Complete intersections in relation to a paper of B.J.Birch

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Abstract. We prove a new property of complete intersections in graded Noetherian rings, and relate it to the Jacobian criterion. We then apply these results to describe the loci of singularities in a family of affine varieties defined by B.J.Birch in his classic paper on forms of many variables.

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1. An assertion of B.J.Birch

In a famous memoir, Professor B.J.Birch concludes as follows (see [2], p.245):

Let $f_1, \ldots, f_R$ be rational forms of degree d in n variables $x_1, \ldots, x_n$, where $n > R \geq 1$. Let $V(\mu)$ be the variety

$$V(\mu)\colon\ f_1(x) = \mu_1, \ldots, f_R(x) = \mu_R, \tag{1}$$

and let V^* be the union of the loci of singularities of the $V(\mu)$ (the μ_i run through some algebraically closed field). Then if $\dim V(0) = n - R$, and if $V(0)$ has non-singular real points and non-singular points in every p-adic completion of the rationals, and if

$$n - \dim V^* > R(R+1)(d-1)2^{d-1},$$

then $V(0)$ certainly has a rational point other than the origin.

As one of the most general statements asserting existence of rational points on a projective algebraic variety, this theorem has been widely cited. An assiduous reader will note however that, in the course of the proof, Professor Birch makes use of the following assertion, [2], p.250 :

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V^* consists of all points x for which

$$\operatorname{rank} \left| \frac{\partial f_i}{\partial x_j} \right|_{i,j} < R. \tag{2}$$

Far from being obvious, this assertion does not seem to have been proved in the literature; some of the authors, [4], [7], [8], just define V^* by the condition (2), while other authors do not feel the cited assertion needs any justification, see, for instance, [9], where Birch's theorem has been generalized to treat projective varieties defined over an algebraic number field. In this note, we prove that this assertion is indeed correct.

To be precise, equations (1) define an affine scheme

$$V(\mu) = \operatorname{Spec} k[x]/\mathfrak{A}_\mu \,,$$

where k is a fixed (algebraically closed field) and $\mathfrak{A}_\mu = (f_1(x) - \mu_1, \ldots, f_R(x) - \mu_R)$ is the ideal in $k[x]$ corresponding to these equations. In general, this scheme is neither reduced, nor irreducible. Condition (2) defines the singular locus of $V(\mu)$ if (i) $\dim V(\mu) = n - R$, and (ii) the Jacobian criterion may be applied. Making use of the tools of modern commutative algebra, [3], [5], [6], we show that each of these conditions follows from the assumption $\dim V(0) = n - R$.

Let J_R be the ideal in $k[x]$ generated by the minors of order R of the Jacobian matrix

$$\left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{1 \le j \le R \\ 1 \le i \le n}},$$

and let $W = \operatorname{Spec} k[x]/J_R$.

Theorem 1. *Let $f_j(x)$, $1 \le j \le R$, be non-constant homogeneous polynomials of the same degree. If $\dim V(0) = n - R$ and k is a perfect field, then*

$$V^* = \operatorname{Spec} k[x]/\mathfrak{c}$$

with $J_R \subseteq \mathfrak{c} \subseteq \sqrt{J_R}$.

This theorem follows from our main result, Theorem 2, and will be proved in §4. As an immediate consequence of the theorem, one obtains the following result.

Corollary 1.

(i) *The set V^* is homeomorphic to W, and in particular $\dim V^* = \dim W$.*

(ii) *The set of k-rational points $V^*(k)$ of V^* coincides with the set of k-rational points $W(k)$ of W, in accordance with the assertion of B.J.Birch cited above.*

In the next section, Theorem 2 is stated and proved; it describes a new property of complete intersections in graded Noetherian rings.

2. Complete intersections and regular sequences

To establish our terminology, let us remind that a finite subset I of a commutative ring A defines a *complete intersection* (see [3], §9, n^o6), if $H_i(I, A) = 0$ for $i > 0$, where $H_i(I, A)$ denotes the i-th homology group of the Koszul complex associated to the set I. Further, a sequence $\{a_1, \ldots, a_\ell\}$ of elements of A is said to be *regular* if a_1 is not a zero divisor of the ring A and if for any $1 \le j \le \ell - 1$ the element a_{j+1} is not a zero divisor in the quotient $A/(a_1, \ldots, a_j)$. The property to be regular depends on the ordering of the elements of the sequence, while the property to define a complete intersection does not. In this section $\mathbf{N}$ stands for set of non-negative rational integers.

Lemma 1. *Let $\{a_1, \ldots, a_\ell\}$ be a sequence of elements in A, and let $(b_{ij}) \in$ $\mathbf{GL}_\ell(A)$, $1 \le i, j \le \ell$. Set*

$$y_i = \sum_{j=1}^{\ell} b_{ij} a_j, \qquad 1 \le i \le \ell.$$

Then the sequence $\{a_1, \ldots, a_\ell\}$ defines a complete intersection if and only if the sequence $\{y_1, \ldots, y_\ell\}$ does.

Proof. See [3], §9, n^o6, Prop.6, b).

Lemma 2. *If $\{a_1, \ldots, a_\ell\}$ is a regular sequence, then the set $\{a_j : 1 \le j \le \ell\}$ defines a complete intersection.*

Proof. See [3], §9, n^o7, Th.1.

The converse of Lemma 2 is true in the following situation.

Lemma 3. *Let A be a Noetherian $\mathbf{N}$-graded ring, $A \neq 0$. If the set $\{a_j : 1 \le j \le \ell\}$ defines a complete intersection and if each of the elements a_j of this set is homogeneous of positive degree, then the sequence $\{a_1, \ldots, a_\ell\}$ is regular.*

Proof. See [6], Theorem 16.5 (β).

As a consequence of Lemmata 1-3, we obtain the following result.

Theorem 2. *Let A be a Noetherian $\mathbf{N}$-graded ring, and let a_j, $1 \le j \le \ell$, be homogeneous elements of degree d , $d \ge 1$. If the set $\{a_1, \ldots, a_\ell\}$ defines a complete intersection, then for any $\mu \in (A^* \cup \{0\})^\ell$ the set $\{a_j + \mu_j : 1 \le j \le \ell\}$ defines a complete intersection; here $\mu = (\mu_1, \ldots, \mu_\ell)$, and A^* stands for the set of units of A.*

Proof. In view of Lemma 1, the set $\{a_1 + b_1 a_\ell, \ldots, a_{\ell-1} + b_{\ell-1} a_\ell, a_\ell\}$ defines a complete intersection for any $b_i \in (A^* \cup \{0\})$, $i = 1, \ldots, \ell - 1$. By Lemma 3, the sequence $\{a_1 + b_1 a_\ell, \ldots, a_{\ell-1} + b_{\ell-1} a_\ell, a_\ell\}$ and, consequently, its subsequence

$\{a_1 + b_1 a_\ell, \ldots, a_{\ell-1} + b_{\ell-1} a_\ell\}$ are regular. Suppose, without loss of generality, $\mu_\ell \neq 0$. Then $a_\ell + \mu_\ell$ is not a zero divisor in the ring $A/(a_1 + b_1 a_\ell, \ldots, a_{\ell-1} + b_{\ell-1} a_\ell)$; therefore the sequence $\{h_1, \ldots, h_\ell\}$, with $h_i = a_i + b_i a_\ell,\ 1 \leq i \leq \ell - 1,\ h_\ell = a_\ell + \mu_\ell$ is regular. By Lemma 2, it defines a complete intersection; therefore (Lemma 1 !) the set $\{h_1 - b_1 h_\ell, \ldots, h_{\ell-1} - b_{\ell-1} h_\ell, h_\ell\}$ defines a complete intersection. It now suffices to choose $b_i = -\mu_i/\mu_\ell,\ 1 \leq i \leq \ell - 1$. This concludes the proof of the theorem.

Remark 1. One may note that to define a complete intersection is an open property in the sense of A.Grothendieck (see, for instance, [1]). Thus, a priori, this property holds for those μ outside a closed subset. In the theorem, we have given some conditions to ensure it is valid for every value of the parameters μ.

3. Jacobian criterion and singular locus

Let us remind that the *depth* of an ideal $\mathfrak{B}$, to be denoted by $\operatorname{dp} \mathfrak{B}$, is equal to the maximal possible length of a regular sequence contained in $\mathfrak{B}$. The *height* of an ideal $\mathfrak{B}$, to be denoted by $\operatorname{ht} \mathfrak{B}$, is defined to be the minimum of the heights of the prime ideals containing $\mathfrak{B}$. In a Cohen-Macaulay ring, $\operatorname{dp} \mathfrak{B} = \operatorname{ht} \mathfrak{B}$. Finally, an ideal $\mathfrak{B}$ is said to have a *pure codimension* r if $\operatorname{ht} \wp = r$ for any *minimal* prime $\wp$ containing $\mathfrak{B}$. Note that in [5] height is called *codimension* and is denoted by "codim".

In what follows, we write $H_i(\mathfrak{B}, A)$ for $H_i(I, A)$, where $\mathfrak{B}$ is the ideal in A generated by the set I. We shall make use of the following theorems of commutative algebra.

Lemma 4. *Let $\mathfrak{B}$ be an ideal in an integral domain A. Then*

$$\dim A/\mathfrak{B} + \operatorname{ht} \mathfrak{B} = \dim A.$$

Proof. See [5], Corollary 13.4, p.286.

Lemma 5. *Let $\mathfrak{B} = (a_1, \ldots, a_\ell)$ be an ideal in a Noetherian ring A, and suppose $\mathfrak{B} \neq A$. Then the sequence $\{a_1, \ldots, a_\ell\}$ is regular if and only if* $\operatorname{dp} \mathfrak{B} = l$.

Proof. See [6], Corollary on p.131.

Lemma 6. *Let $\mathfrak{B} = (a_1, \ldots, a_\ell)$ be an ideal in a Noetherian ring A, suppose $\mathfrak{B} \neq A$, and let*

$$q = \sup\{i : H_i(\mathfrak{B}, A) \neq 0\}.$$

Then any maximal regular sequence contained in $\mathfrak{B}$ has length $n - q$.

Proof. See [6], Theorem 16.8, p. 131.

To deduce Theorem 1 from Theorem 2 we shall make use of the Jacobian criterion in the following form (see [5], Corollary 16.20).

Lemma 7. *Let $A = k[x_1, \ldots, x_n]/I$ be an affine ring over a perfect field k and suppose that I is of pure codimension r; let $I = (g_1, \ldots, g_\ell)$. If J is the ideal of A generated by the $r \times r$ minors of the Jacobian matrix $(\partial g_j/\partial x_i)$, then J defines the singular locus of A in the sense that a prime $\wp$ of A contains J if and only if $A_\wp$ is not a regular local ring.*

Proposition 1. *In the notation of §1, if $\dim V(0) = n - R$ then the ideal $\mathfrak{A}_\mu$ has a pure codimension R for any μ in k^R.*

Proof. By Lemma 4, condition $\dim V(0) = n - R$ shows that $\operatorname{ht}\mathfrak{A}_0 = R$. Since $k[x]$ is a Cohen-Macaulay ring, it follows that $\operatorname{dp}\mathfrak{A}_0 = \operatorname{ht}\mathfrak{A}_0 = R$. Further, $\mathfrak{A}_0 \neq k[x]$, therefore by Lemma 5 the sequence $\{f_1, \ldots, f_R\}$ is regular, and Lemma 3 implies that the set $\{f_j : 1 \leq j \leq R\}$ defines a complete intersection. Thus we can apply Theorem 2 to conclude that the set $\{f_j - \mu_j : 1 \leq j \leq R\}$ defines a complete intersection for any μ. Moreover,

$$H_0(\mathfrak{A}_\mu, k[x]) \cong k[x]/\mathfrak{A}_\mu \neq 0,$$

therefore it follows from Lemma 6 that $\operatorname{dp}\mathfrak{A}_\mu = R$; consequently, $\operatorname{ht}\mathfrak{A}_\mu = R$. An application of the Macaulay unmixedness theorem [5], Corollary 18.14, concludes the proof of the proposition.

4. The proof of the main theorem

Suppose now k be a perfect field. It follows from Proposition 1 and Lemma 7 that

$$\operatorname{Sing} V(\mu) = \operatorname{Spec} k[x]/(\mathfrak{A}_\mu, J_R),$$

where $\operatorname{Sing} V(\mu)$ denotes the singular locus of $V(\mu)$. By definition,

$$V^* = \bigcup_{\mu \in k^R} V(\mu).$$

Let $\mathfrak{C} = \cap_{\mu \in k^R} (\mathfrak{A}_\mu, J_R)$, then

$$V^* = \operatorname{Spec} k[x]/\mathfrak{C}.$$

Clearly, $J_R \subseteq \mathfrak{C}$. Let $h \in \mathfrak{C}, u \in k^n$, and suppose $g(u) = 0$ for every g in J_R. For any μ in k^R, one may write

$$h = \sum_{j=1}^{R} a_j^{(\mu)} (f_j - \mu_j) + b_\mu,$$

where $a_j^{(\mu)} \in k[x],\ b_\mu \in J_R$. Then

$$h(u) = \sum_{j=1}^{R} a_j^{(\mu)}(u)(f_j(u) - \mu_j),$$

therefore $h(u) = 0$ since we may take $\mu_j = f_j(u),\ 1 \leq j \leq R$, and Hilbert's Nullstellensatz shows that $h \in \sqrt{J_R}$. Thus

$$J_R \subseteq \mathfrak{c} \subseteq \sqrt{J_R},$$

as asserted in Theorem 1.

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