A Topological Version of Novikov's Closed Leaf Theorem

U. Hirsch

SFB 40/MPI für Mathematik
Gottfried-Claren-Str. 26
5300 Bonn 3

SFB/MPI 84-28

In 1965 S.P. Novikov published the following theorem which affirms a conjecture of Ehresmann (see [No; p. 286-289]):

Theorem 1. Let M be a closed 3-manifold with finite fundamental group. Then any codimension-one foliation of class C^2 on M admits a compact leaf.

In his list of open problems [Sc; p. 249] P. Schweitzer asks whether there is a C^o version of theorem 1.

Using the methods and results of [HH 2] we are able to prove the following *)

Theorem 2. Let M be a closed 3-manifold with finite fundamental group. Then any codimension-one foliation on M admits a compact leaf.

The proof of theorem 2 breaks into several steps the first of which establishing a C^o version of the Poincaré-Bendixson type argument used by A. Haefliger in this thesis [Ha 1]. We also make use (in a not very essential way) of the notion of averaging sequence of the holonomy pseudogroup of a foliation; cf. [GP] and proposition 1 below.

In what follows we use the terminology of [HH 1] and [HH 2].

Given a 2-dimensional foliation (M,F) we fix once and for all a 1-dimensional transverse foliation $F^{\pitchfork}$ of F. (The existence of $F^{\pitchfork}$ was established by L. Siebenmann in [Si]. Compare also [HH 2; IV,1.1]).

---

*) The author was informed that this result is also contained in V.V. Solodov's paper [So]. Our methods of proof are different.

1. Vanishing cycles.

Definition. i) A (non-trivial) vanishing cycle of (M,F) is a map

$$\nu : S^1 \to L_o , L_o \in F ,$$

together with a homotopy

$$f : S^1 \times [0,1] \to M$$

such that

(1) $f_o = \nu$ and f_t is contained in a leaf L_t of F for any $t \in [0,1]$,

(2) f is transverse to F ,

(3) ν is not null-homotopic in L_o but f_t is null-homotopic in L_t for any $t \in (0,1]$.

We say that ν is supported by L_o or L_o is the support of ν. The homotopy f is referred to as a vanishing deformation of ν.

ii) Call a vanishing deformation f $F^{\pitchfork}$-principal if for any $x \in S^1$ fixed the arc $f(x,t)$, $t \in [0,1]$, lies in some leaf of $F^{\pitchfork}$.

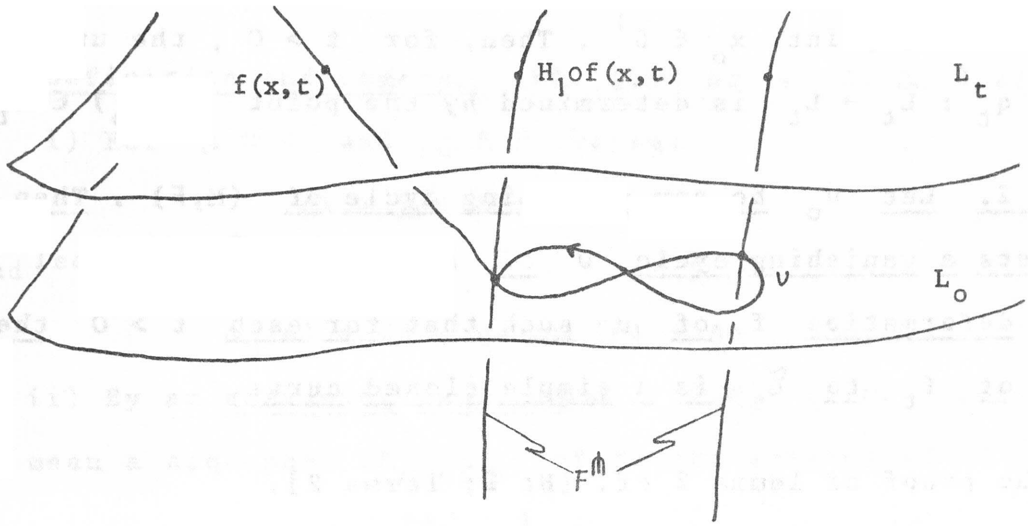
The typical example of a vanishing cycle is provided by the meridional circle on the boundary torus of a 3-dimensional Reeb component $(D^2 \times S^1, R)$. Note, however, that in general a vanishing cycle need not be embedded.

Lemma 1. Let ν be a vanishing cycle of (M,F). Then there exists an $F^{\pitchfork}$-principal vanishing deformation of ν.

Proof: Let $f : S^1 \times [0,1] \to M$ be any vanishing deformation of ν. Using local translation along the leaves of F we can find $t_o \in (0,1]$ and an F-isotopy

$$H : (M,F) \times I \to (M,F)$$

which is stationary outside some open neighbourhood of $f(S^1 \times [0,t_o])$ in M and such that $H_1 \circ f|S^1 \times [0,t_o]$ is an $F^{\pitchfork}$-principal vanishing deformation; see fig. □



Figure

From now on all vanishing cycles are assumed to be $F^{\pitchfork}$-principal.

<u>Remarks.</u> i) If ν is a vanishing cycle supported by $L \in F$ and w is homotopic to ν in L then w is also vanishing cycle. See [HH 2; VII, 1.7, ii)]. Therefore, since every element of $\Pi_1 L$ is represented by a loop whose only multiple points are isolated double points we may work only with vanishing cycles of this type.

We say that such a vanishing cycle is in <u>general position</u>.

ii) Let $p : \hat{M} \to (M,F)$ be a covering map and $\hat{F} = p^* F$. Then ν is a vanishing cycle of F if and only if any lift $\hat{\nu}$ of ν is a vanishing cycle of $\hat{F}$. Consequently, we may assume that all foliations under conderation are transversely orientable. Since all leaves of a transversely orientable foliation are twosided we can fix a transverse orientation of $\mathcal{F}$ (or, equivalently, a local flow defining $\mathcal{F}^{\pitchfork}$) and consider henceforth only right-vanishing cycles; see [HH 2; V, 1.1.3, 1.1.4 and VII, 1.8, ii)].

Now let $f : S^1 \times [0,1] \to M$ be a vanishing deformation of ν. We choose a base point $x_o \in S^1$. Then, for $t > 0$, the universal covering $q_t : \tilde{L}_t \to L_t$ is determined by the point $f_t(x_o) \in L_t$.

<u>Lemma 2. Let ν_o be any vanishing cycle of</u> $(M,\mathcal{F})$. <u>Then there exists a vanishing cycle ν of $\mathcal{F}$ and an $\mathcal{F}^{\pitchfork}$-principal vanishing deformation f of ν such that for each $t > 0$ the lift $\tilde{f}_t$ of f_t to $\tilde{L}_t$ is a simple closed curve.</u>

For the proof of lemma 2 cf. [Ha 2; lemme 2].

The proof of the next lemma uses the so-called holonomy lemma stating that a disk in a leaf of $\mathcal{F}$ can be lifted by means of $\mathcal{F}^{\pitchfork}$ to nearby leaves; see [HH 1; III, 1.2.9] and [Ha 2, lemme 3] for details.

<u>Lemma 3. There is a map</u> $F : D^2 \times (0,1] \to M$ <u>such that:</u>

(1) F <u>is transverse to</u> $\mathcal{F}$ <u>and to</u> $\mathcal{F}^{\pitchfork}$ <u>and</u> $F^*\mathcal{F} = \mathcal{H}$, $F^*\mathcal{F}^{\pitchfork} = \mathcal{V}$, <u>where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal resp. vertical foliation on</u> $D^2 \times (0,1]$.

(2) $F_t | \partial D^2 = f_t$ <u>for</u> $t \in (0,1]$ <u>and each</u> $F_t : D^2 \to L_t$ <u>is the restriction of a covering map</u>.

(3) F <u>cannot be extended continuously to</u> $D^2 \times \{0\}$.

Proof: We use lemma 2 and the Jordan-Schönflies theorem to construct F_1. The holonomy lemma then permits us to construct F_t for all $0 < t \leq 1$. Conditions (1) and (2) are satisfied, and (3) holds because f_o is not null-homotopic in L_o. □

Now choose a nice covering $\mathcal{U}$ of $(M,\mathcal{F})$ by bidistinguished open cubes with respect to $(\mathcal{F},\mathcal{F}^{\pitchfork})$. Denote by $(\mathcal{P},Q)$ the holonomy pseudogroup of $\mathcal{F}$ constructed by means of $\mathcal{U}$ and with canonical set of generators Γ. Note that $\bar{Q}$ is compact and Γ is finite because M is compact.

<u>Definition and remark.</u> (Cf. [GP] or [HH 2; X, 2.2])

i) For $A \subset Q$ and $g \in \mathcal{P}$ we set

$$gA = g(A \cap \text{domain}\, g)$$

and

$$\Delta_g(A) = (A - gA) \cup (gA - A)\ .$$

ii) By an <u>averaging sequence</u> of $(\mathcal{P},Q)$ (with respect to Γ) we mean a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of Q such that

$$\lim_{n\to\infty} \frac{\#\Delta_h(A_n)}{\#A_n} = 0 \quad \text{for every} \quad h \in \Gamma$$

(where $\#$ denotes cardinality).

iii) The Γ-<u>boundary</u> of $A \subset Q$ is by definition the set

$$\partial^\Gamma A = \{x \in A \mid h(x) \notin A \ \text{ for some } \ h \in \Gamma\}\ .$$

If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of finite subsets of Q with $\lim_{n\to\infty} \frac{\#\partial^\Gamma A_n}{\#A_n} = 0$ then $\{A_n\}$ is an averaging sequence.

iv) Given an averaging sequence $\{A_n\}$ of $(\mathcal{P},Q)$ the set of points $x \in \bar{Q}$ for which there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n \in A_n \quad \text{for any} \quad n \quad \text{and} \quad x = \lim_{n\to\infty} x_n$$

is called the <u>limit set</u> of $\{A_n\}$. We denote it by $\lim\{A_n\}$.

The next result is due essentially to Plante and Sullivan.

<u>Proposition 1.</u> <u>If</u> $(M,\mathcal{F})$ <u>supports a vanishing cycle then the holonomy pseudogroup</u> $(\mathcal{P},Q)$ <u>of</u> $\mathcal{F}$ <u>admits an averaging sequence</u>

<u>Proof:</u> Let ν be a vanishing cycle of $\mathcal{F}$ with vanishing deformation f and let $F : D^2 \times (0,1] \to M$ be as in lemma 3. There exists $z \in \overset{\circ}{D}{}^2$ such that the curve $F_z(t) = F(z,t)$ cannot be extended continuously to zero. Since $\mathcal{U}$ is finite this means that there exists a plaque P of $\mathcal{U}$ and points

$$1 \geq t_1 > t_2 > \ldots > t_n > \ldots > 0$$

such that

$$F_z(t_n) \in P \quad \text{for all} \quad n .$$

Next, setting

$$D_n = F_{t_n}(D^2)$$

and using the fact that

$$\partial D_n \cap \partial D_m = \emptyset \quad \text{for} \quad n \neq m$$

we see that either

a) there exists n_o such that $D_n \subset D_{n_o}$ for all $n > n_o$

or

b) there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that

$$D_{n_k} \subset \overset{\circ}{D}_{n_{k+1}} .$$

But in case a) it easily follows that ν would be null-homotopic in its support, so b) must hold, and reindexing, if necessary, we may assume that

$$D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots$$

Now for $x \in Q$ denote by P_x the corresponding plaque of $\mathcal{U}$. Define

$$A_n = \{x \in Q \mid P_x \cap D_n \neq \emptyset\} .$$

Further, let $d \in \mathbb{N}$ be the number of plaques P of $\mathcal{U}$ for which

$$P \cap \nu(S^1) \neq \emptyset .$$

We may assume that

$$L_o \cap \partial \bar{Q} = \emptyset ,$$

where $L_o \in \mathcal{F}$ is the support of ν. Then, for $n \geq n_o$, d is also the number of plaques P of $\mathcal{U}$ such that

$$P \cap f_{t_n}(S^1) \neq \emptyset .$$

Since each F_t is the restriction of a covering map we conclude that

$$\partial D_n \subset F_{t_n}(\partial D^2) .$$

This implies that

$$\#\partial^\Gamma A_n \leq d \quad \text{for} \quad n \geq n_o .$$

Finally, it only remains to show that the numbers $\# A_n$ are unbounded as n tends to infinity. But for $n > m$ any plaque intersecting $f_{t_n}(S^1)$ does not meet $f_{t_m}(S^1)$. This shows that

$$\# A_n > \# A_m \quad \text{for} \quad n > m .$$

Therefore, $\{A_n\}_{n \in \mathbb{N}}$ is an averaging sequence of $(\mathcal{P}, \mathcal{Q})$. □

For the proof of theorem 2 we need one more result that was established in [HH 2; VII, 3.1]:

<u>Proposition 2. If the codimension-one foliation</u> $(M, \mathcal{F})$ <u>admits a closed transversal which is of finite order in</u> $\pi_1 M$ <u>then</u> $\mathcal{F}$ <u>supports a vanishing cycle.</u>

<u>Proof of theorem 2:</u> Every non-compact leaf of $\mathcal{F}$ admits a closed transversal and thus, by proposition 2, $\mathcal{F}$ supports a vanishing cycle. Hence, by proposition 1, any holonomy pseudogroup $(\mathcal{P}, \mathcal{Q})$ of $\mathcal{F}$, with canonical set of generators Γ, admits an averaging sequence $\{A_n\}$. Since Γ is finite the Riesz representation theorem may be used to construct a holonomy invariant measure μ for $(M, \mathcal{F})$ (whose support is contained in $\lim \{A_n\}$); see [HH 2; X, 2.2.4 and 2.2.5]. Since by assumption $H^1(M; \mathbb{R}) = 0$ it follows from [Le] that the support of μ consists of compact leaves (cf. also [HH 2; X, 2.4.7]).

Note that if L is such a leaf then there is no closed transversal passing through L because $H_1(M;\mathbb{R}) = 0$, so L is a torus or Klein bottle; see [Go].

Literature

[Go] Goodman, S.: Closed leaves in foliations of codimension one. Comment. Math. Helv. 50, 383 - 388 (1975)

[GP] Goodman, S. - Plante, J.: Holonomy and averaging in foliated sets. Preprint.

[Ha 1] Haefliger, A.: Structures feuilletées et cohomologie à valeur dans un faisceau de groupoids. Comment. Math. Helv. 32, 249 - 329 (1958)

[Ha 2] Haefliger, A.: Travaux de Novikov sur les feuilletages. Seminaire Bourbaki n° 339, Février 1968

[HH 1] Hector, G. - Hirsch, U.: Introduction to the geometry of foliations. Part A (1981)

[HH 2] Hector, G. - Hirsch, U.: Introduction to the geometry of foliations. Part B (1983)

[No] Novikov, S.P.: Topology of foliations. Trudy Mosk. Math. Obsch. 14, 248 - 278 (1965). Trans. Moscow Math. Soc. 268 - 304

[Sc] Schweitzer, P.: Some problems in foliation theory and related areas. Springer LN 652, 240 - 252 (1978)

[Si] Siebenmann, L.: Deformation of homeomorphisms on stratified sets. Comment. Math. Helv. 47, 123 - 163 (1972)

[So] Solodov, V.V.: Homeomorphisms of the line and a foliation. Math. USSR Izvestiya, 21, No. 2, 341 - 354 (1983)

Bonn, June 1983