New approach to representation theory of semisimple Lie algebras and quantum algebras

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Abstract

A method to construct in explicit form the generators of the simple roots of an arbitrary finite-dimensional representation of a quantum or standard semisimple algebra is found. The method is based on general results from the global theory of representations of semisimple groups. The rank two algebras A_2, $B_2 = C_2$, D_2 and G_2 are considered as examples. The generators of the simple roots are presented as solutions of a system of finite difference equations and given in the form of $N_l \times N_l$ matrices, where N_l is the dimension of the representation.

1 Introduction

The basis vectors of a finite-dimensional irreducible representation $l = (l_1, l_2, \ldots, l_r)$ of a semisimple algebra are usually constructed by repeated application of the lowering generators X_i^- to the highest vector $\mid l\rangle$ with the properties:

$$X_s^+ \mid l\rangle = 0, \quad h_s \mid l\rangle = l_s \mid l\rangle$$

where $X_s^\pm$, h_s are respectively the generators corresponding to the simple roots and the Cartan elements. The obvious problem with such a construction is that not all state vectors arising in this fashion are linearly independent and an additional procedure for excluding linearly dependent components with further orthogonalization of the basis is necessary. Usually this is not a simple matter. Nevertheless, the values which the group element $\exp\tau \equiv \exp\sum h_i\tau_i$ takes on basis vectors may be obtained from the invariant Weyl character formula for the irreducible representation $l = \sum h_i l_i$,

$$\pi^l(\exp\tau) = \frac{\sum_W \delta_W \exp(\tau_W, l + \frac{1}{2}\rho)}{\sum_W \delta_W \exp(\tau_W, \frac{1}{2}\rho)}\ . \tag{1.1}$$

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Presented as a sum of exponents, (the denominator is always a divisor of the numerator!):

$$\pi^l(\exp\tau) = \sum_{n^k}^{N_l} C_{n^k}\exp(\tau, n^k) = \sum_{n^k}^{N_l} C_{n^k}\exp\sum_i^r(\tau_i n_i^k)\ ,$$

this gives answers to many questions about the structure of the basis of the corresponding representation. In (1.1) W is the element of the discrete Weyl group, δ_W is its signature, τ_W is the result of the action of the group element W on τ, C_{n^k} is the multiplicity of corresponding exponent and ρ is the sum of the positive roots of the corresponding algebra.

Recalling the definition,

$$\pi^l(\exp\tau) = Trace(\exp\tau) = \sum_\alpha^{N_l}\langle\alpha \mid (\exp\tau) \mid \alpha\rangle,$$

where $\langle\alpha \mid, \mid \alpha\rangle$ are the bra and ket basis vectors of the representation l, we see that the Weyl formula yields the action of the group element $\exp\tau$ on basis vectors with a given number of lowering operators of various types, namely,

$$e^\tau(X_1^-)^{m_1}...(X_r^-)^{m_r} \mid l\rangle = e^{\sum\tau_i l_i} e^{-\sum_{p,s} m_p\tilde{K}_{p,s}\tau_s}(X_1^-)^{m_1}...(X_r^-)^{m_r} \mid l\rangle \tag{1.2}$$

(of course the ordering of the lowering operators is inessential in this expression). Equating each exponent of the Weyl formula to the corresponding exponent of (1.2) we easily find the indices m_i. The only thing the Weyl formula cannot do is to distinguish among the basis vectors arising from the same set of lowering operators but taken in a different order. Nevertheless, the number of such states is given by the multiplicity of the corresponding exponent in the character formula.

In the present paper the above comments play a key role. We present an alternative way to construct in explicit form the generators of the simple roots. Instead of the procedure of excluding linearly dependent components and subsequent orthogonalization, we need to solve a system of finite difference equations whose solvability is guaranteed by the global theory of representations of semisimple algebras. In principle, these conditions can be found independently (as conditions for the solvability of such system with "fixed" boundary), but they may be seen to be equivalent to known results in global representation theory.

Briefly, our proposed program is as follows. The initial data are the known dimensions and characters of irreducible representations of semisimple groups, given by the famous Weyl formulae [1]. The final ones are the explicit forms of the generators of the simple roots of both quantum and usual semisimple algebras. We do not distinguish between bases for quantum and standard semisimple algebras; and present in explicit form the generators of the simple roots in the form of $(N_l \times N_l)$ matrices (N_l is the dimension of the corresponding representation given by the Weyl dimension formula), passing over the question about the structure of the basis.

The paper is organized in the following way. In section 2 we rewrite defining relations of a quantum algebra in terms of only $2r$ generators instead of the $3r$ generators of the traditional approach. This construction is concretised for algebras of second rank in section 3. In three subsequent sections, 4, 5, and 6, the calculations for the algebras A_2, $B_2 = C_2$ and

G_2 are presented in detail. Concluding remarks and perspectives for further investigation are gathered in section 7.

2 Modified form of equations quantum algebras defined

The customary form of $3r^2$ commutation relations among the $3r$ generators, the simple roots $X_i^{\pm}$ and Cartan elements h_i, of quantum algebras is given by:

$$[h_i, X_j^{\pm}] = \pm K_{j,i} X_j^{\pm}, \quad [X_i^+, X_j^-] = \delta_{j,i} \frac{\sinh(t w_i h_i)}{\sinh(w_i t)} \tag{2.1}$$

where K is the Cartan matrix, $K_{j,i} w_i = w_j K_{j,i} \equiv \tilde{K}_{j,i}$, t is the deformation parameter. The Cartan matrices of the series $A_n, D_n, E_{6,7,8}$ are a priori symmetric with $\tilde{K}_{i,j} = K_{i,j}$, $w_i = 1$.

Let us introduce the alternative set of $3r$ generators belonging to the universal enveloping algebra,

$$T_i^{\pm} = e^{\frac{w_i t h_i}{4}} X_i^{\pm} e^{\frac{w_i t h_i}{4}}, \quad R_i = e^{t w_i h_i}$$

In terms of these the relations determining quantum algebra (2.1) may be rewritten,

$$R_i T_j^{\pm} = e^{\pm \tilde{K}_{j,i} t} T_j^{\pm} R_i, \quad e^{\frac{\tilde{K}_{j,i} t}{2}} T_i^+ T_j^- - e^{\frac{-\tilde{K}_{j,i} t}{2}} T_j^- T_i^+ = \delta_{i,j} \frac{R_i^2 - 1}{2 \sinh w_i t} \ . \tag{2.2}$$

Introducing $2r$ generators,

$$Q_i^{\pm} = T_i^{\pm} \pm \frac{R_i}{2 \sinh w_i t}$$

the system (2.2) takes the form

$$\begin{gathered} e^{w_i t} Q_i^+ Q_i^- - e^{-w_i t} Q_i^- Q_i^+ = -\frac{1}{2 \sinh w_i t} \\ e^{\frac{\tilde{K}_{j,i} t}{2}} (Q_i^+ Q_j^- - Q_j^+ Q_i^-) - e^{\frac{-\tilde{K}_{j,i} t}{2}} (Q_j^- Q_i^+ - Q_i^- Q_j^+) = 0, \quad \tilde{K}_{i,j} \neq 0 \\ [Q_i^+, Q_j^-] = [Q_j^+, Q_i^-] = 0, \quad \tilde{K}_{i,j} = 0 \\ e^{\frac{\tilde{K}_{j,i} t}{2}} (Q_i^+ Q_j^- + Q_j^+ Q_i^-) - e^{\frac{-\tilde{K}_{j,i} t}{2}} (Q_j^- Q_i^+ + Q_i^- Q_j^+) \\ = -\frac{\sinh \frac{\tilde{K}_{j,i}}{2} t}{\sinh w_i t \sinh w_j t} R_i R_j, \quad \tilde{K}_{i,j} \neq 0, \quad i \neq j \\ R_i Q_j^{\pm} = e^{\pm \tilde{K}_{j,i} t} Q_j^{\pm} R_i \mp (e^{\pm \tilde{K}_{j,i} t} - 1) \frac{R_j R_i}{2 \sinh w_j t} \end{gathered} \tag{2.3}$$

We note that the first three rows relate the $Q_i^{\pm}$ among themselves. It is therefore possible to consider these relations as some subalgebra of the universal enveloping algebra (2.1).

Now we would like to show that as a direct corollary of equations (2.3) all generators R_i may be expressed algebraically as functionals of generators $Q_s^{\pm}$. For this purpose let us multiply the equation of the last row of (2.3) (with exchange indices $i \to j$) on R_i from the left. We obtain

$$R_i R_j Q_i^{\pm} = e^{\pm\tilde{K}_{i,j}t} R_i Q_i^{\pm} R_j \mp \; (e^{\pm\tilde{K}_{ij}t} - 1)\frac{R_i^2 R_j}{2\sinh w_i t} =$$

$$e^{\pm(\tilde{K}_{ii}+\tilde{K}_{i,j}t)} Q_i^{\pm} R_i R_j \;\mp\; (e^{\pm(\tilde{K}_{ii}+\tilde{K}_{ij}t)} - 1)\frac{R_i R_j R_i}{2\sinh w_i t} \tag{2.4}$$

Introducing the operator

$$Q_{i,j} \equiv e^{\frac{\tilde{K}_{ij}t}{2}}(Q_i^+ Q_j^- + Q_j^+ Q_i^-) - e^{-\frac{\tilde{K}_{ij}t}{2}}(Q_j^- Q_i^+ + Q_i^- Q_j^+) = Q_{j,i} \;,$$

which in virtue of (2.3) is proportional to $R_i R_j$, we rewrite (2.4) in the form,

$$Q_{ij} Q_i^{\pm} Q_{ij}^{-1} = e^{\pm(\tilde{K}_{ii}+\tilde{K}_{ij})t} Q_i^{\pm} \mp \left(e^{\pm(\tilde{K}_{ii}+\tilde{K}_{ij})t} - 1\right)\frac{R_i}{2\sinh w_i t} \;. \tag{2.5}$$

From (2.5) we conclude that in the case $\tilde{K}_{ii} + \tilde{K}_{ij} \neq 0$ the proposition above is true and the generator R_i may be expressed algebraically in terms of the generators $Q^{\pm}$. In the case $\tilde{K}_{ii} + \tilde{K}_{ij} = 0$, definitely $\tilde{K}_{jj} + \tilde{K}_{ij} \neq 0$ and generator R_j can be expressed in terms of Q generators. Then R_i can be found from the equation relating $Q_{i,j}$ to the product $R_i R_j$. Thus the above assertion is true in all cases.

In the next section we specify ourselves to the rank two cases. The rank one case is well known and we summarize it for later use. Irreducible representations of the A_1^q algebra are labelled by natural or half natural number l, with dimension of the representation given by $2l+1$. The generator H takes all odd or even values between $2l$ and $-2l$; $H_k = 2l - 2k \equiv 2m, 0 \leq k \leq 2l$. The non-zero matrix elements of generators $X^{\pm}$ are

$$X^{\pm}_{m,m\pm1} \;=\; \left(\frac{\sinh(l\mp m)t}{\sinh t}\frac{\sinh(l\pm m+1)t}{\sinh t}\right)^{\frac{1}{2}}$$

where $m{=}l{-}k$ and the condition $X^+ = (X^-)^T$ is satisfied. The structure of the $Q^{\pm}$ generators is as follows: diagonal elements are $Q^{\pm}_{m,m} = \frac{e^{\pm 2(l-k)t}}{2\sinh t}$; and the non-zero nondiagonal elements,

$$Q^{\pm}_{m,m\pm1} = e^{(m\pm\frac{1}{2})t}\left(\frac{\sinh(l\mp m)t}{\sinh t}\frac{\sinh(l\pm m+1)t}{\sinh t}\right)^{\frac{1}{2}} \;.$$

3 The algebras of the second rank

In this section we restrict ourselves indices taking only two values $i = 1, 2$ in the general system (2.3) or to algebras of second rank, $A_2, B_2 = C_2, G_2$. The symmetrical Cartan matrices for these algebras have the form,

$$\tilde{K} = \begin{pmatrix} 2 & -p \\ -p & 2p \end{pmatrix}, \quad w_1 = 1, \quad w_2 = p \;,$$

where $p=1,2,3$ for the cases A_2, B_2, G_2 respectively. The following additional abbreviations are suitable:

$$Q_1^{\pm}=\frac{s^1\pm r^1}{2\sinh t},\quad Q_2^{\pm}=\frac{s^2\pm r^2}{2\sinh pt}$$

The first two rows of (2.3) take the form:

$$\begin{aligned}
[s^1,r^1] &= \tanh t\,((s^1)^2-(r^1)^2+1) \\
[s^2,r^2] &= \tanh pt((s^2)^2-(r^2)^2+1) \\
[s^1,s^2]-[r^1,r^2] &= \tanh \tfrac{pt}{2}(\{r^1,s^2\}-\{s^1,r^2\})
\end{aligned} \tag{3.1}$$

The operator $Q_{1,2}\equiv Q$ is given by (we use a rescaled version of the general definition in the previous section),

$$Q \;=\; \frac{\sinh t\sinh pt}{\sinh\frac{pt}{2}}\left(e^{\frac{-pt}{2}}(Q_1^+Q_2^-+Q_2^+Q_1^-\right)Q-e^{\frac{pt}{2}}(Q_1^-Q_2^++Q_2^-Q_1^+) \;=\; R_1R_2 \tag{3.2}$$

The operators R_1, R_2 are expressed in terms of operators $Q, Q_{1,2}^{\pm}$ in virtue of the relations

$$\begin{aligned}
QQ_1^{\pm}Q^{-1} &= e^{\mp(p-2)t}Q_1^{\pm} \;\mp\; \left(e^{\mp(p-2)t}-1\right)\frac{R_1}{2\sinh t}, \\
QQ_2^{\pm}Q^{-1} &= e^{\pm pt}Q_2^{\pm} \;\mp\; \left(e^{\pm pt}-1\right)\frac{R_2}{\sinh pt}
\end{aligned} \tag{3.3}$$

In terms of $s^{1,2}, r^{1,2}$ these relations may be rewritten in form more suitable for our purposes:

$$\begin{aligned}
\sinh(p-2)tR_1 &= \sinh(p-2)tr^1+Qs^1Q^{-1}-\cosh(p-2)ts^1 \\
-\sinh ptR_2 &= -\sinh ptr^2+Qs^2Q^{-1}-\cosh pts^2\;.
\end{aligned} \tag{3.4}$$

We note that in the case of the $B_2=C_2$ algebra ($p=2$) operator R_1 can't be defined from (3.3). But R_2 is well defined and R_1 may be algebraically expressed after this from the equation for Q operator. Eliminating $R_{1,2}$ from equations (3.3) we conclude that,

$$\begin{aligned}
Q\left(e^{\frac{(p-2)t}{2}}Q_1^+-e^{-\frac{(p-2)t}{2}}Q_1^-\right)Q^{-1} &= \frac{(p-2)t}{2}Q_1^+-e^{\frac{(p-2)t}{2}}Q_1^- \\
Q\left(e^{-\frac{pt}{2}}Q_2^+-e^{\frac{pt}{2}}Q_2^-\right)Q^{-1} &= e^{\frac{pt}{2}}Q_2^+-e^{-\frac{pt}{2}}Q_2^-\;.
\end{aligned}$$

These equations are equivalent to

$$\begin{aligned}
\cosh\frac{(p-2)t}{2}\left(r^1-Qr^1Q^{-1}\right) &= \sinh\frac{(p-2)t}{2}(s^1+Qs^1Q^{-1}), \\
\cosh\frac{pt}{2}\left(r^2-Qr^2Q^{-1}\right) &= -\sinh\frac{pt}{2}\left(s^2+Qs^2Q^{-1}\right)
\end{aligned} \tag{3.5}$$

Using the fact that operators R_1, R_2, Q are mutual commutative as a direct consequence of (3.3) we obtain the following important relations

$$Qs^1Q^{-1}+Q^{-1}s^1Q \;=\; 2\cosh(p-2)ts^1,\quad Qs^2Q^{-1}+Q^{-1}s^2Q \;=\; 2\cosh pts^2 \tag{3.6}$$

From these relations we conclude that in representations with diagonal Q operator, matrix elements of generators $s^{1,2}$ satisfy the equations

$$\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}-e^{(p-2)t}-e^{-(p-2)t}\right)s^1_{i,j} \;=\; 0\;, \quad \left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}-e^{pt}-e^{-pt}\right)s^2_{i,j} \;=\; 0\;. \tag{3.7}$$

In other words the matrix elements $s^1_{i,j}$ are different from zero only in the case when $\frac{\lambda_i}{\lambda_j}=e^{\pm(p-2)t}$, and similarly, the matrix elements $s^2_{i,j}$ are nonzero when $\frac{\lambda_i}{\lambda_j}=e^{\pm pt}$.

In the next sections, we will concretize these relations for the three individual cases of $A_2, B_2=C_2, G_2$ algebras.

4 The A_2 algebra case

We work in the basis with diagonal Q generator. From (3.7) ($p=1$) we can conclude that $\frac{\lambda_i}{\lambda_{i+1}}=e^t$ (keeping in mind irreducibility of s^1, r^1 matrices). But about multiplicity of each λ_i nothing can be deduced and we assume it to be arbitrary, N_i. From (3.6) follows the "row" structure of the $s^{1,2}$ generators. Namely

$$s^1=(\dots.\quad a_{i,i-1}\quad 0\quad a_{i,i+1}\dots)\,,\quad s^2=(\dots.\quad b_{i,i-1}\quad 0\quad b_{i,i+1}\dots)$$

where $a_{i,i-1}, b_{i,i-1}$ are rectangular matrices of the dimension $N_{i-1}\times N_i$ wether as $a_{i,i+1}, b_{i,i+1}$ are rectangular matrices of the dimensions $N_i\times N_{i+1}$ (zero in the center is quadratic $N_i\times N_i$ zero matrix). From equations (3.5) ($p=1$!) it is possible to reconstruct the "row" structure of $r^{1,2}$ matrices, namely,

$$r^1=(\dots.\quad -a_{i,i-1}\quad \alpha_i\quad a_{i,i+1}\dots)\,,\quad r^2=(\dots.\quad -b_{i,i-1}\quad \beta_i\quad b_{i,i+1}\dots)$$

where α_i, β_i are quadratic $N_i\times N_i$ matrices. The system of equations which matrices a, b, α, β satisfy arises after substitution of these ansätze for s, r into (3.1) and (3.2). Equation relating s^1, r^1 is equivalent to a matrix system:

$$e^{-t}a_{n,n-1}\alpha_{n-1}=e^t\alpha_n a_{n,n-1},\quad e^t a_{n,n+1}\alpha_{n+1}=e^{-t}\alpha_n a_{n,n+1},$$
$$2e^{-t}a_{n,n-1}a_{n-1,n}-2e^t a_{n,n+1}a_{n+1,n}=\sinh t(I_n-\alpha_n^2) \tag{4.8}$$

The same for s^2, r^2 leads to:

$$e^{-t}b_{n,n-1}\beta_{n-1}=e^t\beta_n b_{n,n-1},\quad e^t b_{n,n+1}\beta_{n+1}=e^{-t}\beta_n b_{n,n+1},$$
$$2e^{-t}b_{n,n-1}b_{n-1,n}-2e^t b_{n,n+1}b_{n+1,n}=\sinh t(I_n-\beta_n^2) \tag{4.9}$$

where I_n denotes the $N_n\times N_n$ unit matrix. The last equation in (3.1) and the definition of Q (3.2) imply the system,

$$e^{\frac{t}{2}}b_{n,n-1}\alpha_{n-1}=e^{-\frac{t}{2}}\alpha_n b_{n,n-1},\quad e^{-\frac{t}{2}}a_{n,n+1}\beta_{n+1}=e^{\frac{t}{2}}\beta_n a_{n,n+1},$$
$$e^{\frac{t}{2}}a_{n,n-1}\beta_{n-1}=e^{-\frac{t}{2}}\beta_n a_{n,n-1},\quad e^{-\frac{t}{2}}b_{n,n+1}\beta_{n+1}=e^{\frac{t}{2}}\alpha_n b_{n,n+1},$$

$$2e^{-\frac{t}{2}}\left(\frac{\alpha_n\beta_n}{4}+b_{n,n+1}a_{n+1,n}\right)-2e^{\frac{t}{2}}\left(\frac{\beta_n\alpha_n}{4}+a_{n,n-1}b_{n-1,n}\right)=\sinh\frac{t}{2}\lambda_n I_n \qquad (4.10)$$

At first sight the system (4.8),(4.9) and (4.10) is so complicated that all attempts to solve it seem to have a little chance for success. But this is not so, as we shall demonstrate in the next few pages. First of all let us substitute the ansatz for s^1, r^1, s^2, r^2 into (3.4) ($p=1$). Direct calculations show that R_1, R_2 are the block-diagonal matrices with block matrices α_n, β_n, respectively. Moreover the mutual commutativity of R_1, R_2 have as its consequence the mutual commutativity $[\alpha_n, \beta_n]=0$. Equation (3.2) relating Q with R_1R_2 may be rewritten in the form:

$$\lambda_n I_n = \alpha_n\beta_n.$$

Taking into account these circumstances we can eliminate matrices β_n from the system (4.8)-(4.10) and rewrite it in the form,

$$\begin{aligned}
&e^{-t}a_{n,n-1}\alpha_{n-1}=e^{t}\alpha_n a_{n,n-1} \quad , \quad e^{t}a_{n,n+1}\alpha_{n+1}=e^{-t}\alpha_n a_{n,n+1}\\
&e^{\frac{t}{2}}b_{n,n-1}\alpha_{n-1}=e^{-\frac{t}{2}}\alpha_n b_{n,n-1} \quad , \quad e^{-\frac{t}{2}}b_{n,n+1}\alpha_{n+1}=e^{\frac{t}{2}}\alpha_n b_{n,n+1},\\
&2e^{-t}a_{n,n-1}a_{n-1,n}-2e^{t}a_{n,n+1}a_{n+1,n}=\sinh t(I_n-\alpha_n^2)\\
&2e^{-t}b_{n,n-1}b_{n-1,n}-2e^{t}b_{n,n+1}b_{n+1,n}=\sinh t(I_n-\lambda_n^2\alpha_n^{-2})\\
&e^{-\frac{t}{2}}b_{n,n+1}a_{n+1,n}-e^{\frac{t}{2}}a_{n,n-1}b_{n-1,n}=0
\end{aligned} \qquad (4.11)$$

As mentioned above the matrix α_n commutes with β_n and so they both simultaneously can be presented in diagonal form. The diagonal elements of α_n , β_n we will denote by double indices α^s_n , β^s_n , $1\le s\le N_n$. Obviously $\alpha^s_n\beta^s_n=\lambda_n$.

The system (4.11) in the presented form is unlimited and for its solution some additional "boundary" conditions are necessary. To have solution in the form of finite-dimensional matrices we will assume that on its "left end" $a_{1,0}=b_{1,0}=0$ and on its "right end" $a_{N,N+1}=b_{N,N+1}=0$. The numbers N_n and diagonal elements λ_n, α^s_n must be found as conditions for resolving (4.11) under such boundary conditions. We shall use known facts from global representation theory of the A_2 algebra to resolve the system (4.11) explicitly and thus obtain explicit forms for the generators of the simple roots for the irreducible representations (p,q) of the A_2 algebra.

As was mentioned in the introduction basis vectors of irreducible representations of semisimple algebras are constructed from the single highest vector (2.1) by action of lowering operators. From the definition of the highest vector (2.1) it follows immediately that $\lambda_1=e^{(p+q)t}$, $\alpha^1_1=e^{pt}$ and $N_1=1$. We recall that $\alpha=\exp h_1 t$, $\lambda=\exp(h_1+h_2)t$. Let us consider the next bases vectors which arise after action by generators $X^-_{1,2}$ on highest vector. It is obvious that

$$N_2=2, \quad (q\neq 0), \quad \lambda_2=e^{(p+q-1)t}, \quad \alpha^1_2=e^{(p-2)t}, \quad \alpha^2_2=e^{(p+1)t}$$

Let us for the meantime set aside the problem of linearly independent components. Then at each step after application of two generators $X^-_{1,2}$ to each basis vector of the previous step the number of basis vectors will be twice that on the previous step and the following relations become obvious

$$N_s=2^{s-1}, \quad \lambda_s=e^{(p+q-s+1)t}, \quad \alpha^k_s=e^{(p-2(s-1)+3k)t}, \quad (C^k_{s-1}), \quad 0\le k\le(s-1)$$

The multiplicities C_r^k are binomial coefficients.

Now, let us return to a real situation. The character of irreducible representation (p,q) of A_2 algebra is given by the Weyl formula

$$\pi^{(p,q)}(\exp(\tau_1 h_1+\tau_2 h_2)=\frac{Det\begin{pmatrix} e^{\tau_1 l_1} & e^{\tau_1 l_2} & e^{\tau_1 l_3}\\ e^{(\tau_2-\tau_1)l_1} & e^{(\tau_2-\tau_1)l_2} & e^{(\tau_2-\tau_1)l_3}\\ e^{-\tau_2 l_1} & e^{-\tau_2 l_2} & e^{-\tau_2 l_3}\end{pmatrix}}{Det\begin{pmatrix} e^{\tau_l} & 1 & e^{-\tau_1}\\ e^{(\tau_2-\tau_1)} & 1 & e^{-(\tau_2-\tau_1)}\\ e^{-\tau_2} & 1 & e^{\tau_2}\end{pmatrix}} \tag{4.12}$$

where $l_1-l_2=p+1, l_2-l_3=q+1, l_1+l_2+l_3=0$. Evaluating the determinants leads to,

$$\begin{aligned}\pi &= \frac{1}{1-e^{-(\tau_2+\tau_1)}}[e^{(\tau_2-\tau_1)(p-q)}(e^{px}+\cdots+1)(e^{qy}+\cdots+1)\\ &\quad - e^{-\tau_1(q+1)}e^{-\tau_2(p+1)}(e^{qx}+\cdots+1)(e^{py}+\cdots+1)],\end{aligned} \tag{4.13}$$

where $x=2\tau_1-\tau_2$, $y=2\tau_2-\tau_1$, $x+y=\tau_2+\tau_1$. It is obvious that under the condition $\tau_2+\tau_1=0, x+y=0$ the numerator is equal to zero and and thus the last expression passes to the sum of exponentials the arguments of which are different linear combinations of τ with definite coefficients.

The reduction of (4.12) or (4.13) to the A_1 subgroup with the infinitesimal generators $X^+=[X_1^+,X_2^+], X^-=[X_1^-,X_2^-], H=h_1+h_2$ is equivalent to the substitution $\tau_1=\tau_2=t$, yielding the final result:

$$\begin{aligned}\pi^{p,q}(\exp(h_1+h_2)t) &= e^{-(p+q)t}\frac{\sum_{k=0}^{p+q+1}e^{(p+q+1-k)t}\sum_{r=0}^{q}e^{(q-r)t}\sum_{s=0}^{p}e^{(p-s)t}}{(e^t+1)}\\ &= \sum_{m=0}^{2(p+q)}e^{(p+q-m)t}c_m(p,q)\end{aligned} \tag{4.14}$$

Among three natural numbers $p,q,p+q+1$ at least one is odd and by this reason numerator is always divided on denominator. The reduction to the A_1 subgroups with the algebras of the first and second simple roots $X^{\pm}_{1,2}, h_{1,2}$ leads to the same expressions exponents in Weyl formula but in the different order. This result is obvious without any calculations, because Cartan elements of all roots of A_2 algebra are relating by discrete Weyl transformations. Before the general consideration we would like to consider the concrete example of the $(2,1)$ representation of A_2 algebra. This example will reveal the main points of the whole construction.

4.1 $(2,1)$ representation of A_2 algebra

The direct calculation of the character of the $(2,1)$ representation leads to the following sum of exponents (in connection with the (4.13)), which we have written in a definite order the sense of which will be soon understandable:

$$\pi^{(2,1)}(\tau)=e^{(2\tau_1+\tau_2)}+e^{2\tau_2}+e^{(3\tau_1-\tau_2)}+e^{(-2\tau_1+3\tau_2)}+2e^{\tau_1}+2e^{(-\tau_1+\tau_2)}+e^{(-2\tau_1+2\tau_2)}$$

$$e^{-3\tau_1+2\tau_2}+2e^{-\tau_2}+e^{-2\tau_1}+e^{(\tau_1-3\tau_2)}+e^{-(\tau_1+2\tau_2)}$$

Let us now consider reducing of this expression to the $SL(2,R)$ subgroup related to the composite root of A_2. Writing $\tau_1=\tau_2=t$ we obtain,

$$\pi=e^{3t}+2e^{2t}+3e^{t}+3+3e^{-t}+2e^{-2t}+e^{-3t}$$

Since the generator h_1+h_2 takes unit values on each simple root of the A_2 algebra, it follows that the states of highest (lowest) vectors are the singlet ones, subspaces with two or five lowering operators are two- dimensional and so on. The values taken by operator $R_1R_2\equiv\exp(h_1+h_2)t$ on the basis vectors of the $(2,1)$ representation are precisely the exponents of the sum above taken in the same order with the same multiplicity. By the arguments of the same kind we find the values, which generators R_1,R_2 take on this basis. We present them in the row form performing the correct order:

$$R_1=(e^{2t},1,e^{3t},e^{-2t},e^{t},e^{t},e^{-t},e^{-t},e^{2t},e^{-3t},1,1,e^{-2t},e^{t},e^{-t})$$

$$R_2=(e^{t},e^{2t},e^{-t},e^{3t},1,1,e^{t},e^{t},e^{-2t},e^{2t},e^{-t},e^{-t},1,e^{-3t},e^{-2t})$$

These define simultaneously the dimensions of α,β matrices and explicit expressions for them in the diagonal form. The explicit form of α_i matrices (with brackets denoting the diagonal elements) are,

$$\alpha_1=e^{2t},\alpha_2=(1,e^{3t}),\alpha_3=(e^{-2t}e^{t},e^{t}),\alpha_4=(e^{-t},e^{-t},e^{2t}),$$

$$\alpha_5=(e^{-3t},1,1),\alpha_6=(e^{-2t},e^{t}),\alpha_7=e^{-t}.$$

Knowledge of the explicit expressions for α matrices allow without any difficulties to resolve the first row of system (3.7) and present r^1 matrix is the following form:

$$r^1=\begin{pmatrix}
e^{2t} & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-X & 1 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & e^{3t} & 0 & B & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -A & 0 & e^{-2t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -B & 0 & e^{t} & 0 & D & E & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -C & 0 & 0 & e^{t} & F & G & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -D & -F & e^{-t} & 0 & 0 & H & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -E & -G & 0 & e^{-t} & 0 & K & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2t} & 0 & L & M & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -H & -K & 0 & e^{-3t} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -L & 0 & 1 & 0 & N & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -M & 0 & 0 & 1 & P & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -N & -P & e^{-2t} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{t} & Y\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y & e^{-t}
\end{pmatrix}.$$

The ansatz for s^1 matrix arise from the same for r^1, one by deleting diagonal elements and changing the remaining matrix to be symmetrical rather than antisymmetrical. In writing the above ansatz we have assumed additionally that s^1 may be chosen in symmetrical form. For the same reasons the ansatz for r^2 generator has the form:

$$r^2 = \begin{pmatrix}
e^t & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{2t} & 0 & 0 & n & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y & 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{3t} & 0 & 0 & h & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -n & 0 & 0 & 1 & 0 & 0 & 0 & l & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -p & 0 & 0 & 0 & 1 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -h & 0 & 0 & e^t & 0 & 0 & 0 & d & f & 0 & 0 & 0 \\
0 & 0 & 0 & -k & 0 & 0 & 0 & e^t & 0 & 0 & e & g & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -l & -m & 0 & 0 & e^{-2t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2t} & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -d & -e & 0 & 0 & e^{-t} & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -f & -g & 0 & 0 & 0 & e^{-t} & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 1 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & -c & 0 & e^{-3t} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & 0 & e^{-2t}
\end{pmatrix}.$$

The anzates for $s^{1,2}$ obviously follows from the same for $r^{1,2}$. In fact with the help of these ansätze we have resolved first two rows of equations of the system (4.11). We are now ready to solve the system (4.11) as a whole. The first rows of second part of equations (4.11) lead to the following values of X, y ($a_{1,0} = b_{1,0} = 0$):

$$X^2 = e^t \sinh t \sinh 2t, \quad y^2 = \sinh^2 t$$

Resolution of two next rows is the following:

$$A^2 = e^{-t} \sinh t \sinh 2t, \qquad B^2 + C^2 = e^{2t} \sinh t \sinh 3t,$$
$$nB + pC = e^t y X, \qquad n^2 + p^2 = e^t \sinh t \sinh 2t.$$

The last equations define two two-dimensional vectors with given angle between them:

$$(B, C) = e^t (\sinh t \sinh 3t)^{\frac{1}{2}} t (\cos \phi_1, \sin \phi_1),$$
$$(n, p) = e^{\frac{t}{2}} (\sinh t \sinh 2t)^{\frac{1}{2}} (\cos \phi_2, \sin \phi_2),$$
$$\cos(\phi_1 - \phi_2) = \left(\frac{\sinh t}{\sinh 3t} \right)^{\frac{1}{2}}.$$

This is analogue of the quantum angle mixed the states π^0 and Ω mesons in $(1,1)$ representation of A_2 algebra in the initial version of $SU(3)$ symmetry of composite models of elementary particles. In what follows we work in the gauge $\phi_2 = 0$ ($p = 0$). But this choice of gauge is absolutely inessential. The system of equations of the third step is as follows:

$$DF + EG = e^{-2t} BC = \sinh^2 t (2 \cosh 2t)^{\frac{1}{2}},$$
$$D^2 + E^2 = e^{-2t} B^2 + \sinh^2 t = 2 \sinh^2 t,$$
$$F^2 + G^2 = e^{-2t} C^2 + \sinh^2 t = \sinh t \sinh 3t,$$
$$l^2 = e^{-2t} n^2, \qquad m^2 = e^{-2t} p^2 = 0, \qquad ml = e^{-2t} np = 0,$$
$$h^2 + k^2 = e^{2t} \sinh t \sinh 3t, \qquad m = 0,$$
$$Dh + Ek = e^t An = e^t \sinh t \sinh 2t, \qquad Fh + Gk = e^t Ap = 0.$$

The ansätze for $s^{1,2}, r^{1,2}$ are obviously form invariant with respect to two-dimensional rotations in $(5-6),(7-8),(11-12)$ planes (the reflection of this fact was the possibility to choose $p=0$ at the second step of the calculations with the help of rotations in (5,6) plane). Now we can use this invariance choosing $E=0$ in the equations of the fourth step. After this all equations above can be resolved without any difficulties with the result:

$$D=\pm(2)^{\frac{1}{2}}\sinh t,\quad F=\pm\sinh t(\cosh 2t)^{\frac{1}{2}},\quad G^2=\frac{1}{2}\sinh^2 2t,$$

$$l^2=e^{-t}\sinh t\sinh 2t,\quad m=0,\quad h=\pm(2)^{-\frac{1}{2}}e^t\sinh 2t,\quad k=\pm e^t\sinh t(\cosh 2t)^{\frac{1}{2}}$$

In the equations of the fifth step we use the $(11-12)$ invariance to fix $M=0$. Keeping in mind this choice, we have:

$$\begin{gathered}
HK=e^{-2t}FG=\pm 2^{-\frac{1}{2}}\sinh t\sinh 2t(\cosh 2t)^{\frac{1}{2}},\\
H^2=e^{-2t}(D^2+F^2-\sinh^2)=\tfrac{1}{2}e^{-2t}\sinh^2 2t,\\
K^2=e^{-2t}(G^2-\sinh^2 t)=e^{-2t}\sinh^2 t\cosh 2t,\\
L^2=e^2\sinh t\sinh 2t,\qquad dL+fM=e^t(Dl+Fm),\\
eL+gM=e^t(El+Gm),\qquad d=D,\qquad e=E=0,\\
e^2+g^2=e^{-2t}k^2+\sinh^2 t=\tfrac{1}{2}\sinh^2 2t,\\
de+fg=e^{-2t}hk=2^{-\frac{1}{2}}\sinh t\sinh 2t(\cosh 2t)^{\frac{1}{2}},\\
d^2+f^2=e^{-2t}h^2+\sinh^2 t=\sinh^2 t(\cosh 2t+2),\quad l^2=e^{-t}\sinh t\sinh 2t.
\end{gathered}$$

Solution of this system is trivial. We present below the final result:

$$d=D,\quad e=E=0,\quad f=F,\quad g=G.$$

The explicit form of H, K is given above. We will not reproduce here the two remaining steps of calculations because they do not contain anything new and to reconstruct them is a trivial problem.

We note that always on the next following step we have the system of quadratic equations only for unknown matrix elements of $a_{n+1,n}, a_{n,n+1}, b_{n+1,n}, b_{n,n+1}$ matrices, the right hand side of which was calculated on the previous step. On account of the invariance conditions the number of equations exactly equal the number of unknown variables. The selfconsistency of the whole construction is only a consequence of the global representation theory.

4.2 The general case of (p,q) representation

To generalize the results of the previous subsection to the case of arbitrary (p,q) representation of A_2 algebra more detailed information about the structure of $a_{n,n\pm1}, b_{n,n\pm1}$ is necessary. The first two rows of the system (4.11) give additional restrictions on them. Namely different from zero are only those matrix elements $(a_{n,n\pm1})_{s,s'}, (b_{n,n\pm1})_{s,s'}$ for which (s,s') satisfy the conditions,

$$\frac{\alpha_n^s}{\alpha_{n\pm1}^{s'}}=e^{\pm 2t},\quad \frac{\beta_n^s}{\beta_{n\pm1}^{s'}}=e^{\pm 2t}$$

The diagonal elements of α, β matrices are in turn relating by the relations,

$$\frac{\alpha_n^s}{\alpha_n^{s+1}} = e^{-3t}, \quad \frac{\beta_n^s}{\beta_n^{s+1}} = e^{3t}$$

Let us denote the multiplicity of diagonal elements α_n^s, β_n^s as N_n^s. Obviously $N_n = \sum_s N_n^s$. The multiplicity N_n^s is exactly equal to the natural number in the corresponding exponent in the Weyl character formula (1.1). After reducing this to the A_1 subgroup with the algebra of the first (second) simple root this is exactly the coefficient of $\alpha_n^s(\beta_n^s)$.

Each matrix $a_{n,n+1}$ (for definitness we choose the plus sign) is separated into $N_n^s \times N_{n+1}^{s'}$ rectangular block matrices $a_{n,n+1}^{s,s'}$, where (s, s') are the indices of diagonal matrix elements $\alpha_n^s, \alpha_{n+1}^{s'}$ respectively. Only those block matrices for which $\frac{\alpha_n^s}{\alpha_{n+1}^{s'}} = e^{\pm 2t}$ are nonzero. Two important consequences follow from this fact. Firstly, on the line (with the "wide" N_n^s) and on the column (with the "wide" $N_{n+1}^{s'}$) may be only one different from zero block matrix. Secondly, all matrices with shifted on natural number indices $a_{n,n+1}^{s\pm k,s'\pm k}$ (k - natural positive) are different from zero simultaneosly with $a_{n,n+1}^{s,s'}$. Of course if the indices $s \pm k, s' \pm k$ are inside of the domain of their definition.

From the explicit form of ansätze for $s^{1,2}, r^{1,2}$ in the beginning of this section follows their form invariance with respect to each $SL(N_n^s; R)$ canonical transformations, which does not change diagonal matrix elements, transforming a matrices by the law:

$$a \to G(N_n^s, R) a G(N_{n+1}^{s'})$$

If we want to preserve symmetry of $s^{1,2}$ matrices (they are symmetrical by our convention up to now) it is necessary to reduce this transformation up to direct product of orthogonal $O_{N_n^s}$ transformations. With similar considerations holding for the b matrices, we are able to rewrite the remaining equations of the system (4.11) (the three last rows) in terms of only "primitive" block matrices:

$$\begin{gathered} e^{-t} a_{n,n-1}^{s,s+2} a_{n-1,n}^{s+2,s} - e^t a_{n,n+1}^{s,s-2} a_{n+1,n}^{s-2,s} = \alpha_n^s \sinh t \sinh(\ln \alpha_n^s) I_{N_n^s} \\ e^{-\frac{t}{2}} b_{n,n+1}^{s,s+1} a_{n+1,n}^{s+1,s+3} = e^{\frac{t}{2}} a_{n,n-1}^{s,s+2} b_{n-1,n}^{s+2,s+3} \\ e^{-t} bs, s-1_{n,n-1} b_{n-1,n}^{s-1,s} - e^t b_{n,n+1}^{s,s+1} b_{n+1,n}^{s+1,s} = -\sinh t (\lambda_n (\alpha_n^s)^{-1} \sinh(\ln(\lambda_n (\alpha_n^s)^{-1})) I_{N_n^s} \end{gathered} \tag{4.15}$$

where $I_{N_n^s}$ is the unity $N_n^s \times N_n^s$ matrix; the index s runs all values of α_n^s. The initial system (4.11) is therefore split into a chain like system of equations for block matrices and each chain is living absolutely independent from the other ones. On each step of calculations we have to solve the following problem: it is necessary to find the rectangular matrices of the given dimension a, b, which satisfy the following system of algebraic equations:

$$aa^T = A, \quad bb^T = B, \quad ba^T = C, \quad (ab^t = C^T) \tag{4.16}$$

where A, B, C are the known matrices. In connection with (4.15) they are defined on the previous step of calculation (exactly from the system (4.16)) together with α_n^s, β_n^s known from the Weyl formula, which guarantees the consistency of the whole construction.

Now we want to illustrate the consideration above by the the example of the previous subsection. The block structure of $s^{1,2}, r^{1,2}$ ansätze is the following one:

$$a_{1,2}^{2,0} = X, \quad a_{2,3}^{0,-2} = A, \quad a_{2,3}^{3,1} = \begin{pmatrix} B & C \end{pmatrix}, a_{3,4}^{1,-1} = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$$

$$a_{4,5}^{-1,-3} = \begin{pmatrix} H \\ K \end{pmatrix}, \quad a_{4,5}^{2,0} = \begin{pmatrix} L & M \end{pmatrix}, \quad a_{5,6}^{0,-2} = \begin{pmatrix} N \\ P \end{pmatrix}, \quad a_{6,7}^{1,-1} = Y$$

$$b_{1,2}^{2,3} = y, \quad b_{2,3}^{0,1} = \begin{pmatrix} n & p \end{pmatrix}, \quad b_{3,4}^{-2,-1} = \begin{pmatrix} h & k \end{pmatrix}, \quad b_{3,4}^{1,2} = \begin{pmatrix} l \\ m \end{pmatrix}$$

$$b_{4,5}^{-1,0} = \begin{pmatrix} d & f \\ e & g \end{pmatrix}, \quad b_{5,6}^{0,1} = \begin{pmatrix} b \\ c \end{pmatrix}, \quad b_{5,6}^{-3,-2} = a, \quad , b_{6,7}^{-2,-1} = x$$

We would like to demonstrate the chain like structure with the example of equations relating primitive a matrices:

$$-e^t a_{1,2}^{2,0} a_{2,1}^{0,2} = -e^{2t} \sinh t \sinh 2t, \qquad e^{-t} a_{2,1}^{0,2} a_{1,2}^{2,0} - e^t a_{2,3}^{0,-2} a_{3,2}^{-2,0} = 0,$$
$$e^{-t} a_{3,2}^{-2,0} a_{2,3}^{0,-2} = e^{-2t} \sinh t \sinh 2t, \qquad -e^t a_{2,3}^{3,1} a_{3,2}^{1,3} = -e^{3t} \sinh t \sinh 3t,$$
$$e^{-t} a_{3,2}^{1,3} a_{2,3}^{3,1} - e^t a_{3,4}^{1,-1} a_{4,3}^{-1,1} = -e^t \sinh^2 t,$$
$$e^{-t} a_{4,3}^{-1,1} a_{3,4}^{1,-1} - e^t a_{4,5}^{-1,-3} a_{5,4}^{-3,,-1} = e^{-t} \sinh^2 t,$$
$$e^{-t} a_{5,4}^{-3,-1} a_{4,5}^{-1,-3} = e^{-3t} \sinh t \sinh 2t, \qquad -e^t a_{4,5}^{2,0} a_{5,4}^{0,2} = -e^{2t} \sinh t \sinh 2t,$$
$$e^{-t} a_{5,4}^{0,2} a_{4,5}^{2,0} - e^t a_{5,6}^{0,-2} a_{5,6}^{-2,0} = 0, \qquad e^t a_{5,6}^{-2,0} a_{6,5}^{0,-2} = e^{2t} \sinh t \sinh 2t,$$
$$e^t a_{6,7}^{1,-1} a_{7,6}^{-1,1} = e^t \sinh^2 t.$$

Here we have two chains with 3 elements (spin 1), one chain with four elements (spin $\frac{3}{2}$) and one scalar chain.

5 The case of B_2 algebra

In this case $p = 2$. We work again in representation with diagonal Q. From (3.7) it follows that its diagonal matrix elements are relating by the condition $\frac{\lambda_i}{\lambda_{i+1}} = e^{2t}$. It follows also that generators $Q_1^{\pm}$ commute with Q and so s^1, r^1 have the form:

$$s^1 = \begin{pmatrix} \dots & a_i & \dots. \end{pmatrix}, \quad r^1 = \begin{pmatrix} \dots & b_i & \dots \end{pmatrix}$$

s^2, r^2 maintain their previous forms,

$$s^2 = \begin{pmatrix} \dots. & a_{i,i-1} & 0 & a_{i,i+1} \dots \end{pmatrix} \quad r^2 = \begin{pmatrix} \dots. & -a_{i,i-1} & \alpha_i & a_{i,i+1} \dots \end{pmatrix}$$

The first equation (2.5) preserves its form for each component of the ansatz for (s^1, r^1) matrices:

$$[a_i, b_i] = \tanh t (a_i^2 - b_i^2 + 1) \tag{5.1}$$

The second equation (2.5) is equivalent to (4.8) with the replacement $t \to 2t$:

$$e^{-2t} a_{n,n-1} \alpha_{n-1} = e^{2t} \alpha_n a_{n,n-1}, \quad e^{2t} a_{n,n+1} \alpha_{n+1} = e^{-2t} \alpha_n a_{n,n+1},$$

$$2e^{-2t}a_{n,n-1}a_{n-1,n} - 2e^{2t}a_{n,n+1}a_{n+1,n} = \sinh t(I_n - \alpha_n^2) \tag{5.2}$$

The system of equations of mixed unknown functions in the (s^1, r^1) and (s^2, r^2) ansätze has the form:

$$e^{-t}(a_i + b_i)a_{i,i-1} = e^t a_{i,i-1}(a_{i-1} + b_{i-1}) \quad e^t(a_i - b_i)a_{i,i+1} = e^{-t}a_{i,i+1}(a_{i+1} - b_{i+1})$$

$$e^t\alpha_i(a_i + b_i) - e^{-t}(a_i + b_i)\alpha_i = 2\sinh t\lambda_i I_i, \quad e^{-t}\alpha_i(a_i - b_i) - e^t(a_i - b_i)\alpha_i = 2\sinh t\lambda_i I_i \tag{5.3}$$

The Weyl group of B_2 algebra consists of 8 elements. The character of its (p, q) representation calculated with the help of (1.1) has the form:

$$\pi^{(p,q)}(\tau_1, \tau_2) = \frac{\sinh l_1\tilde{\tau}_2 \sinh l_2\tilde{\tau}_2 - \sinh l_1\tilde{\tau}_2 \sinh l_2\tilde{\tau}_1}{\sinh 2\tilde{\tau}_1 \sinh \tilde{\tau}_2 - \sinh \tilde{\tau}_1 \sinh 2\tilde{\tau}_2} \tag{5.4}$$

where $l_1 - l_2 = p + 1, l_2 = q + 1, l_1 = p + q + 2, l_1 + l_2 = p + q + 3, \tilde{\tau}_1 = \tau_1, \tilde{\tau}_2 = \tau_2 - \tau_1$. (Really (5.4) is the character of C_2 algebra).

All diagonal elements of the matrices $R_1, R_2, Q = R_1R_2$ necessary for further calculations have to be obtained from (5.4). Now we would like to consider in details the concrete case of the $(1, 1)$ 16 dimensional representation of B_2 algebra.

5.1 The case of $(1, 1)$ representation

In this case $l_1 = 4, l_2 = 2$ and the character of the $(1, 1)$ representation in correspondence with (5.4) has the form:

$$\pi^{(1,1)}(\tau_1, \tau_2) = (e^{\tilde{\tau}_1} + e^{-\tilde{\tau}_1})(e^{\tilde{\tau}_2} + e^{-\tilde{\tau}2})(e^{\tilde{\tau}_1} + e^{\tilde{\tau}_2})(1 + e^{-(\tilde{\tau}_1+\tilde{\tau}_2)}) =$$

$$e^{2\tau_2-\tau_1} + e^{\tau_2+\tau_1} + e^{2\tau_2-3\tau_1} + 2e^{\tau_2-\tau_1} + 2e^{\tau_1} + e^{-\tau_2+3\tau_1} +$$

$$e^{\tau_2-3\tau_1} + 2e^{-\tau_1} + 2e^{-\tau_2+\tau_1} + e^{-2\tau_2+3\tau_1} + e^{-\tau_2-\tau_1} + e^{-2\tau_2+\tau_1}$$

The order of the exponents in the latter expression is chosen so as to present the final expressions in more attractive form.

After reducing to the A_1 subgroup with the algebra of the first complicate root of B_2 algebra $(X^{\pm}_{1,2}, H = h_1 + 2h_2)$, what is equivalent to substituting $\tau_1 = t, \tau_2 = 2t$ into the expression for the character, we obtain,

$$\pi^{(1,1)}(t, 2t) = 2e^{3t} + 6e^t + 6e^{-t} + 2e^{-3t}$$

Reducing to the A_1 subgroup of the first simple root $(X^{\pm}_1, H = h_1, \tau_1 = t, \tau_2 = 0)$ leads to:

$$\pi^{(1,1)}(t, 0) = e^{-t} + e^t + e^{-3t} + 2e^{-t} + 2e^t + e^{3t} + e^{-3t} + 2e^{-t} + 2e^t + e^{3t} + e^{-t} + e^t$$

Using these details we obtain the explicit form of the α matrices:

$$\alpha_1 = (e^{4t}, e^{2t}), \quad \alpha_2 = (e^{4t}, e^{2t}, e^{2t}, 1, 1, e^{-2t}),$$

$$\alpha_3=(e^{2t},1,1,e^{-2t},e^{-2t},e^{-4t}),\quad \alpha_4=(e^{-2t},e^{-4t})$$

(as above the brackets contain the diagonal elements of the α matrices).

The rectangular matrices from the (s^2,r^2) ansätze have the following dimension:

$$a_{1,2}\to 2\times 6,\quad a_{2,3}\to 6\times 6,\quad a_{3,4}\to 6\times 2.$$

and the a_i,b_i matrices from the (s^1,r^1) ansätze,

$$a_1,b_1\to 2\times 2,\quad a_2,b_2\to 6\times 6,\quad a_3,b_3\to 6\times 6,\quad a_4,b_4\to 2\times 2$$

The first row equations of the system (5.2) has the following solution

$$a_{1,2}=\begin{pmatrix}0&0&0&A&B&0\\0&0&0&0&0&C\end{pmatrix},\qquad a_{2,3}=\begin{pmatrix}0&D&E&0&0&0\\0&0&0&F&G&0\\0&0&0&H&K&0\\0&0&0&0&0&L\\0&0&0&0&0&M\\0&0&0&0&0&0\end{pmatrix},$$

$$a_{3,4}=\begin{pmatrix}N&0\\0&P\\0&R\\0&0\\0&0\\0&0\end{pmatrix}$$

Or in the notations of primitive rectangular matrices, introduced in the previous section:

$$a^{2,-2}_{1,2}=C,\quad a^{4,0}_{1,2}=\begin{pmatrix}A&B\end{pmatrix},\quad a^{4,0}_{2,3}=\begin{pmatrix}D&E\end{pmatrix},\quad a^{2,-2}_{2,3}=\begin{pmatrix}F&G\\H&K\end{pmatrix},$$

$$a^{0,-4}_{2,3}=\begin{pmatrix}L\\M\end{pmatrix},a^{0,-4}_{3,4}=\begin{pmatrix}P\\R\end{pmatrix},a^{2,-2}_{3,4}=N$$

We draw attention to the fact that the notations in terms of primitive rectangular matrices are independent of the choice of the order of exponents in the Weyl character formula.

The second row of the system (5.2) is as follows:

$$\begin{aligned}
-e^{2t}a^{2,-2}_{1,2}a^{-2,2}_{2,1}&=-e^{2t}\sinh^2 2t, & -e^{2t}a^{4,0}_{1,2}a^{0,4}_{2,1}&=-e^{4t}\sinh 2t\sinh 4t,\\
e^{-2t}a^{-2,2}_{2,1}a^{2,-2}_{1,2}&=e^{-2t}\sinh^2 2t, & e^{-2t}a^{0,4}_{2,1}a^{4,0}_{1,2}-e^{2t}a^{0,-4}_{2,3}a^{-4,0}_{3,2}&=0,\\
-e^{2t}a^{2,-2}_{2,3}a^{-2,2}_{3,2}&=-e^{2t}\sinh^2 2tI_2, & e^{-2t}a^{-4,0}_{3,2}a^{0,-4}_{2,3}&=e^{-4t}\sinh 2t\sinh 4t,\\
e^{-2t}a^{-2,2}_{3,2}a^{2,-2}_{2,3}&=e^{-2t}\sinh^2 2tI_2, & e^{-2t}a^{0,4}_{3,2}a^{4,0}_{2,3}-e^{2t}a^{0,-4}_{3,4}a^{-4,0}_{4,3}&=0,\\
e^{-2t}a^{-4,0}_{4,3}a^{0,-4}_{3,4}&=e^{-4t}\sinh 2t\sinh 4t, & e^{-2t}a^{-2,2}_{4,3}a^{2,-2}_{3,4}&=e^{-2t}\sinh^2 2t.
\end{aligned}$$

Not all of the equations above are different but no contradictions between them is observed, what can serve as one additional argument of selfconsistency of the whole construction. We present below only nonrepeated equations:

$$C^2 = \sinh^2 2t, \quad A^2 + B^2 = e^{2t}\sinh 2t \sinh 4t, \quad , L = \pm e^{-2t}A, \quad M = \pm e^{-2t}B,$$
$$\begin{pmatrix} F & G \\ H & K \end{pmatrix} = \sinh 2t \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$
$$D = \pm e^{2t}P, \quad E = \pm e^{2t}R, \quad P^2 + R^2 = e^{-2t}\sinh 2t \sinh 4t, \quad N^2 = \sinh^2 2t$$

From these we conclude that their solution may be obtained uniquely up to orthogonal transformations.

Now we pass to equations of the second row of (5.3) relating α and $a \pm b$ matrices. Direct calculations lead to the result:

$$a_1 + b_1 = \begin{pmatrix} e^{-t} & 0 \\ x & e^t \end{pmatrix}, \quad a_1 - b_1 = \begin{pmatrix} -e^{-t} & y \\ 0 & -e^t \end{pmatrix}$$

$$a_4 + b_4 = \begin{pmatrix} e^{-t} & 0 \\ u & e^t \end{pmatrix}, \quad a_4 - b_4 = \begin{pmatrix} -e^{-t} & v \\ 0 & -e^t \end{pmatrix}$$

$$a_2 + b_2 = \begin{pmatrix} e^{-3t} & 0 & 0 & 0 \\ X_{2,1} & e^{-t} & 0 & 0 \\ 0 & X_{3,2} & e^t & 0 \\ 0 & 0 & X_{4,3} & e^{3t} \end{pmatrix}, \quad a_2 - b_2 = \begin{pmatrix} -e^{-3t} & Y_{1,2} & 0 & 0 \\ 0 & -e^{-t} & Y_{2,3} & 0 \\ 0 & 0 & -e^t & Y_{3,4} \\ 0 & 0 & 0 & -e^{3t} \end{pmatrix}$$

$$a_3 + b_3 = \begin{pmatrix} e^{-3t} & 0 & 0 & 0 \\ U_{2,1} & e^{-t} & 0 & 0 \\ 0 & U_{3,2} & e^t & 0 \\ 0 & 0 & U_{4,3} & e^{3t} \end{pmatrix}, \quad a_3 - b_3 = \begin{pmatrix} -e^{-3t} & V_{1,2} & 0 & 0 \\ 0 & -e^{-t} & V_{2,3} & 0 \\ 0 & 0 & -e^t & V_{3,4} \\ 0 & 0 & 0 & -e^{3t} \end{pmatrix}$$

We have presented the 6×6 matrices $a_2 \pm b_2, a_3 \pm b_3$ in four-dimensional form, keeping in mind that in these expressions $2, 3$ indices indeed are two-dimensional ones. Thus X_{21} and Y_{12} are 2×1 and 1×2 two-dimensional column and row vectors respectively and so on.

Now it is necessary to take into account that a, b matrices satisfy equations (5.1). In terms of $(a \pm b)$ matrices the last equations may be rewritten as:

$$e^t(a+b)(b-a) - e^{-t}(b-a)(a+b) = 2\sinh tI$$

Direct substitution of expressions for $(a_1, b_1), (a_4, b_4)$ obtained above into the last equation yields the following restriction on parameters:

$$xy = 4\sinh^2 t, \quad uv = 4\sinh^2 t$$

If we want a symmetrical s^1 matrix, then we obtain

$$x = y = u = v = 2\sinh t$$

This result is possible to understand without any calculations by consideration of the relations of the section 3, $\frac{a+b}{2\sinh t} = Q^+$, $\frac{a-b}{2\sinh t} = Q^-$. The expressions for $(a_{1,4} \pm b_{1,4}$ coincide with those for spinor (two-dimensional) representation of quantum A_1^q algebra with additional similarity transformation with the two-dimensional matrix $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The situation with the six-dimensional matrices is the same. It is necessary to consider direct sum of four-dimensional ($\frac{3}{2}$ spin) representation of A_1^q, related to 1,2,5,6 indices and two dimensional ($\frac{1}{2}$ spin) one, related to the 3,4 indices. After this do similarity transformation with 6×6 matrix with nonzero unities on the main antidiagonal. Result will be exactly matrices $a_{2,3} \pm b_{2,3}$. Under such a procedure the following limitations on parameters X, Y, U, V arise:

$$X_{2,1} = U_{2,1} = \begin{pmatrix} 2e^{-t}(\sinh 3t \sinh t)^{\frac{1}{2}} \\ 0 \end{pmatrix}, \quad X_{2,3} = U_{2,3} = \begin{pmatrix} 0 & 2\sinh t \\ 2\sinh 2t & 0 \end{pmatrix},$$
$$X_{3,4} = U_{3,4} = (\, 0 \quad 2e^t(\sinh 3t \sinh t)^{\frac{1}{2}}\,)$$

and analogous expressions for Y, V elements ($Y = X^T, V = U^T$). It is necessary to emphasise that we have fixed the gauge and thus all expressions above do not contain any additional parameters. But this was achieved by the definite choice of the form of the x, y, X, Y, u, v, U, V matrices.

At last we have to satisfy the first row of equations (5.3) relating rectangular matrices $a_{n,n+1}$ to the matrices $a_i \pm b_i$. Direct calculation leads to the following final result:

$$e^{-t}(X_{4,3}a_{2,1}^{0,4}) = e^t Cx, \quad e^{-t}(U_{3,2}a_{2,3}^{4,0}) = e^t(a_{23}^{2,-2}X_{2,1}),$$

$$e^{-t}(U_{4,3}a_{23}^{2,-2}) = e^t(a_{3,2}^{-4,0}X_{32}), \quad e^{-t}uN = e^t(a_{4,3}^{-4,0}U_{2,1})$$

These uniquely fixed all parameters in the construction,

$$A = E = e^t \sinh 2t (\frac{\sinh 5t}{\sinh 3t})^{\frac{1}{2}} \quad B = D = e^t \sinh 2t (\frac{\sinh t}{\sinh 3t})^{\frac{1}{2}},$$

$$\sin\phi = -\frac{\sinh 2t}{\sinh 3t}, \quad \cos\phi == \frac{(\sinh 5t \sinh t)^{\frac{1}{2}}}{\sinh 3t}$$

6 The case of G_2 algebra

Repeating the arguments of two previous sections (the cases of A_2 and B_2 algebras) we write down to the following ansätze for the matrices (s^1, r^1), (s^2, r^2), which we present in symbolical row form:

$$s^1 = (\, ... \; a_{i,i-1} \; 0 \; a_{i,i+1} ... \,), \qquad r^1 = (\, ... \; a_{i,i-1} \; \alpha_i \; -a_{i,i+1} ... \,)$$

$$s^2 = (\, ... \; b_{i,i-3} \; 0 \; 0 \; 0 \; 0 \; 0 \; b_{i,i+3} ... \,), \qquad r^2 = (\, ... \; -b_{i,i-3} \; 0 \; 0 \; \beta_i \; 0 \; 0 \; b_{i,i+3} ... \,)$$

The sign differences, $a_{i,i+1}$ instead of $a_{i,i-1}$ in A_2 algebra case, are related to the different signs in the first equation (3.5). In these formulae the notation is the same as in the previous sections: α_i, β_i are quadratic $N_i \times N_i$ matrices, $a_{p,q}, b_{p,q}$ are $N_p \times N_q$ rectangular ones. The equation relating s^1, r^1 is equivalent to a matrix system:

$$e^t a_{n,n-1}\alpha_{n-1} = e^{-t}\alpha_n a_{n,n-1}, \quad e^{-t}a_{n,n+1}\alpha_{n+1} = e^t \alpha_n a_{n,n+1},$$

$$-2e^t a_{n,n-1}a_{n-1,n} + 2e^{-t}a_{n,n+1}a_{n+1,n} = \sinh t(I_n - \alpha_n^2) \qquad (6.5)$$

and that for s^2, r^2 leads to:

$$e^{-3t}b_{n,n-3}\beta_{n-3} = e^{3t}\beta_n b_{n,n-3}, \quad e^{3t}b_{n,n+3}\beta_{n+3} = e^{-3t}\beta_n b_{n,n+3},$$

$$2e^{-3t}b_{n,n-3}b_{n-3,n} - 2e^{3t}b_{n,n+3}b_{n+3,n} = \sinh t(I_n - \beta_n^2) \qquad (6.6)$$

where I_n is quadratic unity $N_n \times N_n$ matrix. The last equation (3.1) and definition of Q (3.2) imply the system for mixed a, α and b, β matrices:

$$\begin{array}{l}
e^{\frac{3t}{2}}b_{n,n-3}\alpha_{n-3} \;=\; e^{-\frac{3t}{2}}\alpha_n b_{n,n-3} \quad , \quad e^{\frac{3t}{2}}a_{n,n+1}\beta_{n+1} = e^{-\frac{3t}{2}}\beta_n a_{n,n+1} \ , \\
e^{-\frac{3t}{2}}a_{n,n-1}\beta_{n-1} = e^{\frac{3t}{2}}\beta_n a_{n,n-1} \quad , \quad e^{-\frac{3t}{2}}b_{n,n+3}\beta_{n+3} = e^{\frac{3t}{2}}\alpha_n b_{n,n+3} \ , \\
e^{-\frac{3t}{2}}b_{n,n+3}a_{n+3,n+4} = e^{\frac{3t}{2}}a_{n,n+1}b_{n+1,n+4} \quad , \quad e^{\frac{3t}{2}}b_{n,n-3}a_{n-3,n-4} = e^{-\frac{3t}{2}}a_{n,n-1}b_{n-1,n-4} \ , \\
\alpha_n\beta_n = \lambda_n I_n
\end{array} \qquad (6.7)$$

The last equation allows the elimination of matrices β_n to rewrite (6.5), (6.6),(6.7) in more compact form. We assume also that matrices s^1, s^2 are symmetrical in that,

$$a_{i+1,i} = a^T_{i,i+1}, \quad b_{i+3,i} = b^T_{i,i+3} \ .$$

Finally the system of equations to be solved in the case of G_2 algebra takes the form:

$$\begin{array}{l}
e^t a_{n,n-1}\alpha_{n-1} = e^{-t}\alpha_n a_{n,n-1}, \quad e^{-t}a_{n,n+1}\alpha_{n+1} = e^t\alpha_n a_{n,n+1}, \\
e^{\frac{3t}{2}}b_{n,n-3}\alpha_{n-3} = e^{-\frac{3t}{2}}\alpha_n b_{n,n-3}, \quad e^{-\frac{3t}{2}}b_{n,n+3}\alpha_{n+3} = e^{\frac{3t}{2}}\alpha_n b_{n,n+3}, \\
-2e^t a_{n,n-1}a_{n-1,n} + 2e^{-t}a_{n,n+1}a_{n+1,n} = \sinh t(I_n - \alpha_n^2) \\
2e^{3t}b_{n,n-3}b_{n-3,n} - 2e^{-3t}b_{n,n+3}b_{n+3,n} = \sinh t(I_n - \beta_n^2) \\
e^{\frac{3t}{2}}b_{n,n-3}a_{n-3,n-4} = e^{-\frac{3t}{2}}a_{n,n-1}b_{n-1,n-4}
\end{array} \qquad (6.8)$$

Here, as in the previous sections, the diagonal matrices α_n, β_n are considered to be known from the Weyl formula (1.1); unknown are the rectangular matrices a, b of the corresponding dimension.

As in the case of A_2 algebra rectangular matrices a, b have block structure in terms of primitive component of which the system (6.8) may be presented. We omit this general consideration restricting ourselves to the simplest example of the first fundamental representation of quantum G_2 algebra only, with the aim of demonstrating the consistency of the proposed construction.

6.1 The case of $(1, 0)$ representation

The standard basis, using lowering operators, of the $(1, 0)$ representation is given by

$$X_1^- \mid 1\rangle, \mid 1\rangle, X_1^-X_1^-X_2^-X_1^- \mid 1\rangle, X_1^-X_2^-X_1^- \mid 1\rangle,$$

$$X_2^-X_1^- \mid 1\rangle, X_1^-X_2^-X_1^-X_1^-X_2^-X_1^- \mid 1\rangle, X_2^-X_1^-X_1^-X_2^-X_1^- \mid 1\rangle$$

On this basis the generator $Q = \exp(h_1 + 3h_2)t$ takes the values

$$Q = (e^{2t}, e^t, e^t, 1, e^{-t}, e^{-t}, e^{-2t})$$

and $R_{1,2}$,

$$R_1 \equiv \exp h_1 t = (e^{-t}, e^t, e^{-2t}, 1, e^{2t}, e^{-t}, e^t)$$

$$R_2 \equiv \exp 3h_2 t = (e^{3t}, 1, e^{3t}, 1, e^{-3t}, 1, e^{-3t})$$

The explicit expressions for the α matrices are as follows:

$$\alpha_1 = e^{-t} \ , \ \alpha_2 = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}, \ \alpha_3 = 1 \ , \ \alpha_4 = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}, \ \alpha_5 = e^t$$

According to the general scheme nonzero are the following nondiagonal elements of (s^1, r^1) matrices:

$$a_{1,2} = (A, B), \quad a_{3,2} = (C, D), \quad a_{3,4} = (E, F), \quad a_{5,4} = (G, H)$$

and elements of (s^2, r^2) ones:

$$b_{1,4} = (x, y), \quad b_{5,2} = (u, v)$$

where all the parameters are unknown and need to be determined from the system (6.8). After substitution into equations of the first row of the system (6.8) using the explicit expressions for α matrices we obtain:

$$a_{1,2} = (A, 0), \quad a_{3,2} = (0, D), \quad a_{3,4} = (E, 0), \quad a_{5,4} = (0, H)$$

$$b_{1,4} = (x, 0), \quad b_{5,2} = (0, v)$$

From the next two rows of the system (6.8) it uniquely follows that

$$A^2 = \sinh^2 t, \qquad D^2 = e^{-t} \sinh t \sinh 2t,$$
$$E^2 = e^t \sinh t \sinh 2t, \qquad H^2 = \sinh^2 t, \qquad x^2 = v^2 = \sinh^2 3t.$$

By the direct check one can verify that equations of the last row of the system (6.8) are satisfied automatically.

7 Outlook

The results of the present paper are simultaneously absolutely unexpected and surprising at least to the author. Long time peoples were sure that the solution of the problem of explicit form of infinitesimal generators for an arbitrary representation must be in a deep connection with the bases defined by the eigenvalues of the necessary number of mutually commuting operators constructed from the generators of the corresponding algebra. The values of Casimir operators in such construction define the indices of the irreducible representation. The solution of the problem in this way was found in the famous papers of I. M. Gelfand and M. L. Tsetlin fifty years ago [2]. These authors were able to represent in explicit form the infinitesimal operators for an arbitrary representation of classical semisimple series A_n, B_n, and D_n. However all other numerous attempts to generalize these result to all remaining semisimple series were unsuccessful.

The way proposed in the present paper deals neither with the Casimir operators nor with a family of mutually commutating operators and their eigenvalues. Only knowledge of the result of the action of the group operator e^{τ} on the basis state vectors is essential and this yields a possibility to solve the problem also for the case of quantum algebras.

The case of the usual semisimple algebras arises in the limit of the deformation parameter going to zero. Usually the back way of the thought is used: it is necessary firstly to construct a representation of a semisimple algebra and then generalize it to the quantum algebra case.

The algorithm of the present paper is similar to a computer program in that it is necessary to iterate the same operation: to solve the algebraical system of equations the inhomogeneous part of which is known from the previous steps of calculations.

Of course the key point of the whole construction is the Weyl formula for the characters of finite-dimensional irreducible representations of semisimple groups. It guarantees the self-consistency of the whole construction and gives the necessary number of initial parameters (the explicit values of diagonal elements of α, β matrices and their multiplicities) through which the generators of the simple roots are expressed. The invariant character of the Weyl formula allows the hope that the problem of explicit realisation of the generators of the simple roots for quantum algebras may also be solved in invariant terms. Unfortunately at this moment these terms are unknown to the author.

Acknowledgements.

The author would like to thank C. Devchand, A.V. Razumov and Nodari Vakhania for fruitful discussions and big help in preparation of the manuscript for publication. This paper was done under the partial support of the Russian Foundation for Fundamental Researches (RFFI) GRANT–N 98-01-00330.

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