Inverse monodromy problem for Hurwitz Frobenius manifolds: regular singularities

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Abstract. In this paper we systematically study the Fuchsian Riemann-Hilbert (inverse monodromy) problem introduced by Dubrovin to describe Frobenius structures on Hurwitz spaces. We find a fundamental solution to this Riemann-Hilbert problem in terms of integrals of certain meromorphic differentials over a basis of an appropriate relative homology group over a Riemann surface. We study the corresponding monodromy group and compute the monodromy matrices explicitly for various examples.

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1 Introduction

The matrix Riemann-Hilbert problems (or inverse monodromy problems) appear in mathematical physics in many different ways, from the theory of integrable systems [1, 9] to random matrices [2, 5]. Historically, the main origin of these problems is the theory of systems of linear differential equations with meromorphic coefficients.

In the analytic aspects of the theory of Frobenius manifolds [6, 7], the Riemann-Hilbert problems also play an important role: the corresponding monodromy data provide a way of classification of Frobenius manifolds.

To each Frobenius manifold one can naturally associate two systems of linear differential equations: a Fuchsian system (where the coefficients have poles of the first order only) and a non-Fuchsian one, when the coefficients have both first and second order poles. These two systems are related by a formal Laplace transform. For the class of Frobenius manifolds associated to the Hurwitz spaces, the non-Fuchsian systems were recently solved in [18] (although many essential elements of this construction were already given by Dubrovin in [6, 7]); the corresponding Stokes and monodromy matrices were also computed in [18]. In principle, one can apply the formal Laplace transform to the solution from [18] and get solutions to the corresponding Fuchsian systems, however, this does not give a satisfactory final result due to a non-trivial superposition of various Laplace transforms.

In this paper we construct solutions to the Fuchsian Riemann-Hilbert problems corresponding to the Hurwitz Frobenius manifolds; these solutions are not related in an obvious way to the solutions to the non-Fuchsian systems found in [18].

The coefficients of the system of Fuchsian linear ODE's with meromorphic coefficients corresponding to a given Frobenius manifold are written in terms of rotation coefficients Γ_{ij} of the Darboux-Egoroff metric on the manifold. This linear system has the form:

$$\frac{\partial \Phi}{\partial \lambda} = \sum_{j=1}^{L} \frac{A_j}{\lambda - \lambda_j} \Phi, \tag{1.1}$$

where $\Phi(\lambda)$ is an $L \times L$ matrix (L is the dimension of the Frobenius manifold); $\lambda \in \mathbb{CP}^1$; $\lambda_i,\ i = 1, \dots, L$, are the canonical coordinates on the manifold; $A_j = -E_j(V + \alpha I)$, where $E_j = \mathrm{diag}(0, \dots, 1, \dots, 0)$ is the diagonal $L \times L$ matrix with 1 on jth place; $\alpha \in \mathbb{C}$ is an arbitrary constant (in this paper we consider the case $\alpha = -1/2$; in [7] the case $\alpha = 1/2$ was considered). The matrix V is defined as follows: $V := [\Gamma, U]$, where Γ is the matrix of rotation coefficients: $(\Gamma)_{jk} = \Gamma_{jk}$ if $j \neq k$ and $(\Gamma)_{jj} = 0$; $U = \mathrm{diag}(\lambda_1, \dots, \lambda_L)$. Each matrix A_j in (1.1) has only one non-trivial row (the jth row).

The Hurwitz spaces are the spaces of equivalence classes of pairs $(\mathcal{L}, f)$, where $\mathcal{L}$ is a Riemann surface of genus g, and f is a meromorphic function of degree N on $\mathcal{L}$; two pairs $(\mathcal{L}_1, f_1)$ and $(\mathcal{L}_2, f_2)$ are equivalent if there exists a biholomorphic map $h : \mathcal{L}_1 \to \mathcal{L}_2$, such that $f_1 = f_2 \circ h$. Using the function f, we can realize the Riemann surface $\mathcal{L}$ as an N-sheeted branched covering of the Riemann sphere; the branch points of this covering are given by critical values of the function f. The Hurwitz space is stratified according to the type of branching over the branch points. The Frobenius structures can be defined on any stratum for which the branching over the point at infinity is arbitrary, while all finite branch points of the covering $\mathcal{L}$ are simple. The branch points (we denote them by $\lambda_1, \dots, \lambda_L$, while the corresponding ramification points on $\mathcal{L}$ are denoted by $P_1, \dots, P_L$) can be used as local coordinates on such a stratum; they also play the role of canonical coordinates on the corresponding Frobenius manifold.

Let us introduce the canonical meromorphic bidifferential $W(P,Q)$ on the Riemann surface $\mathcal{L}$. This bidifferential is symmetric, has a quadratic pole on the diagonal $P = Q$ with biresidue 1 and has vanishing a-periods with respect to both P and Q. The rotation coefficients of Frobenius structures on the Hurwitz spaces are given by

$$\Gamma_{jk} = \frac{1}{2} W(P_j, P_k) := \frac{1}{2} \frac{W(P,Q)}{d(\sqrt{f(P)-\lambda_j})\; d(\sqrt{f(Q)-\lambda_k})}\Big|_{P=P_j,\, Q=P_k} . \tag{1.2}$$

To construct the corresponding solution of the Fuchsian linear system (1.1) we introduce, for any $\lambda \in \mathbb{C}$, the homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$ with coefficients in $\mathbb{Z}$ of the punctured Riemann surface $\mathcal{L}$ punctured at the poles of the function f relative to the set of (generically N) points on $\mathcal{L}$ where the value of f equals λ. The dimension of this relative homology group equals $2g + N + K - 2$, where K is the number of poles of the function f, i.e. the number of points in the set $f^{-1}(\infty)$.

For any contour $\mathbf{s} \in H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$ the vector function with the components

$$\Phi_j^{(\mathbf{s})}(\lambda) := \lambda \int_{\mathbf{s}} W(P, P_j) - \int_{\mathbf{s}} f(P) W(P, P_j)\ ,$$

where $j = 1, \dots, L$, and

$$W(P, P_j) := \frac{W(P,Q)}{d\sqrt{f(Q)-\lambda_j}}\Big|_{P=P_j},$$

satisfies the linear system (1.1). Choosing $\mathbf{s}$ to run through a basis in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$, we get the full set of $2g + N + K - 2$ independent solutions to (1.1); our proof of this independence is a tedious exercise involving analysis of the behaviour of the bidifferential $W(P,Q)$ at the boundary of the Hurwitz space.

Let us choose a neighbourhood D of a point λ_0 which contains no branch points λ_k.

A set of basis contours in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$ can be chosen as follows: a canonical basis of $2g$ cycles on $\mathcal{L}$ (this canonical basis does not necessarily coincide with the set of cycles on $\mathcal{L}$ which enter the definition of the bidifferential W); small contours around $K - 1$ points which can be arbitrarily chosen from the set $f^{-1}(\infty)$ consisting of K points. The remaining $N - 1$ contours can be chosen to connect pairwise the N points from $f^{-1}(\lambda)$; for the linear independence of these contours one has to require connectedness of the graph whose edges are given by these contours and vertices are the N points from $f^{-1}(\lambda)$. The bases of cycles can be naturally identified for any two values of $\lambda \in D$. In this way we get a non-degenerate matrix-valued matrix $\Phi(\lambda)$ solving (1.1) and analytic for $\lambda \in D$.

Being analytically continued along generators of the fundamental group $\pi_1(\mathcal{L} \setminus \{\lambda_1, \dots, \lambda_L, \infty\})$, the function Φ is multiplied from the right by monodromy matrices M_k, $k = 1, \dots, L, \infty$.

The monodromy matrices describe the transformation of a chosen basis in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$ under a natural action of an element of $\pi_1(\mathcal{L} \setminus \{\lambda_1, \dots, \lambda_L, \infty\})$; thus all entries of the monodromy matrices are integer numbers.

If a basis in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$ is chosen as described above, the monodromy matrices possess the following structure:

$$M_k = \begin{pmatrix} I & S_k \\ 0 & T_k \end{pmatrix},$$

where T_k are square $(N-1)\times(N-1)$ matrices; they generate a subgroup of $GL(N-1,\mathbb{Z})$ given by the image of a group homomorphism from the monodromy group of the covering $\mathcal{L}$ to $GL(N-1,\mathbb{Z})$. The unit matrices in the upper diagonal block are of the size $(2g+K-1)\times(2g+K-1)$; the matrices S_k of the size $(2g+K-1)\times(N-1)$ depend on the choice of a basis in $H_1(\mathcal{L}\setminus f^{-1}(\infty))$. However, the change of a basis in $H_1(\mathcal{L}\setminus f^{-1}(\infty))$ results in a simultaneous conjugation of all monodromy matrices M_k by the same matrix; thus the monodromy group is in fact independent of the choice of the basis of cycles.

Definition of the canonical bidifferential $W(P,Q)$ depends on the choice of a symplectic basis in $H_1(\mathcal{L})$. A change of the basis changes the rotation coefficients Γ_{ij}, and, therefore, the coefficients of the linear system (1.1), as well as the corresponding solution Φ. We show that, however, such a change of W does not change the monodromy matrices of the system (1.1). Therefore, this change of $W(P,Q)$ corresponds to a Schlesinger transformation of the solution to the linear system. We construct this Schlesinger transformation explicitly.

An important object associated to any Riemann-Hilbert problem is the isomonodromic Jimbo-Miwa tau-function, which is a function of $\{\lambda_k\}$; the divisor of zeros of the tau-function corresponds to a configuration of poles $\{\lambda_k\}$ where the Riemann-Hilbert problem loses its solvability. In the context of Frobenius manifold structures on Hurwitz spaces, the tau-function determines the G-function of the Frobenius manifold, which plays the role of genus one free energy of the corresponding topological field theory. This tau-function coincides with the so-called Bergman tau-function on the Hurwitz space [13]. The Bergman tau-function plays a key role in the computation of the determinant of the Laplacian in flat metrics on Riemann surfaces [14] and of the genus one free energy in the Hermitian two-matrix models [15].

The paper is organized as follows. Section 2 contains a few basic facts about the Fuchsian and non-Fuchsian Riemann-Hilbert problems appearing in the theory of Frobenius manifolds. In Section 3 we construct a solution to the Fuchsian system and describe its monodromy matrices. In Section 4 compute monodromy matrices explicitly for various Hurwitz spaces. Section 5 is devoted to the proof of non-degeneracy of our solution.

2 The Fuchsian Riemann-Hilbert problem in Frobenius manifolds theory

For the reader's convenience and to set up the notations we shall review here the links between solutions to systems of linear differential equations with meromorphic coefficients, matrix Riemann-Hilbert (inverse monodromy) problems, and Frobenius manifolds.

Consider a matrix linear differential equation (1.1); depending on the context we shall understand Φ as either a vector solution to this equation, or a square $L\times L$ matrix of linearly independent vector solutions to this equation. Generically, a solution to equation (1.1) has non-trivial monodromy under the analytical continuation around singularities $\{\lambda_i\}$ and around the point $\lambda=\infty$. Let us choose a set of generators $\gamma_1,\dots,\gamma_L,\gamma_\infty$ of the fundamental group of the punctured sphere $\mathbb{C}P^1\setminus\{\lambda_1,\dots,\lambda_L,\infty\}$ such that each generator γ_j encloses only the point λ_j, the generator γ_∞ goes around the point at infinity, and the following relation is fulfilled:

$$\gamma_1\dots\gamma_L\gamma_\infty = id\,. \tag{2.1}$$

Suppose that the solution Φ, being analytically continued along γ_j, gains the right multiplier M_j (which is called the monodromy matrix). Being analytically continued along γ_∞, the solution Φ gains the right multiplier M_∞. As a corollary of relation (2.1) the monodromy matrices satisfy the relation

$$M_\infty M_L \dots M_1 = I, \tag{2.2}$$

i.e. they give an anti-representation of the fundamental group.

At the poles λ_j of the coefficients of the system (1.1), the function Φ has regular singularities (i.e. $\Phi(\lambda)$ grows at these points not faster than some power of $\lambda - \lambda_j$). If the matrices A_j are diagonalizable (this is the only case considered in this paper), the behaviour of Φ in a neighbourhood of λ_i looks as follows:

$$\Phi(\lambda) = G(\lambda)(\lambda - \lambda_j)^{T_j} C_j\,,$$

where T_i is a diagonal matrix, $G(\lambda) = G_j + O(\lambda - \lambda_i)$ is a function holomorphic in a neighbourhood of λ_j. If some matrix A_j is non-diagonalizable, the asymptotics of Φ near λ_j contains logarithmic terms.

The monodromy matrices can be expressed in terms of C_j and T_j as follows:

$$M_j = C_j^{-1} e^{2\pi \mathrm{i} T_j} C_j\,. \tag{2.3}$$

The Riemann-Hilbert (or inverse monodromy) problem is the problem of reconstruction of the function Φ knowing its monodromy matrices $\{M_j\}$ and positions of singularities $\{\lambda_j\}$. Obviously, a solution to the Riemann-Hilbert problem is not unique: multiplying one solution to such a problem from the left with an arbitrary matrix-valued rational function of λ, we again get a solution to the same Riemann-Hilbert problem. On the other hand, assuming that Φ has at $\{\lambda_j\}$ regular singularities of the form (2.3) with given $\{T_j,\, C_j\}$, and has no other singularities (including zeros of $\det\Phi$) we guarantee the uniqueness of the solution of the Riemann-Hilbert problem.

Let us now impose the isomonodromy condition, i.e. the condition of independence of the monodromy data $\{T_j,\, C_j\}$ on the positions of singularities $\{\lambda_j\}$. The isomonodromy condition implies a system of differential equations, called the Schlesinger equations, for the residues A_j as functions of $\{\lambda_j\}$. The Schlesinger equations of a special type and the corresponding Riemann-Hilbert problem play a significant role in the theory of Frobenius manifolds.

We shall skip here the complete description of the notion of a Frobenius manifold and associated objects, referring the reader to [6, 7]. We recall only that to each Frobenius manifold one can associate a Darboux-Egoroff (i.e. diagonal flat potential) metric. The poles λ_j, $j = 1, \dots, L$, of the coefficients in (1.1) coincide with canonical coordinates on the Frobenius manifold. Introduce the following two differential operators: $\mathbf{e} = \sum_{j=1}^L \frac{\partial}{\partial \lambda_j}$, called the unit vector field, and $\mathbf{E} = \sum_{j=1}^L \lambda_j \frac{\partial}{\partial \lambda_j}$, called the Euler vector field.

For the Darboux-Egoroff metrics appearing in the theory of Frobenius manifolds the rotation coefficients satisfy the following system of equations:

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \Gamma_{ik}\Gamma_{jk}, \tag{2.4}$$

where all i, j, k are distinct, and

$$\mathbf{e}(\Gamma_{ij}) = 0\,, \qquad \mathbf{E}(\Gamma_{ij}) = -\Gamma_{ij}\,. \tag{2.5}$$

The non-linear system (2.4), (2.5) is the compatibility condition for the following system of linear differential equations [6, 7]:

$$\frac{d\Phi}{d\lambda} = -\sum_{j=1}^{L} \frac{E_j(V + \alpha I)}{\lambda - \lambda_j}\Phi, \tag{2.6}$$

$$\frac{d\Phi}{d\lambda_j} = \left(\frac{E_j(V+\alpha I)}{\lambda-\lambda_j} + [\Gamma, E_j] \right) \Phi, \tag{2.7}$$

where Φ is an $L \times L$ matrix-valued function of λ and $\{\lambda_j\}$; $\alpha \in \mathbb{C}$ is an arbitrary constant; matrices V, Γ and E_j are defined after (1.1).

The system (2.7) provides the isomonodromy condition for the Fuchsian system (2.6).

The Fuchsian linear system introduced in the original papers [6, 7] corresponds to the value $\alpha = 1/2$. In this paper, we shall study the case $\alpha = -1/2$; below we discuss the relationship between the linear systems (2.6), (2.7) with the values of α which differ by an integer.

In the sequel we shall use the following convenient alternative formulation of the linear system (2.6), (2.7).

Proposition 1 *A vector $\Phi := (\varphi_1, \dots, \varphi_L)^T$ satisfies the linear system (2.6), (2.7) if and only if the following equations are fulfilled*

$$\lambda \frac{\partial \varphi_j}{\partial \lambda} + \mathbf{E}(\varphi_j) = -\alpha \varphi_j \tag{2.8}$$

$$\frac{\partial \varphi_j}{\partial \lambda} + \mathbf{e}(\varphi_j) = 0 \tag{2.9}$$

$$\frac{\partial \varphi_j}{\partial \lambda_k} = \Gamma_{jk}\varphi_k \,, \qquad j \neq k. \tag{2.10}$$

Proof. Let us first prove the sufficiency of equations (2.8) - (2.10) for the linear system (2.6), (2.7). Equation (2.6) for the vector $(\varphi_1, \dots, \varphi_n)^T$ reads in the components:

$$\frac{\partial \varphi_j}{\partial \lambda} = -\frac{1}{\lambda-\lambda_j} \left(\alpha\varphi_j + \sum_{k=1,k\neq j}^{L} \Gamma_{kj}(\lambda_k - \lambda_j)\varphi_k \right). \tag{2.11}$$

Similarly, equation (2.7) for the vector $(\varphi_1, \dots, \varphi_n)^T$ is equivalent to

$$\frac{\partial \varphi_j}{\partial \lambda_k} = \Gamma_{jk}\varphi_k, \qquad j \neq k, \tag{2.12}$$

$$\frac{\partial \varphi_j}{\partial \lambda_j} = \frac{1}{\lambda-\lambda_j} \left(\alpha\varphi_j + \sum_{k=1,k\neq j}^{L} \Gamma_{kj}(\lambda_k - \lambda_j)\varphi_k \right) - \sum_{k=1,k\neq j}^{L} \Gamma_{kj}\varphi_k.$$

The latter equation rewrites due to (2.11) as

$$\frac{\partial \varphi_j}{\partial \lambda_j} = -\frac{\partial \varphi_j}{\partial \lambda} - \sum_{k=1,k\neq j}^{L} \Gamma_{kj}\varphi_k,$$

which, by virtue of (2.12), coincides with (2.9).

We thus need to show the equivalence of equations (2.8) and (2.11) provided (2.9) and (2.10) hold. Using (2.10), we rewrite (2.11) as follows:

$$\frac{\partial \varphi_j}{\partial \lambda} = -\frac{1}{\lambda-\lambda_j} \left(\alpha\varphi_j + \sum_{k=1,k\neq j}^{L} (\lambda_k - \lambda_j)\partial_{\lambda_k}\varphi_j \right).$$

Adding and subtracting $\lambda_j \partial_{\lambda_j} \varphi_j$ in the right hand side and using the unit and Euler vector fields, we obtain

$$(\lambda - \lambda_j)\frac{\partial \varphi_j}{\partial \lambda} = -\alpha\varphi_j - \mathbf{E}(\varphi_j) + \lambda_j \mathbf{e}(\varphi_j). \tag{2.13}$$

Plugging equation (2.9) into the above relation (2.13), we obtain (2.8). □

Remark 1 Using Theorem 1 we can easily deduce that the solutions of the linear systems (2.6), (2.7) corresponding to values of α which differ by integers are related by a simple transformation. Namely, let us indicate explicitly the dependence of a solution to the system (2.6), (2.7) on α, i.e. we denote Φ by Φ^α. Then

$$\Phi^{\alpha+1} = \frac{\partial \Phi^\alpha}{\partial \lambda} \equiv A^\alpha(\lambda)\Phi^\alpha(\lambda), \tag{2.14}$$

where $A^\alpha(\lambda) = -\sum_{i=1}^n \frac{E_i(V+\alpha I)}{\lambda - \lambda_i}$ is the matrix of coefficients of (2.6).

In this paper we find a complete system of linearly independent solutions to the system (2.6), (2.7) for the case $\alpha = -1/2$. Several columns of our solution Φ turn out to be independent of λ, therefore formula (2.14) can not be used to generate fundamental solutions to the system with $\alpha = -1/2 + m$ for integer $m \geq 1$. However, from our solution for $\alpha = -1/2$ we can obtain the complete system of solutions for any negative half-integer value of α.

Remark 2 The same system of equations (2.4), (2.5) describes isomonodromic deformations of the non-Fuchsian equation

$$\frac{d\Psi}{dz} = (U + \frac{1}{z}V)\Psi\ . \tag{2.15}$$

A solution Ψ to the system (2.15) has an irregular singularity of Poincaré rank 1 at $z = \infty$, and a regular singularity at the origin.

Solutions to the Fuchsian system (2.6) and the non-Fuchsian system (2.15) are related by a formal Laplace transform (see [6], p. 87, (3.149)).

3 Solution to the Fuchsian system corresponding to Hurwitz Frobenius manifolds

3.1 Preliminaries

Let $\mathcal{L}$ be a Riemann surface of genus g and f be a meromorphic function on $\mathcal{L}$ of degree N. Let us fix the degrees of the poles of f to be $k_1, \dots, k_K$ ($k_1 + \cdots + k_K = N$), and assume that all finite critical points of the function f are simple (we denote them by $P_1, \dots, P_L$, where, according to the Riemann-Hurwitz formula, $L = 2g+N+K-2$. We denote by $\mathcal{H}_{g,N}(k_1, \dots, k_K)$ the Hurwitz space i.e. the space of equivalence classes of pairs $(\mathcal{L}, f)$ (two pairs $(\mathcal{L}_1, f_1)$ and $(\mathcal{L}_2, f_2)$ are called equivalent if there exists a biholomorphic isomorphism $h : \mathcal{L}_1 \to \mathcal{L}_2$ such that $f_1 = f_2 \circ h$). The local coordinates $\{\lambda_k\}_{k=1}^L$ on this Hurwitz space can be chosen to be the critical values of the function f, i.e. $\lambda_k := f(P_k)$, $k = 1, \dots, L$.

Using the function f, we can represent $\mathcal{L}$ as an N-sheeted covering of $\mathbb{C}P^1$ ramified at the points $P_1, \dots, P_L$ as well as at those poles of f whose degrees are higher than 1. The critical values $\{\lambda_k\}$ are the finite branch points of the ramified covering. In a neighbourhood of the ramification point P_k we introduce the standard local parameter $x_k(P) := \sqrt{f(P) - \lambda_k}$.

Introduce the canonical meromorphic bidifferential $W(P,Q)$ on $\mathcal{L} : P, Q \in \mathcal{L}$. This bidifferential is symmetric; it has a quadratic pole on the diagonal with the singular part given by $dx(P)dx(Q)(x(P)-x(Q))^{-2}$ in any local parameter x, and is normalized by the requirement that all of its a-periods with respect to some symplectic basis $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ in $H_1(\mathcal{L})$ vanish. Let us also introduce the canonical basis of holomorphic differentials $w_1, \dots, w_g$ on $\mathcal{L}$ normalized by $\oint_{\mathbf{a}_\alpha} w_\beta = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker symbol and $\alpha, \beta = 1, \dots, g$. Integrals of these differentials over the cycles $\mathbf{b}_\alpha$ give the Riemann matrix $\mathbb{B}$ of the surface: $\mathbb{B}_{\alpha\beta} = \oint_{\mathbf{b}_\alpha} w_\beta$.

We shall need the following Rauch variational formulas, which describe the dependence of w_α, W and $\mathbb{B}$ on the branch points $\{\lambda_k\}$ (see [16, 13]):

$$\frac{d}{d\lambda_k}\{B_{\alpha\beta}\} = \pi \mathrm{i} w_\alpha(P_k) w_\beta(P_k); \tag{3.1}$$

$$\frac{d}{d\lambda_k}\Big|_{f(P)}\{w_\alpha(P)\} = \frac{1}{2} w_\alpha(P_k) W(P, P_k); \tag{3.2}$$

$$\frac{d}{d\lambda_k}\Big|_{f(P),\,f(Q)}\{W(P,Q)\} = \frac{1}{2} W(P,P_k) W(Q,P_k). \tag{3.3}$$

Here the derivative with respect to λ_k is taken keeping the projections $f(P)$ and $f(Q)$ of the points P and Q to $\mathbb{C}P^1$ constant;

$$w_\alpha(P_k) := \frac{w_\alpha(P)}{dx_k(P)}\Big|_{P=P_k}, \qquad W(P,P_k) := \frac{W(P,Q)}{dx_k(Q)}\Big|_{P=P_k}. \tag{3.4}$$

Below we solve the linear system (2.6), (2.7), where the rotation coefficients are given by (1.2), i.e. $\Gamma_{jk} = \frac{W(P,Q)}{dx_j(P)dx_k(Q)}\Big|_{P=P_j\,,Q=P_k}$. These coefficients satisfy the system (2.4), (2.5) as a simple corollary of the Rauch formulas (3.3).

3.2 Construction of a solution to the Fuchsian system

Let us fix some $\lambda \in \mathbb{C}P^1$ which does not coincide with any of λ_j, i.e. such that its pre-image $f^{-1}(\lambda)$ consists of N different points $\lambda^{(k)}$, $k = 1, \dots, N$. Let us also enumerate in some way the points of $f^{-1}(\infty)$, which we denote by $\infty^{(s)}$, $s = 1, \dots, K$ (if some of $\infty^{(s)}$ are ramification points then $K < N$).

Introduce the homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$, with coefficients in $\mathbb{Z}$, of the Riemann surface $\mathcal{L}$ punctured at K points $\infty^{(s)}$, $s = 1, \dots, K$, relative to the set $f^{-1}(\lambda)$ of N points $\lambda^{(k)}$, $k = 1, \dots, N$. The dimension of $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$ equals $2g + N + K - 2$. We notice that this dimension equals the number L of the branch points $\{\lambda_j\}$. The set of basic contours $\mathbf{s}_k$, $k = 2g + N + K - 2$ in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$ can be chosen as follows:

$$\mathbf{s}_{2\alpha-1} := a_\alpha \qquad \mathbf{s}_{2\alpha} := b_\alpha\,, \qquad \alpha = 1, \dots, g, \tag{3.5}$$

where (a_α, β_α) is a canonical basis of cycles in the homology group $H_1(\mathcal{L}, \mathbb{Z})$;

$$\mathbf{s}_{2g+s} := l_s\,, \qquad s = 1, \dots, K-1, \tag{3.6}$$

where l_s is the closed contour encircling $\infty^{(s)}$ in the positive direction (in $H_1(\mathcal{L}, \mathbb{Z})$ the contour l_s is trivial);

$$\mathbf{s}_{2g+K-1+n} := \gamma_{n,n+1}(\lambda)\,, \qquad n = 1, \dots, N-1, \tag{3.7}$$

where $\gamma_{n,n+1}(\lambda)$ is some contour connecting the points $\lambda^{(n)}$ and $\lambda^{(n+1)}$.

It is sometimes convenient to choose the basis (3.5) which forms a part of the basis in the space of relative homologies $H_1(\mathcal{L}\backslash f^{-1}(\infty)\,;\, f^{-1}(\lambda)))$ independently of the basis $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ used for normalization of the bidifferential W (see Section 3.1).

From $W(P,Q)$ we can construct the meromorphic differentials $W(P,P_j)$ on $\mathcal{L}$. A differential $W(P,P_j)$ is the Abelian differential of the second kind, having a second order pole at P_j with the singular part $x_j(P)^{-2}dx_j(P)$ and all vanishing periods over the cycles $\mathbf{a}_\alpha$. The meromorphic differential $f(P)W(P,P_j)$ is not normalized; it has a second order pole at P_j and poles at all poles $\infty^{(s)}$, $s = 1,\dots,K$ of the function f.

Now we are going to construct a solution to the Fuchsian system in terms of integrals of the differentials $W(P,P_j)$ and $f(P)W(P,P_j)$ over the basis in the group of relative homologies $H_1(\mathcal{L}\setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$.

Consider some point $\lambda_0 \in \mathbb{C}$ which does not coincide with any of λ_j. Consider an open neighbourhood $D \subset \mathbb{C}$ of λ_0 such that for all $\lambda \in D$ one can naturally identify the corresponding groups $H_1(\mathcal{L}\setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$ with $H_1(\mathcal{L}\setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda_0))$ (this concerns in fact only the contours $\gamma_{n,n+1}(\lambda)$ (3.7): we require that for all $\lambda \in D$ these contours differ from $\gamma_{n,n+1}(\lambda_0)$ only by paths connecting the endpoints $[\lambda_0^{(n)}, \lambda^{(n)}]$ and $[\lambda_0^{(n+1)}, \lambda^{(n+1)}]$ within $f^{-1}(D)$).

For any contour $\mathbf{s} \in H_1(\mathcal{L}\setminus f^{-1}(\infty);\ f^{-1}(\lambda))$ we introduce the column vector-function $\Phi^{(\mathbf{s})}$ with values in $\mathbb{C}^L$ whose jth component ($j = 1,\dots,L$) is given by:

$$\Phi_j^{(\mathbf{s})}(\lambda) := \lambda \int_{\mathbf{s}} W(P,P_j) - \int_{\mathbf{s}} f(P)W(P,P_j). \tag{3.8}$$

Let us choose for a moment the canonical basis of cycles $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$, used for normalizing (see Section 3.1) the meromorphic bidifferential W, to coincide with the canonical basis of cycles (a_α, b_α) from the basis (3.5) in $H_1(\mathcal{L}\setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$. Then the vectors $\Phi^{(a_\alpha)}$, $\alpha = 1,\dots,g$, do not depend on λ, since a-periods of the differentials $W(P,P_j)$ vanish:

$$\Phi_j^{(a_\alpha)}(\lambda) = -\oint_{a_\alpha} \lambda(P)W(P,P_j)\ .$$

The vectors $\Phi^{(b_\alpha)}$, $\alpha = 1,\dots,g$ are linear in λ; since b-periods of W are given by the holomorphic normalized differentials $\{w_\alpha\}$:

$$\Phi_j^{(b_\alpha)}(\lambda) = 2\pi\mathrm{i}\,\lambda\, w_\alpha(P_j) - \oint_{b_\alpha} f(P)W(P,P_j)\ .$$

The columns corresponding to the contours l_s do not depend on λ either, since the differentials $W(P,P_j)$ are non-singular at $\infty^{(s)}$:

$$\Phi_j^{(l_s)}(\lambda) = 2\pi\mathrm{i}\,\mathrm{res}|_{P=\infty^{(s)}}[f(P)W(P,P_j)]\ , \qquad s = 1,\dots,K-1. \tag{3.9}$$

In particular, if all $\infty^{(s)}$ are not ramification points, i.e. $K = N$, the residues in (3.9) can be easily computed to give

$$\Phi_j^{(l_s)}(\lambda) = 2\pi\mathrm{i}\, W(\infty^{(s)}, P_j)\ , \qquad s = 1,\dots,N-1. \tag{3.10}$$

The columns $\Phi^{(\gamma_{n,n+1}(\lambda))}$ depend on λ non-trivially since the integration contours $\gamma_{n,n+1}(\lambda)$ depend on λ.

Theorem 1 *For any contour* $\mathbf{s} \in H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$*, the vector function* $\Phi^{(\mathbf{s})}$ *defined by (3.8), satisfies the linear system (2.6), (2.7) with* $\alpha = -1/2$ *and* $\lambda \in D$*.*

Proof. We shall check that the vector $\Phi^{(\mathbf{s})}(\lambda) = (\Phi_1^{(\mathbf{s})}(\lambda), \dots, \Phi_L^{(\mathbf{s})}(\lambda))^T$ satisfies the system (2.8), (2.9), (2.10) with $\alpha = -1/2$, which is equivalent to the original system (2.6), (2.7) with the same value of the parameter α.

The validity of equations (2.10) is an immediate consequence of the Rauch variational formulas for the bidifferential $W(P,Q)$.

To verify (2.9) we lift the functions $\Phi_j^{(\mathbf{s})}(\lambda)$, $\lambda \in D \subset \mathbb{CP}^1$, (3.8) to the function $\Phi_j^{(\mathbf{s})}(f(P))$ on the covering $\mathcal{L}$. We shall study the behaviour of the functions $\Phi_j^{(\mathbf{s})}(f(P))$ under biholomorphic transformations of the Riemann surface $\mathcal{L}$.

The equation (2.9) is an infinitesimal form of the invariance the function $\Phi^{(\mathbf{s})}(f(P))$ under a simultaneous translation of all λ_j and $\lambda = f(P)$ by a constant. Namely, consider a biholomorphic mapping of the Riemann surfaces $\mathcal{L} \to \mathcal{L}^\delta$ which acts in every sheet of $\mathcal{L}$ by sending the point P with the projection $\lambda = f(P)$ to the point P^δ projecting to $\lambda^\delta := f(P^\delta) = f(P) + \delta$ on the base of the covering. The branch points $\{\lambda_i\}$ are then mapped to $\{\lambda_i + \delta\}$. Due to the invariance of the local parameters $x_i(P) = \sqrt{f(P) - \lambda_i}$ under the mapping and the invariance of the bidifferential W under all biholomorphic mappings of the surfaces, the equality $W(P, P_i) = W^\delta(P^\delta, P_i^\delta)$ holds, where W^δ is the bidifferential W defined on $\mathcal{L}^\delta$. Therefore, for the function $\Phi_j^{(\mathbf{s})}(f(P))$ we have:

$$
\begin{aligned}
(\Phi_j^{(\mathbf{s})})^\delta(f(P^\delta)) &:= f(P^\delta)\int_{\mathbf{s}^\delta} W^\delta(Q, P_j^\delta) - \int_{\mathbf{s}^\delta} f(Q) W^\delta(Q, P_j^\delta) \\
&= (f(P) + \delta)\int_{\mathbf{s}} W(Q, P_j) - \int_{\mathbf{s}} (f(Q) + \delta) W(Q, P_j),
\end{aligned}
$$

where the second equality is obtained by changing the variable of integration $Q \mapsto Q^\delta$ and using the invariance $W(P, P_j) = W^\delta(P^\delta, P_j^\delta)$. Differentiating the above relation with respect to δ at $\delta = 0$ we get $\partial_\lambda \Phi_j^{(\mathbf{s})}(\lambda) + \mathbf{e}(\Phi_j^{(\mathbf{s})}(\lambda)) = 0$, i.e. the first equation in (2.9).

Finally, the equation in (2.8) with $\alpha = -1/2$ can be verified by considering the transformation of the function $\Phi^{(\mathbf{s})}(f(P))$ under the biholomorphic mapping of the Riemann surfaces $\mathcal{L} \to \mathcal{L}^\epsilon$ which maps the point P with the projection $f(P)$ to the point P^ϵ belonging to the same sheet and projecting to $f(P^\epsilon) = (1+\epsilon) f(P)$ on the base. The local parameters $x_j(P)$ get multiplied by $\sqrt{1+\epsilon}$ and the bidifferential W stays invariant, i.e. $W(P,Q) = W^\epsilon(P^\epsilon, Q^\epsilon)$. Thus for the differential $W(Q, P_j)$ we have $W^\epsilon(Q^\epsilon, P_j^\epsilon) = W(Q, P_j)/\sqrt{1+\epsilon}$, see (3.4). Therefore, for the function $\Phi_j^{(\mathbf{s})}(f(P))$ (3.8) we have:

$$
\begin{aligned}
(\Phi_j^{(\mathbf{s})})^\epsilon(f(P^\epsilon)) &:= f(P^\epsilon)\int_{\mathbf{s}^\epsilon} W^\epsilon(Q, P_j^\epsilon) - \int_{\mathbf{s}^\epsilon} f(Q) W^\epsilon(Q, P_j^\epsilon) \\
&= \sqrt{1+\epsilon}\left[f(P)\int_{\mathbf{s}} W(Q, P_j) - \int_{\mathbf{s}} f(Q) W(Q, P_j) \right],
\end{aligned}
$$

where the second equality is obtained by changing the variable of integration $Q \mapsto Q^\epsilon$ and using the relation $W^\epsilon(Q^\epsilon, P_j^\epsilon) = W(Q, P_j)/\sqrt{1+\epsilon}$. This implies for the function $\Phi_j^{(\mathbf{s})}(\lambda(P))$:

$$
(\Phi_j^{(\mathbf{s})})^\epsilon(f(P^\epsilon)) = \sqrt{1+\epsilon}\,\Phi_j^{(\mathbf{s})}(f(P)).
$$

Differentiating this relation with respect to ϵ at $\epsilon = 0$ we get

$$\lambda\,\partial_\lambda \Phi_j^{(\mathbf{s})}(\lambda) + \mathbf{E}(\Phi_j^{(\mathbf{s})}(\lambda)) = d_\epsilon\,|_{\epsilon=0}\,(\Phi_j^{(\mathbf{s})})^\epsilon(\lambda^\epsilon) = \frac{1}{2}\Phi_j^{(\mathbf{s})}(\lambda).$$

$\square$

Now from L vectors $\Phi^{(\mathbf{s}_k)}$, $k = 1, \dots, L$, corresponding to the basis (3.5), (3.7), (3.6) of $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda))$, we construct the $L \times L$ matrix

$$\Phi(\lambda) := (\Phi^{(\mathbf{s}_1)}, \Phi^{(\mathbf{s}_2)}, \dots, \Phi^{(\mathbf{s}_L)}) \qquad \text{for } \lambda \in D. \tag{3.11}$$

Theorem 2 *The matrix $\Phi(\lambda)$ (3.11) gives a complete set of linearly independent solutions to the Fuchsian linear system (2.6) for $\lambda \in D$ with $\alpha = -1/2$. The matrix $\Phi(\lambda)$ also satisfies the isomonodromy deformation equations (2.7).*

Proof. The matrix Φ satisfies equations (2.6) and (2.7) since each of its columns satisfies these equations. The proof of linear independence of its columns is rather tedious. We postpone it to Section 5 which is entirely devoted to this proof. $\square$

The next section is devoted to a description of the monodromy group of the function Φ (3.11).

3.3 Monodromy group

For any set of N points $Q_1, \dots, Q_N$ on a Riemann surface $\mathcal{L}$ introduce the surface braid group $B_N(\mathcal{L}\,,\,\{Q_j\}_{j=1}^N)$ (see [3]; if $\mathcal{L}$ is the complex plane, the surface braid group coincides with the Artin braid group).

For a description of the monodromy group of the Fuchsian system (2.6) we introduce the surface braid group $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\, f^{-1}(\lambda_0))$. The corresponding strands end at N points from $f^{-1}(\lambda_0)$, i.e. at $\lambda_0^{(1)}, \dots, \lambda_0^{(N)}$.

The lift $f^{-1}(\gamma)$ of a path $\gamma \in \pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ from $\mathbb{C}P^1$ to $\mathcal{L} \setminus f^{-1}(\infty)$ consists of N non-intersecting paths on $\mathcal{L}$ which start and end in the set $\{\lambda_0^{(1)}, \dots, \lambda_0^{(N)}\}$. Therefore, $f^{-1}(\gamma)$ naturally defines an element of the group $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\, f^{-1}(\lambda_0))$. Obviously, for any two elements γ and $\tilde{\gamma}$ of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ the element of the surface braid group corresponding to $f^{-1}(\gamma \circ \tilde{\gamma})$ coincides with that corresponding to the product $f^{-1}(\gamma) \circ f^{-1}(\tilde{\gamma})$. Therefore, we get the following

Proposition 2 *The map f^{-1} from $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ to $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\, f^{-1}(\lambda_0))$ defined above is a group homomorphism.*

There exists also the standard homomorphism from the surface braid group $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\, f^{-1}(\lambda_0))$ to the symmetric group S_N acting on the set of N points $\lambda_0^{(1)}, \dots, \lambda_0^{(N)}$. The superposition of this homomorphism with the homomorphism f^{-1} from Proposition 2 gives the standard group homomorphism $\mathbf{h}$ from $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ to the symmetric group S_N; the image of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ under the homomorphism $\mathbf{h}$ is called the *monodromy group of the covering.*

Now, for any Riemann surface $\mathcal{L}$ and the set of N points $\{Q_n \in \mathcal{L}\}_{n=1}^N$ one can define a natural action of the surface braid group $B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ on the relative homology group $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$. Namely, on the space of absolute homologies $H_1(\mathcal{L})$ (which is a linear subspace of $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$) the group $B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ acts identically. Let us describe the action of an element $G \in B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ on an element of $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ represented by a contour γ_{mn} which connects the points Q_m and

Q_n. Denote an element of S_N defined by G by $(i_1, \dots, i_N)$; then G is defined by N paths $\{l_n\}$ on $\mathcal{L}$; path l_n connects points Q_n and Q_{i_n}.

Denote the classes of the paths l_n in the relative homology group $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$, by $\mu_1, \dots, \mu_N$. Consider some system of contours $\gamma_{mn} \in H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ connecting the points Q_m and Q_n, $m, n = 1, \dots, N$. Here we speak of contours and of the elements of the corresponding homology groups which they represent interchangeably. The natural action of $G \in B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ on a contour γ_{mn} is defined by

$$\gamma_{mn} \to \gamma_{mn} - \mu_m + \mu_n \,. \tag{3.12}$$

The contour μ_m connects the points Q_m and Q_{i_m}; thus $\mu_m = \gamma_{m\,i_m} + C_m$, where $C_m \in H_1(\mathcal{L})$; also $\mu_n = \gamma_{n\,i_n} + C_n$, where $C_n \in H_1(\mathcal{L})$. Therefore, the action (3.12) of G on γ_{mn} has the form: $\gamma_{mn} \to \gamma_{i_m\,i_n} + C_{mn}$, where $C_{mn} \in H_1(\mathcal{L})$ (i.e. C_{mn} correspond to some closed contours in $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$.

In this way we assign to each $G \in B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ a linear automorphism of $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$.

Proposition 3 *This map from $B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ to the group of linear automorphisms of $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ is a group homomorphism.*

The proof is geometrically obvious: it is easy to see that the action of the product of two elements of $B_N(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ on $H_1(\mathcal{L}\,,\,\{Q_n\}_{n=1}^N)$ corresponds to the superposition of the automorphisms corresponding to each of these elements.

Let us now denote by R the homomorphism from the surface braid group $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\,f^{-1}(\lambda_0))$ to the group of linear automorphisms of the linear vector space $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda_0))$.

The superposition $F := R \circ f^{-1}$ (the homomorphism f^{-1} from $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ to $B_N(\mathcal{L} \setminus f^{-1}(\infty)\,,\,f^{-1}(\lambda_0))$ is described before Proposition 2) defines a group homomorphism from $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ to $\mathrm{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda_0))]$.

The next theorem states that, essentially, the image of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ in $\mathrm{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda_0))]$ under F coincides with monodromy group of the Fuchsian system (2.15).

Consider a standard system of generators $\gamma_1, \dots, \gamma_L, \gamma_\infty$ (2.1) in the fundamental group $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ based at λ_0.

Theorem 3 *Let a solution $\Phi(\lambda)$ to the Fuchsian system (2.6) in the neighbourhood D of a base point λ_0 be given by (3.8), (3.11), where the basis $\{\mathbf{s}_k\}$ in the relative homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda))$ is given by (3.5), (3.6) and (3.7). Let the automorphisms $F(\gamma_k) \in \mathrm{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda_0))]$ (where the homomorphism F is defined before the theorem) be defined in the basis $\{\mathbf{s}_k\}$ by the matrices F_k. Then the solution $\Phi(\lambda)$ transforms under the analytical continuation along the path γ_k as follows: $\Phi \to \Phi M_k$, where the monodromy matrices M_k are related to the matrices F_k by:*

$$M_k = (F_k)^t\,, \qquad k = 1, \dots, L, \infty\,. \tag{3.13}$$

Proof. To prove the theorem one has to remember that the neighbourhood D of λ_0 was chosen such that the contours $\mathbf{s}_k(\lambda)$ can be naturally identified with $\mathbf{s}_k(\lambda_0)$ for any $\lambda \in D$. Then the statement of the theorem is just a corollary of the definition of the function Φ (3.8), (3.11) in terms of integrals of certain meromorphic differentials over the contours $\mathbf{s}_k(\lambda)$, as well as of the definitions of monodromy matrices and the homomorphism F. □

The transposition in the relation (3.13) between the matrices M_k and F_k appears since the cycles $\mathbf{s}_k$ label the *columns* of matrix Φ. Thus the map from $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ to $GL(L, \mathbb{C})$ given by monodromy map is an *anti*-homomorphism (i.e. the monodromy matrices multiply in the order opposite to the order of multiplication of the corresponding paths in $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0))$, see (2.1), (2.2).

In our situation, when all finite branch points are simple and the covering is connected, the monodromy group of the covering $\mathcal{L}$ (i.e. the image of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_L, \infty\}, \lambda_0)$ in S_N under the homomorphism $\mathbf{h}$) coincides with the whole symmetric group S_N. Let us denote the permutations corresponding to the points λ_k by σ_k, i.e. $\sigma_k = \mathbf{h}(\gamma_k)$, $k = 1, \dots, L, \infty$. The permutations satisfy the relation

$$\sigma_1\sigma_2 \dots \sigma_L\sigma_\infty = id.$$

One can make the following statement about the structure of the monodromy matrices:

Theorem 4 *The monodromy matrices of the function Φ defined by (3.8), (3.11) have the following block structure:*

$$M_k = \begin{pmatrix} I & S_k \\ 0 & T_k \end{pmatrix}, \tag{3.14}$$

where I is the $2g + K - 1 \times 2g + K - 1$ identity matrix; 0 is the $2g + K - 1 \times N - 1$ matrix with zero entries; S and T are matrices with integer entries of size $2g + K - 1 \times N - 1$ and $N - 1 \times N - 1$, respectively. Moreover, the matrix T depends only on the element σ_k of the monodromy group of the covering.

Proof. The diagonal unit block of the size $2g + K - 1 \times 2g + K - 1$ and the zero matrix in the left lower corner of M_k appear since the first $2g + K - 1$ columns of the matrix Φ are either linear functions of λ or constant with respect to λ; these $2g + K - 1$ columns remain thus invariant under the analytical continuation of Φ along any γ_k (this can also be seen from the fact that the basic contours $\mathbf{s}_k$, $k = 1, \dots, 2g + K - 1$, are independent of λ and, therefore, do not change under the analytical continuation). The matrices S_k and T_k define the transformation of the contours $\gamma_{n,n+1}(\lambda_0)$, $n = 1, \dots, n-1$ under the analytical continuation along γ_k. The contour $\gamma_{n,n+1}(\lambda_0)$ gets mapped under such a transformation to some contour connecting the points $\lambda_0^{(i_n)}$ and $\lambda_0^{(i_{n+1})}$ (where $(i_1, \dots, i_N) \in S_N$ is an element $\mathbf{h}(\gamma_k)$ of the monodromy group of the covering $\mathcal{L}$ corresponding to γ_k). This contour can be expressed in $H_1(\mathcal{L} \setminus f^{-1}(\infty)\,;\, f^{-1}(\lambda_0))$ as a linear combination of the contours $\gamma_{n,n+1}(\lambda_0)$, $n = 1, \dots, N-1$, basic a- and b-cycles, and cycles around $\infty^{(s)}$. The coefficients in front of $\{\gamma_{n,n+1}(\lambda_0)\}$ are given by the matrix T_k; clearly, they depend only on the permutation $\mathbf{h}(\gamma_k)$; thus the matrices T_k are entirely determined by the monodromy group of the covering $\mathcal{L}$. The matrices S_k, which determine the coefficients in front of the a- and b-cycles, and the cycles around $\infty^{(s)}$, depend also on the choice of a canonical basis of cycles in $H_1(\mathcal{L})$. □

It is thus easy to see that under a change of the basis $(a_\alpha, b_\alpha, l_s)$ in $H_1(\mathcal{L} \setminus f^{-1}(\infty))$ the matrices T_k do not change; the matrices S_k transform in an obvious way given by the next proposition.

Proposition 4 *Let $2g + K - 1 \times 2g + K - 1$ matrix Q define a transformation between the canonical basis $(a_\alpha, b_\alpha, l_s)$ in $H_1(\mathcal{L} \setminus f^{-1}(\infty)$ and a new basis $(\tilde{a}_\alpha, \tilde{b}_\alpha, \tilde{l}_s)$, i.e.*

$$\begin{pmatrix} a_\alpha \\ b_\alpha \\ l_s \end{pmatrix} = Q \begin{pmatrix} \tilde{a}_\alpha \\ \tilde{b}_\alpha \\ \tilde{l}_s \end{pmatrix}. \tag{3.15}$$

Then the new monodromy matrices have the form (3.14) with the same matrices T_k and new matrices S_k given by:

$$\tilde{S}_k = Q^t S_k\,. \tag{3.16}$$

The proof is an immediate corollary of the definition of the matrices S_k; it is also easy to observe that the simultaneous transformation (3.16) of all matrices S_k preserves the relation (2.2) between the monodromy matrices.

Remark 3 We would like to stress that in Proposition 4 we only consider the dependence of Φ on the change of some of the integration contours $\mathbf{s}_k$ in (3.8); the canonical basis of cycles $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ used in the definition of the bidifferential W (see Section 3.1) is assumed to remain the same. The dependence of Φ on the choice of a basis $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ (i.e. on the normalization of $W(P,Q)$) is discussed in the next section.

The transformed monodromy matrices $\tilde{M}_k$ (3.14), (3.16) are related to the monodromy matrices M_k by a simultaneous conjugation :

$$\tilde{M}_k = \begin{pmatrix} Q^t & 0 \\ 0 & I \end{pmatrix} M_k \begin{pmatrix} (Q^t)^{-1} & 0 \\ 0 & I \end{pmatrix}^{-1} ;$$

the corresponding solutions of the Fuchsian system are related by

$$\tilde{\Phi} = \Phi \begin{pmatrix} (Q^t)^{-1} & 0 \\ 0 & I \end{pmatrix}^{-1} . \tag{3.17}$$

3.4 Dependence of the solution on the normalization of $W(P,Q)$

In this section we discuss the dependence of the solution Φ (3.8), (3.11) on the choice of a canonical homology basis $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ used to normalize the bidifferential W.

Denote by $\mathbf{a}$ and $\mathbf{b}$ the vectors of basis cycles: $\mathbf{a} := (\mathbf{a}_1, \dots, \mathbf{a}_g)^T$ and $\mathbf{b} := (\mathbf{b}_1, \dots, \mathbf{b}_g)^T$. Two canonical homology bases $(\mathbf{a}, \mathbf{b})$ and $(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ are related by a symplectic transformation:

$$\begin{pmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} . \tag{3.18}$$

The corresponding transformation of the bidifferential W is given by [11], p.10:

$$\widehat{W}(P,Q) = W(P,Q) - 2\pi \mathrm{i}\, w^T(P)(C\mathbb{B} + D)^{-1} C w(Q), \tag{3.19}$$

where w is the vector of holomorphic differentials, $w := (w_1, \dots, w_g)^T$, normalized by $\oint_{\mathbf{a}_\alpha} w_\beta = \delta_{\alpha\beta}$, and $\mathbb{B}$ is the matrix of $\mathbf{b}$-periods: $\mathbb{B}_{\alpha\beta} := \oint_{\mathbf{b}_\alpha} w_\beta$.

Let us denote by $\widehat{\Phi}(\lambda)$ the matrix function constructed as in (3.8), (3.11) from the transformed bidifferential $\widehat{W}$ using the same basis $\{\mathbf{s}_k\}$ of $H_1(\mathcal{L} \setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$. The function $\widehat{\Phi}(\lambda)$ solves the system (2.6), (2.7) with the matrix V built from the rotation coefficients given by the deformed bidifferential: $\beta_{ij} = \widehat{W}(P_i, P_j)/2$. As can be seen from Section 4, the monodromy matrices M_k are the same for the functions Φ and $\widehat{\Phi}$. Therefore, these functions are related by a Schlesinger transformation, which we describe in the next theorem.

Theorem 5 *Let the matrix $\Phi(\lambda)$ be, as before, defined by formulas (3.8), (3.11) and the integration contours (3.5) - (3.7). Let the matrix $\widehat{\Phi}(\lambda)$ be defined by the same formulas and contours with the bidifferential W replaced by the transformed bidifferential $\widehat{W}$ (3.19). Then the following relation holds:*

$$\widehat{\Phi}(\lambda) = (\mathbf{1} - \mathbf{T}(\lambda))\, \Phi(\lambda), \tag{3.20}$$

where $\mathbf{1}$ *denotes the identity matrix; the matrix* $\mathbf{T}$ *is a symmetric matrix with the entries:*

$$(\mathbf{T})_{ij} = \pi \mathrm{i}\,(\lambda_j - \lambda) \sum_{\alpha,\beta=1}^{g} \left[(C\mathbb{B}+D)^{-1}C\right]_{\alpha\beta} w_\alpha(P_i) w_\beta(P_j). \tag{3.21}$$

Here w_α *are the holomorphic differentials normalized with respect to the cycles* $\mathbf{a}$*, their value at the ramification point* P_j *is defined by* $w_\alpha(P_j) := \frac{w_\alpha(P)}{d\sqrt{\lambda-\lambda_j}}|_{P=P_j}$*;* $\mathbb{B}$ *is the matrix of their* $\mathbf{b}$*-periods; the constant matrices* C *and* D *are blocks of the symplectic transformation (3.18) between the two canonical homology bases.*

Remark 4 Notice that we can rewrite the transformation (3.20) in the form $\widehat{\Phi}(\lambda) = (\mathbf{1} + \mathbf{T}_1 - \lambda \mathbf{T}_2)\,\Phi(\lambda)$, where the matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ do not depend on λ.

Proof. The theorem can be proved by a direct computation as follows. Relation (3.20) is equivalent to

$$\lambda \int_{\mathbf{s}} \widehat{W}(P,P_i) - \int_{\mathbf{s}} f(P)\widehat{W}(P,P_i) = \sum_{j=1}^{L} (\mathbf{1}-\mathbf{T})_{ij} \left(\lambda \int_{\mathbf{s}} W(P,P_j) - \int_{\mathbf{s}} f(P) W(P,P_j)\right). \tag{3.22}$$

Using the definition (3.21) of the matrix $\mathbf{T}$ and the Rauch variational formula (3.2) for the holomorphic differentials w_α, we obtain:

$$\sum_{j=1}^{L} (\mathbf{1}-\mathbf{T})_{ij} \int_{\mathbf{s}} f(P) W(P,P_j)$$
$$= \int_{\mathbf{s}} f(P)W(P,P_i) - 2\pi\mathrm{i} \sum_{\alpha,\beta=1}^{g} \left[(C\mathbb{B}+D)^{-1}C\right]_{\alpha\beta} w_\alpha(P_i) \left[\mathbf{E}\left(\int_{\mathbf{s}} f(P) w_\beta(P)\right) - \lambda \mathbf{e}\left(\int_{\mathbf{s}} f(P) w_\beta(P)\right)\right], \tag{3.23}$$

where $\mathbf{E} = \sum_{j=1}^{L} \lambda_j \partial_{\lambda_j}$ is the Euler vector field and $\mathbf{e} = \sum_{j=1}^{L} \partial_{\lambda_j}$ is the unit vector field on the Frobenius manifold. We compute the action of these fields on our integrals using the invariance of the holomorphic differentials w_k with respect to the biholomorphic mappings of Riemann surfaces $\mathcal{L} \to \mathcal{L}^\epsilon$ and $\mathcal{L} \to \mathcal{L}^\delta$ from the proof of Theorem 1:

$$\mathbf{E}\left(\int_{\mathbf{s}} f(P) w_\beta(P)\right) = \frac{d}{d\epsilon}\,|_{\epsilon=0} \int_{\mathbf{s}^\epsilon} f(P) w_\beta^\epsilon(P) = \frac{d}{d\epsilon}\,|_{\epsilon=0} \int_{\mathbf{s}} f(P)(1+\epsilon) w_\beta(P) = \int_{\mathbf{s}} f(P) w_\beta(P). \tag{3.24}$$

$$\mathbf{e}\left(\int_{\mathbf{s}} f(P) w_\beta(P)\right) = \frac{d}{d\delta}\,|_{\delta=0} \int_{\mathbf{s}^\delta} f(P) w_\beta^\delta(P) = \frac{d}{d\delta}\,|_{\delta=0} \int_{\mathbf{s}} (f(P)+\delta) w_\beta(P) = \int_{\mathbf{s}} w_\beta(P). \tag{3.25}$$

To obtain the second equalities in the above lines we used the invariance $w_\beta^\epsilon(P^\epsilon) = w_\beta(P)$ and $w_\beta^\delta(P^\delta) = w_\beta(P)$ of the normalized holomorphic differentials under the biholomorphic mappings.

Similarly, for the first summand in the right hand side of (3.22) we get:

$$\sum_{j=1}^{L}(\mathbf{1}-\mathbf{T})_{ij}\,\lambda\int_{\mathbf{s}}W(P,P_j)=\lambda\int_{\mathbf{s}}W(P,P_i)$$

$$-2\pi\mathrm{i}\lambda\sum_{\alpha,\beta=1}^{g}\left[(C\mathbb{B}+D)^{-1}C\right]_{\alpha\beta}w_\alpha(P_i)\left[\mathbf{E}\left(\int_{\mathbf{s}}w_\beta(P)\right)-\lambda\mathbf{e}\left(\int_{\mathbf{s}}w_\beta(P)\right)\right]=\lambda\int_{\mathbf{s}}W(P,P_i). \quad (3.26)$$

The last equality in (3.26) follows from the easily verified fact that the integrals of differentials w_k over the contours $\mathbf{s}$ (3.5) - (3.7) are invariant under biholomorphic mappings of the Riemann surface, and therefore, the actions of the vector fields $\mathbf{E}$ and $\mathbf{e}$ on these integrals give zero.

Thus, plugging relations (3.23), (3.24), (3.25) and (3.26) into (3.22) and using the expression (3.19) for the transformed bidifferential W, we get (3.22). □

Lemma 1 *The matrix* $\mathbf{1}-\mathbf{T}$ *from Theorem 5 is non-degenerate. Its inverse is given by* $\mathbf{1}+\mathbf{T}$.

Proof. The statement of the lemma follows from the relation $\mathbf{T}^2=0$, which holds due to the following identity:

$$\sum_{j=1}^{L}(\lambda_j-\lambda)w_\alpha(P_j)w_\beta(P_j)=0 \quad \text{for any } \alpha,\beta=1,\dots,g. \quad (3.27)$$

Using the Rauch variational formulas (3.1) for the Riemann matrix we note that the left hand side of (3.27) is a multiple of the quantity $\mathbf{E}(\mathbb{B}_{kl})-\lambda\mathbf{e}(\mathbb{B}_{kl})$. The constancy of the Riemann matrix $\mathbb{B}$ along the Euler and the unit vector fields, $\mathbf{E}(\mathbb{B}_{\alpha\beta})=0$ and $\mathbf{e}(\mathbb{B}_{\alpha\beta})=0$, is proved as in (3.26) choosing the contour of integration to be $\mathbf{s}=b_\beta$. □

Corollary 1 *Let* Φ *and* $\widehat{\Phi}$ *be the solutions (3.8), (3.11) to the systems (2.6), (2.7) built from the canonical meromorphic bidifferentials* W *and* $\widehat{W}$, *normalized using the canonical homology bases* $(\mathbf{a}_\alpha,\,\mathbf{b}_\alpha)$ *and* $(\hat{\mathbf{a}}_\alpha,\,\hat{\mathbf{b}}_\alpha)$, *respectively (the integration contours* $\{\mathbf{s}_k\}$ *(3.5) - (3.7) are taken to be the same for* Φ *and* $\widehat{\Phi}$*). Then the non-degeneracy of the matrix* Φ, $\det\Phi\neq 0$, *implies the non-degeneracy of the matrix* $\widehat{\Phi}$, $\det\widehat{\Phi}\neq 0$.

Remark 5 Note that while the transformation (3.18) of the homology basis is done by a symplectic matrix with integer entries, we can construct a bidifferential $\widehat{W}^{\mathbb{C}}$ as in (3.19) with C and D being the corresponding blocks of a symplectic matrix with complex entries. Such a bidifferential $\widehat{W}^{\mathbb{C}}$ gives a "deformation" of the original bidifferential W.

Namely, let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in \mathrm{Sp}(2g,\mathbb{C})$$

and assume the matrix $C\mathbb{B}+D$ is non-degenerate. Then the bidifferential $\widehat{W}(P,Q)$, $P,Q\in\mathcal{L}$, given by

$$\widehat{W}^{\mathbb{C}}(P,Q)=W(P,Q)-2\pi\mathrm{i}\,w^T(P)(C\mathbb{B}+D)^{-1}Cw(Q), \quad (3.28)$$

can be characterized as a unique symmetric bidifferential with a second order pole at the diagonal $P=Q$ with biresidue 1, normalized by the conditions:

$$\sum_{\alpha=1}^{g}C_{\beta\alpha}\oint_{\mathbf{b}_\beta}\widehat{W}^{\mathbb{C}}(P,Q)+\sum_{\alpha=1}^{g}D_{\beta\alpha}\oint_{\mathbf{a}_\alpha}\widehat{W}^{\mathbb{C}}(P,Q)=0,$$

the integration being done with respect to either of the arguments. (Notice that due to the non-degeneracy of the matrix $C\mathbb{B}+D$, the vanishing of the above combinations of periods of a holomorphic differential w, namely, $\sum_{\alpha=1}^{g} C_{\beta\alpha}\oint_{\mathbf{b}_\alpha} w + \sum_{\alpha=1}^{g} D_{\beta\alpha}\oint_{\mathbf{a}_\alpha} w = 0$ for all $\beta = 1,\dots,g$ implies $w = 0$.)

The variational formulas for $\widehat{W}^{\mathbb{C}}$ have the same form as the Rauch variational formulas (3.3) for the W. The deformed bidifferential $\widehat{W}^{\mathbb{C}}$ is also invariant with respect to biholomorphic transformations of the Riemann surface.

Thus the matrix $\widehat{\Phi}^{\mathbb{C}}(\lambda)$ given by (3.11), (3.8), (3.5)-(3.7) with the W replaced by its deformation $\widehat{W}^{\mathbb{C}}$ solves the system (2.6), (2.7) with $\alpha = -1/2$ and the matrix V built from the entries $V_{ij} = \widehat{W}^{\mathbb{C}}(P_i, P_j)(\lambda_i - \lambda_j)/2$. The deformed system is related to the original one by the Schlesinger transformation of the form (3.20), (3.21) with the matrices C and D having complex-valued entries.

If the matrix C is invertible, the definition (3.28) yields the bidifferential $W_{\mathbf{q}}(P,Q) = W(P,Q) - 2\pi\mathrm{i}\, w^T(P)(\mathbb{B}+\mathbf{q})^{-1}Cw(Q)$, where $\mathbf{q} = C^{-1}D$. This is the deformation of the bidifferential W considered in [17], where the corresponding deformations of Frobenius structures were built - the Frobenius structures with rotation coefficients $\beta_{ij} = W_{\mathbf{q}}(P_i, P_j)/2$. Apparently, one can generalize the deformations from [17] to Frobenius structures with rotation coefficients $\beta_{ij} = \widehat{W}^{\mathbb{C}}(P_i, P_j)/2$.

4 Explicit form of monodromy matrices

4.1 Meromorphic functions with simple poles

Consider the Hurwitz space $\mathcal{H}_{g;N}(1,\dots,1)$ of functions with N simple poles and simple critical points on a Riemann surface of genus g. Then the branched covering $\mathcal{L}$ of genus g has L finite branch points λ_j and no branching at $\lambda = \infty$; the covering $\mathcal{L}$ is defined by a set of L elements of the symmetric group S_N assigned to the branch points. For an explicit computation of monodromy matrices of the solution (3.8), (3.11) to the Fuchsian system (2.6), (2.7) it is useful to represent the branched covering $\mathcal{L}$ in a standard form. For that purpose we make use of Clebsch's result ([4], see [8] for the modern exposition) stating that one can always choose generators $\{\gamma_j\}$ of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1,\dots,\lambda_L,\infty\},\lambda_0)$ satisfying (2.1) in such a way that the loop γ_j encircles only the point λ_j and the set of the corresponding elements σ_k of the monodromy group of the covering has the form:

$$\sigma_1,\dots,\sigma_L = (1,2),(1,2),\dots,(1,2),(1,2),(2,3),(2,3),(3,4),(3,4),\dots,(N-1,N),(N-1,N)\ , \quad (4.1)$$

where the first transposition $(1,2)$ occurs $2g+2$ times at the beginning and the other transpositions $(j,j+1)$, $j \geq 2$, each occur twice, in order.

The covering $\mathcal{L}$ can be visualized as a hyperelliptic Riemann surface of genus g with $N-2$ Riemann spheres attached to it.

4.1.1 Space of hyperelliptic coverings with no branching at infinity

Let us consider the Hurwitz space $\mathcal{H}_{g;2}(1,1)$ of two-fold ramified coverings with $2g+2$ simple finite ramification points, i.e. the coverings represented by the Hurwitz diagram from Figure 1.

Assume the canonical homology basis on the Riemann surface to be chosen in the standard way, i.e. the cycle a_α encircles the ramification points $P_{2\alpha+1}$, $P_{2\alpha+2}$ on the second sheet, and the cycle b_α goes around the points P_2 and $P_{2\alpha+1}$, see Figure 2. Assume also that the branch cuts are chosen to connect the points P_{2k-1} and P_{2k} for $k = 1,\dots,g+1$.

For $\lambda_1,\dots,\lambda_{2g+2}$ denoting, as before, the branch points, we pick the base point λ_0 and the standard generators $\gamma_1,\dots,\gamma_{2g+2},\gamma_\infty$ of the fundamental group $\pi_1(\mathbb{C}\setminus\{\lambda_1,\dots,\lambda_{2g+2}\},\lambda_0)$ satisfying the relation

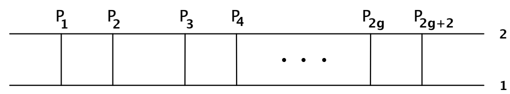

Figure 1: A Hurwitz diagram for the space $\mathcal{H}_{g;2}(1,1)$.

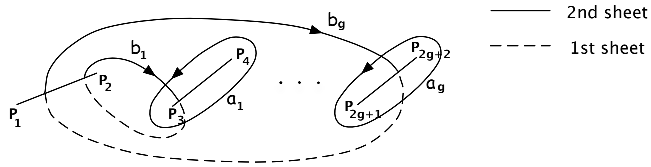

Figure 2: Canonical homology basis for a hyperelliptic curve.

(2.1) with $L = 2g+2$, and the following assumptions. Each generator encircles only one puncture. The loop γ_k going counterclockwise once around the point λ_k on the base of the covering crosses the projection of the branch cut ending at P_k and does not cross projections of other branch cuts.

Corresponding to the Hurwitz space $\mathcal{H}_{g;2}(1,1)$ and the above setting of this section is the solution $\Phi(\lambda)$ to the Fuchsian system (2.6), (2.7), given by (3.5) - (3.8), (3.11) in a neighbourhood of the base point λ_0. In this section we compute monodromy matrices of the solution.

Recall that monodromy matrices have the structure (3.14); they are determined by the transformations of the basis (3.5) - (3.7) in the relative homology group $H_1(\mathcal{L}\setminus f^{-1}(\infty);\ \pi^{-1}(\lambda))$ which occur as the point λ describes the loops γ_k on the base of the covering. The first $2g+1$ columns of the matrix Φ remain unchanged under these transformations. The matrices S_k in (3.14) are thus vectors of the length $2g+1$ and the matrices T_k are scalars.

We now look at the transformations of the last column of the matrix $\Phi(\lambda)$ given by the integral (3.8) over the contour $\gamma_{1,2}$ and find the corresponding S_k and T_k for $k = 1,\dots,2g+2,\infty$. Assume that the contour $\gamma_{1,2}$ goes around the point P_1 when passing from the first sheet to the second.

When λ goes along the loop γ_1 on the base, the contour $\gamma_{1,2}$ transforms to $-\gamma_{1,2}$, as shown in Figure 3. Note that the sum of the two contours in Figure 3 is the closed contour encircling the point P_1; this contour is trivial in the space $H_1(\mathcal{L}\setminus f^{-1}(\infty);\ \pi^{-1}(\lambda))$. Thus S_1 is the zero vector and $T_1 = -1$.

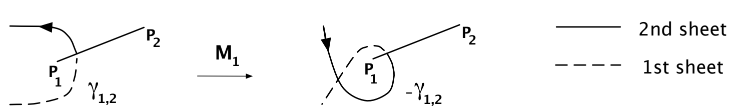

Figure 3: The transformation of the contour $\gamma_{1,2}$ corresponding to the monodromy matrix M_1.

The transformation of the contour $\gamma_{1,2}$ corresponding to the monodromy M_2 is shown in Figure 4. The contour encircling the points P_1 and P_2 on the first sheet is equivalent to the sum $\sum_{\alpha=1}^{g} a_\alpha - l_1$, where the contour l_1 (3.6) encircles counterclockwise the point at infinity on the first sheet. If to the non-closed contour in the right hand side in Figure 4 we add $\gamma_{1,2}$ (the contour $-\gamma_{1,2}$ from Figure 3 with inverse orientation), we obtain a closed contour encircling clockwise the branch cut $[P_1, P_2]$ on the second sheet, i.e. again the contour $\sum_{\alpha=1}^{g} a_\alpha - l_1$. The transformed $\gamma_{1,2}$ is thus equivalent to

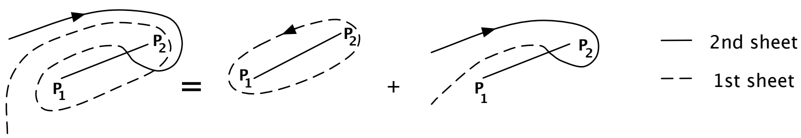

Figure 4: The transformation of the contour $\gamma_{1,2}$ corresponding to monodromy around λ_2.

$2\sum_{\alpha=1}^{g} a_\alpha - 2l_1 - \gamma_{1,2}$. Therefore, $S_2 = (2, 0, \ldots, 2, 0, -2)^T$ and $T_2 = -1$.

The contour $\gamma_{1,2}$ as its end points go counterclockwise around the point λ_{2k+1} results in the contour shown in Figure 5 for $k = 1, \ldots, g+1$. As before, the paths on the first sheet are drawn with dash line and solid line corresponds to the second sheet. If we add the original contour $\gamma_{1,2}$ to the non-closed

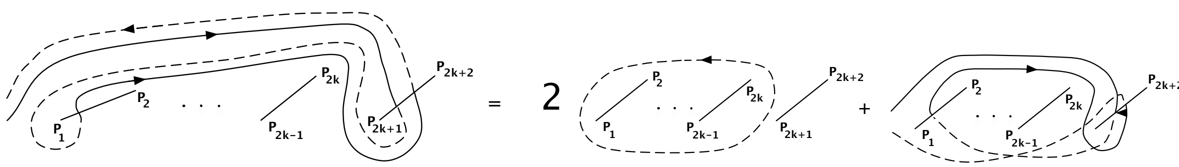

Figure 5: The transformation of the contour $\gamma_{1,2}$ corresponding to monodromy around $\lambda_{2k+1}, k > 1$.

component in the right hand side in Figure 5, we get a closed contour equivalent to $2b_k$. The contour encircling the first $2k$ ramification points on the first sheet is equivalent to the sum $\sum_{\alpha=k}^{g} a_\alpha - l_1$. Therefore, the transformed contour $\gamma_{1,2}$ is equivalent to $-\gamma_{1,2} + 2b_k - 2l_1 + 2\sum_{\alpha=k}^{g} a_\alpha$. The components of the monodromy matrix M_{2k+1} are thus $T_{2k+1} = -1$ and $S_{2k+1} = (\underbrace{0, \ldots, 0}_{2k-2}, 2, 2, \underbrace{2, 0, \ldots, 2, 0}_{2g-2k}, -2)^T$.

Analogously, for M_{2k}, $k = 2, \ldots, g+1$, we find that $\gamma_{1,2}$ becomes to $-\gamma_{1,2} + 2b_{k-1} - 2l_1 + 2\sum_{\alpha=k}^{g} a_\alpha$, which gives $S_{2k} = (\underbrace{0, \ldots, 0}_{2k-3}, 2, \underbrace{2, 0, \ldots, 2, 0}_{2g-2k+2}, -2)^T$ and $T_{2k} = -1$.

As λ goes around ∞ on the base of the covering, the contour $\gamma_{1,2}$ transforms to $\gamma_{1,2} + l_2 - l_1 = \gamma_{1,2} - 2l_1$, thus the monodromy matrix M_∞ is built from $S_\infty = (0, \ldots, 0, -2)^T$ and $T_\infty = 1$.

4.1.2 Space of rational functions with simple poles

In this section we compute the monodromy matrices of the solution Φ (3.5) - (3.8), (3.11) corresponding to the Hurwitz space $\mathcal{H}_{0;N}(1, \ldots, 1)$ of N-fold simple ramified coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$, represented by the Hurwitz diagram in Figure 6. Let $N \geq 3$ in this section; for the case $N = 2$ see Section 4.3, (4.16).

The solution $\Phi(\lambda)$ in this case contains $N-1$ columns which change as λ goes around the points $\lambda = \lambda_k$ on the base of the covering. We compute these changes under the assumptions similar to those in the previous section. Namely, we assume the ramification points to be ordered so that the points P_{2k-1} and P_{2k} belong to the sheets number k and $k+1$ for $k = 1, \ldots, N-1$ and a branch cut is made between them. The generator γ_k of the fundamental group $\pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_{2g+2}\}, \lambda_0)$ is chosen in a standard way to encircle the point λ_k counterclockwise and we assume that it crosses only the projection of the branch cut ending at the point P_k.

As λ describes the loop γ_{2k-1}, $k > 1$, going around λ_{2k-1} on the base of the covering, the columns

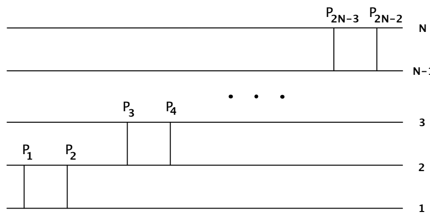

Figure 6: A Hurwitz diagram for the space $\mathcal{H}_{0;N}(1,\dots,1)$.

of the matrix $\Phi(\lambda)$ given by the integrals (3.8) over the contours $\gamma_{k-1,k}$, $\gamma_{k,k+1}$ and $\gamma_{k+1,k+2}$ transform following the transformation of the contours.

Let us assume the contour $\gamma_{k,k+1}$ goes around the point P_{2k-1} to pass from the kth sheet to the next. Then, as is easy to see, when λ goes around λ_{2k-1} counterclockwise, the contour $\gamma_{k-1,k}$ turns into $\gamma_{k-1,k}+\gamma_{k,k+1}$, the contour $\gamma_{k+1,k+2}$ becomes $\gamma_{k,k+1}+\gamma_{k+1,k+2}$, and $\gamma_{k,k+1}$ transforms into $-\gamma_{k,k+1}$ as in Figure 3. Thus the components S and T (3.14) of the corresponding monodromy matrix are the $(N-1)\times(N-1)$ matrices of the form: $S_{2k-1}=0$ and

$$T_{2k-1}=\begin{pmatrix} I_{k-2} & 0 & 0\\ 0 & A & 0\\ 0 & 0 & I_{N-k-2}\end{pmatrix}, \qquad k>1, \tag{4.2}$$

where the block A at the diagonal is given by

$$A=\begin{pmatrix} 1 & 0 & 0\\ 1 & -1 & 1\\ 0 & 0 & 1\end{pmatrix}. \tag{4.3}$$

Similarly, for $k=1$ we have: $S_1=0$ and

$$T_1=\begin{pmatrix} -1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & I_{N-3}\end{pmatrix}. \tag{4.4}$$

The same columns of the matrix $\Phi(\lambda)$ transform, when λ follows the loop γ_{2k}, $k>1$, on the base. The transformation of $\gamma_{k,k+1}$ is analogous to that in Figure 4, where the ramification points are P_{2k-1} and P_{2k} instead of P_1 and P_2, respectively. The contour encircling the branch cut $[P_{2k-1},P_{2k}]$ counterclockwise on the kth sheet is equivalent to the sum $-\sum_{i=1}^k l_i$. Thus after the transformation the contour $\gamma_{k,k+1}$ becomes $-\gamma_{k,k+1}-2\sum_{i=1}^k l_i$.

The transformations of $\gamma_{k-1,k}$ and $\gamma_{k+1,k+2}$ are shown in Figures 7 and 8, respectively.

The sum of the contour in the right hand side of Figure 7 and the contour $-\gamma_{k,k+1}$ is equivalent to $\gamma_{k-1,k}$ plus the contour encircling the branch cut $[P_{2k-1},P_{2k}]$ clockwise on the kth sheet (we use the triviality of the contour encircling one ramification point). Therefore, the contour in Figure 7 is equivalent to $\gamma_{k-1,k}+\gamma_{k,k+1}+\sum_{i=1}^k l_i$.

Analogously, adding $-\gamma_{k,k+1}$ to the contour in the right hand side of Figure 8 we get $\gamma_{k+1,k+2}$ plus a closed contour around the branch cut $[P_{2k-1},P_{2k}]$ oriented clockwise on the kth sheet. Thus the contour $\gamma_{k+1,k+2}$ transforms to $\gamma_{k+1,k+2}+\gamma_{k,k+1}+\sum_{i=1}^k l_i$ as λ goes describes the loop γ_{2k} around λ_{2k}.

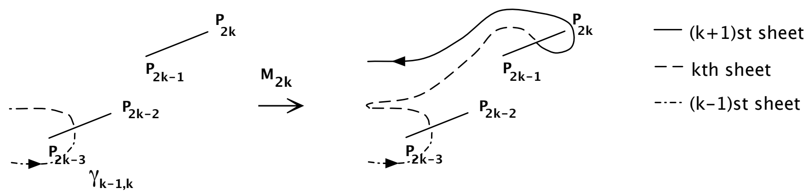

Figure 7: The transformation of the contour $\gamma_{k-1,k}$ corresponding to the monodromy around λ_{2k}.

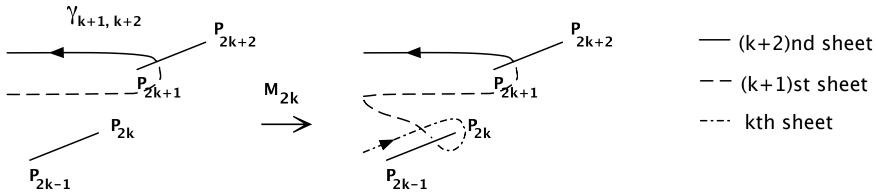

Figure 8: The transformation of the contour $\gamma_{k+1,k+2}$ corresponding to the monodromy around λ_{2k}.

We conclude that the matrix T_{2k} for the monodromy matrix M_{2k} coincides with T_{2k-1} given by (4.2) or (4.4), and the $(N-1)\times(N-1)$ matrix S_{2k} is

$$S_{2k} = \begin{pmatrix} 0_{[1,k-2]} & 1 & -2 & 1 & 0_{[1,N-k-2]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{[1,k-2]} & 1 & -2 & 1 & 0_{[1,N-k-2]} \\ 0_{[N-k-1,k-2]} & 0_{[N-k-1,1]} & 0_{[N-k-1,1]} & 0_{[N-k-1,1]} & 0_{[N-k-1,N-k-2]} \end{pmatrix} \quad \text{if } k>1, \tag{4.5}$$

and in the case $k=1$, S_2 is the $(N-1)\times(N-1)$ matrix with two non-zero columns:

$$S_2 = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

As λ goes counterclockwise around the point at infinity, each contour $\gamma_{k,k+1}$ transforms to $\gamma_{k,k+1} - l_k + l_{k+1}$, where $l_N = -\sum_{i=1}^{N-1} l_i$.

4.2 Space of polynomials

Here we consider the Hurwitz space $\mathcal{H}_{0;N}(N)$ with a degenerate ramification over $\lambda=\infty$ where all N sheets are glued together. This space can be regarded as a space of polynomial functions on $\mathbb{CP}^1$. The Hurwitz diagram for the coverings from the space $\mathcal{H}_{0;N}(N)$ is given in Figure 9.

As before, we assume the generator γ_k of the fundamental group $\pi_1(\mathbb{C}\backslash\{\lambda_1,\dots,\lambda_{2g+2}\},\lambda_0)$ to cross only the projection of the branch cut going from the ramification point P_k to the point at infinity. The kth column of the solution Φ corresponding to this Hurwitz space is given by the integral (3.8) over the contour $\gamma_{k,k+1}$ for $k=1,\dots,N-1$. Let the contour $\gamma_{k,k+1}$ be going around the point P_k when passing from the kth sheet to the next. Then, as is easy to see, as λ describes the loop γ_k on

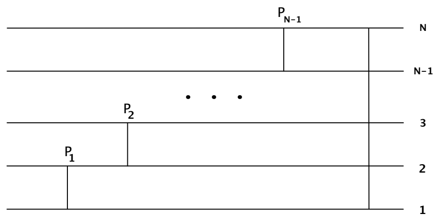

Figure 9: A Hurwitz diagram for the space $\mathcal{H}_{0;N}(N)$.

the base of the covering, the contours change as follows: $\gamma_{k-1,k}$ becomes $\gamma_{k-1,k}+\gamma_{k,k+1}$; the contour $\gamma_{k,k+1}$ turns into its negative $-\gamma_{k,k+1}$, and $\gamma_{k+1,k+2}$ becomes $\gamma_{k,k+1}+\gamma_{k+1,k+2}$ for $k=1,\dots,N-1$ and contours from $\gamma_{1,2}$ to $\gamma_{N-1,N}$. Thus the monodromy matrices have the following form:

$$M_k = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I_{N-k-2} \end{pmatrix}, \qquad 1 \geq k \geq N-1, \tag{4.6}$$

where the block M is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the components S_k from (3.14) do not exist in this case. The monodromies around λ_1 and λ_{N-1} are given by

$$M_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{N-3} \end{pmatrix}, \qquad M_{N-1} = \begin{pmatrix} I_{N-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

To compute the monodromy around $\lambda=\infty$ we note that since the covering surface is of genus zero and since the preimage $f^{-1}(\infty)$ consists of just one point, all closed contours on the covering are trivial in the relative homology space $H_1(\mathcal{L}\setminus f^{-1}(\infty)\ ,\ f^{-1}(\lambda))$. Therefore, the non-closed contours from this space can be characterized by their end points, i.e. any contour connecting points from $f^{-1}(\lambda)$ on the kth and $(k+1)$th sheet is equivalent $\gamma_{k,k+1}$ up to orientation. Then it is easy to see that the monodromy matrix corresponding to the loop γ_∞ based at λ_0 and going around $\lambda=\infty$ counterclockwise has the form:

$$M_\infty = \begin{pmatrix} 0 & \dots & 0 & -1 \\ & I_{N-2} & & \vdots \\ & & & -1 \end{pmatrix}. \tag{4.7}$$

4.3 Example: two sheets, two branch points

In this section we discuss the simplest case of rational functions f of degree two with simple poles, whose equivalence classes form the Hurwitz space $\mathcal{H}_{0,2}(1,1)$. Up to a Möbius transformation in the γ-plane, any degree two rational function with critical values λ_1 and λ_2 is equivalent to the function

$$f(\gamma) = \frac{\lambda_1-\lambda_2}{4}\left(\gamma+\frac{1}{\gamma}\right)+\frac{\lambda_1+\lambda_2}{2}. \tag{4.8}$$

The function f (4.8) defines the two-sheeted genus zero branched covering $\mathcal{L}$ of the Riemann sphere with two branch points λ_1 and λ_2; this covering is the Riemann surface of the function $\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}$. For simplicity in this section we shall identify the ramification points $P_{1,2}$ with the corresponding branch points $\lambda_{1,2}$.

The uniformisation map, i.e. the map from this covering to the Riemann sphere, is given by the function

$$h(\lambda)=\frac{2}{\lambda_1-\lambda_2}\left\{\lambda-\frac{\lambda_1+\lambda_2}{2}+\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}\right\}\,; \tag{4.9}$$

the value of λ together with the sign of the square root $\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}$ determines the point $P\in\mathcal{L}$. The functions f (4.8) and h (4.9) are related by $f\circ h\,(\lambda)=\lambda$. In terms of the function h the bidifferential W has the form:

$$W(\lambda,\mu)=\frac{dh(\lambda)\,dh(\mu)}{(h(\lambda)-h(\mu))^2}\,. \tag{4.10}$$

The relative homology group $H_1(\mathcal{L}\setminus f^{-1}(\infty)\,;\,f^{-1}(\lambda))$ is in this case two-dimensional; a basis in this group can be chosen to consist of a closed contour $\mathbf{s}_1:=l_1$ around $\infty^{(1)}$ (3.6), and a contour $\mathbf{s}_2:=\gamma_{1,2}(\lambda)$ (3.7) connecting in some way the points $\lambda^{(1)}$ and $\lambda^{(2)}$; we shall choose $\gamma_{1,2}(\lambda)$ to consist of two segments: the first segment lies on the first sheet and connects the points $\lambda^{(1)}$ with the branch point λ_1; the second interval lies on the second sheet and connects the points λ_1 and $\lambda^{(2)}$.

If one of the arguments of the W is fixed to coincide with a branch point (see (3.4)), we get from (4.9) and (4.10):

$$W(\lambda,\lambda_1)=\frac{\sqrt{\lambda_1-\lambda_2}}{2}\,\frac{d\lambda}{(\lambda-\lambda_1)^{3/2}(\lambda-\lambda_2)^{1/2}}\,; \tag{4.11}$$

$$W(\lambda,\lambda_2)=\frac{\sqrt{\lambda_2-\lambda_1}}{2}\,\frac{d\lambda}{(\lambda-\lambda_2)^{3/2}(\lambda-\lambda_1)^{1/2}}\,.$$

Therefore, according to (3.10), for the first column of the matrix Φ we get:

$$\Phi_1^{(\mathbf{s}_1)}=2\pi\mathrm{i}\,W(\infty^{(1)},\lambda_1)=-2\pi\mathrm{i}\,\frac{\sqrt{\lambda_1-\lambda_2}}{2}, \tag{4.12}$$

$$\Phi_2^{(\mathbf{s}_1)}=2\pi\mathrm{i}\,W(\infty^{(1)},\lambda_2)=-2\pi\mathrm{i}\,\frac{\sqrt{\lambda_2-\lambda_1}}{2}. \tag{4.13}$$

Integration over the contour $\gamma_{12}(\lambda)$ gives the following expressions for the second column of the matrix Φ:

$$\Phi_1^{(\mathbf{s}_2)}=-\frac{2}{\sqrt{\lambda_1-\lambda_2}}\left\{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}+\frac{1}{2}(\lambda_1-\lambda_2)\log h(\lambda)\right\}\,, \tag{4.14}$$

$$\Phi_2^{(\mathbf{s}_2)}=-\frac{2}{\sqrt{\lambda_2-\lambda_1}}\left\{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}+\frac{1}{2}(\lambda_2-\lambda_1)\log h(\lambda)\right\}\,. \tag{4.15}$$

Computing the determinant of the matrix function Φ (4.12) - (4.15), we get

$$\det\Phi=\pm 8\pi\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}.$$

The monodromy matrices M_1, M_2 and M_∞ are as follows:

$$M_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad M_2=\begin{pmatrix}1&-2\\0&-1\end{pmatrix},\qquad M_\infty=\begin{pmatrix}1&-2\\0&1\end{pmatrix}. \tag{4.16}$$

5 Completeness of the set of solutions to the Fuchsian system

Here we are going to prove the completeness of the set of solutions to the system (2.6), (2.7) given by formula (3.8) with the integration contours given by the basis in $H_1(\mathcal{L} \setminus f^{-1}(\infty);\ \pi^{-1}(\lambda))$ defined by (3.5) - (3.7).

The whole section will be devoted to the proof of the following theorem:

Theorem 6 *The determinant of the matrix function Φ defined by (3.8), (3.11) is given by:*

$$\det \Phi = C \prod_{j=1}^{L} (\lambda - \lambda_j)^{1/2}, \tag{5.1}$$

where $C \neq 0$ is a constant independent of λ and $\{\lambda_j\}$.

Proof. Since the function Φ satisfies the linear system (2.6) with $\alpha = -1/2$, we have:

$$\frac{d}{d\lambda} \log \det\Phi = \operatorname{tr} \left\{ -\sum_{j=1}^{L} \frac{E_j(V - \frac{1}{2}I)}{\lambda - \lambda_j} \right\} = \frac{1}{2} \sum_{j=1}^{L} \frac{1}{\lambda - \lambda_j} \ ,$$

where we used the relation $\operatorname{tr} V = 0$. Analogously, from (2.7) we get

$$\frac{d}{d\lambda_j} \log \det\Phi = -\frac{1}{\lambda - \lambda_j} \ .$$

Therefore, $\det \Phi$ has the form (5.1) with some constant C. What remains to check is that C is not equal to 0, i.e. the columns of the matrix $\Phi(\lambda)$ form a complete set of linearly independent solutions to (2.6), (2.7).

For simplicity we restrict ourselves to the space of coverings with no branching at infinity, i.e. $K = N$. According to the Riemann-Hurwitz formula we have in this case $L = 2g + 2N - 2$.

Let us choose generators of the fundamental group in such a way that the corresponding generators of the monodromy group of the covering are given by (4.1).

The branch cuts can then be chosen to connect the branch points P_{2k+1} and P_{2k+2}, $k = 0, \ldots, g + N - 1$. The branch cuts $[P_1, P_2], \ldots, [P_{2g+1}, P_{2g+2}]$ connect the sheets number $N-1$ and N; the branch cut $[P_{2g+3}, P_{2g+4}]$ connects sheets number $N - 1$ and $N - 2$ etc; the branch cut $[P_{L-1}, P_L]$ connects sheets number 2 and 1. In this way we realize the branch covering $\mathcal{L}$ as a hyperelliptic Riemann surface of genus g with $N - 2$ Riemann spheres attached to it.

Due to Corollary 1 and relations (3.15), (3.17), the completeness of the set of our solutions to the system (2.6), (2.7) depends neither on the choice of a symplectic basis $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ used in the normalization the bidifferential W, nor on the choice of a symplectic basis (a_α, b_α) in (3.5) used as integration contours in (3.8). Therefore, we shall verify the completeness choosing these two bases to our convenience. First, we choose them to coincide: $(a_\alpha, b_\alpha) = (\mathbf{a}_\alpha, \mathbf{b}_\alpha)$. Second, we choose these contours to lie on the "hyperelliptic part" of the covering as shown in Figure 2: the cycle a_α encircles the ramification points $P_{2\alpha+1}$, $P_{2\alpha+2}$ on the Nth sheet, and the cycle b_α goes around the points P_2 and $P_{2\alpha+1}$.

Our proof of the non-vanishing of the constant C will be inductive: first we check that $C \neq 0$ for any covering with $N = 2$ (i.e. a hyperelliptic covering) of any genus. Second, we check that C remains non-vanishing when we attach any number of Riemann spheres to the 2-sheeted covering keeping the genus of the covering unchanged.

5.1 Completeness for $N = 2$

We start by proving a few auxiliary facts related to degeneration of hyperelliptic Riemann surfaces. Consider a hyperelliptic Riemann surface $\mathcal{L}_g$ defined by the equation

$$\nu^2 = \Pi_{2g+2}(\lambda) := \prod_{k=1}^{2g+2} (\lambda - \lambda_k).$$

We are going to study behaviour of the bidifferential W under the degeneration of one of the branch cuts: we put $\lambda_0 := \lambda_{2g+1}$ and consider the limit $\lambda_{2g+2} \to \lambda_0$.

As a result of the degeneration of the surface $\mathcal{L}_g$ there arises the hyperelliptic Riemann surface $\mathcal{L}_{g-1}$ of genus $g - 1$ defined by the equation

$$\nu^2 = \Pi_{2g}(\lambda) := \prod_{k=1}^{2g} (\lambda - \lambda_k). \tag{5.2}$$

Due to the choice of a canonical basis of cycles $\{a_\alpha, b_\alpha\}_{\alpha=1}^{g}$ on $\mathcal{L}_g$ as shown in Figure 2, the cycles $\{a_\alpha, b_\alpha\}_{\alpha=1}^{g-1}$ in the limit $\lambda_{2g+2} \to \lambda_{2g+1}$ provide a canonical basis of cycles on $\mathcal{L}_{g-1}$.

Let us denote by $W_g(P, Q)$ the canonical meromorphic bidifferential W on the surface $\mathcal{L}_g$ of genus g. Consider the behaviour of $W_g(P, Q)$ in the limit $\lambda_{2g+2} \to \lambda_{2g+1} \equiv \lambda_0$. Since all a-periods of $W_g(P, Q)$ with respect to both of its arguments vanish, and in the limit the a_g period becomes the residue at P_0, the bidifferential $W_g(P, Q)$ does not gain any singularity at P_0 on $\mathcal{L}_{g-1}$. At all other points, the singularity structure of $W_g(P, Q)$ under the degeneration coincides with that of $W_{g-1}(P, Q)$. Therefore, if $f(P)$ and $f(Q)$ remain independent of λ_{2g+2} and lie outside of a fixed neighbourhood of λ_0, we have as $\lambda_{2g+2} \to \lambda_0$:

$$W_g(P, Q) = W_{g-1}(P, Q) + o(1) \; . \tag{5.3}$$

The analysis becomes more subtle if one of the arguments of W coincides with P_{2g+1} or P_{2g+2} :

Lemma 2 *Let $f(P)$ lie outside of a fixed neighbourhood of $\lambda_0 := \lambda_{2g+1}$ and be independent of λ_{2g+2}. Then*

$$W_g(P, P_{2g+2}) = \frac{\sqrt{\lambda_{2g+2} - \lambda_0}}{2} \{W_{g-1}(P, P_0) - W_{g-1}(P, P_0^*) + o(1)\}, \tag{5.4}$$

$$W_g(P, P_{2g+1}) = \frac{\sqrt{\lambda_0 - \lambda_{2g+2}}}{2} \{W_{g-1}(P, P_0) - W_{g-1}(P, P_0^*) + o(1)\}, \tag{5.5}$$

as $\lambda_{2g+2} \to \lambda_0$, where P_0 and P_0^* are the points on the 1st and 2nd sheets of $\mathcal{L}_{g-1}$, respectively, projecting to λ_0 on the λ-plane.

Proof. The proof of this lemma can be obtained analogously to ([10], p.51, 52) using the Rauch variational formulas. Consider for example (5.4). In the hyperelliptic case considered here, the asymptotics (5.4) can alternatively be derived from an explicit formula for $W_g(P, P_{2g+2})$. Namely, the differential $W_g(P, P_{2g+2})$ can be written as follows:

$$W_g(P, P_{2g+2}) = W^0(P) - \sum_{\alpha=1}^{g} \left\{ \oint_{a_\alpha} W^0 \right\} w_\alpha(P), \tag{5.6}$$

where

$$W^0(P) := \frac{1}{\lambda - \lambda_{2g+2}} \frac{\sqrt{\Pi_{2g}(\lambda_{2g+2})}\sqrt{\lambda_{2g+2} - \lambda_0}}{2\sqrt{\Pi_{2g+2}(\lambda)}} d\lambda \tag{5.7}$$

(with $\lambda = f(P)$) is a non-normalized meromorphic differential having the same singular part as $W_g(P, P_{2g+2})$; a linear combination of holomorphic differentials in (5.6) provides the vanishing of all a-periods of the right hand side.

In the limit $\lambda_{2g+2} \to \lambda_0$ we have

$$\frac{W^0(P)}{\sqrt{\lambda_{2g+2} - \lambda_0}} \to \frac{d\lambda}{2(\lambda - \lambda_0)^2} \frac{\sqrt{\Pi_{2g}(\lambda_0)}}{\sqrt{\Pi_{2g}(\lambda)}}. \tag{5.8}$$

The holomorphic terms in (5.6) guarantee the vanishing of all periods of the differential $(\lambda_{2g+2} - \lambda_0)^{1/2} W_g(P, P_{2g+2})$, as well as the vanishing of the residues at P_0 and P_0^* of the differential in the limit considered. The coefficient in front of $(\lambda - \lambda_0)^{-2}$ in the expansion at P_0 and P_0^* of the differential in the limit coincides with that in (5.8); therefore, taking into account the normalization condition $\oint_{a_k} W = 0$, $k = 1, \ldots, g$, we arrive at (5.4). □

Below we use also the following

Lemma 3 *In the limit $\lambda_{2g+2} \to \lambda_{2g+1} := \lambda_0$, the following asymptotics hold true:*

$$\oint_{a_g} f(P) W_g(P, P_{2g+2}) = \pi \mathrm{i} (\lambda_{2g+2} - \lambda_0)^{1/2} (1 + o(1)), \tag{5.9}$$

$$2\pi \mathrm{i}\, w_g(P_{2g+2}) = (\lambda_{2g+2} - \lambda_0)^{-1/2} (2 + o(1)), \tag{5.10}$$

$$\oint_{b_g} f(P) W_g(P, P_{2g+2}) = (\lambda_{2g+2} - \lambda_0)^{-1/2} (2\lambda_0 + o(1))). \tag{5.11}$$

and

$$\oint_{a_g} f(P) W_g(P, P_{2g+1}) = \pi \mathrm{i} (\lambda_0 - \lambda_{2g+2})^{1/2} (1 + o(1)),$$

$$2\pi \mathrm{i}\, w_g(P_{2g+1}) = (\lambda_0 - \lambda_{2g+2})^{-1/2} (2 + o(1)),$$

$$\oint_{b_g} f(P) W_g(P, P_{2g+1}) = (\lambda_0 - \lambda_{2g+2})^{-1/2} (2\lambda_0 + o(1))).$$

Proof. We shall prove only the formulas involving P_{2g+2}. To prove (5.9) we make use of the asymptotics (5.4), which implies, as $\lambda_{2g+2} \to \lambda_0$,

$$\frac{1}{\pi \mathrm{i} (\lambda_{2g+2} - \lambda_0)^{1/2}} \oint_{a_g} f(P) W_g(P, P_{2g+2}) \to \operatorname*{res}_{P = P_0} \{ f(P) W_{g-1}(P, P_0) \} = 1\ ,$$

which yields (5.9).

To prove (5.10), let us write the differential w_g in the form:

$$w_g(P) = \frac{1}{2\pi \mathrm{i}} \frac{d\lambda}{\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_{2g+2})}} \frac{Q_{g-1}(\lambda)}{\sqrt{\Pi_{2g}(\lambda)}}, \qquad \lambda = f(P), \tag{5.12}$$

where $Q_{g-1}(\lambda)$ is a polynomial of degree $g - 1$ with coefficients depending on $\{\lambda_k\}$. In the limit $\lambda_{2g+2} \to \lambda_0$, the differential w_g becomes the normalized abelian differential of the third kind with poles at P_0 and P_0^* and residues $+1$ and -1, respectively (this follows from the normalization condition

$\oint_{a_g} w_\alpha = \delta_{\alpha,g}$). Therefore, if we first take the limit $\lambda_{2g+2} \to \lambda_0$, and then put $\lambda = \lambda_0$, we get $Q_{g-1}(\lambda_0) = \sqrt{\Pi_{2g}(\lambda_0)}$. Since from (5.12) we have

$$w_g(P_{2g+2}) = \frac{1}{\pi \mathrm{i}} \frac{1}{\sqrt{\lambda_{2g+2} - \lambda_{2g+1}}} \frac{Q_{g-1}(\lambda_{2g+2})}{\sqrt{\Pi_{2g}(\lambda_{2g+2})}} ,$$

in the limit $\lambda_{2g+2} \to \lambda_0$ we arrive at (5.10).

The asymptotics (5.11) can be deduced from (5.10) and (5.9) by noticing that the integral $\oint_{b_g}(f(P) - \lambda_0)W(P, P_{2g+2})$ remains finite in the limit $\lambda_{2g+2} \to \lambda_0$. One should also use the relation $2\pi \mathrm{i}\, w_g(P_{2g+2}) = \oint_{b_g} W(P, P_{2g+2})$. □

Now we are in a position to prove the following

Proposition 5 *The constant C in (5.1) is non-vanishing for $N = 2$, i.e. for all hyperelliptic coverings of genus g (with no branching at ∞).*

Proof. For $N = 2$ the number of ramification points is $L = 2g + 2$. We prove the proposition by reducing the computation of the determinant of the $2g + 2 \times 2g + 2$ dimensional matrix Φ_g to the computation of the determinant of the $2g \times 2g$ dimensional matrix Φ_{g-1} arising from Φ_g in the limit $\lambda_{2g+2} \to \lambda_{2g+1} \equiv \lambda_0$.

Consider the $2g \times 2g$ matrix obtained from Φ_g by crossing out the columns and rows number $2g - 1$ and $2g$. This matrix, due to (5.3), tends in the limit $\lambda_{2g+2} \to \lambda_{2g+1}$ to a solution Φ_{g-1} given by (3.8) to the Riemann-Hilbert problem associated to the hyperelliptic curve (5.2) of genus $g - 1$. According to the assumption of our induction, $\det \Phi_{g-1}(\lambda) \neq 0$ for $\lambda \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_L\}$.

Due to (5.4), $W_g(P, P_{2g+2})$, as well as $W_g(P, P_{2g+1})$, tend to 0 as $\lambda_{2g+2} \to \lambda_{2g+1}$ if $f(P)$ is independent of λ_{2g+1} and λ_{2g+2}. Therefore, the entries of the $(2g-1)$th and $(2g)$th rows of the matrix Φ_g not belonging to the diagonal 2×2 block tend to 0 as $\lambda_{2g+2} \to \lambda_{2g+1}$.

Therefore, in our limit, $\det \Phi_g$ tends to the product of $\det \Phi_{g-1}$ and the determinant of the 2×2 block at the diagonal:

$$\det\Phi_g \to \det \mathbf{A}\, \det\Phi_{g-1},$$

where

$$\mathbf{A} = \lim_{\lambda_{2g+2} \to \lambda_0} \begin{pmatrix} \oint_{a_g} f(P)W(P, P_{2g+1}) & 2\pi \mathrm{i}\, w_g(P_{2g+1})\lambda - \oint_{b_g} f(P)W(P, P_{2g+1}) \\ \oint_{a_g} f(P)W(P, P_{2g+2}) & 2\pi \mathrm{i}\, w_g(P_{2g+2})\lambda - \oint_{b_g} f(P)W(P, P_{2g+2}) \end{pmatrix}.$$

Using Lemma 3, we find the behaviour of $\det \mathbf{A}$ in the limit:

$$\det \begin{pmatrix} \pi \mathrm{i}(\lambda_{2g+1} - \lambda_{2g+2})^{1/2} & 2(\lambda_{2g+1} - \lambda_{2g+2})^{-1/2}(\lambda - \lambda_0) \\ \pi \mathrm{i}(\lambda_{2g+2} - \lambda_{2g+1})^{1/2} & 2(\lambda_{2g+2} - \lambda_{2g+1})^{-1/2}(\lambda - \lambda_0) \end{pmatrix} = \pm 4\pi(\lambda - \lambda_0).$$

The corresponding constants in (5.1) are thus related by $C_g = \pm 4\pi C_{g-1}$ and $C_g \neq 0$ if $C_{g-1} \neq 0$. □

5.2 Completeness for any N

Here we shall perform an induction over the number of sheets without changing the genus of the covering $\mathcal{L}$ (in this section we denote it by $\mathcal{L}_N$); on each step we detach one sheet by a degeneration of one branch cut. Put $P_0 := P_{L-1}$ (and $\lambda_0 := \lambda_{L-1}$) and take the limit $P_L \to P_0$. In this limit the first sheet of $\mathcal{L}_N$ detaches and the N-sheeted covering splits into an $(N - 1)$-sheeted covering $\mathcal{L}_{N-1}$

of the same genus with the ramification points $\{P_k\}_{k=1}^{L-2}$, and a Riemann sphere, which we denote by $\mathcal{L}_1$. Denote the bidifferential W on $\mathcal{L}_N$ by W_N, on $\mathcal{L}_{N-1}$ by W_{N-1} and on $\mathcal{L}_1$ by W_1 (note that $W_1(\lambda,\mu) = (\lambda-\mu)^{-2}d\lambda d\mu$). The points in the set $f^{-1}(\lambda_0)$ on the covering we denote by $\lambda_0^{(k)}$ (the upper index indicates the sheet number).

Let us prove a few auxiliary facts about this type of degeneration. First, we determine the behaviour of the bidifferential $W_N(P,Q)$ in our limit. Assuming that $f(P)$ and $f(Q)$ are independent of λ_L and λ_{L-1} we have the following obvious asymptotics (see [10]):

$$W_N(P,Q) \to W_{N-1}(P,Q)\;, \qquad P,Q\in\mathcal{L}_{N-1}; \tag{5.13}$$

$$W_N(P,Q) \to W_1(P,Q) \equiv \frac{d\mu(P)\,d\mu(Q)}{(\mu(P)-\mu(Q))^2}\;, \qquad P,Q\in\mathcal{L}_1,$$

where μ is a coordinate on the Riemann sphere $\mathcal{L}_1$; and

$$W_N(P,Q)\to 0\;, \qquad P\in\mathcal{L}_{N-1}\quad Q\in\mathcal{L}_1.$$

The next lemma is less trivial.

Lemma 4 *There are the following asymptotic expansions as $P_L\to P_{L-1}=P_0$:*

$$W_N(P,P_0) = \frac{\sqrt{\lambda_0-\lambda_L}}{2}\{W_{N-1}(P,\lambda_0^{(2)}) + O(\lambda_0-\lambda_L)\}, \tag{5.14}$$

where $P\in\mathcal{L}_{N-1}$ and

$$W_{N-1}(P,\lambda_0^{(2)}) := \frac{W_{N-1}(P,Q)}{df_0(Q)}\Big|_{Q=\lambda_0^{(2)}}\;,$$

where f_0 is the meromorphic function on $\mathcal{L}_{N-1}$ arising from f in our limit (this is nothing but projection from $\mathcal{L}_{N-1}$ to the λ-plane);

$$W_N(P,P_0) = \frac{\sqrt{\lambda_0-\lambda_L}}{2}\{W_1(P,\lambda_0^{(1)}) + O(\lambda_0-\lambda_L)\}, \tag{5.15}$$

where $P\in\mathcal{L}_1$ and

$$W_1(P,\lambda_0^{(1)}) := \frac{d\mu(P)}{(\mu(P)-\lambda_0)^2}\;,$$

μ being the coordinate on the Riemann sphere $\mathcal{L}_1$.

Proof. Following [10], Chapter 3, consider a domain $D\subset\mathcal{L}_N$, which contains the segment $[P_0,P_L]$ on both 1st and 2nd sheets, and can be conformally mapped to an annulus by the map

$$h(\lambda) = \frac{1}{\lambda_0-\lambda_L}\left\{\lambda - \frac{\lambda_0+\lambda_L}{2} + \sqrt{(\lambda-\lambda_0)(\lambda-\lambda_L)}\right\}\;;$$

the union of two banks of the branch cut $[P_0,P_L]$ is mapped by the function $h(\lambda)$ to the unit circle. The Laurent series for $W_N(P,P_0)$ in the coordinate $h(\lambda)$ in a neighbourhood of the unit circle can be written as follows in terms of the coordinate λ within the domain D [10]:

$$W_N(P,P_0) = \frac{1}{\sqrt{(\lambda-\lambda_0)(\lambda-\lambda_L)}}\sum_{k=-1}^{\infty} a_k(\tau)(\lambda-\lambda_0)^k\,d\lambda + \sum_{k=0}^{\infty} b_k(\tau)(\lambda-\lambda_0)^k\,d\lambda\,, \tag{5.16}$$

where $\lambda = f(P)$; $\tau = \sqrt{\lambda_L - \lambda_0}$; coefficients $a_k(\tau)$ and $b_k(\tau)$ are holomorphic at $\tau = 0$. The first sum in (5.16) starts from $k = -1$ since $W_N(P, P_0)$ has a quadratic pole at P_0. Since the singular part of $W(P, P_0)$ at $P = P_0$ has the form $(\lambda - \lambda_0)^{-1} d\sqrt{\lambda - \lambda_0}$, we have $a_{-1}(\tau) = \sqrt{\lambda_0 - \lambda_L}/2$. The term in the second sum in (5.16) corresponding to $k = -1$ is absent since the residue of $W_N(P, P_0)$ at $P = P_0$ equals zero.

Therefore, the differential

$$\lim_{\lambda_L \to \lambda_0} \frac{2}{\sqrt{\lambda_0 - \lambda_L}} W_N(P, P_0), \qquad P \in D \tag{5.17}$$

has a singular part of the form

$$\frac{d\lambda}{(\lambda - \lambda_0)^2}, \qquad \lambda = f(P)$$

in neighbourhoods of $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$. The term containing the first order pole must vanish since the integral of (5.17) over the (homologous to zero) contour on $\mathcal{L}_N$ encircling the branch cut $[P_0, P_L]$ is zero; thus the residues of (5.17) at $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$ vanish.

The differential (5.17) does not have any other singularities neither on $\mathcal{L}_{N-1}$ nor on $\mathcal{L}_1$; this differential has all vanishing a-periods on $\mathcal{L}_{N-1}$. Therefore, we arrive at (5.14), (5.15). □

Lemma 5 *There are the following asymptotic expansions as $\lambda_L \to \lambda_{L-1} \equiv \lambda_0$:*

$$\sqrt{\lambda_0 - \lambda_L} \int_P^Q W_N(R, P_0) = 2 + O(\lambda_L - \lambda_0); \tag{5.18}$$

$$\sqrt{\lambda_0 - \lambda_L} \int_P^Q f(R) W_N(R, P_0) = 2\lambda_0 + O(\lambda_L - \lambda_0), \tag{5.19}$$

where $P \in \mathcal{L}_1$, $Q \in \mathcal{L}_{N-1}$; $f(P)$ and $f(Q)$ are assumed to be independent of λ_L.

Proof. The proof is similar to the proof of the previous lemma. Consider (5.18). The integral of $W_N(R, S)$ with respect to R between the points P and Q is an abelian differential of the third kind in S with simple poles at $S = P$ and $S = Q$ and residues -1 and 1, respectively. We denote this differential by $W_N^{P,Q}(S) := \int_P^Q W_N(\cdot, S)$. Since the sum of the residues of the differential $W_1(S) := \lim_{\lambda_L \to \lambda_0} W_N^{P,Q}(S)$ on $\mathcal{L}_1$ must vanish, we conclude that $W_1(S)$ has two simple poles on $\mathcal{L}_1$: the pole of residue -1 at $S = P$, inherited from $W_N^{P,Q}(S)$, and a new pole at $\lambda_0^{(1)}$, arising as a result of the degeneration, with the residue $+1$ (the absence of higher order terms of $W_1(S)$ at $\lambda_0^{(1)}$ follows from the expansion (5.16) for $W(P, P_0)$). Similarly, on $\mathcal{L}_{N-1}$, the differential $W_N^{P,Q}(S)$ tends to the normalized abelian differential of the third kind with simple poles at $S = \lambda_0^{(2)}$ and $S = Q$ and residues -1 and $+1$, respectively.

Let us now write down an analog of the expansion (5.16) for $W_N^{P,Q}(S)$, when $S \in D$:

$$W_N^{P,Q}(S) = \frac{1}{\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_L)}} \sum_{k=0}^{\infty} c_k(\tau)(\lambda - \lambda_0)^k \, d\lambda + \sum_{k=0}^{\infty} d_k(\tau)(\lambda - \lambda_0)^k \, d\lambda, \tag{5.20}$$

where $\lambda = f(S)$; as before, $\tau := \sqrt{\lambda_0 - \lambda_L}$; the coefficients $c_k(\tau)$ and $d_k(\tau)$ are holomorphic at $\tau = 0$. Both sums in (5.20) start from $k = 0$ since the differential $W_N^{P,Q}(S)$ is holomorphic at $S = P_0 \equiv P_{L-1}$ and $S = P_L$. Since in our limit the differential $W_N^{P,Q}(S)$ gains simple poles at $S = \lambda_0^{(2)}$ and $S = \lambda_0^{(1)}$

with residues -1 and $+1$, respectively, we conclude that $c_0 = 1 + o(\tau)$ as $\tau \to 0$. Now, taking $S = P_0$, and evaluating $W_N^{P,Q}$ at P_0 with respect to the local parameter $\sqrt{\lambda - \lambda_0}$ similarly to (3.4), we arrive at (5.18).

The asymptotics (5.19) easily follows from (5.18) since the integral $\int_P^Q (f(R) - \lambda_0) W_N(R, P_0)$ behaves as $o(1)$ in our limit. $\square$

We notice that all the asymptotics computed in the above lemmas are symmetric under the interchange of λ_L and λ_{L-1}.

Let us now assume that the constant C_{N-1} in relation (5.1) corresponding to the branch covering $\mathcal{L}_{N-1}$ is non-vanishing. One needs to prove the non-vanishing of the constant C_N corresponding to the covering $\mathcal{L}_N$.

Denote the function Φ (3.8) corresponding to the N-sheeted covering $\mathcal{L}_N$ by Φ_N, and the function Φ corresponding to the $(N-1)$-sheeted covering $\mathcal{L}_{N-1}$ by Φ_{N-1}. The columns of Φ_N given by the integrals over the contours l_1 encircling $\infty^{(1)}$, and the contour $\gamma_{1,2}(\lambda)$ have, according to (3.8) and (3.10), the form:

$$\Phi_k^{(\gamma_{1,2}(\lambda))} = -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_k) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_k) \ ,$$

and

$$\Phi_k^{(l_1)} = -2\pi \mathrm{i}\, W(\infty^{(1)}, P_k).$$

The contours l_1 and $\gamma_{1,2}(\lambda)$ are absent from the integration contours determining Φ_{N-1}. The rows corresponding to P_{L-1} and P_L are also missing in Φ_{N-1}. The 2×2 block on the intersection of these rows and columns in the matrix Φ_N looks as follows:

$$\mathbf{B} = \begin{pmatrix} -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_{L-1}) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_{L-1}) & -2\pi\mathrm{i}\, W(P_{L-1}, \infty^{(1)}) \\ -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_L) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_L) & -2\pi\mathrm{i}\, W(P_L, \infty^{(1)}) \end{pmatrix}.$$

According to (5.13), the $(2L-2)\times(2L-2)$ minor in the matrix Φ_N obtained by deleting these two rows and two columns tends to Φ_{N-1} in our limit. Since all other entries of the two rows of Φ_N corresponding to P_{L-1} and P_L, tend to 0 as $P_L \to P_0 = P_{L-1}$, we see that in this limit $\det \Phi_N \to \det \mathbf{B} \det \Phi_{N-1}$.

Now, due to Lemmas 4 and 5, in this limit

$$\det \mathbf{B} \to \begin{pmatrix} -2\frac{\lambda-\lambda_0}{\sqrt{\lambda_{L-1}-\lambda_L}} & -\frac{\sqrt{\lambda_{L-1}-\lambda_L}}{2} \\ -2\frac{\lambda-\lambda_0}{\sqrt{\lambda_L-\lambda_{L-1}}} & -\frac{\sqrt{\lambda_L-\lambda_{L-1}}}{2} \end{pmatrix} = \left\{ \sqrt{\frac{\lambda_L - \lambda_{L-1}}{\lambda_{L-1} - \lambda_L}} - \sqrt{\frac{\lambda_{L-1} - \lambda_L}{\lambda_L - \lambda_{L-1}}} \right\} (\lambda - \lambda_0) = \pm 2\mathrm{i}(\lambda - \lambda_0),$$

where $\lambda_0 = f(P_0)$; therefore, $C_N = \pm 2\mathrm{i} C_{N-1}$, i.e. $C_{N-1} \neq 0$ implies $C_N \neq 0$. $\square$

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