum_{j=1}^{\ell} a_j x_{i,j} = b,\ 1 \leq i \leq k\}$.

Now let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, a finite collection of (affine) hyperplanes. Via the above process, there is an arrangement $\mathcal{A}^k = \{H^k \mid H \in \mathcal{A}\}$ of codimension k affine subspaces in $\mathbb{C}^{k\ell}$ obtained from $\mathcal{A}$. For evident reasons, call the subspace arrangement $\mathcal{A}^k$ *redundant*. A brief description of the relationship between the combinatorics and topology of the hyperplane arrangement $\mathcal{A} = \mathcal{A}^1$ and the redundant subspace arrangement $\mathcal{A}^k$ is given in this section.

Example 2.1. Let $\mathcal{A}_n$ be the braid arrangement in $\mathbb{C}^n$, consisting of the hyperplanes $H_{i,j} = \{\mathbf{x} \in \mathbb{C}^n \mid x_i = x_j\}$. As is well known, the complement $M(\mathcal{A}_n) = F(\mathbb{C}, n)$ is the configuration space of n points in $\mathbb{C}$.

For each positive integer k, the associated redundant arrangement $\mathcal{A}_n^k$ consists of subspaces $H_{i,j}^k = \{(\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbb{C}^n)^k \mid x_{p,i} = x_{p,j},\ 1 \leq p \leq k\}$. These subspaces may be realized as $H_{i,j}^k = \{(\mathbf{y}_1, \dots, \mathbf{y}_n) \in (\mathbb{C}^k)^n \mid \mathbf{y}_i = \mathbf{y}_j\}$. Thus the complement $M(\mathcal{A}_n^k) = F(\mathbb{C}^k, n)$ is the configuration space of n points in $\mathbb{C}^k$.

For an arbitrary hyperplane arrangement $\mathcal{A}$, and for each k, let $\mathsf{L}(\mathcal{A}^k)$ be the intersection poset of the arrangement $\mathcal{A}^k$, the partially ordered set of non-empty multi-intersections of elements of $\mathcal{A}^k$. Order $\mathsf{L}(\mathcal{A}^k)$ by reverse inclusion, and include the ambient space $\mathbb{C}^{k\ell}$ in $\mathsf{L}(\mathcal{A}^k)$ as the minimal element, corresponding to the intersection of no elements of $\mathcal{A}^k$. For the hyperplane arrangement $\mathcal{A} = \mathcal{A}^1$, it is known that $\mathsf{L}(\mathcal{A})$ is a geometric poset, see [20, Section 2.3]. This need not be the case for an arbitrary subspace arrangement. However, for redundant arrangements, the following holds.

Proposition 2.2. *If $\mathcal{A}$ is a hyperplane arrangement, then $\mathsf{L}(\mathcal{A}) \cong \mathsf{L}(\mathcal{A}^k)$ for all k.*

Proof. It will be shown that the bijection between $\mathcal{A}$ and $\mathcal{A}^k$ given by $H \leftrightarrow H^k$ induces an isomorphism of posets $\mathsf{L}(\mathcal{A}) \cong \mathsf{L}(\mathcal{A}^k)$. To establish this, it suffices to show that to each codimension r flat $X \in \mathsf{L}(\mathcal{A})$ there corresponds a codimension kr flat $X^k \in \mathsf{L}(\mathcal{A}^k)$.

Write $\mathcal{A} = \{H_1, \dots, H_n\}$, where $H_i = \left\{\mathbf{x} \in \mathbb{C}^\ell \mid \sum_{j=1}^{\ell} a_{i,j}x_j = b_i\right\}$, and let $X = H_1 \cap \cdots \cap H_m$. The flat X may be realized as the set of solutions of the system of equations $A\mathbf{x} = \mathbf{b}$, where $A = (a_{i,j})$ is $m \times \ell$ and $\mathbf{b} = (b_1, \dots, b_m)$. Then, X has codimension r in $\mathbb{C}^\ell$ if and only if $\operatorname{rank}[A \mid \mathbf{b}] = \operatorname{rank} A = \ell - r$ if and only if

$$\operatorname{rank}\begin{bmatrix} A & & & \mid & \mathbf{b} \\ & A & & \mid & \mathbf{b} \\ & & & \mid & \\ & & A & \mid & \mathbf{b} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & & \\ & A & \\ & & \\ & & A \end{bmatrix} = k(\ell - r)$$

if and only if $X^k = H_1^k \cap \cdots \cap H_m^k$ has codimension kr in $\mathbb{C}^{k\ell}$. □

For each k, let $M(\mathcal{A}^k) = \mathbb{C}^{k\ell} \setminus \bigcup_{H^k \in \mathcal{A}^k} H^k$ denote the complement of the (subspace) arrangement $\mathcal{A}^k$. In the case $k = 1$, the cohomology of the hyperplane complement $M(\mathcal{A}) = M(\mathcal{A}^1)$ is well known. It is isomorphic to the Orlik-Solomon algebra $\mathsf{A}(\mathcal{A})$, see [20, Sections 3.1, 3.2]. A family of algebras which includes $\mathsf{A}(\mathcal{A})$ is defined next.

For each positive integer k, let $\mathsf{E}_{2k-1}[k] = \bigoplus_{H \in \mathcal{A}} \mathbb{Z}e_H^k$ be a free $\mathbb{Z}$-module generated by degree $2k-1$ elements e_H^k in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let $\mathsf{E}[k] = \bigwedge \mathsf{E}_{2k-1}[k]$ be the exterior algebra of $\mathsf{E}_{2k-1}[k]$, and denote by $\mathsf{I}[k]$ the ideal of $\mathsf{E}[k]$ generated by the homogeneous elements

$$\sum_{p=1}^{q} (-1)^{p-1} e_{H_1}^k \wedge \cdots \widehat{e_{H_p}^k} \cdots \wedge e_{H_q}^k \quad \text{if} \quad 0 \le \operatorname{codim} H_1 \cap \cdots \cap H_q < q,$$

$$e_{H_1}^k \wedge \cdots \wedge e_{H_q}^k \quad \text{if} \quad H_1 \cap \cdots \cap H_q = \emptyset.$$

Let $\mathsf{A}[k] = \mathsf{E}[k]/\mathsf{I}[k]$. The Orlik-Solomon algebra is then given by $\mathsf{A}(\mathcal{A}) = \mathsf{A}[1]$.

Proposition 2.2 may be used to determine the cohomology of $M(\mathcal{A}^k)$ for $k > 1$ in terms of that of $M(\mathcal{A})$. Let $P(\mathcal{A}^k, t) = \sum_{q \ge 0} b_q(M(\mathcal{A}^k)) \cdot t^q$ be the Poincaré polynomial of $M(\mathcal{A}^k)$, where $b_q(X)$ is the q-th Betti number of X. Results of Goresky and MacPherson [12], and Yuzvinsky [23], see also Feichtner and Ziegler [11], together with Proposition 2.2, yield the following.

Corollary 2.3. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$.*

1. *For each k, the integral (co)homology of $M(\mathcal{A}^k)$ is torsion free, and we have $P(\mathcal{A}^k, t) = P(\mathcal{A}, t^{2k-1})$.*
2. *For each k, the cohomology algebra of $M(\mathcal{A}^k)$ isomorphic to the algebra $\mathsf{A}[k]$, $H^*(M(\mathcal{A}^k); \mathbb{Z}) \cong \mathsf{A}[k]$.*

An explicit basis for the first non-zero (reduced) homology group, $H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$, of the complement of the subspace arrangement $\mathcal{A}^k$ is recorded next. Let $L \subset \mathbb{C}^\ell$ be a complex line that is transverse to the hyperplane arrangement $\mathcal{A}$. Write $L = \{t \cdot \mathbf{u} + \mathbf{v}\}$

where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^\ell$ are fixed and $t \in \mathbb{C}$ varies. For each hyperplane H of $\mathcal{A}$, the intersection $L \cap H$ is a point, say $\mathbf{q}_H = \tau_H \cdot \mathbf{u} + \mathbf{v}$ for some $\tau_H \in \mathbb{C}$. The following is immediate.

Lemma 2.4. *The subspace $L^k = \{(t_1 \cdot \mathbf{u} + \mathbf{v}, \ldots, t_k \cdot \mathbf{u} + \mathbf{v}) \mid t_1, \ldots, t_k \in \mathbb{C}\}$ of $\mathbb{C}^{k\ell}$ is transverse to the subspace arrangement $\mathcal{A}^k \subset \mathbb{C}^{k\ell}$. For each subspace H^k of $\mathcal{A}^k$, the intersection $L^k \cap H^k$ is the point $(\mathbf{q}_H, \ldots, \mathbf{q}_H) = (\tau_H \cdot \mathbf{u} + \mathbf{v}, \ldots, \tau_H \cdot \mathbf{u} + \mathbf{v})$.*

Let S^{2k-1} be the unit sphere in $\mathbb{C}^k$. For $\epsilon > 0$ sufficiently small, the point

$$\big((\tau_H + \epsilon' z_1) \cdot \mathbf{u} + \mathbf{v}, \ldots, (\tau_H + \epsilon' z_k) \cdot \mathbf{u} + \mathbf{v}\big) \in L^k$$

lies in the intersection $L^k \cap M(\mathcal{A}^k)$ for all ϵ', $0 < \epsilon' \leq \epsilon$. Fix such an ϵ, and define a map $c_H^k : S^{2k-1} \to L^k \cap M(\mathcal{A}^k)$ using the above formula:

$$c_H^k(\mathbf{z}) = c_H^k(z_1, \ldots, z_k) = \big((\tau_H + \epsilon z_1) \cdot \mathbf{u} + \mathbf{v}, \ldots, (\tau_H + \epsilon z_k) \cdot \mathbf{u} + \mathbf{v}\big). \tag{2.1}$$

Let ι_{2k-1} be the fundamental class of $H_{2k-1}(S^{2k-1}; \mathbb{Z})$, and denote the image of $(c_H^k)_*(\iota_{2k-1}) \in H_{2k-1}(L^k \cap M(\mathcal{A}^k); \mathbb{Z})$ under the map induced by the natural inclusion $L^k \cap M(\mathcal{A}^k) \hookrightarrow M(\mathcal{A}^k)$ by $C_H^k \in H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$.

Proposition 2.5. *The classes $\{C_H^k \mid H \in \mathcal{A}\}$ form a basis for $H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$.*

Proof. For $H \in \mathcal{A}$, define $p_H^k : L^k \cap M(\mathcal{A}^k) \to S^{2k-1}$ by

$$p_H^k(t_1 \cdot \mathbf{u} + \mathbf{v}, \ldots, t_k \cdot \mathbf{u} + \mathbf{v}) = \frac{\mathbf{t} - \tau_H \cdot \mathbf{e}}{\|\mathbf{t} - \tau_H \cdot \mathbf{e}\|},$$

where $\mathbf{t} = (t_1, \ldots, t_k)$ and $\mathbf{e} = (1, \ldots, 1)$ are in $\mathbb{C}^k$. It is then readily checked that $p_H^k \circ c_H^k = \mathrm{id} : S^{2k-1} \to S^{2k-1}$ is the identity map. Furthermore, if $H' \neq H$ is another hyperplane of $\mathcal{A}$, the composition $p_H^k \circ c_{H'}^k$ is given by

$$p_H^k \circ c_{H'}^k(\mathbf{z}) = \frac{\mathbf{z} + \frac{1}{\epsilon}(\tau_{H'} - \tau_H) \cdot \mathbf{e}}{\|\mathbf{z} + \frac{1}{\epsilon}(\tau_{H'} - \tau_H) \cdot \mathbf{e}\|},$$

so is null-homotopic. Consequently, the classes $(c_H^k)_*(\iota_{2k-1}) \in H_{2k-1}(L^k \cap M(\mathcal{A}^k); \mathbb{Z})$ form a basis. Finally, using stratified Morse theory, one can show that the relative homology group $H_i(M(\mathcal{A}^k), L^k \cap M(\mathcal{A}^k); \mathbb{Z})$ vanishes for $i < 4k-2$, see [12, Parts II, III]. It follows that the natural inclusion $L^k \cap M(\mathcal{A}^k) \hookrightarrow M(\mathcal{A}^k)$ induces an isomorphism $H_{2k-1}(L^k \cap M(\mathcal{A}^k); \mathbb{Z}) \xrightarrow{\sim} H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$. So the classes C_H^k form a basis for $H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$ as asserted. □

Remark 2.6. The cohomology classes $(C_H^k)^* \in H^{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$ dual to the classes $C_H^k \in H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$ generate the cohomology algebra $H^*(M(\mathcal{A}^k); \mathbb{Z})$. Let $a_H^k \in \mathsf{A}[k]$ denote the image of $e_H^k \in \mathsf{E}[k]$ under the natural projection. Then the map $H^{2k-1}(M(\mathcal{A}^k); \mathbb{Z}) \to \mathsf{A}_{2k-1}[k]$, $(C_H^k)^* \mapsto a_H^k$, induces an isomorphism of algebras $H^*(M(\mathcal{A}^k); \mathbb{Z}) \xrightarrow{\sim} \mathsf{A}[k]$, see Corollary 2.3.

3. Linearly Fibered Arrangements

In this section, the topology of those redundant arrangements arising from strictly linearly fibered and fiber-type hyperplane arrangements is studied further. Recall the definition of arrangements of the former type from [9, 20].

Definition 3.1. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^{\ell+1}$ is *strictly linearly fibered* if there is a choice of coordinates $(\mathbf{x}, z) = (x_1, \ldots, x_\ell, z)$ on $\mathbb{C}^{\ell+1}$ so that the restriction, p, of the projection $\mathbb{C}^{\ell+1} \to \mathbb{C}^\ell$, $(\mathbf{x}, z) \mapsto \mathbf{x}$, to the complement $M(\mathcal{A})$ is a fiber bundle projection, with base $p(M(\mathcal{A})) = M(\mathcal{B})$, the complement of an arrangement $\mathcal{B}$ in $\mathbb{C}^\ell$, and fiber the complement of finitely many points in $\mathbb{C}$. Refer to the hyperplane arrangement $\mathcal{A}$ as strictly linearly fibered over $\mathcal{B}$.

The complements of hyperplane arrangements of this type are closely related to configuration spaces, as we now illustrate. For each hyperplane H of $\mathcal{A}$, let f_H be a linear polynomial with $H = \ker f_H$. Then $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H$ is a defining polynomial for $\mathcal{A}$. From the definition, if $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and $|\mathcal{A}| = |\mathcal{B}| + n$, there is a choice of coordinates for which a defining polynomial for $\mathcal{A}$ factors as

$$Q(\mathcal{A}) = Q(\mathcal{B}) \cdot \phi(\mathbf{x}, z), \tag{3.1}$$

where $Q(\mathcal{B}) = Q(\mathcal{B})(\mathbf{x})$ is a defining polynomial for $\mathcal{B}$, and $\phi(\mathbf{x}, z)$ is a product

$$\phi(\mathbf{x}, z) = (z - g_1(\mathbf{x}))(z - g_2(\mathbf{x})) \cdots (z - g_n(\mathbf{x})),$$

with $g_j(\mathbf{x})$ linear. Define $g : \mathbb{C}^\ell \to \mathbb{C}^n$ by

$$g(\mathbf{x}) = \big(g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_n(\mathbf{x})\big), \tag{3.2}$$

Since $\phi(\mathbf{x}, z)$ necessarily has distinct roots for any $\mathbf{x} \in M(\mathcal{B})$, the restriction of g to $M(\mathcal{B})$ takes values in the configuration space $F(\mathbb{C}, n)$. The following result was proven by the first author, see [2, Theorem 1.1.5, Corollary 1.1.6].

Theorem 3.2. *Let $\mathcal{B}$ be an arrangement of m hyperplanes, and let $\mathcal{A}$ be an arrangement of $m + n$ hyperplanes which is strictly linearly fibered over $\mathcal{B}$. Then the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of the bundle of configuration spaces $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ along the map g. Consequently, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ admits a cross-section and has trivial local coefficients in homology.*

Since it is relevant to the subsequent discussion, a proof is included.

Proof. Denote points in $F(\mathbb{C}, n + 1)$ by $(\mathbf{y}, z)$, where $\mathbf{y} = (y_1, \ldots, y_n) \in F(\mathbb{C}, n)$ and $z \in \mathbb{C}$ satisfies $z \neq y_j$ for each j. Similarly, denote points in $M(\mathcal{A})$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in M(\mathcal{B})$ and $\phi(\mathbf{x}, z) \neq 0$. In this notation, we have $p_{n+1}(\mathbf{y}, z) = \mathbf{y}$ and $p(\mathbf{x}, z) = \mathbf{x}$. Let $E = \big\{\big(\mathbf{x}, (\mathbf{y}, z)\big) \in M(\mathcal{B}) \times F(\mathbb{C}, n + 1) \mid g(\mathbf{x}) = \mathbf{y}\big\}$ be the total space of the pullback of $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ along the map g. It is then readily checked that the map $h : M(\mathcal{A}) \to E$ defined by $h(\mathbf{x}, z) = \big(\mathbf{x}, (g(\mathbf{x}), z)\big)$ is an equivalence of bundles.

Since the bundle $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ admits a cross-section, so does the pullback $p : M(\mathcal{A}) \to M(\mathcal{B})$. Furthermore, the structure group of the latter is the pure braid group P_n. Consequently, the action of the fundamental group of the base $M(\mathcal{B})$ on that of the fiber $\mathbb{C}\setminus\{n \text{ points}\}$ is by pure braid automorphisms. As such, this action is by conjugation (see for instance [1] or [13]), hence is trivial in homology. ☐

It is now shown that redundant strictly linearly fibered arrangements admit (linear) fibrations, just as their codimension one progenitors do.

Theorem 3.3. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^{\ell+1}$ which is strictly linearly fibered over $\mathcal{B}$, with projection $p : M(\mathcal{A}) \to M(\mathcal{B})$ induced by the map $\mathbb{C}^{\ell+1} \to \mathbb{C}^\ell$ given by $(x_1, \ldots, x_\ell, z) \mapsto (x_1, \ldots, x_\ell)$. Then for each k, the map $\mathbb{C}^{k(\ell+1)} \to \mathbb{C}^{k\ell}$ given by $(\mathbf{x}_1, \ldots, \mathbf{x}_\ell, \mathbf{z}) \mapsto (\mathbf{x}_1, \ldots, \mathbf{x}_\ell)$ induces a fiber bundle projection $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$. Furthermore, the bundle $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$ admits a cross-section.*

Proof. By the previous result, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of $p_{n+1} : F(\mathbb{C}, n+1) \to F(\mathbb{C}, n)$ along the map g of (3.2). An analogous result for the complements of the subspace arrangements $\mathcal{A}^k$ and $\mathcal{B}^k$ is established next. For $k \geq 2$, view $\mathbb{C}^{k\ell}$ as $(\mathbb{C}^\ell)^k$ and $\mathbb{C}^{kn}$ as $(\mathbb{C}^k)^n$. Denote points in the configuration space $F(\mathbb{C}^k, n + 1)$ by $(\mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{z})$, where $(\mathbf{y}_1, \ldots, \mathbf{y}_n) \in F(\mathbb{C}^k, n)$ and $\mathbf{z} \neq \mathbf{y}_j$ for each j. Define $g^k : \mathbb{C}^{k\ell} \to \mathbb{C}^{kn}$ by

$$g^k(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Big(\big(g_1(\mathbf{x}_1), \ldots, g_1(\mathbf{x}_k)\big), \ldots\ldots, \big(g_n(\mathbf{x}_1), \ldots, g_n(\mathbf{x}_k)\big)\Big). \tag{3.3}$$

where $\big(g_i(\mathbf{x}_1),\dots,g_i(\mathbf{x}_k)\big) \in \mathbb{C}^k$ for each i. It is readily checked that the restriction of g^k to $M(\mathcal{B}^k)$ takes values in the configuration space $F(\mathbb{C}^k,n)$. Let $\pi^k : E^k \to M(\mathcal{B}^k)$ be the pullback of the bundle $p^k_{n+1} : F(\mathbb{C}^k,n+1) \to F(\mathbb{C}^k,n)$ along this restriction, with total space E^k consisting of all points

$$\big((\mathbf{x}_1,\dots,\mathbf{x}_k),(\mathbf{y}_1,\dots,\mathbf{y}_n,\mathbf{z})\big) \in M(\mathcal{B}^k) \times F(\mathbb{C}^k,n+1)$$

for which $g^k(\mathbf{x}_1,\dots,\mathbf{x}_k) = p^k_{n+1}(\mathbf{y}_1,\dots,\mathbf{y}_n,\mathbf{z}) = (\mathbf{y}_1,\dots,\mathbf{y}_n)$.

Since the hyperplane arrangement $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$, the complement of the subspace arrangement $\mathcal{A}^k$ may be realized as

$$M(\mathcal{A}^k) = \{(\mathbf{x}_1,\dots,\mathbf{x}_k,\mathbf{z}) \in M(\mathcal{B}^k)\times\mathbb{C}^k \mid \mathbf{z} \neq \big(g_i(\mathbf{x}_1),\dots,g_i(\mathbf{x}_k)\big) \text{ for } 1 \le i \le n\}.$$

Define $h^k : M(\mathcal{A}^k) \to E^k$ by $h^k(\mathbf{x}_1,\dots,\mathbf{x}_k,\mathbf{z}) = \big((\mathbf{x}_1,\dots,\mathbf{x}_k),(g^k(\mathbf{x}_1,\dots,\mathbf{x}_k),\mathbf{z})\big)$.

The map h^k is a homeomorphism. Moreover, the following diagram commutes.

$$\begin{array}{ccc} M(\mathcal{A}^k) & \xrightarrow{h^k} & E^k \\ \Big\downarrow{\scriptstyle p^k} & & \Big\downarrow{\scriptstyle \pi^k} \\ M(\mathcal{B}^k) & \xrightarrow{\mathrm{id}} & M(\mathcal{B}^k) \end{array} \tag{3.4}$$

It follows that $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$ is a bundle which is equivalent to the pullback of the bundle of configuration spaces $p^k_{n+1} : F(\mathbb{C}^k,n+1) \to F(\mathbb{C}^k,n)$ along the map $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k,n)$, and therefore has a cross-section. □

An analysis of map in homology induced by the map $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k,n)$ defined in (3.3) is given next. For $1 \le i < j \le n$, define $p_{i,j} : F(\mathbb{C}^k,n) \to S^{2k-1}$ by $p_{i,j}(\mathbf{y}_1,\dots,\mathbf{y}_n) = (\mathbf{y}_j - \mathbf{y}_i)/\|\mathbf{y}_j - \mathbf{y}_i\|$. Recall that $\iota_{2k-1} \in H_{2k-1}(S^{2k-1};\mathbb{Z})$ denotes the fundamental class. The classes $p^*_{i,j}(\iota_{2k-1})$ form a basis for $H^{2k-1}(F(\mathbb{C}^k,n))$, and generate the cohomology algebra $H^*(F(\mathbb{C}^k,n))$, see [4, 5]. Denote the dual classes in $H_{2k-1}(F(\mathbb{C}^k,n)$ by $A_{i,j}$, $1 \le i < j \le n$. Note that the classes $A_{i,j}$ may be represented by spheres linking the subspaces $H^k_{i,j} = \{\mathbf{y}_i = \mathbf{y}_j\}$ in $\mathbb{C}^{kn}$, as in (2.1).

As in Section 2, let $L = \{t\cdot\mathbf{u}+\mathbf{v}\} \subset \mathbb{C}^\ell$ be a line transverse to the hyperplane arrangement $\mathcal{B}$, and L^k the corresponding codimension k subspace of $\mathbb{C}^{k\ell}$, transverse to the subspace arrangement $\mathcal{B}^k$. Recall the maps $c^k_H : S^{2k-1} \to L^k \cap M(\mathcal{B}^k)$ from (2.1), and the resulting basis $\{C^k_H \mid H \in \mathcal{B}\}$ for $H_{2k-1}(M(\mathcal{B}^k))$ exhibited in Proposition 2.5.

Proposition 3.4. *Let $\mathcal{B} \subset \mathbb{C}^\ell$ be an arrangement of complex hyperplanes, and let $g : \mathbb{C}^\ell \to \mathbb{C}^n$ be an affine transformation whose restriction, $g : M(\mathcal{B}) \to F(\mathbb{C},n)$, to the complement of $\mathcal{B}$ takes values in the configuration space $F(\mathbb{C},n)$. Then for every $k \ge 1$, the induced map $(g^k)_* : H_{2k-1}(M(\mathcal{B}^k);\mathbb{Z}) \to H_{2k-1}(F(\mathbb{C}^k,n);\mathbb{Z})$ is given by $(g^k)_*(C^k_H) = \sum A_{i,j}$ for each hyperplane H of $\mathcal{B}$, where the sum is over all distinct i and j for which $g(H)$ is contained in the hyperplane $H_{i,j} = \{y_i = y_j\}$ in $\mathbb{C}^n$.*

Proof. For each hyperplane H of $\mathcal{B}$, let $\tilde{c}^k_H : S^{2k-1} \to M(\mathcal{B}^k)$ denote the composition of $c^k_H : S^{2k-1} \to L^k \cap M(\mathcal{B}^k)$ and the natural inclusion $L^k \cap M(\mathcal{B}^k) \hookrightarrow M(\mathcal{B}^k)$. It will be shown that the composition $p_{i,j} \circ g^k \circ \tilde{c}^k_H : S^{2k-1} \to S^{2k-1}$ induces the identity in homology if $g(H) \subset H_{i,j}$, and induces the trivial homomorphism if $g(H) \not\subset H_{i,j}$, thereby establishing the result.

For $\mathbf{x} \in \mathbb{C}^\ell$, write $g(\mathbf{x}) = (g_1(\mathbf{x}),\dots,g_n(\mathbf{x}))$ as in (3.2). Then $g^k : \mathbb{C}^{k\ell} \to \mathbb{C}^{kn}$ is given by $g^k(\mathbf{x}_1,\dots,\mathbf{x}_k) = (\mathbf{y}_1,\dots,\mathbf{y}_n)$, where $\mathbf{y}_i = (g_i(\mathbf{x}_1),\dots,g_i(\mathbf{x}_k))$, see (3.3). Since the restriction of g to $M(\mathcal{B})$ takes values in $F(\mathbb{C},n)$, the restriction of g^k to $M(\mathcal{B}^k)$ takes values in $F(\mathbb{C}^k,n)$.

From (2.1), the map $\tilde{c}^k_H : S^{2k-1} \to M(\mathcal{B}^k)$ is given by $\tilde{c}^k_H(\mathbf{z}) = (\mathbf{w}_1,\dots,\mathbf{w}_k)$, where $\mathbf{w}_j = (\tau_H + \epsilon z_j)\cdot\mathbf{u}+\mathbf{v}$, and $L \cap H$ is the point $\mathbf{q}_H = \tau_H\cdot\mathbf{u}+\mathbf{v}$. Let $\alpha_i = g_i(\mathbf{q}_H)$, and define β_i by the equation

$$g_i(\mathbf{w}_j) = g_i((\tau_H + \epsilon z_j)\cdot \mathbf{u} + \mathbf{v}) = g_i(\mathbf{q}_H + \epsilon z_j \cdot \mathbf{u}) = \alpha_i + \epsilon\beta_i z_j.$$

Then, a calculation yields $g^k \circ \tilde{c}^k_H(\mathbf{z}) = (\alpha_1 \cdot \mathbf{e} + \epsilon\beta_1 \cdot \mathbf{z}, \dots, \alpha_n \cdot \mathbf{e} + \epsilon\beta_n \cdot \mathbf{z})$ and

$$p_{i,j} \circ g^k \circ \tilde{c}^k_H(\mathbf{z}) = \frac{\epsilon(\beta_j - \beta_i)\mathbf{z} + (\alpha_j - \alpha_i)\mathbf{e}}{\|\epsilon(\beta_j - \beta_i)\mathbf{z} + (\alpha_j - \alpha_i)\mathbf{e}\|},$$

where, as before, $\mathbf{e} = (1, \dots, 1)$. Recall that $\epsilon > 0$ was chosen sufficiently small so as to insure that the point $(\mathbf{w}'_1, \dots, \mathbf{w}'_k)$, where $\mathbf{w}'_j = (\tau_H + \epsilon' z_j)\cdot\mathbf{u} + \mathbf{v}$, lies in $L^k \cap M(\mathcal{B}^k)$ for all ϵ', $0 < \epsilon' \leq \epsilon$. Since $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k, n)$, it follows that $g^k \circ \tilde{c}^k_H(\mathbf{z}) \in F(\mathbb{C}^k, n)$ for all $\mathbf{z} \in S^{2k-1}$. In other words, $\epsilon(\beta_j - \beta_i)\mathbf{z} + (\alpha_j - \alpha_i)\mathbf{e} \neq \mathbf{0}$ for all distinct i and j.

If $g(H) \not\subset H_{i,j}$, then $g(\mathbf{q}_H) \notin H_{i,j}$ since $\mathbf{q}_H = L \cap H$ is generic in H. Thus, $\alpha_i = g_i(\mathbf{q}_H) \neq g_j(\mathbf{q}_H) = \alpha_j$, and the point $(\alpha_i \mathbf{e} + \epsilon'\beta_i\mathbf{z}, \alpha_j\mathbf{e} + \epsilon'\beta_j\mathbf{z})$ lies in the configuration space $F(\mathbb{C}^k, 2)$ for all $\epsilon' \leq \epsilon$, including $\epsilon' = 0$. It follows that $p_{i,j} \circ g^k \circ c^k_H$ is trivial in homology in this instance.

If, on the other hand, $g(H) \subset H_{i,j}$, then $\alpha_i = \alpha_j$ and thus $\beta_j - \beta_i$ is necessarily non-zero. In this instance, $p_{i,j} \circ g^k \circ \tilde{c}^k_H(\mathbf{z}) = \lambda \cdot \mathbf{z}$, where $\lambda \in S^1 \subset \mathbb{C}^*$ is given by $\lambda = (\beta_j - \beta_i)/|\beta_j - \beta_i|$, which clearly induces the identity in homology. □

These results extend immediately to fiber-type arrangements, defined next.

Definition 3.5. An arrangement $\mathcal{A} = \mathcal{A}_1$ of finitely many points in $\mathbb{C}^1$ is *fiber-type*. An arrangement $\mathcal{A} = \mathcal{A}_\ell$ of hyperplanes in $\mathbb{C}^\ell$ is *fiber-type* if $\mathcal{A}$ is strictly linearly fibered over a fiber-type hyperplane arrangement $\mathcal{A}_{\ell-1}$ in $\mathbb{C}^{\ell-1}$.

Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$. Then there is a choice of coordinates $(x_1, \dots, x_\ell)$ on $\mathbb{C}^\ell$ so that a defining polynomial for $\mathcal{A}$ factors as $Q(\mathcal{A}) = \prod_{j=1}^{\ell} Q_j(x_1, \dots, x_j)$, see (3.1). Write $Q_j = \prod_{i=1}^{d_j}(x_j - g_{i,j}(x_1, \dots, x_{j-1}))$, where d_j is the degree of Q_j and each $g_{i,j}$ is linear. The non-negative integers $\{d_1, \dots, d_\ell\}$ are called the exponents of $\mathcal{A}$. For each $j \leq \ell$, the polynomial $\prod_{i=1}^{j} Q_i$ defines a fiber-type arrangement $\mathcal{A}_j$ in $\mathbb{C}^j$ with exponents $\{d_1, \dots, d_j\}$, and $\mathcal{A}_j$ is strictly linearly fibered over $\mathcal{A}_{j-1}$. Furthermore, the map $g_j = (g_{1,j}, \dots, g_{d_j,j}) : \mathbb{C}^{j-1} \to \mathbb{C}^{d_j}$ gives rise to maps $g^k_j : M(\mathcal{A}^k_{j-1}) \to F(\mathbb{C}^k, d_j)$ for each k. Theorems 3.2 and 3.3 yield

Theorem 3.6. *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with defining polynomial $Q(\mathcal{A}) = \prod_{j=1}^{\ell} Q_j$. Then, for each j, $2 \leq j \leq \ell$, and each $k \geq 1$, the projection $\mathbb{C}^j \to \mathbb{C}^{j-1}$, $(x_1, \dots, x_{j-1}, x_j) \mapsto (x_1, \dots, x_{j-1})$, gives rise to a bundle map $p^k_j : M(\mathcal{A}^k_j) \to M(\mathcal{A}^k_{j-1})$. This bundle is equivalent to the pullback of the bundle of configuration spaces $F(\mathbb{C}^k, d_j + 1) \to F(\mathbb{C}^k, d_j)$ along the map $g^k_j : M(\mathcal{A}^k_{j-1}) \to F(\mathbb{C}^k, d_j)$. Consequently, the bundle $p^k_j : M(\mathcal{A}^k_j) \to M(\mathcal{A}^k_{j-1})$ admits a cross-section, has trivial local coefficients in homology, and the fiber is the complement of d_j points in $\mathbb{C}^k$.*

Proposition 3.4 also extends to fiber-type arrangements. The specific statement is omitted.

4. The Descending Central Series

In this section, the structure of the Lie algebra $E^*_0(G)$ associated to the descending central series of the fundamental group G of the complement of a fiber-type arrangement is analyzed. Additively, this structure is given by well known results of Falk and Randell [9, 10] stated below. Moreover, as shown by Jambu and Papadima [14], this Lie algebra is isomorphic to the (integral) holonomy Lie algebra of the arrangement $\mathcal{A}$ defined by Kohno [16]. An alternate description of $E^*_0(G)$, which will facilitate comparison with the Lie algebra of primitives in the homology of the loop space of the complement of the subspace arrangement $\mathcal{A}^k$ in Section 5, is given here.

Example 4.1. Let P_n be the Artin pure braid group, the fundamental group of the configuration space $F(\mathbb{C},n)$. The structure of the Lie algebra $E_0^*(P_n)$ was first determined rationally by Kohno [17]. As observed by many authors, the following description holds over the integers as well. For each $n \geq 2$, let $L[n]$ be the free Lie algebra generated by elements $A_{1,n+1},\dots,A_{n,n+1}$. Then the Lie algebra $E_0^*(P_n)$ is additively isomorphic to the direct sum $\bigoplus_{j=1}^{n-1} L[j]$, and the Lie bracket relations in $E_0^*(P_n)$ are the infinitesimal pure braid relations, given by

$$\begin{aligned} [A_{i,j}+A_{i,k}+A_{j,k},A_{m,k}] &= 0 \quad \text{for } m=i \text{ or } m=j, \text{ and} \\ [A_{i,j},A_{k,l}] &= 0 \quad \text{for } \{i,j\}\cap\{k,l\}=\emptyset. \end{aligned} \tag{4.1}$$

Note that this description realizes the Lie algebra $E_0^*(P_{n+1})$ as the semidirect product of $E_0^*(P_n)$ by $L[n]$ determined by the Lie homomorphism $\theta_n : E_0^*(P_n) \to \mathrm{Der}(L[n])$ given by $\theta_n(A_{i,j}) = \mathrm{ad}(A_{i,j})$. From the infinitesimal pure braid relations, one has

$$\mathrm{ad}(A_{i,j})(A_{m,n+1}) = \begin{cases} [A_{m,n+1},A_{i,n+1}+A_{j,n+1}] & \text{if } m=i \text{ or } m=j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

This extension of Lie algebras arises topologically from the bundle of configuration spaces $F(\mathbb{C},n+1) \to F(\mathbb{C},n)$.

The additive structure noted above may be obtained from the following result of Falk and Randell [9, 10].

Theorem 4.2. *Let $1 \to H \to G \to K \to 1$ be a split extension of groups such that the conjugation action of K is trivial on H_1/H_2. Then there is a short exact sequence of Lie algebras $0 \to E_0^*(H) \to E_0^*(G) \to E_0^*(K) \to 0$ which is split as a sequence of abelian groups. Furthermore, if the descending central series quotients of H and K are free abelian, then so are those of G.*

Now let $\mathcal{A} = \mathcal{A}_\ell$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$. The complement of $\mathcal{A}_\ell$ sits atop a tower of fiber bundles

$$M(\mathcal{A}_\ell) \xrightarrow{p_\ell} M(\mathcal{A}_{\ell-1}) \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} M(\mathcal{A}_1) = \mathbb{C}\setminus\{d_1 \text{ points}\},$$

where the fiber of p_j is homeomorphic to the complement of d_j points in $\mathbb{C}$. For simplicity, write $\mathcal{B} = \mathcal{A}_{\ell-1}$ and $n = d_\ell$. Then $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$, and by Theorem 3.2, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of the configuration space bundle $p_{n+1} : F(\mathbb{C},n+1) \to F(\mathbb{C},n)$ along the map g of (3.2).

Application of the homotopy exact sequence of a bundle (and induction) shows that $M(\mathcal{A})$ is a $K(G,1)$ space, where $G = G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$. In light of Theorem 3.2, there is also a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & G(\mathcal{A}) & \longrightarrow & G(\mathcal{B}) & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle g_\#} & & \\ 1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & 1 \end{array} \tag{4.3}$$

where $g_\# : G(\mathcal{B}) \to P_n$ is induced by $g : M(\mathcal{B}) \to F(\mathbb{C},n)$, and the fundamental group of the fiber $\mathbb{C}\setminus\{n \text{ points}\}$ is identified with the free group $\mathbb{F}_n$ on n generators. Since the underlying bundles admit cross-sections, the rows in the diagram above are split exact.

Theorem 4.3. *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement. If the exponents of $\mathcal{A}$ are $\{d_1,\dots,d_\ell\}$, then $E_0^*(G(\mathcal{A})) \cong L[d_1]\oplus\cdots\oplus L[d_\ell]$ as abelian groups.*

Proof. The proof is by induction on ℓ.

In the case $\ell = 1$, $\mathcal{A}$ is an arrangement of $d = d_1$ points in $\mathbb{C}$, the fundamental group of the complement is $\mathbb{F}_d$, the free group on d generators, and it is well known that $E_0^*(\mathbb{F}_d)$ is isomorphic to the free Lie algebra $L[d]$, see for instance [21, Chapter IV].

In general, assume that the fiber-type arrangement $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and that $d_\ell = n$ as above. Then there is a split, short exact sequence of groups $1 \to \mathbb{F}_n \to G(\mathcal{A}) \to G(\mathcal{B}) \to 1$, and by Theorem 3.2, the action of $G(\mathcal{B})$ on $\mathbb{F}_n$ is by pure braid automorphisms. As such, this action is by conjugation, hence is trivial on $H_*(\mathbb{F}_n;\mathbb{Z})$. By Theorem 4.2, the descending central series quotients of $G(\mathcal{A})$ are free abelian, and there is a short exact sequence of Lie algebras

$$0 \to E_0^*(\mathbb{F}_n) \to E_0^*(G(\mathcal{A})) \to E_0^*(G(\mathcal{B})) \to 0, \tag{4.4}$$

which splits as a sequence of abelian groups. The result follows by induction. □

The additive decomposition provided by this result does not, in general, preserve the underlying Lie algebra structure. An inductive description of the Lie algebra structure of $E_0^*(G(\mathcal{A}))$ is given next.

Theorem 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be fiber-type hyperplane arrangements with $|\mathcal{A}| = |\mathcal{B}| + n$, and suppose that $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$. Then the Lie algebra $E_0^*(G(\mathcal{A}))$ is isomorphic to the semidirect product of $E_0^*(G(\mathcal{B}))$ by $L[n]$ determined by the Lie homomorphism $\Theta = \theta_n \circ g_* : E_0^*(G(\mathcal{B})) \to \mathrm{Der}(L[n])$, where $g_* : E_0^*(G(\mathcal{B})) \to E_0^*(P_n)$ is induced by the map $g : M(\mathcal{B}) \to F(\mathbb{C}, n)$, and $\theta_n : E_0^*(P_n) \to \mathrm{Der}(L[n])$ is given by $\theta_n(A_{i,j}) = \mathrm{ad}(A_{i,j})$.*

Proof. From the exact sequence of Lie algebras (4.4) noted above, it follows that $E_0^*(G(\mathcal{A}))$ is isomorphic to the semidirect product of $E_0^*(G(\mathcal{B}))$ by $L[n]$ determined by the Lie homomorphism $\Theta : E_0^*(G(\mathcal{B})) \to \mathrm{Der}(L[n])$ given by $\Theta(b) = \mathrm{ad}_{L[n]}(b)$ for $b \in E_0^*(G(\mathcal{B}))$. It suffices to show that the homomorphism Θ factors as asserted.

From the diagram (4.3), and the results of Falk and Randell stated in Theorem 4.2, there is a commutative diagram of Lie algebras with split exact rows

$$\begin{tikzcd}
0 \arrow[r] & L[n] \arrow[r] \arrow[d, "\mathrm{id}"] & E_0^*(G(\mathcal{A})) \arrow[r] \arrow[d] & E_0^*(G(\mathcal{B})) \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & L[n] \arrow[r] & E_0^*(P_{n+1}) \arrow[r] & E_0^*(P_n) \arrow[r] & 0
\end{tikzcd}$$

Via the splittings, view $E_0^*(G(\mathcal{B}))$ and $E_0^*(P_n)$ as Lie subalgebras of $E_0^*(G(\mathcal{A}))$ and $E_0^*(P_{n+1})$ respectively. Then for $a \in L[n]$ and $b \in E_0^*(G(\mathcal{B}))$, we have $[b, a] = [g_*(b), a]$ in $L[n]$. Thus $\mathrm{ad}_{L[n]}(b) = \mathrm{ad}_{L[n]}(g_*(b))$ in $\mathrm{Der}(L[n])$ and $\Theta = \theta_n \circ g_*$. □

This result, together with Proposition 3.4, provides an inductive description of the Lie bracket structure of $E_0^*(G(\mathcal{A}))$. Recall the basis $\{C_H^1 \mid H \in \mathcal{B}\}$ for $H_1(M(\mathcal{B});\mathbb{Z}) = E_0^1(G(\mathcal{B}))$ exhibited in Proposition 2.5, and recall that the free Lie algebra $L[n]$ is generated by $A_{1,n+1}, \dots, A_{n,n+1}$.

Corollary 4.5. *For generators C_H^1 of $E_0^1(G(\mathcal{B}))$ and $A_{m,n+1}$ of $L[n]$, one has*

$$\Theta(C_H^1)(A_{m,n+1}) = \sum_{g(H) \subset H_{i,j}} [A_{i,j}, A_{m,n+1}].$$

Proof. By Proposition 3.4, one has $g_*(C_H^1) = \sum A_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. The result follows. □

This corollary can be used to explicitly record the Lie bracket relations in $E_0^*(G(\mathcal{A}))$, and to show that these relations are combinatorial, that is, completely determined by the intersection poset $\mathsf{L}(\mathcal{A})$. The Lie algebra $E_0^*(G(\mathcal{A}))$ is generated by $\{C_H^1 \mid H \in \mathcal{A}\}$.

For a flat $X \in \mathsf{L}(\mathcal{A})$, write $C^1_X = \sum_{X \subset H} C^1_H$. The following was proven by Jambu and Papadima [14], see also [2].

Theorem 4.6. *If $\mathcal{A}$ is a fiber-type hyperplane arrangement with exponents $\{d_1, \dots, d_\ell\}$, then the Lie bracket relations in $E^*_0(G(\mathcal{A}))$ are given by*

$$[C^1_X, C^1_H] = 0,$$

for codimension two flats $X \in \mathsf{L}(\mathcal{A})$ and hyerplanes $H \in \mathcal{A}$ containing X.

Proof. The proof is by induction on ℓ.

In the case $\ell = 1$, there is nothing to show since $G(\mathcal{A})$ is a free group on $d = d_1$ generators, $E^*_0(G(\mathcal{A}))$ is isomorphic to the free Lie algebra $L[d]$, and there are no codimension two flats in $\mathsf{L}(\mathcal{A})$.

In general, assume that $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and that $d_\ell = n$ as before. Then $\mathcal{A}$ has a defining polynomial of the form $Q(\mathcal{A}) = Q(\mathcal{B}) \cdot \prod_{j=1}^n (z - g_j(\mathbf{x}))$, see (3.1). View $\mathcal{B}$ as a subarrangement of $\mathcal{A} = \{H \mid H \in \mathcal{B}\} \cup \{H_j \mid 1 \le j \le n\}$, where $H_j = \ker(z - g_j(\mathbf{x}))$. Then the set $\{C^1_H \mid H \in \mathcal{B}\} \cup \{C^1_{H_j} \mid 1 \le j \le n\}$ generates $E^*_0(G(\mathcal{A}))$, where the generators $C^1_{H_j}$ correspond to the hyperplanes H_j of $\mathcal{A} \setminus \mathcal{B}$, and to the generators $A_{j,n+1}$ of the free Lie algebra $L[n]$ under the additive isomorphism $E^*_0(G(\mathcal{A})) \cong E^*_0(G(\mathcal{B})) \oplus L[n]$.

By Theorem 4.4, $E^*_0(G(\mathcal{A}))$ is isomorphic to an extension of $E^*_0(G(\mathcal{B}))$ by $L[n]$. Consequently, the Lie bracket relations in $E^*_0(G(\mathcal{A}))$ consist of those of $E^*_0(G(\mathcal{B}))$, and those arising from the extension. By induction, the Lie bracket relations in $E^*_0(G(\mathcal{B}))$ are given by $[C^1_X, C^1_H] = 0$ for codimension two flats X contained only in hyperplanes $H \in \mathcal{B} \subset \mathcal{A}$. So it remains to analyze those relations in $E^*_0(G(\mathcal{A}))$ arising from the extension. These are given implicitly in Corollary 4.5.

Recall from (4.1) that $[A_{i,j}, A_{m,n+1}] = [A_{m,n+1}, A_{i,n+1} + A_{j,n+1}]$ if $m \in \{i, j\}$, and is zero otherwise. Thus the results of Corollary 4.5 may be recorded as

$$\Theta(C^1_H)(A_{m,n+1}) = [C^1_H, A_{m,n+1}] = \sum_{g(H) \subset H_{i,j}} [A_{m,n+1}, (\delta_{i,m} + \delta_{j,m})(A_{i,n+1} + A_{j,n+1})],$$

where $C^1_H \in E^*_0(G(\mathcal{B})) \subset E^*_0(G(\mathcal{A}))$ and $\delta_{i,m}$ is the Kronecker delta. Note that the expression on the right lies in $L[n]$. Under the above identifications, these relations take the form

$$[C^1_H, C^1_{H_m}] = \sum_{g(H) \subset H_{i,j}} [C^1_{H_m}, (\delta_{i,m} + \delta_{j,m})(C^1_{H_i} + C^1_{H_j})] \tag{4.5}$$

Now one can check that $g(H) \subset H_{i,j}$ if and only if $H \cap H_i \cap H_j$ is a codimension two flat in $\mathsf{L}(\mathcal{A})$ if and only if $H_i \cap H_j \subset H$. Using this observation, the relation (4.5) may be expressed as

$$[C^1_H, C^1_{H_m}] = [C^1_{H_m}, \sum_{H_m \cap H_j \subset H} C^1_{H_j}].$$

A calculation then shows that this relation is equivalent to $[C^1_X, C^1_{H_m}] = 0$, where X is the codimension two flat in $\mathsf{L}(\mathcal{A})$ contained in H and H_m. Since this relation holds for all $H_m \in \mathcal{A} \setminus \mathcal{B}$ for which $X \subset H \cap H_m$, it follows that $[C^1_X, C^1_H] = 0$ as well. ☐

5. Homology of the Loop Space

The structure of the Lie algebra of primitive elements in the homology of the loop space of the complement of a redundant subspace arrangement associated to a fiber-type hyperplane arrangement is analyzed in this section. In analogy with the previous section, begin by recalling this structure for the classical configuration spaces $F(\mathbb{C}^{k+1}, n)$ for $k \ge 1$.

Example 5.1. The integral homology of the loop space $\Omega F(\mathbb{C}^{k+1}, n)$ was calculated by Fadell and Husseini [7]. The structure of the Lie algebra $\operatorname{Prim} H_*(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z})$ was subsequently determined by Cohen and Gitler [3]. For brevity, denote this Lie algebra by $\mathcal{L}(n)_k$. The structure of $\mathcal{L}(n)_k$ may be described as follows.

For each $n \geq 2$, let $L[n]_k$ denote the free Lie algebra generated by elements $B_{i,n+1}$, $1 \leq i \leq n$, of degree $2k$. Then $\mathcal{L}(n)_k$ is additively isomorphic to the direct sum $\bigoplus_{j=1}^{n-1} L[j]_k$, and the Lie bracket relations in $\mathcal{L}(n)_k$ are given by the infinitesimal pure braid relations on the $B_{i,j}$, see (4.1). Thus, there is an isomorphism of graded Lie algebras $\mathcal{L}(n)_k \cong E_0^*(P_n)_k$, see Definition 1.1, Theorem 1.2, and Example 4.1.

Furthermore, as is the case for the descending central series of the pure braid group, the Lie algebra $\mathcal{L}(n+1)_k$ is isomorphic to the semidirect product of $\mathcal{L}(n)_k$ by $L[n]_k$ determined by the Lie homomorphism $\theta_n^k : \mathcal{L}(n)_k \to \operatorname{Der}(L[n]_k)$ given by $\theta_n^k(B_{i,j}) = \operatorname{ad}(B_{i,j})$. From the infinitesmial pure braid relations, there is a formula for $\operatorname{ad}(B_{i,j})$ analogous to that given in (4.2). As before, this extension of Lie algebras arises topologically from the bundle of configuration spaces $F(\mathbb{C}^{k+1}, n+1) \to F(\mathbb{C}^{k+1}, n)$.

Now let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with exponents $\{d_1, \ldots, d_\ell\}$. Then, for each k, there is a tower of fiber bundles

$$M(\mathcal{A}_\ell^k) \xrightarrow{p_\ell^k} M(\mathcal{A}_{\ell-1}^k) \xrightarrow{p_{\ell-1}^k} \cdots \xrightarrow{p_2^k} M(\mathcal{A}_1^k) = \mathbb{C}^k \setminus \{d_1 \text{ points}\},$$

where the fiber of p_j^k is homeomorphic to the complement of d_j points in $\mathbb{C}^k$, see Theorem 3.6. Furthermore, each of the fiber bundles $p_j^k : M(\mathcal{A}_j^k) \to M(\mathcal{A}_{j-1}^k)$ involving the complements of the redundant subspace arrangements $\mathcal{A}_j^k \subset \mathbb{C}^{jk}$ admits a cross-section, and, as indicated above, $M(\mathcal{A}_1^k)$ is the complement of d_1 points in $\mathbb{C}^k$. By work of the second two authors [6, Theorem 1], the following holds.

Theorem 5.2. *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with exponents $\{d_1, \ldots, d_\ell\}$. Then, for each $k \geq 1$,*

(a) *There is a homotopy equivalence $\Omega M(\mathcal{A}^{k+1}) \to \prod_{j=1}^{\ell} \Omega(\mathbb{C}^{k+1} \setminus \{d_j \text{ points}\})$;*

(b) *The integral homology of $\Omega M(\mathcal{A}^{k+1})$ is torsion free, and is isomorphic to the tensor product $\bigotimes_{j=1}^{\ell} H_*(\Omega(\mathbb{C}^{k+1} \setminus \{d_j \text{ points}\}); \mathbb{Z})$ as a coalgebra;*

(c) *The module of primitives in the integral homology of $\Omega M(\mathcal{A}^{k+1})$ is isomorphic to $\pi_*(\Omega M(\mathcal{A}^{k+1}))$ modulo torsion as a Lie algebra.*

Remark 5.3. The homotopy groups of a loop space admit a bilinear pairing, which satisfies the axioms for a graded Lie algebra in case there is no 2 or 3 torsion in the homotopy groups. The graded analogue of the symmetry law can fail in case 2-torsion is present, while the graded analogue of the Jacobi identity can fail if 3-torsion is present. Thus, forming the quotient of the homotopy groups by the torsion gives a graded module which satisfies the axioms for a graded Lie algebra. Analogous properties of iterated loop spaces yield a graded Poisson algebra, see Section 6 below.

Proof of Theorem 5.2. Given a fibration $F \xrightarrow{i} E \longrightarrow B$ with a section σ, there is a homotopy equivalence $\Omega E \simeq \Omega B \times \Omega F$ given by the composite:

$$\Omega B \times \Omega F \xrightarrow{\Omega\sigma \times \Omega i} \Omega E \times \Omega E \xrightarrow{\mu} \Omega E,$$

where μ is the loop space multiplication and such that the inclusions of ΩB and ΩF into ΩE are maps of H-spaces. Morever, if the spaces involved have torsion free homology then $H_*(\Omega E) \cong H_*(\Omega B) \otimes H_*(\Omega F)$. By a theorem of Milnor and Moore, one obtains

$$\operatorname{Prim} H_*(\Omega E) \cong \operatorname{Prim} H_*(\Omega B) \oplus \operatorname{Prim} H_*(\Omega F) \tag{5.1}$$

upon passing to the Lie algebra of primitives. This result is a topological analogue of Theorem 4.2 as the underlying Lie algebra structure is "twisted."

Now apply these considerations to the fiber bundle $p_j^{k+1} : M(\mathcal{A}_j^{k+1}) \to M(\mathcal{A}_{j-1}^{k+1})$. The fiber in this case is $F = \mathbb{C}^{k+1} \setminus \{d_j \text{ points}\} \simeq \bigvee_{d_j} S^{2k+1}$. Assertion (a) follows by induction, and then (b) by the Künneth theorem. By the Bott-Samelson Theorem, $H_*(\Omega F)$ is isomorphic to $T[d_j]_k$, a tensor algebra on d_j generators of degree $2k$.

Thus the module of primitive elements is generated as a Lie algebra by the primitive elements in degree $2k$ which are in the image of the Hurewicz map. Since the Hurewicz map takes values in the the module of primitive elements, that module is generated as a Lie algebra by those spherical classes given by the homology classes of degree $2k$.

Next notice that the homology groups here are torsion free. Hence the Hurewicz map factors through $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\operatorname{Torsion}$. Furthermore, the homotopy groups of a loop space modulo torsion give a graded Lie algebra where the Lie bracket is induced by the classical Samelson product, and the Hurewicz map is a morphism of graded Lie algebras. Thus the induced map $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\operatorname{Torsion} \to \operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ is an epimorphism of Lie algebras.

Since all spaces are simply connected, and are of finite type, the homotopy groups modulo torsion are finitely generated free abelian groups in any fixed degree. By a classical theorem of Milnor and Moore concerning rational homotopy groups, the induced map $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\operatorname{Torsion} \to \operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ is also a monomorphism. The result follows. □

By (5.1) above, there is an isomorphism of graded abelian groups

$$\operatorname{Prim} H_*(\Omega M(\mathcal{A}_j^{k+1})) \cong \operatorname{Prim} H_*(\Omega M(\mathcal{A}_{j-1}^{k+1})) \oplus L[d_j]_k.$$

Proceeding inductively, this implies that the Lie algebra $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ is isomorphic to $L[d_1]_k \oplus \cdots \oplus L[d_\ell]_k$ as a graded abelian group, where $\{d_1, \ldots, d_\ell\}$ are the exponents of $\mathcal{A}$. Thus the Lie algebras $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ and $E_0^*(G(\mathcal{A}))_k$ are additively isomorphic, see Theorem 4.3. To show that they are are isomorphic as Lie algebras, thereby completing the proof of Theorem 1.3, it remains to show that the Lie bracket structure of $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ coincides with that of $E_0^*(G(\mathcal{A}))_k$. This analysis parallels the determination of the Lie algebra structure of $E_0^*(G(\mathcal{A}))$ in Section 4.

The fiber-type hyperplane arrangement $\mathcal{A} = \mathcal{A}_\ell$ is strictly linearly fibered over $\mathcal{A}_{\ell-1}$, and $|\mathcal{A}| = |\mathcal{A}_{\ell-1}| + d_\ell$. As before, write $\mathcal{B} = \mathcal{A}_{\ell-1}$ and $n = d_\ell$. Recall the map $g^{k+1} : M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$ from (3.3). Recall also that the Lie algebra $\operatorname{Prim} H_*(\Omega F(\mathbb{C}^{k+1}, n);\mathbb{Z})$ is denoted by $\mathcal{L}(n)_k$. Analogously, denote the Lie algebra $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ by $\mathcal{L}(\mathcal{A})_k$.

Theorem 5.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be fiber-type hyperplane arrangements with $\mathcal{A}$ strictly linearly fibered over $\mathcal{B}$ and $|\mathcal{A}| = |\mathcal{B}| + n$. Then the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is isomorphic to the semidirect product of $\mathcal{L}(\mathcal{B})_k$ by the free Lie algebra $L[n]_k$ determined by the Lie homomorphism $\Theta^k = \theta_n^k \circ \gamma_*^k : \mathcal{L}(\mathcal{B})_k \to \operatorname{Der}(L[n]_k)$, where $\gamma_*^k : \mathcal{L}(\mathcal{B})_k \to \mathcal{L}(n)_k$ is the map in loop space homology induced by $g^{k+1} : M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$, and $\theta_n^k : \mathcal{L}(n)_k \to \operatorname{Der}(L[n]_k)$ is given by $\theta_n^k(B_{i,j}) = \operatorname{ad}(B_{i,j})$.*

Proof. The realization of the bundle $p^{k+1} : M(\mathcal{A}^{k+1}) \to M(\mathcal{B}^{k+1})$ as the pullback of the bundle of configuration spaces $p_{n+1}^{k+1} : F(\mathbb{C}^{k+1}, n+1) \to F(\mathbb{C}^{k+1}, n)$ along the map $g^{k+1} : M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$ from Theorem 3.3 yields a commutative diagram of

Hopf algebras

$$\begin{array}{ccccc} H_*(\Omega(\mathbb{C}^{k+1}\setminus\{n \text{ points}\})) & \longrightarrow & H_*(\Omega M(\mathcal{A}^{k+1})) & \longrightarrow & H_*(\Omega M(\mathcal{B}^{k+1})) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow & & \Big\downarrow \gamma^k_* \\ H_*(\Omega(\mathbb{C}^{k+1}\setminus\{n \text{ points}\})) & \longrightarrow & H_*(\Omega F(\mathbb{C}^{k+1},n+1)) & \longrightarrow & H_*(\Omega F(\mathbb{C}^{k+1},n)) \end{array}$$

with exact rows, and, on the level of primitives, a commutative diagram of Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L[n]_k & \longrightarrow & \mathcal{L}(\mathcal{A})_k & \longrightarrow & \mathcal{L}(\mathcal{B})_k & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{id} & & \Big\downarrow & & \Big\downarrow \gamma^k_* & & \\ 0 & \longrightarrow & L[n]_k & \longrightarrow & \mathcal{L}(n+1)_k & \longrightarrow & \mathcal{L}(n)_k & \longrightarrow & 0 \end{array}$$

where $\gamma^k_* : \mathcal{L}(\mathcal{B})_k \to \mathcal{L}(n)_k$ is induced by $g^{k+1} : M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$. Since the underlying bundles admit cross-sections, the rows in the above diagrams are split exact. Via these splittings, view $\mathcal{L}(\mathcal{B})_k$ and $\mathcal{L}(n)_k$ as Lie subalgebras of $\mathcal{L}(\mathcal{A})_k$ and $\mathcal{L}(n+1)_k$ respectively.

From the above considerations, it follows that the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is isomorphic to the semidirect product of $\mathcal{L}(\mathcal{B})_k$ by $L[n]_k$ determined by the Lie homomorphism $\Theta^k : \mathcal{L}(\mathcal{B})_k \to \mathrm{Der}(L[n]_k)$ given by $\Theta^k(b) = \mathrm{ad}_{L[n]_k}(b)$ for $b \in \mathcal{L}(\mathcal{B})_k$. Moreover, for $a \in L[n]_k$, we have $[b, a] = [\gamma^k_*(b), a]$ in $L[n]_k$. Thus $\mathrm{ad}_{L[n]_k}(b) = \mathrm{ad}_{L[n]_k}(\gamma^k_*(b))$ in $\mathrm{Der}(L[n]_k)$ and $\Theta^k = \theta^k_n \circ \gamma^k_*$. □

This result, together with Proposition 3.4, provides an inductive description of the Lie bracket structure of $\mathcal{L}(\mathcal{A})_k$. The space $M(\mathcal{B}^{k+1})$ is $2k$-connected, and the cohomology algebra $H^*(M(\mathcal{B}^{k+1}); \mathbb{Z})$ is generated by classes a^{k+1}_H in one-to-one correspondence with the hyperplanes $H \in \mathcal{B}$, see Corollary 2.3. These classes are of degree $2k + 1$, and are dual to the elements of the basis $\{C^{k+1}_H \mid H \in \mathcal{B}\}$ for $H_{2k+1}(M(\mathcal{B}^{k+1}); \mathbb{Z})$ exhibited in Proposition 2.5. See also Remark 2.6.

The above observations imply that homology suspension induces an isomorphism

$$\sigma_* : H_{2k}(\Omega M(\mathcal{B}^{k+1}); \mathbb{Z}) \to H_{2k+1}(M(\mathcal{B}^{k+1}); \mathbb{Z}).$$

Let $\beta^k_H \in H_{2k}(\Omega M(\mathcal{B}^{k+1}); \mathbb{Z})$ be the unique class satisfying $\sigma_*(\beta^k_H) = C^{k+1}_H$. Recall that the free Lie algebra $L[n]_k$ is generated by $B_{1,n+1}, \dots, B_{n,n+1}$.

Corollary 5.5. *For generators β^k_H of $\mathcal{L}(\mathcal{B})_k$ and $B_{m,n+1}$ of $L[n]_k$, one has*

$$\Theta^k(\beta^k_H)(B_{m,n+1}) = \sum_{g(H)\subset H_{i,j}} [B_{i,j}, B_{m,n+1}].$$

Proof. By Proposition 3.4, one has $g^{k+1}_*(C^{k+1}_H) = \sum A_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. Since the homology suspension σ_* is an isomorphism and γ^k_* is the map in loop space homology induced by g^{k+1}, one has $\gamma^k_*(\beta^k_H) = \sum B_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. The result follows. □

To complete the proof of Theorem 1.3, assume inductively that the Lie algebras $E^*_0(G(\mathcal{B}))_k$ and $\mathcal{L}(\mathcal{B})_k$ are isomorphic. By Theorem 4.4, the Lie algebra $E^*_0(G(\mathcal{A}))$ is the extension of $E^*_0(G(\mathcal{B}))$ by the free Lie algebra $L[n]$ (generated in degree one) determined by the Lie homomorphism $\Theta = \theta_n \circ g_*$. Thus $E^*_0(G(\mathcal{A}))_k$ may be realized as the extension of $E^*_0(G(\mathcal{B}))_k$ by the free Lie algebra $L[n]_k$ (generated in degree $2k$) determined by Θ as specified in Definition 1.1. Similarly, by Theorem 5.4, the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is the extension of $\mathcal{L}(\mathcal{B})_k$ by the free Lie algebra $L[n]_k$ determined by the Lie homomorphism $\Theta^k = \theta^k_n \circ \gamma^k_*$. A comparison of the results of Corollary 4.5 and Corollary 5.5 reveals that these extensions coincide. Therefore, the Lie algebras $E^*_0(G(\mathcal{A}))_k$ and $\mathcal{L}(\mathcal{A})_k$ are isomorphic.

Alternatively, Corollary 5.5 may be used to explicitly determine the Lie bracket structure in $\mathcal{L}(\mathcal{A})_k$. As the argument is completely analogous to that which established Theorem 4.6, the result stated below without proof. The Lie algebra $\mathcal{L}(\mathcal{A})_k$ is generated by $\{\beta_H^k \mid H \in \mathcal{A}\}$. For a flat $X \in \mathsf{L}(\mathcal{A})$, write $\beta_X^k = \sum_{X \subset H} \beta_H^k$.

Theorem 5.6. *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement. Then, for each $k \geq 1$, the Lie bracket relations in $\mathcal{L}(\mathcal{A})_k$ are given by*

$$[\beta_X^k, \beta_H^k] = 0,$$

for codimension two flats $X \in \mathsf{L}(\mathcal{A})$ and hyerplanes $H \in \mathcal{A}$ containing X.

6. Homology of Iterated Loop Spaces

In this final section, the Poisson algebra structure on the homology of an iterated loop space of the complement of a redundant subspace arrangement associated to a fiber-type hyperplane arrangement is briefly analyzed.

For $q > 1$, the homology of an q-fold loop space, $\Omega^q X$, admits the structure of a graded Poisson algebra. Namely, there is a bilinear map given by the Browder operation

$$\lambda_{q-1} : H_i(\Omega^q X) \otimes H_j(\Omega^q X) \to H_{i+j+q-1}(\Omega^q X)$$

which satisfies properties listed in [4, pages 215–217]. In particular, this pairing satisfies the axioms of a (graded) Poisson algebra, and is compatible with the Whitehead product structure for the classical Hurewicz homomorphism.

In the case where $X = X_\ell(\mathbb{C}^{k+1})$, $k \geq 1$, satisfies conditions (1)–(3) from the Introduction, these structures are analogues of classical constructions in homotopy theory. First, note that the single suspension $\Sigma X_\ell(\mathbb{C}^{k+1})$ is homotopy equivalent to a bouquet of spheres. Thus there is an induced map $\sigma^2 : \Sigma^2 X_\ell(\mathbb{C}^{k+1}) \to X_\ell(\mathbb{C}^{k+2})$ which induces an isomorphism on the first non-trivial homology group. The adjoint $E^2 : X_\ell(\mathbb{C}^{k+1}) \to \Omega^2 X_\ell(\mathbb{C}^{k+2})$ also induces an isomorphism on the first non-trivial homology group. This last map is an analogue of the classical Freudenthal double suspension where the spaces $X_\ell(\mathbb{E})$ are replaced by single odd dimensional spheres. Looping E^2 is given by $\Omega(E^2) : \Omega X_\ell(\mathbb{C}^{k+1}) \to \Omega^3 X_\ell(\mathbb{C}^{k+2})$.

Theorem 6.1. *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with exponents $\{d_1, \ldots, d_\ell\}$. Then, for each $k \geq 1$,*

(a) *The multiplicative map $\Omega(E^2) : \Omega M(\mathcal{A}^{k+1}) \to \Omega^3 M(\mathcal{A}^{k+2})$ induces an isomorphism on $H_{2k}(-;\mathbb{Z})$, and is zero in degrees greater than $2k$.*

(b) *If $q > 1$, the homology of $\Omega^q M(\mathcal{A}^{k+1})$, with any field coefficients, is a graded Poisson algebra with Poisson bracket given by the Browder operation for the homology of a q-fold loop space.*

(c) *If $q > 1$, then $\Omega^q M(\mathcal{A}^{k+1})$ is homotopy equivalent to $\prod_{j=1}^{\ell} \Omega^q(\bigvee_{d_j} S^{2k+1})$.*

(d) *If $1 < q < 2k+1$, the homology of $\Omega^q M(\mathcal{A}^{k+1})$, with coefficients in a field $\mathbb{F}$ of characteristic zero, is generated as a Poisson algebra by elements β_H of degree $2k+1-q$ for $H \in \mathcal{A}$. The Poisson bracket is given by the Browder operation λ_{q-1}, and satisfies the relations*

$$\lambda_{q-1}[\beta_X, \beta_H],$$

for codimension two flats $X \in \mathsf{L}(\mathcal{A})$ and hyerplanes $H \in \mathcal{A}$ containing X, where $\beta_X = \sum_{X \subset H} \beta_H$.

Sketch of Proof. Part (a) follows from the fact that the homology of $\Omega^3 M(\mathcal{A}^{k+2})$ is abelian while the homology of $\Omega M(\mathcal{A}^{k+1})$ is generated by Lie brackets of weight at least 2 in homological degrees greater than $2k$.

Part (b) follows from the remarks at the beginning of this section.

Part (c) follows at once from the fact that the result holds in case $q = 1$, which was established in Theorem 5.2.

In case $q = 1$, the Browder operation λ_{q-1} is precisely the natural Lie bracket in the homology of a 1-fold loop space, $\Omega M(\mathcal{A}^{k+1})$. These Lie bracket relations are recorded in Theorem 5.6. As shown in [4, pages 215–217], a further property of the operation λ_{q-1} is that $\sigma_* \lambda_{q-1}(x, y) = \lambda_{q-2}(\sigma_* x, \sigma_* y)$, where σ_* denotes the homology suspension. Thus by induction on q, the asserted Poisson bracket relations are satisfied modulo elements in the kernel of the suspension. Furthermore, $\lambda_{q-1}(x, y)$ is primitive in case the classes x and y are primitive.

In characteristic zero, and in case q is greater than 1, the homology suspension induces an isomorphism on the module of primitives. Thus the asserted Poisson bracket relations are satisfied. □

Remark 6.2. Let $\mathcal{A} = \mathcal{A}_n$ be the braid arrangement in $\mathbb{C}^n$. As noted in Example 2.1, one then has $M(\mathcal{A}_n^{k+1}) = F(\mathbb{C}^{k+1}, n)$ for all k. For the braid arrangement, the codimension two flats in $\mathsf{L}(\mathcal{A}_n)$ (the partition lattice) are of the forms

$$H_{i,j} \cap H_{i,k} \cap H_{j,k} \text{ for } 1 \leq i < j < k \leq n, \quad \text{and} \quad H_{i,j} \cap H_{k,l} \text{ for } \{i,j\} \cap \{k,l\} = \emptyset.$$

Thus by Theorem 6.1, for $1 < q < 2k + 1$, the homology of $\Omega^q F(\mathbb{C}^{k+1}, n)$, with coefficients in a field $\mathbb{F}$ of characteristic zero, is generated as a Poisson algebra by elements $B_{i,j} = \beta_{H_{i,j}}$ of degree $2k + 1 - q$ for $1 \leq i < j \leq n$. Moreover, the Poisson bracket relations are given by the universal infinitesimal Poisson braid relations:

$$\begin{aligned} \lambda_{q-1}[B_{i,j} + B_{i,k} + B_{j,k}, B_{m,k}] &= 0 \quad \text{for } m = i \text{ or } m = j, \text{ and} \\ \lambda_{q-1}[B_{i,j}, B_{k,l}] &= 0 \quad \text{for } \{i,j\} \cap \{k,l\} = \emptyset. \end{aligned}$$

As shown in [3], these are precisely the infinitesimal pure braid relations in case $q = 1$, see also Examples 4.1 and 5.1.

It seems likely that, via the natural universal mapping property, one could define the "universal infinitesimal Poisson braid algebra," and that the homology of $\Omega^q F(\mathbb{C}^{k+1}, n)$ with coefficients in a field $\mathbb{F}$ of characteristic zero is that algebra over $\mathbb{F}$.

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