QUASISYMMETRIC SEWING IN RIGGED TEICHMÜLLER SPACE

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Abstract. One of the basic geometric objects in conformal field theory (CFT) is the moduli space of Riemann surfaces whose boundaries are "rigged" with analytic parametrizations. The fundamental operation is the sewing of such surfaces using the parametrizations. We generalize this picture to quasisymmetric boundary parametrizations. By using tools such as the extended λ-lemma and conformal welding we prove: (1) The universal Teichmüller space induces complex manifold structures on the Riemann and Teichmüller moduli spaces of rigged surfaces. (2) The border and puncture pictures of the rigged moduli and rigged Teichmüller spaces are biholomorphically equivalent. (3) The sewing operation is holomorphic.

Because of the simplified picture we obtain it appears this is the natural setting for the geometric objects in CFT.

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1. Introduction

1.1. **Statement and discussion of results.** This paper is devoted to the rigorous construction of some of the fundamental objects and operations in the geometric approach to two-dimensional conformal field theory. The central object is the *rigged Riemann surface.* We use this term to refer to both: (1) a bordered Riemann surface with parametrizations of its boundary components, and (2) a punctured surface with specified local biholomorphic coordinates at the punctures. The space of conformal equivalence classes of rigged Riemann surfaces is called the *rigged moduli space.* In this paper it is shown that the operation of sewing two Riemann surfaces with quasisymmetric boundary parametrizations is holomorphic (Theorem 6.7). This requires the construction of mathematically consistent and rigorous models of these rigged moduli spaces and their complex structures. The motivation for this problem along with applications are discussed in Section 1.2.

In the classical theory of moduli spaces of Riemann surfaces, Teichmüller spaces are instrumental in defining and investigating the complex structures. Thus to construct a complex structure on the rigged Riemann moduli space, we introduce a *rigged Teichmüller space* for both the border and puncture models. Roughly, these rigged Teichmüller spaces have the same relation to the rigged moduli spaces as the Teichmüller space of a surface of finite type has to its Riemann moduli space.

Although intuitively the border and puncture models of rigged moduli spaces are the same thing, the relation between the rigged Teichmüller spaces is less obvious. We will show that the puncture and border models of the rigged Teichmüller spaces are indeed biholomorphically equivalent, and this also provides a rigorous proof that the two models of the rigged moduli space are biholomorphically equivalent. Furthermore, somewhat surprisingly, the rigged Teichmüller spaces can be seen to lie in the ordinary Teichmüller space of a bordered Riemann surface. The collection of theorems which embody this construction are summarized in Diagram (5.10) and Theorem 5.26.

An unexpected consequence of our results is that our model of the rigged moduli spaces gives an apparently new description of the ordinary Teichmüller space of a bordered Riemann surface. Given a bordered Riemann surface Σ^B, it can be completed to a punctured surface Σ^P by sewing on punctured disks. We show that up to Dehn twists of the boundary curves, the ordinary Teichmüller space of Σ^B is essentially the rigged Teichmüller space of Σ^P. This can be thought of as an application of Segal's definition of conformal field theory to Teichmüller theory.

As outlined below, the standard approach in conformal field theory is to use analytic boundary parametrizations. It was shown by the first author in [46] that sewing is holomorphic in this setting. The genus-zero case was first done in [22]. In this paper we generalize to quasisymmetric boundary parametrizations. This is not a pointless generalization; in fact it reveals a greatly simplified picture. The choice of quasisymmetric parametrizations arises naturally, since quasisymmetric functions are the boundary values of quasiconformal maps.

The generalization to quasisymmetric boundary data for a bordered Riemann surface is essential for several reasons. First, it allows one to recognize that the rigged Teichmüller spaces are contained in the ordinary Teichmüller space of a bordered surface, as mentioned above. In other words, the boundary data is naturally included in the ordinary Teichmüller space. As a consequence one can use standard theory in order to construct the complex structures on all of the spaces involved. Secondly, the generalization to quasisymmetric maps makes it possible to define and relate the puncture and border models, not only at the moduli space level, but also at the Teichmüller space level. This was not previously possible and it enables us to fill out a conceptually satisfying commutative diagram relating all these spaces (see Section 5.5). Finally, the entire picture can be embedded into the universal Teichmüller space, which is now fairly well understood. This can be thought of as an application of geometric function theory to conformal field theory.

The idea of applying constructions in geometric function theory to conformal field theory and string theory is not new (e.g. [10], [35], [39], [43], [50] and [51]). It has been suggested (see the comprehensive review of Pekonen [43] and references therein) that the universal Teichmüller space would provide the natural arena for conformal field theory, and the results of this paper indicate that this is in fact correct.

We also hope that this paper will encourage more interaction between conformal field theory and geometric function theory. To further this goal we have outlined in Section 7.3 the relation of our model to the standard conformal field theory model involving punctured surfaces and analytic local coordinates. This problem in itself is not entirely trivial and the full details will appear in a forthcoming article.

1.2. **Background.** In this section we motivate the problem from the point of view of conformal field theory (CFT).

This paper can be seen as part of the program of constructing the geometric objects and operations in CFT. The results will have applications to the program of rigorously constructing CFT from vertex operator algebras. We briefly discuss the basic concepts of CFT in order to provide context.

Conformal field theory originally arose in physics from various two-dimensional statistical mechanics models. In the seminal paper of Belavin, Polyakov and Zamolodchikov [6] much of the structure of CFT was encoded in the notion of a chiral algebra, at the physical level of rigor. These algebras are essentially equivalent to vertex operator algebras which were developed independently in mathematics by Borcherds [9] and Frenkel, Lepowsky and Meurman [14].

At around the same time in string theory the study of the geometry of CFT was introduced by Friedan and Shenker [15]. In this context the two dimensional objects of study are the world sheets of strings which are Riemann surfaces with boundary.

In the path integral approach to quantum field theory, one must "sum" over all possible paths. In the case of string interactions the possible paths are the possible Riemann surfaces joining the prescribed boundaries (strings). The conformal invariance inherent in the physics requires that the path integrals be taken over the moduli space. However, it is doubted that these path integrals can be made rigorous in a direct way.

In 1987, Segal [47] and Kontsevich independently extracted the mathematical properties such a theory should have and gave a purely mathematical definition of CFT. Substantial work was done recently by Hu and Kriz in [18] and [19] to rigorize the categorical structures in this definition. Details of the analytic aspects have been worked out by Radnell [46] and will appear in a forthcoming article.

In CFT each boundary circle must be associated with a Hilbert space and each interaction (Riemann surface with parametrized boundary components) must be associated with an operator. Such Riemann surfaces can be sewn using the boundary parametrizations and certain geometric properties of this operation translate into relations between the corresponding operators.

Collecting these ideas we present an outline of the definition of a *conformal field theory* in the sense of Segal (see [47] or the review article [25]). Consider the category, $\mathcal{C}$, whose objects are ordered sets of copies of the unit circle S^1 and whose morphisms are conformal equivalence classes of Riemann surfaces with oriented, ordered, and analytically parametrized boundaries such that the negatively (positively) oriented boundaries are parametrized by the copies of S^1 in the domain (co-domain). Composition of morphisms is defined by the sewing of oppositely oriented boundary components in the unique way specified by the parametrizations. A conformal field theory is a projective functor from this category to the category of complete locally convex vector spaces over $\mathbb{C}$, satisfying certain natural axioms.

Although this definition has existed since 1987, no general construction for arbitrary genus has been given. This attests to the richness of the mathematical structure of CFT and the difficulties faced in its construction. The genus-zero theory has been completely worked out by Huang in, [20], [21], [22], [23], and [26]. The genus-one theory is also essentially complete due to the work of Zhu [55] and Huang [24]. Free fermion theories were outlined by Segal [47] and have recently been elaborated on by Kriz [31]. Many people have worked on the algebro-geometric and topological aspects of the higher-genus theory. Some key works are in this direction are [3], [4], [5], [13], [16], [27], [40], [52], and [53].

To construct CFT completely however, many holomorphicity issues must be addressed. A richer mathemtical structure is contained in the chiral and anti-chiral parts of CFT, and in the construction of CFT from vertex operator algebras it is actually these parts which are constructed first. Axiomatically such structures are *weakly conformal field theories* as defined by Segal [47]. In the chiral case, the operators in the CFT are required to depend holomorphically on the associated Riemann surface with parametrized boundary components. For such a statement to make sense the moduli space of Riemann surfaces with parametrized boundaries must be a complex manifold and the sewing operation is required to be holomorphic.

A particular problem in higher-genus CFT is completing the modules of vertex operator algebras to obtain the required Hilbert spaces and constructing trace-class maps, associated to Riemann surfaces with parametrized boundaries, between tensor powers of the Hilbert spaces. In this completion process and the construction of the maps, being able to perform the sewing operation with parametrizations that are more general than analytic is necessary. Our generalization to quasisymmetric boundary parametrizations in this paper was partly motivated by this application. It is crucial that the sewing operation be holomorphic, and we prove this fact in Section 3.

1.3. **Outline.** In order that this paper be accessible to both those working in conformal field theory and geometric function theory, brief sketches of material considered standard are provided to make the paper self-contained. Also, although all definitions are given in the main body of the paper, a partial list of notation appears in Section 9 for the convenience of the reader.

To get a conceptual picture of the results as quickly as possible, it may be best to look at Diagram (5.10) once the appropriate definitions have been absorbed. The holomorphicity of the sewing is indicated in Diagram (6.2), and this can be understood once the sewing operation from Section 3 is read along with Theorems 5.12 and 5.13.

An outline of the contents of the paper follows.

Section 2 contains some basic concepts, definitions and standard results. We present the definitions and background on mapping class groups, Teichmuller theory, the extended λ-lemma and quasisymmetric maps (with some extensions of the standard terminology). In Section 3, we generalize the 'conformal welding' construction of geometric function theory in order to sew general Riemann surfaces with quasisymmetric boundary parametrization.

The main technical results appear in Section 4. We derive various lemmas regarding the construction of quasiconformal mappings with specified properties. These lemmas are first applied to give the relation between the mapping class groups of bordered Riemann surfaces and the mapping class groups of the punctured Riemann surfaces obtained by sewing on 'caps' (that is, copies of the disk). Although this relation has been given in the case of the homeomorphic and diffeomorphic setting, it does not seem to exist in the quasiconformal setting. The technical lemmas of this section are also crucial to proving the relation between the Teichmüller and Riemann moduli spaces of the border and puncture model.

Sections 5 and 6 contain the main results. In Section 5, we present the definitions of the of the rigged Teichmüller and Riemann moduli spaces of bordered and punctured surfaces. The relation between all the spaces and the construction of their complex structures is given. These results are summarized in Diagram (5.10) and Theorem 5.26. Once the complex structures are constructed, it is proved that the sewing operation is holomorphic in Section 6. This is the content of Theorem 6.7.

In Section 7 we take an important step towards understanding the local (fiber) structure of the rigged Teichmüller space. In particular we prove in Corollary 7.8 that a holomorphic family of riggings gives a holomorphic family in the rigged Teichmüller space.

Although the results of this paper clearly indicate that the quasisymmetric approach to the riggings is both natural and necessary, it is of interest to establish the exact relation between this new approach and the standard analytic rigged moduli space. A sketch of the relation is given in Section 7.3, with details to appear in a later publication. In particular the compatibility of the complex structure on the space of germs of holomorphic functions with that on rigged Teichmüller space is demonstrated.

Finally, Section 8 contains some concluding remarks, and a notation key is provided in Section 9.

2. Preliminaries

Let n^- and n^+ be non-negative integers and let $n = n^- + n^+$. An *oriented point* on a Riemann surface is a point together with an element of $\{+, -\}$. Let Σ be a compact Riemann surface of genus g, and choose a set of ordered, oriented and distinct points $\mathbf{p} = (p_1, \ldots, p_n)$ where n^- points are negatively oriented and n^+ points are positively oriented. Let $\Sigma^P = \Sigma \backslash \mathbf{p}$

be the corresponding punctured surface. We say Σ^P is of type (g, n^-, n^+). Where it causes no confusion we will sometimes think of Σ^P as a surface with marked points rather than punctures.

Let Σ^B be a Riemann surface bounded by n closed curves which are homeomorphic to S^1 and such that sewing in n disks would result in a compact Riemann surface of genus g. We assign an order to the set of boundary components and assign an element of $\{+, -\}$ to each boundary component such that n^- components are negative and n^+ are positive. We will also say Σ^B is of type (g, n^-, n^+) in this case. We denote the boundary of Σ^B by $\partial\Sigma^B$ and the ith boundary component by $\partial_i\Sigma^B$. When we write $\partial\Sigma^B$ for $\partial_1\Sigma^B \cup \cdots \cup \partial_n\Sigma^B$ an ordering of the components is implicit.

The orientation is often not important and we will simply refer to the surface as having n punctures or boundary components.

Remark 2.1. Each boundary curve has an *orientation* which is the (topological) orientation induced from the Riemann surface structure. To avoid a conflict in terminology we will refer to boundary components with an assignment of '$-$' (respectively, '$+$') as *incoming* (respectively, *outgoing*). For the relation to conformal field theory and boundary parametrizations see Remark 3.2.

2.1. **Mapping class groups.** When dealing with surfaces with punctures or boundary there are several different mapping class (or modular) groups that can be considered, depending on how the boundaries and punctures are to be preserved under homeomorphisms and homotopies. In this section some standard definitions and results are presented. As is natural in Teichmüller theory we work solely with quasiconformal homeomorphisms. The boundary curves and punctures will always be ordered and all maps are required to preserve the given ordering. Some general sources for this material are [8], [28], [38] and [42].

Definition 2.2. For a Riemann surface Σ^B bounded by an ordered set of n curves, let $\mathrm{PQC}^B(\Sigma^B)$ be the space of quasiconformal self-mappings of Σ^B which preserve the ordering of the boundary components, and $\mathrm{PQC}_0^B(\Sigma^B)$ be the subspace of these which are isotopic to the identity relative to the boundary (that is, so that the isotopy fixes the boundary components pointwise). Finally, let

$$\mathrm{PMod}^B(\Sigma^B) = \frac{\mathrm{PQC}^B(\Sigma^B)}{\mathrm{PQC}_0^B(\Sigma^B)}.$$

We often abbreviate "isotopic relative to the boundary" by "isotopy rel $\partial\Sigma^B$". The "P" in the notation stands for "pure", which refers to the fact that the mapping class group preserves the order of the boundary curves. $\mathrm{PMod}^B(\Sigma^B)$ is often called the *pure (quasiconformal) mapping class group* or *Teichmüller modular group.*

Definition 2.3. For a Riemann surface Σ^P with n ordered punctures, let $\mathrm{PQC}^P(\Sigma^P)$ be the space of quasiconformal self-mappings of Σ^P which preserve the punctures and their ordering, and $\mathrm{PQC}_0^P(\Sigma^P)$ be the subspace of these which are isotopic to the identity. Finally, let

$$\mathrm{PMod}^P(\Sigma^P) = \frac{\mathrm{PQC}^P(\Sigma^P)}{\mathrm{PQC}_0^P(\Sigma^P)}.$$

We emphasize that throughout an isotopy the punctures must remain fixed. This is automatic for a punctured surface but must be imposed as an extra condition if one instead thinks of a surface with marked points.

Remark 2.4. For a surface of finite topological type, the Baer-Mangler-Epstein Theorem states that two orientation preserving self-homeomorphisms are homotopic if and only if they are isotopic (see [38, Theorem 1.5.4]). With this in mind we will use 'isotopy' throughout this paper. It is proved in [12] that any homotopy can be replaced with an isotopy such that the maps are uniformly quasiconformal, but we will not need this fact.

Definition 2.5. Let $\mathrm{PModI}(\Sigma^B)$ be the subgroup of $\mathrm{PMod}^B(\Sigma^B)$ consisting of equivalence classes of quasiconformal mappings of Σ^B whose representatives are the identity on $\partial\Sigma^B$.

Definition 2.6. Let $\mathrm{DB}(\Sigma^B)$ be the subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by the eq-uivalence classes of mappings which are Dehn twists around curves that are isotopic to boundary curves.

Explicitly, let $\partial_i\Sigma^B$ be a boundary curve and γ_i be a curve isotopic to $\partial_i\Sigma^B$. Let $\mathbb{A}_{\gamma_i}$ be the annular neighborhood bounded by γ_i and $\partial_i\Sigma^B$. $\mathrm{DB}(\Sigma^B)$ consists of maps which are equivalent in $\mathrm{PModI}(\Sigma^B)$ to quasiconformal maps which are the identity on $\Sigma^B \setminus \cup_i \mathbb{A}_{\gamma_i}$.

Definition 2.7. Let $\mathrm{DI}(\Sigma^B)$ be the subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by Dehn twists around curves γ which are not isotopic to a boundary curve or a point. These are the non-separating curves.

As an aside, we note that the groups $\mathrm{PMod}^P(\Sigma^P)$ and $\mathrm{PModI}(\Sigma^B)$ are isomorphic to their analogues which are defined using homeomorphism or diffeomorphisms.

Proposition 2.8. *The mapping class group* $\mathrm{PModI}(\Sigma^B)$ *is generated by Dehn twists about finitely many non-separating closed curves together with Dehn twists about curves isotopic to the boundary components.*

Proposition 2.9. *The subgroup* $\mathrm{DB}(\Sigma^B)$ *generated by boundary Dehn twists is contained in the center of* $\mathrm{PModI}(\Sigma^B)$ *and is isomorphic to* $\mathbb{Z}^n$.

Proof. Dehn twists about disjoint curves commute because the twist homeomorphisms can be taken to have disjoint support. The second part is the content of [42, Theorem 3.8]. $\square$

Corollary 2.10. $\mathrm{PModI}(\Sigma^B)/\mathrm{DB}(\Sigma^B) \simeq \mathrm{DI}(\Sigma^B)$

2.2. **Teichmüller spaces.** We define the ordinary Teichmüller spaces (i.e. without riggings) of punctured and bordered surfaces and describe their complex structure, as well as discuss the mapping class groups. Since this material is standard we only provide a sketch, and refer the reader to [32] or [38] for details.

The Teichmüller space of a bordered Riemann surface is defined as follows. Let Σ^B be a fixed base surface, which establishes the genus and number of boundary components. Consider the set of triples $(\Sigma^B, f_1, \Sigma_1^B)$ where Σ_1^B is a Riemann surface, and $f_1 : \Sigma^B \to \Sigma_1^B$ is a quasiconformal mapping. We say that

$$(\Sigma^B, f_1, \Sigma_1^B) \sim_T (\Sigma^B, f_2, \Sigma_2^B)$$

if there exists a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity 'rel $\partial\Sigma^B$'. Recall that the term 'rel $\partial\Sigma^B$' means that the isotopy is constant on $\partial\Sigma^B$; in particular it is the identity there.

Remark 2.11. If we impose an ordering of the boundary components of Σ^B then a map $f_1 : \Sigma^B \to \Sigma_1^B$ induces an ordering on the boundary components of Σ_1^B. In the definition of the equivalence relation the 'isotopy rel boundary' condition implies that σ automatically preserves the ordering.

Definition 2.12. The Teichmüller space of a bordered Riemann surface Σ^B is

$$T^B(\Sigma^B) \cong \{(\Sigma^B, f_1, \Sigma_1^B)\}/\sim_T .$$

Taking the quotient by the weaker equivalence relation that $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity (not necessarily rel boundary), produces the *reduced* Teichmüller space $T_{\#}^B(\Sigma^B)$.

The case of punctured surfaces is similar. Let Σ^P be the punctured base surface. We say

$$(\Sigma^P, f_1, \Sigma_1^P) \sim_T (\Sigma^P, f_2, \Sigma_2^P)$$

if and only if there exists a biholomorphism $\sigma : \Sigma_1^P \to \Sigma_2^P$ such that $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity. Note that the punctures are necessarily fixed throughout a isotopy.

Definition 2.13. The Teichmüller space $T^P(\Sigma^P)$ of punctured Riemann surfaces is

$$T^P(\Sigma^P) \cong \{(\Sigma^P, f_1, \Sigma_1^P)\}/\sim_T .$$

These two definitions are special cases of a more general definition but we will not discuss this here.

The *Beltrami equation* is the partial differential equation

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial \bar{z}}.$$

The following two theorems are crucial.

Theorem 2.14. *For any $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < 1$, there exists a unique solution to the Beltrami equation fixing $0, 1$ and ∞. This normalized quasiconformal map will be denoted w^μ.*

Theorem 2.15. *For every fixed $z \in \mathbb{C}$, the map $\mu \mapsto w^\mu(z)$ is holomorphic. In particular, if μ depends on a parameter t holomorphically, then $t \mapsto w^\mu(z)$ is holomorphic.*

Let X be a Riemann surface with punctures or boundary. We temporarily denote its Teichmüller space by $T(X)$ and its mapping class group by $\mathrm{PMod}(X)$. We have dropped the superscript "B" or "P" as the following considerations apply in both cases.

Let $L^\infty_{(-1,1)}(X)_1$ be the unit ball in the complex Banach space of differentials of type $(-1, 1)$ on X, which as a linear space possesses a complex structure. Elements $\mu\, d\bar{z}/dz \in L^\infty_{(-1,1)}(X)_1$ are called *Beltrami differentials.* If $f : X \to X_1$ is quasiconformal then, in terms of a local parameter z,

$$\mu(f) = \frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z}$$

is called the *complex dilation* of f. The existence and uniqueness of solutions to the Beltrami equation guarantees a well-defined association of an element $[\Sigma^B, f_1, \Sigma_1^B]$ to each element of $L^\infty_{(-1,1)}(X)_1$. This association is called the *fundamental projection* and is denoted by

$$\Phi : L^\infty_{(-1,1)}(X)_1 \longrightarrow T(X).$$

The Teichmüller space $T(X)$ possesses a natural complex structure, and this complex structure has the following crucial properties.

Theorem 2.16. *The fundamental projection $\Phi : L^\infty_{(-1,1)}(X)_1 \longrightarrow T(X)$ is holomorphic.*

Theorem 2.17. *The fundamental projection possesses local holomorphic sections; i.e. for any point $p \in T(X)$ there is a holomorphic map $\sigma : U \to L^\infty_{(-1,1)}(X)_1$ on a neighborhood U of p such that $\Phi \circ \sigma$ is the identity.*

Next, we define the 'Teichmüller distance', a metric on $T(X)$. Any topological statements about the various Teichmüller spaces in this paper refer to the unique topology compatible with this distance.

For any quasiconformal mapping f defined on a Riemann surface X, we define its 'maximal dilatation' K_f by

$$||\mu(f)||_\infty = \frac{K_f - 1}{K_f + 1}$$

where $\mu(f)$ is the complex dilatation of μ.

Definition 2.18. The Teichmüller distance τ between two elements $[X, f, X_1]$ and $[X, g, X_2]$ of $T(X)$ is given by

$$\tau\left([X, f, X_1], [X, g, X_2]\right) = \frac{1}{2}\inf\{\log K_{g\circ f^{-1}}\}$$

where the infimum is taken over all representatives f and g of the equivalence classes $[X, f, X_1]$ and $[X, g, X_2]$.

A full exposition of the preceding material can be found in [38, Chapter 3] or [32, Chapter V].

A quasiconformal map $h : X \to X$ induces an bijection $h_* : T(X) \longrightarrow T(X)$ defined by $[X, f, X_1] \mapsto [X, f \circ h, X_1]$. For an element $[\rho] \in \mathrm{PMod}(X)$ we denote its corresponding action by $[\rho]\cdot[X, f, X_1] = [X, f\circ\rho, X_1]$. (It is actually an anti-action but there is no need to dwell upon this fact.) It is not hard to see that this action is well defined and the quotient of $T(X)$ by this action of the mapping class group is isomorphic to the Riemann moduli space. The following Lemma is a deeper result (see for example [38, page 225]).

Lemma 2.19. *The group* $\mathrm{PMod}(X)$ *acts as a group of biholomorphisms on* $T(X)$*. That is,* $\rho_* : T(X) \to T(X)$ *is a biholomorphism for each* $\rho \in \mathrm{PMod}(X)$*.*

Since there are many definitions of proper discontinuity to choose from, we include the following definition for definiteness.

Definition 2.20. The action of a group G on a topological space S is called *properly discontinuous* if for any $s \in S$ there exists a neighborhood V_s of s such that $(g \cdot V_s) \cap V_s = \emptyset$ except for finitely many $g \in G$.

The next result can be found in [38, page 152].

Lemma 2.21. *Let* Σ^P *be a punctured Riemann surface of finite type. The action of the group* $\mathrm{PMod}^P(\Sigma^P)$ *on* $T^P(\Sigma^P)$ *is properly discontinuous.*

The behavior of the action of $\mathrm{PModI}(\Sigma^B)$ and its subgroups on $T^B(\Sigma^B)$ is an important part of this paper. The relation to $\mathrm{PMod}^P(\Sigma^P)$ will also be discussed and utilized. For the structure of certain quotient spaces we will need the following (see [54, Proposition 5.3] or [38, page 160]).

Proposition 2.22. *Let* M *be a complex manifold. Let* G *be a group acting properly discontinuously and fixed-point freely by biholomorphisms on* M*. Then* M/G *is a Hausdorff topological space which can be given a unique complex structure, so that the projection mapping* $p : M \to M/G$ *is holomorphic. Moreover,* p *possess local holomorphic sections.*

Remark 2.23. Proposition 2.22 holds in the case that M is infinite-dimensional. Uniqueness relies on the fact that a homeomorphism is holomorphic if it is holomorphic on all finite-dimensional affine subspaces (see [38, page 87]).

2.3. **Holomorphic motions and the λ-lemma.** The λ-lemma and its extension are discussed. This result will be used in a fundamental way in this section to produce quasiconformal maps with specified boundary values. The material in this section is taken from [2] and [7]. Originally the λ-lemma is due to Mañé, Sad and Sullivan [36] where it was used in the context of complex dynamics.

Remark 2.24. The λ-lemma was used in a different way in [46] for proving holomorphicity of the sewing operation. Similar ideas will be used in Section 7.3 to demonstrate the compatibility of the complex structure on the rigged moduli space with those given in [22] and [46].

Let Δ be the open unit disk in $\mathbb{C}$.

Definition 2.25. Let A be a subset of $\hat{\mathbb{C}}$. A holomorphic motion of A is a map $f : \Delta \times A \to \hat{\mathbb{C}}$ such that:

(1) for any fixed $z \in A$, the map $t \mapsto f(t, z)$ is holomorphic on Δ,
(2) for any fixed $t \in \Delta$, the map $z \mapsto f(t, z)$ is an injection, and
(3) the mapping $f(0, z)$ is the identity on A.

Since t is a kind of deformation parameter we often use the notation $f_t(z)$ for $f(t, z)$. Also, as f_0 is the identity, we think of $f_t(z)$ as a holomorphic perturbation of the identity. The following theorem is the λ-lemma of [36]. It says that any holomorphic perturbation of the identity must be a quasiconformal map.

Theorem 2.26 (λ-lemma)**.** *If f is a holomorphic motion as above then f has an extension to $F : \Delta \times \overline{A} \to \hat{\mathbb{C}}$ such that:*

(1) *F is a holomorphic motion of $\overline{A}$,*
(2) *each $F_t(\cdot) : \overline{A} \to \hat{\mathbb{C}}$ is quasiconformal, and*
(3) *F is jointly continuous in (t, z).*

In fact the holomorphic motion extends to the whole plane. This was originally proved by Slodkowski in [48] although other proofs now exist.

Theorem 2.27 (Extended λ-lemma)**.** *If f is a holomorphic motion as above then f has an extension to $F : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:*

(1) *F is a holomorphic motion of $\hat{\mathbb{C}}$,*
(2) *each $F_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal with dilatation not exceeding $(1 + |t|)/(1 - |t|)$, and*
(3) *F is jointly continuous in (t, z).*

The Beltrami differential of the quasiconformal extension $\mu(F_t)$ is holomorphic in t. See for example [7, Theorem 2].

Theorem 2.28. *Let f be a holomorphic motion of a set A with non-empty interior A^0. Then in A^0, the Beltrami coefficient $\mu(t, z)$ of $f(t, z)$ is a holomorphic function of $t \in \Delta$. That is, the map $\Delta \to L^\infty(A^0)_1$ given by $t \mapsto \mu(t, z)$ is holomorphic.*

Note that in particular this theorem applies to $F_t(z)$ in the extended λ-lemma and so $\mu(F_t) \in L^\infty(\hat{\mathbb{C}})_1$ is holomorphic in t.

2.4. **Quasisymmetric maps.** We briefly review some standard definitions and adjust them to our purposes. Useful facts about extensions of quasiconformal maps are stated. Let $\overline{\mathbb{R}}$ denote the extended real line $\mathbb{R} \cup \infty$.

Definition 2.29. An (orientation preserving) homeomorphism

$$h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$$

is k-quasisymmetric if there exists a constant k such that

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k$$

for all x, $t \in \overline{\mathbb{R}}$. If h is quasisymmetric for some unspecified k it is simply called quasisymmetric.

We find it more convenient to work on S^1 than on $\overline{\mathbb{R}}$. It is also necessary to speak of quasisymmetry of a mapping on a closed boundary curve of a Riemann surface. The map $T(z) = i(1+z)/(1-z)$ sends the unit circle to $\overline{\mathbb{R}}$ with $T(1) = \infty$.

Definition 2.30. Let $h : S^1 \to S^1$ be a homeomorphism.

(1) Let $e^{i\theta}$ be chosen so that $e^{i\theta} h(1) = 1$. Then we say that h is quasisymmetric if $T \circ e^{i\theta} h \circ T^{-1}$ is quasisymmetric according to Definition 2.29.
(2) Let C be a connected component of the border of a Riemann surface, and h a homeomorphism of C into S^1. We say that h is quasisymmetric if, for any biholomorphism $F : \mathbb{A}_C \to \mathbb{A}_r^1$ from a neighborhood $\mathbb{A}_C$ of C into a standard annulus with outer radius 1, $h \circ F^{-1}$ is quasisymmetric on S^1 in the sense of part one.

Three observations should be made at this point.

Remark 2.31 (Regularity of the inner boundary curve). Given F, we can always restrict to a smaller neighborhood such that the inside boundary of $\mathbb{A}_C$ is an analytic curve.

Remark 2.32 (Independence of the choice of F). The boundary values of a biholomorphism of a sufficiently nice domain are quasisymmetric. In particular, if $H : \mathbb{A}^1_{C_1} \to \mathbb{A}^1_{C_2}$ is a biholomorphic map between doubly connected domains bounded by S^1 and analytic curves C_i, then H extends to a quasisymmetric map on S^1. Also the composition of two quasisymmetric functions is quasisymmetric [34, II.7]. Thus if part two of Definition 2.30 holds for one biholomorphism F_1, it holds for any biholomorphism F_2 of an annular neighborhood, since $h \circ F_2^{-1} = h \circ F_1^{-1} \circ F_1 \circ F_2^{-1}$. In fact, this is true in greater generality: it holds if H is quasiconformal (see Theorem 2.40 ahead), and further if C_i are quasicircles.

Remark 2.33 (On k-quasisymmetry). These definitions cannot be refined to k-quasisymmetry in a canonical way.

The following theorem explains the importance of quasisymmetric mappings in Teichmüller theory (see [34, II.7]).

Theorem 2.34. *A homeomorphism $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is quasisymmetric if and only if there exists a quasiconformal map of the upper half plane with boundary values h.*

Since quasiconformality is preserved under biholomorphisms, a map $h : S^1 \to S^1$ is quasisymmetric if and only if there is a quasiconformal map from the unit disk to itself with boundary values h.

We require a local version of this statement, involving only a doubly connected neighborhood of a closed boundary curve. Towards this end we collect some more results on extensions of quasiconformal mappings. We will also require a result on continuation of quasiconformal maps across certain sets of measure zero. The following results are taken directly from Theorems 8.1, 8.2 and 8.3 in [34, II.8].

Definition 2.35. A curve $\gamma \subset \mathbb{C}$ in called a *quasicircle* if it is the image of a circle under a quasiconformal mapping of the plane. A *quasiarc* is a subarc of a quasicircle.

Note that in [34] quasiarcs are called quasiconformal curves.

Theorem 2.36. *Let $w_0 : G \to G'$ be a quasiconformal mapping and F a compact subset of the domain G. Then $w_0|_F$ extends to a quasiconformal mapping of the whole plane.*

Theorem 2.37. *Let G and G' be two domains with free boundary curves C and C'. Let $w : G \longrightarrow G'$ be a quasiconformal mapping such that $w(C) = C'$. If C and C' are quasicircles then w can be extended to a quasiconformal mapping of any domain G_1 containing $G \cup C$.*

Combining the above two theorems we obtain the following.

Theorem 2.38. *Let G and G' be two n-tuply connected domains whose boundary curves are quasicircles. Then every quasiconformal mapping $w : G \longrightarrow G'$ can be extended to a quasiconformal mapping of the whole plane.*

The next theorem states that quasiarcs are removable for quasiconformal mappings. See for example [37], [49] or [34, V.3].

Theorem 2.39. *Let G be an open subset of $\mathbb{C}$ and let E be a closed subset of G such that the two-dimensional measure of E is zero. If w is a homeomorphism of G that is K-quasiconformal on $G \setminus E$, then w is K-quasiconformal on G. In particular this holds when E is a quasiarc.*

We now return to the problem of giving a version of Theorem 2.34 for boundary curves of a Riemann surface. Consider any quasiconformal map $f : \mathbb{A}^1_{C_1} \to \mathbb{A}^1_{C_2}$ where C_i are Jordan curves enclosed by S^1. Theorems 2.36 and 2.37 guarantee that for any r such that $\mathbb{A}^1_r \subset \mathbb{A}^1_{C_1}$, there exists an extension of f to the disk, Δ, which agrees with f on $\mathbb{A}^1_r$. So we have the following theorem.

Theorem 2.40.

1. *A map $h : S^1 \to S^1$ is quasisymmetric if and only if it is the restriction of some quasiconformal map $f : \mathbb{A}^1_{C_1} \to \mathbb{A}^1_{C_2}$ for Jordan curves C_1 and C_2 enclosed by S^1.*
2. *Let C be a connected component of the border of a Riemann surface and $h : C \to S^1$ be a homeomorphism. The map h is quasisymmetric if and only if it is the restriction of some quasiconformal map $f : \mathbb{A}_C \to \mathbb{A}^1_{C_1}$ on some doubly connected neighborhood $\mathbb{A}_C$ of C.*

Remark 2.41. Definition 2.30 and Theorem 2.40 can be extended in the obvious way to mappings between connected components of a bordered Riemann surface.

3. Quasisymmetric sewing

As discussed in the introduction, the operation of sewing Riemann surfaces along analytically parametrized boundary components is a fundamental operation in conformal field

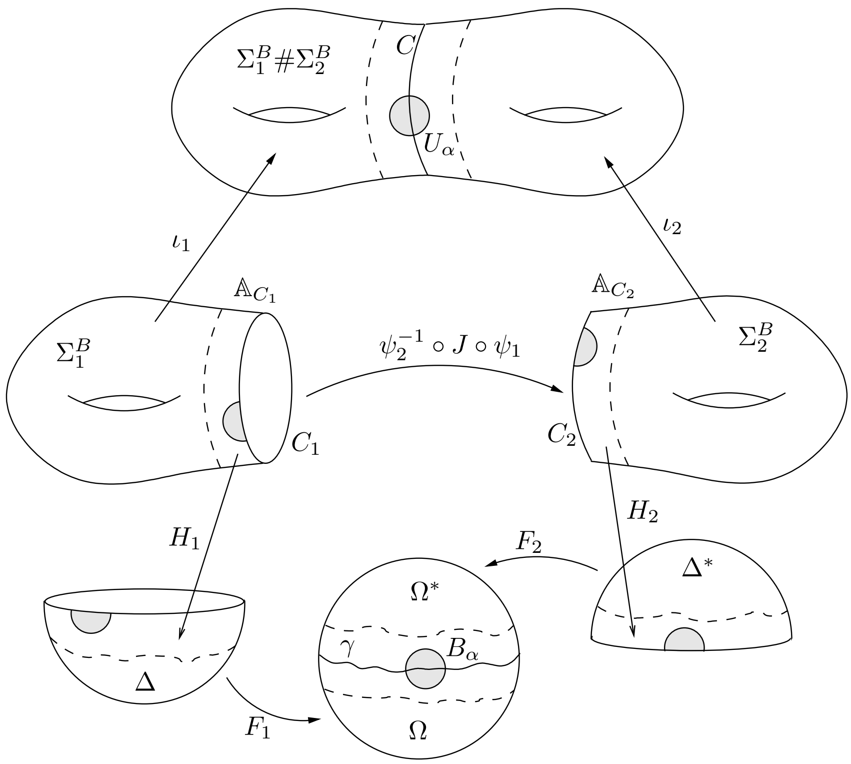

FIGURE 3.1. Sewing using quasisymmetric boundary identification.

theory. In this section it is shown how to sew Riemann surfaces where the identification of the boundary components is by quasisymmetric maps. For the case of disk this is a well-known construction in geometric function theory called *conformal welding* [32, III.1.4], [33], [44]. Sewing the boundaries of a strip to produce annuli was investigated by Oikawa [41]. The case of higher-genus surfaces is no more difficult but, at least in the context of conformal field theory, it has not been discussed in the literature.

A second fundamental use of sewing is to produce punctured surfaces by sewing caps onto bordered surfaces. Being able to do this with quasisymmetric maps enables us to relate the puncture and border models of rigged Teichmüller space.

3.1. **The sewing operation.** Let $\Delta^* = \hat{\mathbb{C}} \setminus \Delta$ be the upper-hemisphere. The following theorem (see [32, III.1.4]) describes the classical conformal welding of disks.

Theorem 3.1. *If $h : S^1 \longrightarrow S^1$ is quasisymmetric then there exists conformal maps F and G from Δ and Δ^* into complementary Jordan domains Ω and Ω^* of $\hat{\mathbb{C}}$ such that $G^{-1} \circ F|_{S^1} = h$. Moreover, the Jordan curve separating Ω and Ω^* is a quasicircle.*

We now describe the sewing of arbitrary Riemann surfaces using the conformal welding idea. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces of type (g_1, n_1^-, n_1^+) and (g_2, n_2^-, n_2^+) respectively where $n_1^+ > 0$ and $n_2^- > 0$. Let C_1 be an outgoing boundary component of Σ_1^B and C_2 be an incoming boundary component of Σ_2^B. Let ψ_1 and ψ_2 be quasisymmetric parametrizations of C_1 and C_2 (that is, quasisymmetric maps $\psi_i : C_i \to S^1$, for $i = 1, 2$, in the sense of Definition 2.30). Define $J : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $J(z) = 1/z$ and note that $\psi_2^{-1} \circ J \circ \psi_1 : C_1 \to C_2$ is an orientation reversing map.

Remark 3.2. In conformal field theory it is customary to include the information of 'incoming/outgoing' (that is, choice of sign) in the orientation of the parametrization. For our purposes it is easier to work solely with positively oriented quasisymmetric maps and record the orientation separately (see Remark 2.1). This is the view taken in [22] and is the most natural approach when working in the puncture picture of the rigged moduli space. We could recover the standard picture by left-composing all the parametrizations, of the incoming boundary, with J.

Let $\Sigma_1^B \# \Sigma_2^B = \Sigma_1^B \sqcup \Sigma_2^B / \sim$ where $x \sim y$ if and only if $x \in C_1$, $y \in C_2$ and $(\psi_2^{-1} \circ J \circ \psi_1)(x) = y$. Let ι_i be the inclusion maps $\Sigma_i^B \longrightarrow \Sigma_1^B \# \Sigma_2^B$. Both C_1 and C_2 map to a common curve on $\Sigma_1^B \# \Sigma_2^B = \Sigma_1^B \sqcup \Sigma_2^B / \sim$ which we denote by C and call the *seam*.

Remark 3.3. There is a natural way to make $\Sigma_1^B \# \Sigma_2^B$ into a topological space. If ψ_1 and ψ_2 are analytic parametrizations then $\Sigma_1^B \# \Sigma_2^B$ becomes a Riemann surface in a standard way using ψ_1 and ψ_2 to produce charts on the seam, C. See for example Ahlfors and Sario [1, Section II.3D].

We now describe how to put a complex structure on the sewn surface. Consulting Figure 3.1 may be helpful. Let $\mathbb{A}_{C_1}$ and $\mathbb{A}_{C_2}$ be annular neighborhoods of C_1 and C_2 and choose biholomorphic maps $H_1 : \mathbb{A}_{C_1} \longrightarrow \Delta$ and $H_2 : \mathbb{A}_{C_2} \longrightarrow \Delta^*$ such that $H_i(C_i) = S^1$. Note that the images of H_1 and H_2 are annular neighborhoods of S^1. With the orientation on C_i induced from Σ_i^B, $H_1|_{C_1} : C_1 \to S^1$ is orientation preserving while $H_2|_{C_2} : C_2 \to S^1$ is orientation reversing.

The map

$$h = H_2 \circ \psi_2^{-1} \circ J \circ \psi_1 \circ H_1^{-1}|_{S^1} : S^1 \longrightarrow S^1 \tag{3.1}$$

is orientation preserving and is quasisymmetric since it is the composition of quasisymmetric maps. By Theorem 3.1 there exist conformal maps F_1 and F_2 from Δ and Δ^* into complementary Jordan domains in $\hat{\mathbb{C}}$ such that $h = F_2^{-1} \circ F_1|_{S^1}$. Let γ be the quasicircle separating the Jordan domains Ω and Ω^*.

The charts on $\Sigma_1^B \# \Sigma_2^B$ will now be described. On $\iota_1(\Sigma_1^B)$ or $\iota_2(\Sigma_2^B)$ we take the original (interior) charts on Σ_1^B and Σ_2^B. To be precise we really have to compose with the inclusions ι_i. On the join we must be more careful. We consider all sets, B_α, of the following form. Let B_α be a simply connected domain in $\hat{\mathbb{C}}$ such that $B_\alpha \cap \gamma \neq \emptyset$ and $B_\alpha \subset F_1(H_1(\mathbb{A}_{C_1})) \cup F_2(H_2(\mathbb{A}_{C_2}))$. Let $B_\alpha^+ = B_\alpha \cap \overline{\Omega}$ and $B_\alpha^- = B_\alpha \cap \overline{\Omega^*}$. We can use these half-balls to define a chart in a neighborhood of a point in C. Let $A = \iota_1(\mathbb{A}_{C_1}) \cup \iota_2(\mathbb{A}_{C_2})$, and note that it is an annular neighborhood of C. For $i = 1, 2$, define $\zeta_i : A \to \hat{\mathbb{C}}$ by

$$\zeta_i = F_i \circ H_i \circ \iota_i^{-1}, \tag{3.2}$$

and let $U_\alpha^+ = \zeta_1^{-1}(B_\alpha^+)$ and $U_\alpha^- = \zeta_2^{-1}(B_\alpha^-)$. Let $U_\alpha = U_\alpha^+ \cup U_\alpha^-$ and define $\zeta_\alpha : U_\alpha \longrightarrow B_\alpha$ by

$$\zeta_\alpha(p) = \begin{cases} \zeta_1(p) & \text{if } p \in U_\alpha^+ \\ \zeta_2(p) & \text{if } p \in U_\alpha^-. \end{cases} \tag{3.3}$$

We take all such U_α together with the original (interior) open sets on Σ_1^B and Σ_2^B to form a basis of open sets and hence a topology on $\Sigma_1^B \# \Sigma_2^B$. In this topology we have:

Lemma 3.4. *The map $\zeta_\alpha : U_\alpha \longrightarrow B_\alpha$ is a homeomorphism.*

Proof. Since ζ_α is constructed from two homeomorphisms we need only check that it is well defined on $U_\alpha^+ \cap U_\alpha^- \subset C$. Let $w \in U_\alpha^+ \cap U_\alpha^-$ and let x and y be points in C_1 and C_2 respectively such that $\iota_1(x) = \iota_2(y) = w$. This implies $\psi_1(x) = J \circ \psi_2(y)$. We need to show that $\zeta_1(w) = \zeta_2(w)$.

By definition of ζ_α we must show that $(F_1 \circ H_1)(x) = (F_2 \circ H_2)(y)$. The conformal welding maps F_1 and F_2 have the property that $F_2^{-1} \circ F_1(z) = h(z)$ where h is defined in (3.1). Therefore

$$\begin{aligned}(F_1 \circ H_1)(x) &= ((F_2 \circ h) \circ H_1)(x)\\ &= (F_2 \circ (H_2 \circ \psi_2^{-1} \circ J \circ \psi_1 \circ H_1^{-1}) \circ H_1)(x)\\ &= ((F_2 \circ H_2) \circ (\psi_2^{-1} \circ J \circ \psi_1))(x)\\ &= (F_2 \circ H_2)(y)\end{aligned}$$

as required. □

Theorem 3.5. *The charts (U_α, ζ_α) together with the original (interior) charts from Σ_1^B and Σ_2^B give $\Sigma_1^B \# \Sigma_2^B$ a complex manifold structure. That is, $\Sigma_1^B \# \Sigma_2^B$ with these charts is a Riemann surface.*

Proof. The transition function corresponding to a chart (U_α, ζ_α), and a chart from Σ_1^B or Σ_2^B, is holomorphic because it is a composition of functions that are holomorphic with respect to the complex structures on Σ_1^B or Σ_2^B.

Let (U_α, ζ_α) and (U_β, ζ_β) be two charts with $U_\alpha \cap U_\beta \neq \emptyset$. From equations (3.2) and (3.3) we see that the ζ_α are defined using the globally defined maps F_i, H_i and ι_i. Moreover, the transition function $\zeta_\alpha \circ \zeta_\beta^{-1}$ is the identity and is thus holomorphic. □

Remark 3.6. If ψ_1 and ψ_2 are analytic then we can choose $H_1 = \psi_1$ and $H_2 = J \circ \psi_2$. In this case $h = \mathrm{id}$, $F_1 = \mathrm{id}$ and $F_2 = \mathrm{id}$. So our more general sewing procedure reduces to the standard one outlined in Remark 3.3.

Theorem 3.7. *The complex structure on $\Sigma_1^B \# \Sigma_2^B$ defined in Theorem 3.5 is the unique complex structure which is compatible with the original complex structures on Σ_1^B and Σ_2^B.*

Proof. Let $\{(V_\beta, \xi_\beta)\}$ be an analytic atlas for $\Sigma_1^B \# \Sigma_2^B$ that is compatible with the complex structures on Σ_1^B and Σ_2^B. The compatibility means that if $V_\beta \subset \iota_i(\Sigma_i^B)$ then $\xi_\beta \circ \iota_i$ is a holomorphic function from Σ_i^B to $\mathbb{C}$.

The case that needs attention is for the transition functions for the charts (U_α, ζ_α) and (V_β, ξ_β) where $U_\alpha \cap V_\beta \cap C \neq \emptyset$. The compatibility of ζ_α and ξ_β with the complex structures on Σ_1^B and Σ_2^B means that at points not in C, the holomorphicity of the transition function $\xi_\beta \circ {\zeta_\alpha}^{-1}$ is immediate. On the other hand let $x \in C \cap (U_\alpha \cap V_\beta)$ and let $p = \zeta_\alpha(x)$. We need to show that $\xi_\beta \circ {\zeta_\alpha}^{-1}$ is holomorphic at p. Let $N_\alpha = \zeta_\alpha(U_\alpha \cap V_\beta)$ and $N_\beta = \xi_\beta(U_\alpha \cap V_\beta)$. By Lemma 3.4 the transition map $\xi_\beta \circ \zeta_\alpha^{-1} : N_\alpha \to N_\beta$ is a homeomorphism and its restrictions to $N_\alpha^+ = N_\alpha \cap \Omega$ and $N_\alpha^- = N_\alpha \cap \Omega^*$ are holomorphic. By Theorem 3.1 we know that the boundary $\partial N_\alpha^+ \cap \Omega$ is a quasiarc. Theorem 2.39 therefore implies that $\zeta_\alpha^{-1} \circ \xi_\beta$ is in fact holomorphic on all of N_α. □

Remark 3.8. In the case of welding to produce annuli, an analogous result is proved in [41, Lemma 2 and Theorem 1]. A key ingredient in that case is also the removability of quasiarcs for quasiconformal maps.

3.2. **Sewing on caps.** We describe in detail the sewing of caps onto a surface with boundary to produce a punctured surface. The general setup and notation will be used throughout the paper.

The punctured disk $\overline{\Delta}_0 = \{z \in \mathbb{C} \,|\, 0 < |z| \leq 1\}$ will be considered as a bordered Riemann surface whose boundary is parametrized by the identity map. Given Σ^B of type (g, n^-, n^+) we choose a collection of 'boundary trivializations' $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$; each component is a quasiconformal map from a collared neighborhood of a boundary curve to the annulus $\mathbb{A}_r^1$ for some $r < 1$. By Theorem 2.40 we could equivalently say $\tau_i : \partial_i \Sigma^B \to S^1$ is a quasisymmetric boundary parametrization.

At each boundary curve $\partial_i \Sigma^B$ we sew in the punctured disk $\overline{\Delta}_0$ using $\mathrm{id} \circ J \circ \tau_i$ as the identification map as described in Section 3.1. We denote the simultaneous sewing by $\Sigma^B \#_\tau (\overline{\Delta}_0)^n$ and let $\Sigma^P = \Sigma^B \#_\tau (\overline{\Delta}_0)^n$ be the resultant punctured surface. The images of the punctured disks in Σ^P will be called *caps*. Let D_i denote the ith cap, $\mathbf{D} = D_1 \cup \ldots \cup D_n$ and $\mathbf{D}^0$ be the interior of $\mathbf{D}$. Note that $\Sigma^B = \Sigma^P \setminus \mathbf{D}^0$. With some slight abuse of notation we use $\partial_i \Sigma^B$ to denote the image of the ith boundary curve in Σ^P.

4. Quasiconformal extensions

Here we collect some non-standard results on quasiconformal maps and prove some new ones that we need. A key tool is the extended λ-lemma.

Two of the important technical results of this section are the related Lemmas 4.14 and 4.16, which address the problem of deforming quasiconformal maps to take specified boundary values. Their immediate application is to the relation between the mapping class groups of a bordered surface and its corresponding punctured surface. Moreover, these results are crucial in showing the equivalence between the border and puncture models of rigged Teichmüller space in Section 5.

4.1. **Holomorphic families and extension results.** The main idea here is that any quasisymmetric/quasiconformal map can be embedded in a holomorphic motion. Combining this with the extended λ-lemma, we are able to show that a quasisymmetric mapping of the boundary of an annulus extends to the entire annulus.

A version of the following Lemma can be found in [45, Example 1]. Our inclusion of the normalization is important in the application of the result.

Lemma 4.1. *Any quasiconformal map $u : \mathbb{C} \to \mathbb{C}$ can be embedded in a holomorphic motion. That is, there exists a holomorphic motion $u^t(z)$ such that $u(z) = u^{t_0}(z)$ for some $t_0 \in \Delta$. Furthermore, if $u(0) = 0$, we may take holomorphic motion to satisfy $u^t(0) = 0$ for all t.*

Proof. We proceed in two steps. First we normalize u and then find a holomorphic family relating u to the normalizing map. Second we produce a holomorphic motion of the normalizing map. Combining these produces the required holomorphic motion of u.

Let σ be the Möbius transformation such that $\sigma \circ u$ fixes 0, 1 and ∞. Actually since u already fixes ∞, $\sigma(z) = az + b$ for some $a, b \in \mathbb{C}$.

The complex dilation $\mu(\sigma \circ u)$ is equal to $\mu(u)$ because σ is conformal (but actually this is not important in our case). By the definition of quasiconformality, $||\mu(u)||_\infty = k$ for some $k < 1$. Choose ℓ such that $k < \ell < 1$ and let $n = 1/\ell$.

For $t \in \Delta$, $\mu^t = nt\mu(u)$ defines a holomorphic family of Beltrami coefficients. This follows since $||nt\mu(u)||_\infty < |n||t||k| < 1$. The factor n ensures that $\mu^t = \mu(u)$ when $t = \ell$, which is in the interior of Δ.

Let w^t be the normalized solution to the Beltrami equation with coefficient μ^t (see Theorem 2.14). The following properties hold:

(1) for fixed z, $w^t(z)$ is holomorphic in t (see Theorem 2.15),
(2) $w^0(z) = z$, and
(3) $w^\ell = \sigma \circ u$.

We now need to embed σ^{-1} in a holomorphic motion. Let

$$\rho^t(z) = \frac{(\ell - t)z + t\sigma^{-1}(z)}{\ell}$$

and note that ρ^t is also holomorphic in z. We claim that $u^t(z) = (\rho^t \circ w^t)(z)$ is the desired holomorphic motion. To see this note that:

(1) for fixed z, $u^t(z)$ is holomorphic in t because $\rho^t(z)$ is holomorphic in t and z,
(2) $u^0(z) = z$, and
(3) $u^\ell = \sigma^{-1} \circ w^\ell = u$.

The final statement of the theorem follows from the fact that if $u(0) = 0$, then $\sigma(0) = 0$. Since w_t is normalized so that $w_t(0) = 0$ for all t, we have $u_t(0) = 0$ for all t. □

Remark 4.2. This proves that any quasicircle can be embedded in a holomorphic motion of S^1.

Choose $R > 1$ and consider the standard annuli $\mathbb{A}_1^R$. Recall that S_R is the circle of radius R and $\overline{B(0,R)}$ is the closed ball of radius R.

Lemma 4.3. *Let $f : S_R \to S_R$ be a quasisymmetric mapping and let $\iota : S_1 \to S_1$ be the identity map. There exists a quasiconformal mapping $F : \mathbb{A}_1^R \to \mathbb{A}_1^R$ extending f_1 and ι. That is, $F|_{S_1} = \iota$ and $F|_{S_R} = f$. In fact F can be extended to $\hat{\mathbb{C}}$.*

Proof. The idea is to embed f in a holomorphic motion and apply the extended λ-lemma to the motion of $S_1 \cup S_R$, with the motion being the identity on S_1. However this cannot be done directly as the motion of S_R may intersect S_1 and so the motion of $S_1 \cup S_R$ may not be injective. We use compactness of the motion and a rescaling to avoid this problem.

Let $u_f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal extension of f. We may assume that $u_f(0) = 0$ (this can be achieved by composing u_f with an appropriate linear transformation).

From Lemma 4.1 there exists a holomorphic motion u^t such that $u^{t_0} = u_f$ for some $t_0 \in \Delta$. We know from the λ-lemma that $u^t(z) : \Delta \times \mathbb{C} \to \mathbb{C}$ is continuous and thus takes compact sets in z to compact sets. We also have $u^t(0) = 0$. Therefore there exists $\epsilon > 0$ such that $u^t(S_R) \cap \overline{B(0,\epsilon)} = \emptyset$ for all t with $|t| \leq t_0$.

Consider the function $v^t(z) = (1/\epsilon)u^t(\epsilon z)$. For $|z| = R/\epsilon$ and $|t| \leq |t_0|$, $|v^t(z)| > (1/\epsilon)\epsilon = 1$. So the motion of $S_{R/\epsilon}$ under v^t is disjoint from S_1. Also $v^0(z) = z$. The function

$$g^t(z) = \begin{cases} z & \text{for } z \in S_1 \\ v^t(z) & \text{for } z \in S_{R/\epsilon} \end{cases}$$

is injective and thus a holomorphic motion of $S_1 \cup S_{R/\epsilon}$. The extended λ-lemma (see Theorem 2.27) guarantees an extension of g^t to a holomorphic motion $G^t : \Delta \times \mathbb{C} \to \mathbb{C}$. Note that $G^t|_{S_1}(z) = z$ and $G^{t_0}(S_{R/\epsilon}) = S_{R/\epsilon}$. Now G^{t_0} must be modified to map S_R to itself. Write $z = re^{i\theta}$ and let

$$w(re^{i\theta}) = \begin{cases} re^{i\theta} & \text{for } r \leq 1 \\ (ar + b)e^{i\theta} & \text{for } r \geq 1 \end{cases}$$

where a and b are defined by $a + b = 1$ and $aR + b = R/\epsilon$. Note that w is quasiconformal and $w(Re^{i\theta}) = (R/\epsilon)e^{i\theta}$.

We now claim that $F = w^{-1} \circ G^{t_0} \circ w$ is the desired extension. That is, $F|_{S_1} = \iota$ and $F|_{S_R} = f$. The map F is quasiconformal since it is the composition of quasiconformal maps. When $z \in S_1$,

$$F(z) = w^{-1}(G^{t_0}(w(z))) = w^{-1}(G^{t_0}(z)) = w^{-1}(z) = z.$$

If $z = Re^{i\theta}$ then $u_f(z) = f(z)$ and

$$\begin{aligned} F(z) &= w^{-1}(G^{t_0}(w(Re^{i\theta}))) \\ &= w^{-1}(v^{t_0}((R/\epsilon)e^{i\theta}) \\ &= w^{-1}((1/\epsilon)u^{t_0}(\epsilon(R/\epsilon)e^{i\theta})) \\ &= w^{-1}((1/\epsilon)(u_f(z))) \\ &= u_f(z) \end{aligned}$$

The penultimate equality follows from equalities $|u_f(z)| = R$ and $w^{-1}(Re^{i\theta}/\epsilon) = Re^{i\theta}$. □

Remark 4.4. The roles of the inner and outer boundary of the annuli can be interchanged without difficulty.

Corollary 4.5. *Let A and B be doubly-connected regions bounded by quasicircles α_1 and α_2, and β_1 and β_2 respectively. Let $f_1 : \alpha_1 \to \beta_1$ and $f_2 : \alpha_2 \to \beta_2$ be quasisymmetric maps. There exists a quasiconformal map $F : A \to B$ extending f_1 and f_2. That is, $F|_{\gamma_1} = f_1$ and $F|_{\gamma_2} = f_2$.*

Proof. Choose numbers R_1 and R_2 and quasiconformal maps $g_1 : A \to \mathbb{A}_{R_1}^{R_2}$ and $g_2 : B \to \mathbb{A}_{R_1}^{R_2}$ Pick R' such that $R_1 < R' < R_2$. By applying Lemma 4.3 on $\mathbb{A}_{R_1}^{R'}$ we get a quasiconformal map extending $g_2 \circ f_1 \circ g_1^{-1}$ which is the identity on $S_{R'}$. Similarly $g_2 \circ f_1 \circ g_1^{-1}$ can be extended to the identity on $S_{R'}$. Gluing these maps and pulling back by g_1 and g_2 gives the desired map $F : A \to B$. □

Let Σ^B be a Riemann surface of type (g, n^-, n^+) with boundary $\partial\Sigma^B = \partial_1\Sigma^B \cup \cdots \cup \partial_n\Sigma^B$ as usual.

Corollary 4.6. *For $i = 1, \ldots, n$, let $f_i : \partial_i\Sigma^B \to \partial_i\Sigma^B$ be a quasisymmetric self-map of the i^{th} boundary component of Σ^B. There exists a quasiconformal map $f : \Sigma^B \to \Sigma^B$ such that $f|_{\partial_i\Sigma^B} = f_i$.*

Proof. For each i, choose γ_i to be a quasicircle that is isotopic to the boundary component $\partial_i\Sigma^B$. The curves γ_i can be taken to be mutually disjoint. Let $\mathbb{A}_{\partial_i\Sigma^B}^{\gamma_i}$ be the annulus bounded by γ_i and $\partial_i\Sigma^B$. After conformally mapping $\mathbb{A}_{\partial_i\Sigma^B}^{\gamma_i}$ to the plane we can apply Corollary 4.5 to obtain a quasiconformal map $F_i : \mathbb{A}_{\partial_i\Sigma^B}^{\gamma_i} \to \mathbb{A}_{\partial_i\Sigma^B}^{\gamma_i}$ with $F_i|_{\gamma_i} = \mathrm{id}$ and $F_i|_{\partial_i\Sigma^B} = f_i$. Combining the F_i and extending by the identity gives the desired extension. □

Corollary 4.7. *Let γ_1 and γ_2 be quasicircles in Δ with 0 contained in their interiors. Let $f_1 : \partial\Delta \to \partial\Delta$ and $f_2 : \gamma_1 \to \gamma_2$ be quasisymmetric maps. Then there exists a quasiconformal map $f : \Delta \to \Delta$ such that $f|_{\partial\Delta} = f_1$, $f|_{\gamma_1} = f_2$, and $f(0) = 0$.*

Proof. Choose $\epsilon > 0$ such that $S_\epsilon \cap \gamma_1 = \emptyset$ and $S_\epsilon \cap \gamma_2 = \emptyset$. Apply Corollary 4.5 to the doubly connected domains bounded by S^1 and γ_1, and S^1 and γ_2 to get an extension of f_1

and f_2. Now apply Corollary 4.5 again to the connected domains bounded by γ_1 and S_ϵ, and γ_2 and S_ϵ to get an extension of f_2 and the identity map on S_ϵ. Gluing these two extensions and the identity map on $B(0,\epsilon)$ produces the desired extension f. □

Remark 4.8. One can extend f to the entire plane.

4.2. **The correcting map and extension of** $\mathrm{PMod}^B(\Sigma^B)$ **to** $\mathrm{PMod}^P(\Sigma^P)$**.** Given a Riemann surface Σ^B bounded by n closed curves, we can extend it to a punctured Riemann surface $\Sigma^P = \Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n$ by sewing on caps as described in Section 3.2. In the current section the orientation of the punctures and boundary components plays no role and can be safely ignored.

In this section we establish various extension theorems for quasiconformal mappings. The main result is the following: any quasiconformal self-map of Σ^P is isotopic to a quasiconformal self-map of Σ^P which preserves Σ^B. In fact, the boundary values of the restriction of the new map can be arbitrarily assigned. These results are contained in the key Lemmas 4.14 and 4.16, which will be crucial in establishing the bijection between the border and puncture models of Teichmüller space.

It is also necessary to relate the mapping class group of Σ^B to that of Σ^P. These results, summarized by Theorem 4.21, are present in the literature for the mapping class groups of diffeomorphisms or homeomorphisms (see for example [17, Section 2.1], [29, section 3] and [11, Section 6]). For the most part we are able to apply these known results to our quasiconformal setting. Where this is not possible we supply short proofs. These proofs appear to be new and rely on the extended λ-lemma in an essential way.

Lemma 4.9. *If f_1 and f_2 are quasiconformal self-maps of $\overline{\Delta}$ which agree on $\partial\Delta$ then f_1 and f_2 are isotopic rel $\partial\Delta$.*

Proof. This follows from [32, Theorem V.1.4], with the trivial group. Alternatively, one can explicitly construct the homotopy as in [32, Theorem IV.3.5]: Let $f_t(z)$ be the point dividing the geodesic joining $f_1(z)$ to $f_2(z)$ into segments whose hyperbolic lengths have the ratio $t : 1-t$. We note again that homotopies can always be replaced with isotopies (see Remark 2.4). □

Remark 4.10. If $f_1(0) = 0$ and $f_2(0) = 0$, one can take the homotopy to satisfy $f_t(0) = 0$ for all t. That is, f_1 and f_2 are homotopic rel $\partial\Delta \cup \{0\}$. This follows from the explicit construction in the proof.

Corollary 4.11. *Any $f \in \mathrm{PQC}^B(\Sigma^B)$ has an extension $\tilde{f} \in \mathrm{PQC}^P(\Sigma^P)$. Any two such extensions of f are isotopic.*

Proof. Apply Lemma 4.9 and Remark 4.10 on the (punctured) caps with boundary values determined by f. □

Proposition 4.12. *A map $f \in \mathrm{PQC}^B(\Sigma^B)$ is isotopic to the identity via an isotopy keeping each boundary component setwise (but not pointwise) fixed, if and only if any extension $\tilde{f} \in \mathrm{PQC}^P(\Sigma^P)$ is isotopic to the identity on the punctured surface Σ^P.*

Proof. This appears in [8, Proposition 1.3], and references therein, for the more general case of homeomorphisms. □

Corollary 4.13. *The map $\mathrm{PMod}^B(\Sigma^B) \to \mathrm{PMod}^P(\Sigma^P)$ given by taking $[f]$ to an extension $[\tilde{f}]$ is well defined.*

Recall that D_i is the ith cap on Σ^P and is bounded by $\partial_i\Sigma^B$ (considered as a curve in Σ^P).

Lemma 4.14. *Let $f \in \mathrm{PQC}^P(\Sigma^P)$ and for $i = 1, \ldots, n$, let N_i be an open neighborhood of $D_i \cup f(D_i)$. There exists a quasiconformal mapping $\alpha : \Sigma^P \to \Sigma^P$ with the following properties:*

(1) *α is the identity outside $\cup N_i$,*
(2) *α takes the curves $\partial_i\Sigma^B$ to the curves $f(\partial_i\Sigma^B)$, and*
(3) *α is homotopic to the identity.*

Proof. We want to construct a separate homotopy on an open neighborhood of $\partial_i\Sigma^B \cup f(\partial_i\Sigma^B)$ for each i, but unfortunately the curves $f(\partial_i\Sigma^B)$ may intersect $\partial_j\Sigma^B$ for $i \neq j$. So the first step is to produce a map that separates $\partial_i\Sigma^B$ from $f(\partial_j\Sigma^B)$ for $i \neq j$. Since f preserves the punctures, $f(D_i)$ contains only the puncture p_i, and for each fixed i there exists a neighborhood $B_i \subset D_i \cap f(D_i)$ of p_i such that $B_i \cap f(D_j) = \emptyset$ for all $j \neq i$. We choose a punctured domain V_i such that $D_i \subset V_i \subset N_i$ and V_i maps to a quasidisk under a local coordinate. Let $s_i : V_i \to V_i$ be a quasiconformal map that is the identity on ∂V_i and shrinks D_i so that it lies inside B_i (that is, $s_i(D_i) \subset B_i$). The maps s_i can easily be constructed after mapping V_i to the plane and constructing a suitable map using Corollary 4.7. By choosing the V_i to be mutually disjoint we can glue the s_i with the identity map to produce a quasiconformal map $s : \Sigma^P \to \Sigma^P$ with the property that $s(\partial_i\Sigma^B) \cap f(\partial_j\Sigma^B) = \emptyset$ for all i, j with $i \neq j$. The fact that s is quasiconformal on ∂V_i follows from Theorem 2.39. It follows from Lemma 4.9 that s is homotopic to the identity.

By applying s we have now reduced the problem to finding a map $\beta : \Sigma^P \to \Sigma^P$ that takes the curves $s(\partial_i\Sigma^B)$ to the curves $f(\partial_i\Sigma^B)$. For each i, choose a punctured domain $U_i \subset N_i$ that contains $f(D_i)$ (and thus also $s(D_i)$), and that maps to a quasidisk under a local coordinate. Moreover we can choose the U_i to be mutually disjoint (this is why we needed step one). Choose conformal mappings $g_i : U_i \to \Delta_0$. On each copy of the disk we apply Corollary 4.7 to obtain a map $\beta_i : \Delta_0 \to \Delta_0$ which is the identity on $\partial\Delta$, and takes $g_i(s_i(\partial_i\Sigma^B))$ to $g_i(f(\partial_i\Sigma^B))$. The maps β_i must be homotopic to the identity rel $\partial\Delta$ by Lemma 4.9. Pulling back under the maps g_i and gluing to the identity, we obtain β. Theorem 2.39 ensures that β is quasiconformal across the joins.

Now $\alpha = \beta \circ s$ is the required map. It is homotopic to the identity and equal to the identity outside $U_i \cup V_i \subset N_i$. □

Corollary 4.15. *Every mapping $f \in \mathrm{PQC}^P(\Sigma^P)$ is homotopic to a mapping $\tilde{f} \in \mathrm{PQC}^P(\Sigma^P)$ which restricts to an element of $\mathrm{PQC}^B(\Sigma^B)$. That is, $\tilde{f}|_{\Sigma^B} \in \mathrm{PQC}^B(\Sigma^B)$.*

Proof. Apply Lemma 4.14 to obtain α. The mapping $\tilde{f} = \alpha^{-1} \circ f$ is homotopic to f and preserves the boundary curves of Σ^B. □

Lemma 4.16 (Strengthening of Lemma 4.14)**.** *For $i = 1, \ldots n$, let $h_i : \partial_i\Sigma^B \to \partial_i\Sigma^B$ be a quasisymmetric self-map of the boundary component $\partial_i\Sigma^B$. If $g \in \mathrm{PQC}^P(\Sigma^P)$ then there exists $\alpha \in \mathrm{PQC}_0^P(\Sigma^P)$ such that the map $g' = \alpha \circ g$ preserves $\partial_i\Sigma^B$ (that is, g' restricts to an element of $\mathrm{PQC}^B(\Sigma^B)$) and $g'|_{\partial_i\Sigma^B} = h_i$ for each i.*

Proof. Rather than refining the proof of Lemma 4.14 we instead use Lemma 4.14 directly, followed by an application of Corollary 4.7 to obtain the specified boundary values.

Applying Lemma 4.14 with $f = g$, we obtain a map $\alpha_0 \in \mathrm{PQC}_0^P(\Sigma^P)$ such that $\alpha_0(\partial_i\Sigma^B) = g(\partial_i\Sigma^B)$. The quasiconformal map $g_0 = \alpha_0^{-1} \circ g$ is homotopic to g and preserves the boundaries. Let $U_i \subset \Sigma^P$ be a mutually disjoint collection of punctured quasidisks containing D_i

(i.e., U_i map to quasidisks under a local biholomorphic chart.) Let $G_i : U_i \to \Delta_0$ be biholomorphic (or quasiconformal) mappings . By Corollary 4.7 there is a mapping $\beta_i : \Delta \to \Delta$ such that $\beta_i|_{S^1} = \mathrm{id}$, $\beta_i(0) = 0$, and

$$\beta_i|_{G_i(\partial_i \Sigma^B)} = G_i \circ g_0 \circ h_i^{-1} \circ G_i^{-1};$$

in particular, β_i maps the quasicircle $G_i(\partial_i \Sigma^B)$ to itself. By Lemma 4.9 and Remark 4.10, β_i is homotopic to the identity rel $\partial\Delta \cup \{0\}$. Set

$$\alpha_1 = \begin{cases} G_i^{-1} \circ \beta_i^{-1} \circ G_i & \text{on } \overline{U}_i, \quad i = 1, \ldots, n \\ \mathrm{id} & \text{on } (\cup_i U)^c \end{cases}$$

and note that α_1 is homotopic to the identity and is quasiconformal on ∂U_i by Theorem 2.39. Let $\alpha = \alpha_1 \circ \alpha_0^{-1}$ and check that on $\partial_i \Sigma^B$,

$$\alpha \circ g = \left(G_i^{-1} \circ (G_i \circ h_i \circ g_0^{-1} \circ G_i^{-1}) \circ G_i\right) \circ g_0 = h_i.$$

So $g' = \alpha \circ g$ is homotopic to g and agrees with h_i on the boundary curves. □

Corollary 4.17. *If $g \in \mathrm{PQC}^P(\Sigma^P)$ then g is isotopic to a map which is the identity on $\partial\Sigma^B$.*

Proof. Apply Lemma 4.16 with h the identity map. □

Corollary 4.18. *The map $\chi : \mathrm{PModI}(\Sigma^B) \to \mathrm{PMod}^P(\Sigma^P)$ sending $[f]$ to any extension $[\tilde{f}]$ is surjective.*

Proof. The map is well defined by Corollary 4.13 and surjective by Corollary 4.17. □

Proposition 4.19. *Let $f \in \mathrm{PQC}(\Sigma^B)$ and assume $f|_{\partial\Sigma^B}$ is the identity. An extension $\tilde{f} \in \mathrm{PQC}(\Sigma^P)$ is isotopic to the identity if and only if the isotopy class of f is an element of $\mathrm{DB}(\Sigma^B)$.*

Proof. This is a special case of [42, Theorem 4.1(iii)]. □

Corollary 4.20. *The kernel of χ is $\mathrm{DB}(\Sigma^B)$.*

Theorem 4.21. *The sequence*

$$1 \longrightarrow \mathrm{DB}(\Sigma^B) \longrightarrow \mathrm{PModI}(\Sigma^B) \xrightarrow{\chi} \mathrm{PMod}^P(\Sigma^P) \longrightarrow 1$$

is exact.

Proof. This follows directly from the above results. □

5. The Moduli spaces and their complex structures

In this section we define two models of the rigged Riemann and Teichmüller moduli spaces. The two models can be described as the 'puncture' and 'border' models. In either picture, the relevant Riemann moduli space consists of Riemann surfaces together with a specification of how to sew them together; this is called the 'rigged' Riemann moduli space. In the border model, the boundary data or 'rigging' consists of a collection of mappings of the connected components of the border into S^1. In the puncture model, this boundary data takes the form of local biholomorphic coordinates around distinguished points. These two models are described in Section 5.1.

We give a generalization of the standard versions of these models. This generalization is easiest to state in the border model: we allow the boundary data of the rigging to be quasisymmetric rather than analytic. This allows us to show that the familiar Teichmüller space of a bordered Riemann surface covers the rigged Riemann moduli space. The complex structure of Teichmüller space then projects down to rigged Riemann moduli space. This is accomplished in Section 5.2.

It is an interesting fact that the boundary data is contained in the standard Teichmüller space of a bordered Riemann surface. In order to connect with the more familiar approach in conformal field theory, we construct two 'rigged' Teichmüller spaces corresponding to the puncture and border model in Section 5.3. The satisfying relation between these spaces is completed in Section 5.4. The entire picture is summarized in Section 5.5.

5.1. **Puncture and border models of the rigged Riemann moduli space.**

Puncture model: As in Section 2, let Σ^P be a Riemann surface of type (g, n^-, n^+) with oriented and ordered punctures $\mathbf{p} = (p_1, \dots, p_n)$. The non-negative integers n^-, n^+ and g are fixed throughout. We need to describe the rigging.

For ease of language, we temporarily think of marked points instead of punctures. For any point $q \in \Sigma^P$, let $\mathcal{O}(q)$ denote the set of germs of mappings which are holomorphic maps from a neighborhood of q into a neighborhood of $0 \in \mathbb{C}$, mapping q to 0.

Definition 5.1. Let $\mathcal{O}^{\Delta}_{qc}(q)$ be the set of $\phi \in \mathcal{O}(q)$ such that $\Delta \subset \mathrm{Im}(\phi)$, ϕ is biholomorphic on $\phi^{-1}(\Delta)$, and ϕ extends quasiconformally to a neighborhood of $\phi^{-1}(\Delta)$. For the set of punctures $\mathbf{p}$, let

$$\mathcal{O}^{\Delta}_{qc}(\mathbf{p}) = \big\{(\phi^1(p_1), \dots, \phi^n(p_n)) \in \mathcal{O}^{\Delta}_{qc}(p_1) \times \cdots \times \mathcal{O}^{\Delta}_{qc}(p_n) \;|\; (\phi^i)^{-1}(\overline{\Delta}) \cap (\phi^j)^{-1}(\overline{\Delta}) = \emptyset, \forall i \neq j\big\}$$

For clarification we note that ϕ and ϕ' in $\mathcal{O}^{\Delta}_{qc}(q)$ are equivalent if and only if they are equal on some neighborhood of q (and thus on $\phi^{-1}(\overline{\Delta})$). An element

$$\boldsymbol{\phi}(\mathbf{p}) = \big(\phi^1(p_1), \dots, \phi^n(p_n)\big) \in \mathcal{O}^{\Delta}_{qc}(\mathbf{p})$$

will be referred to as the *local coordinates* or *rigging* of Σ^P. For brevity we will denote the data of the surface, punctures, and local coordinates by $(\Sigma^P, \boldsymbol{\phi})$. We will refer to $(\Sigma^P, \boldsymbol{\phi})$ as a *rigged Riemann surface.*

Remark 5.2. It should be observed that $\mathcal{O}^{\Delta}_{qc}(q)$ is strictly smaller than the set $\{\phi \in \mathcal{O}(q) \,|\, \phi^{-1}$ is biholomorphic on $\Delta\}$. Considering ϕ^{-1}, a conformal map of the disk need not have a quasiconformal extension to a neighborhood.

We define an equivalence relation on the set, $\{(\Sigma^P, \boldsymbol{\phi})\}$, of rigged Riemann surfaces of type (g, n^-, n^+): we say $(\Sigma^P_1, \boldsymbol{\phi}_1) \sim_P (\Sigma^P_2, \boldsymbol{\phi}_2)$ if and only if there exists a biholomorphism $\sigma : \Sigma^P_1 \to \Sigma^P_2$ such that on $\boldsymbol{\phi}_1^{-1}(\overline{\Delta}_0)$, we have $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 \circ \sigma$. That is, for $i = 1, \dots, n$, $\phi^i_1 = \phi^i_2 \circ \sigma$ on each domain $(\phi^i_1)^{-1}(\overline{\Delta}_0)$. Note that this requires σ to take the ith puncture of Σ^P_1 to the ith puncture of Σ^P_2. The equivalence class of $(\Sigma^P, \boldsymbol{\phi})$ will be denoted by $[\Sigma^P, \boldsymbol{\phi}]$.

Definition 5.3. The puncture model of the moduli space of rigged Riemann surfaces is

$$\widetilde{\mathcal{M}}^P(g, n^-, n^+) = \{(\Sigma^P, \boldsymbol{\phi})\} / \sim_P .$$

Remark 5.4. This equivalence relation can be stated in the seemingly weaker form that there exists a biholomorphism $\sigma : \Sigma_1^P \setminus \boldsymbol{\phi}_1^{-1}(\Delta_0) \to \Sigma_2^P \setminus \boldsymbol{\phi}_2^{-1}(\Delta_0)$ such that on $\boldsymbol{\phi}_1^{-1}(S^1)$, we have $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 \circ \sigma$. This gives the same relation, since if $(\phi_1^i)^{-1}$ and $(\phi_2^i)^{-1}$ are analytic on Δ for each i, then the map σ extends to a biholomorphism $\hat{\sigma} : \Sigma_1 \to \Sigma_2$. Explicitly

$$\hat{\sigma} = \begin{cases} \sigma & \text{on } \Sigma_1 \setminus \boldsymbol{\phi}_1^{-1}(\Delta_0) \\ \boldsymbol{\phi}_2^{-1} \circ \boldsymbol{\phi}_1 & \text{on } \boldsymbol{\phi}_1^{-1}(\overline{\Delta}_0). \end{cases}$$

This map is well defined because $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 \circ \sigma$.

Remark 5.5. In this definition we are consciously imitating the 'complex analytic' model of the universal Teichmüller space due to Bers. His key insight was to extend the complex dilatation of a quasiconformal self-map of the disk to the entire plane by setting it to zero outside the disk.

Border model: As in Section 2 let Σ^B be a Riemann surface of type (g, n^-, n^+). That is, $\partial\Sigma = (\partial_1\Sigma^B \cup \cdots \cup \partial_n\Sigma^B)$ where $n = n^- + n^+$, there are n^- incoming boundary components, n^+ outgoing boundary components, and sewing in n disks would result in a compact Riemann surface of genus g. We fix n^-, n^+ and g throughout.

A *rigging* of Σ^B is an assignment of a quasisymmetric map $\psi_i : \partial_i\Sigma^B \to S^1$ for each boundary component. Note that according to Definition 2.30 a quasisymmetric map is orientation preserving. We denote this ordered set of maps concisely by $\boldsymbol{\psi} = (\psi^1, \ldots, \psi^n)$. The pair $(\Sigma^B, \boldsymbol{\psi})$ will be called a *rigged Riemann surface*. The notation should prevent any confusion with the puncture case where the same terminology is used.

Remark 5.6. See Remark 3.2 for the relation to the orientation of parametrizations in conformal field theory.

We define an equivalence relation on the set $\{(\Sigma^B, \psi)\}$ of type (g, n^-, n^+) rigged Riemann surfaces: $(\Sigma_1^B, \boldsymbol{\psi}_1) \sim_B (\Sigma_2^B, \boldsymbol{\psi}_2)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 \circ \sigma$. The equivalence class of $(\Sigma^P, \boldsymbol{\psi})$ will be denoted $[\Sigma^B, \boldsymbol{\psi}]$.

Definition 5.7. The border model of the moduli space of rigged Riemann surfaces is

$$\widetilde{\mathcal{M}}^B(g, n^-, n^+) = \{(\Sigma^B, \boldsymbol{\psi})\}/ \sim_B .$$

We will sometimes write $\widetilde{\mathcal{M}}^P(g, n^-, n^+)$ and $\widetilde{\mathcal{M}}^B(g, n^-, n^+)$ as $\widetilde{\mathcal{M}}^P(\Sigma^P)$ and $\widetilde{\mathcal{M}}^B(\Sigma^B)$ or even $\widetilde{\mathcal{M}}^P$ and $\widetilde{\mathcal{M}}^B$, where the type is assumed to be specified by a base space Σ^B or Σ^P.

We now describe how to convert a punctured surface with local coordinates into a surface with boundary and quasisymmetric parametrizations and vice versa. This produces a bijection between $\widetilde{\mathcal{M}}^P(g, n^-, n^+)$ and $\widetilde{\mathcal{M}}^B(g, n^-, n^+)$; this is the content of Theorem 5.9. Recall that the inversion map $J : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined by $J(z) = 1/z$.

Consider a rigged surface $(\Sigma^P, \boldsymbol{\phi})$ and for notational convenience let

$$\boldsymbol{\phi}^{-1}(\Delta_0) = \bigcup_{i=1}^{n} (\phi^i)^{-1}(\Delta_0).$$

We form the surface $\Sigma^B = \Sigma^P \setminus \boldsymbol{\phi}^{-1}(\Delta_0)$ whose n boundary components are specified to be incoming (respectively, outgoing) if the corresponding puncture is negatively (respectively, positively) oriented. The map

$$J \circ \phi^i|_{\partial_i\Sigma^B} : \partial_i\Sigma^B \longrightarrow S^1$$

is a quasisymmetric parametrization by Theorem 2.40. The map J is needed to correct the orientation (see Remark 5.8 below). So

$$\left(\Sigma^P \setminus \boldsymbol{\phi}^{-1}(\Delta_0), J \circ \boldsymbol{\phi}|_{\boldsymbol{\phi}^{-1}(S^1)}\right)$$

is a rigged Riemann surface.

Considering the converse situation, we begin with the rigged surface $(\Sigma^B, \boldsymbol{\psi})$. Choose a set of disjoint annular neighborhoods $\mathbb{A}_{\partial_i \Sigma^B}$ of $\partial_i \Sigma^B$, $i = 1, \ldots, n$ and let $\mathbb{A}_{\partial \Sigma^B} = \bigcup_{i=1}^n \mathbb{A}_{\partial_i \Sigma^B}$. Using Theorem 2.40, we can choose a quasiconformal extension of ψ^i to $\mathbb{A}_{\partial_i \Sigma^B}$. Let $\boldsymbol{\psi}_{\text{ext}}$ be such an extension of $\boldsymbol{\psi}$ to $\mathbb{A}_{\partial \Sigma^B}$. Note that each annular neighborhood maps to the interior of S^1. As in Section 3.2 we use $\boldsymbol{\psi}$ to sew disks onto Σ^B to produce the punctured surface $\Sigma^P = \Sigma^B \#_{\boldsymbol{\psi}} (\overline{\Delta}_0)^n$. The orientation of a puncture is determined by whether the corresponding boundary component is incoming or outgoing. Recall that the set of caps is denoted by $\mathbf{D} = (D_1 \cup \cdots \cup D_n)$. We claim that

$$\tilde{\boldsymbol{\psi}} = \begin{cases} J \circ \boldsymbol{\psi}_{\text{ext}} & \text{on } \mathbb{A}_{\partial \Sigma^B} \\ \text{id} & \text{on } \mathbf{D} \end{cases} \tag{5.1}$$

is quasiconformal and thus $(\Sigma^P, \tilde{\boldsymbol{\psi}})$ is a rigged Riemann surface. A direct check using the definition of sewing shows that $\tilde{\boldsymbol{\psi}}$ is well defined. In terms of local coordinates on Σ^P given by equation (3.3) in Section 3.1, the image of $\partial_i \Sigma^B \subset \Sigma^P$ is a Jordan curve on $\hat{\mathbb{C}}$. This curve is guaranteed to be a quasicircle by Theorem 3.1. Thus $\tilde{\boldsymbol{\psi}}$ is quasiconformal by Theorem 2.39.

Remark 5.8. The reason for using J in $J \circ \phi^i$ and $J \circ \tilde{\boldsymbol{\psi}}$ in the constructions above can be seen most clearly in the case of the sphere. Let Σ^P be the Riemann sphere with a puncture at 0, and consider the local coordinate $\phi = \text{id}$. Then $\Sigma^B = \Sigma^P \setminus \phi^{-1}(\Delta_0)$ is the upper-hemisphere and $\partial \Sigma^B$ is S^1 but with clockwise orientation. So $\phi|_{S^1} : \partial \Sigma^B \to S^1$ is orientation reversing. Equivalently, this can be understood in terms of which side of S^1 the extension of the boundary parametrization maps to.

Theorem 5.9. *The map* $\mathcal{I} : \widetilde{\mathcal{M}}^P(g, n^-, n^+) \to \widetilde{\mathcal{M}}^B(g, n^-, n^+)$ *defined by*

$$\mathcal{I}([\Sigma^P, \boldsymbol{\phi}]) = [\Sigma^P \backslash \boldsymbol{\phi}^{-1}(\Delta),\, J \circ \boldsymbol{\phi}|_{\boldsymbol{\phi}^{-1}(S^1)}]$$

is a bijection.

Proof.
$\mathcal{I}$ is well defined: Assuming $(\Sigma_1^P, \boldsymbol{\phi}_1) \sim (\Sigma_2^P, \boldsymbol{\phi}_2)$ in $\widetilde{\mathcal{M}}^P(g, n^-, n^+)$, there exists a biholomorphism $\sigma : \Sigma_1^P \to \Sigma_1^P$ such that $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 \circ \sigma$ on $\boldsymbol{\phi}_1^{-1}(\overline{\Delta}_0)$. After restricting σ to $\Sigma_1^P \backslash \boldsymbol{\phi}_1(\Delta_0)$ this is exactly the condition that $(\Sigma_1^P \backslash \boldsymbol{\phi}_1^{-1}(\Delta_0),\, J \circ \boldsymbol{\phi}_1|_{\boldsymbol{\phi}_1^{-1}(S^1)}) \sim (\Sigma_2^P \backslash \boldsymbol{\phi}_2^{-1}(\Delta_0),\, J \circ \boldsymbol{\phi}_2|_{\boldsymbol{\phi}_2^{-1}(S^1)})$ in $\widetilde{\mathcal{M}}^B(g, n^-, n^+)$.

$\mathcal{I}$ is injective: Assuming $\mathcal{I}([\Sigma_1^P, \boldsymbol{\phi}_1]) = \mathcal{I}([\Sigma_2^P, \boldsymbol{\phi}_2])$, there exists a biholomorphism σ : $\Sigma_1^P \backslash \boldsymbol{\phi}_1^{-1}(\Delta) \to \Sigma_1^P \backslash \boldsymbol{\phi}_2^{-1}(\Delta)$ such that $J \circ \boldsymbol{\phi}_1 = J \circ \boldsymbol{\phi}_2 \circ \sigma$ on $\boldsymbol{\phi}_1^{-1}(S^1)$. Thus by Remark 5.4. $[\Sigma_1^P, \boldsymbol{\phi}_1] = [\Sigma_2^P, \boldsymbol{\phi}_2]$.

$\mathcal{I}$ is surjective: Let $[\Sigma^B, \boldsymbol{\psi}] \in \widetilde{\mathcal{M}}^B(g, n^-, n^+)$. By the procedure described above, sewing in unit disks with $\boldsymbol{\psi}$ produces a punctured surface $\Sigma^P = \Sigma^B \#_{\boldsymbol{\psi}} (\overline{\Delta}_0)^n$ and an element $[\Sigma^P, \tilde{\boldsymbol{\psi}}]$ in $\widetilde{\mathcal{M}}^P(g, n^-, n^+)$. It follows directly that $\mathcal{I}([\Sigma^P, \tilde{\boldsymbol{\psi}}]) = [\Sigma^B, \boldsymbol{\psi}]$. $\square$

5.2. **Teichmüller space of bordered Riemann surfaces and the complex structure on the rigged Riemann moduli space.** The (usual) Teichmüller space of bordered Riemann surfaces actually contains the data of the boundary parametrizations. See Section 2.2 for the definition of Teichmüller space, its complex structure, and for the properties of the mapping class group action.

As subgroups of $\mathrm{PMod}^B(\Sigma^B)$, the groups $\mathrm{PModI}(\Sigma^B)$, $\mathrm{DB}(\Sigma^B)$ and $\mathrm{DI}(\Sigma^B)$ also act on $T^B(\Sigma^B)$. Each element is a biholomorphism of $T^B(\Sigma^B)$.

Lemma 5.10. *The action of* $\mathrm{PModI}(\Sigma^B)$ *on* $T^B(\Sigma^B)$ *is fixed-point free. In particular,* $\mathrm{DI}(\Sigma^B)$ *and* $\mathrm{DB}(\Sigma^B)$ *also act fixed-point freely on* $T^B(\Sigma^B)$*.*

Proof. Only the group $\mathrm{PModI}(\Sigma^B)$ needs to be considered as the other two are subgroups. Assume that $[\rho] \in \mathrm{PModI}(\Sigma^B)$ fixes $[\Sigma^B, f, \Sigma_1^B]$. Then $[\Sigma^B, f \circ \rho, \Sigma_1^B] = [\Sigma^B, f, \Sigma_1^B]$ and so there exists a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_1^B$ such that $(f \circ \rho)^{-1} \circ \sigma \circ f$ is isotopic to the identity rel $\partial\Sigma^B$. Since ρ is the identity on $\partial\Sigma^B$ we see that σ must be the identity on $\partial\Sigma_1^B$. Therefore σ is the identity on Σ_1^B and thus ρ is isotopic to the identity. So $[\rho]$ is the identity in $\mathrm{PModI}(\Sigma^B)$. □

Lemma 5.11. *The action of* $\mathrm{PModI}(\Sigma^B)$ *on* $T^B(\Sigma^B)$ *is properly discontinuous. In particular,* $\mathrm{DI}(\Sigma^B)$ *and* $\mathrm{DB}(\Sigma^B)$ *also act properly discontinuously on* $T^B(\Sigma^B)$*.*

Proof. Since the action of $\mathrm{PModI}(\Sigma^B)$ on $T^B(\Sigma^B)$ is fixed-point free, $\mathrm{PModI}(\Sigma^B)$ acts on $T^B(\Sigma^B)$ properly discontinuously (see Definition 2.20) if and only if

$$[\rho_n] \cdot [\Sigma^B, \mathrm{id}, \Sigma^B] \longrightarrow [\Sigma^B, \mathrm{id}, \Sigma^B]$$

implies that there exists $N \in \mathbb{N}$ such that $[\rho_n] = [\mathrm{id}]$ for all $n \geq N$.

By sewing on tori to the boundary components we will reduce the problem to the compact surface case. Let Y be a genus-one Riemann surface with one boundary component. Let Σ^Y be the Riemann surface (without punctures) obtained by sewing copies of Y to the boundary components of Σ^B. The parametrizations we use for sewing are not important here. Define $i_* : \mathrm{PModI}(\Sigma^B) \to \mathrm{PMod}^P(\Sigma^Y)$ by $i_*([\rho]) = [\tilde{\rho}]$ where

$$\tilde{\rho} = \begin{cases} \rho & \text{on } \Sigma^B \\ \mathrm{id} & \text{on the copies of } Y \end{cases}$$

From [42, Theorem 4.1] we know that i_* is injective. Note that this is not true if we sew on caps rather than tori.

It follows directly from the definition of the Teichmüller metric (see Definition 2.18) that

$$\tau_B\left([\Sigma^B, \rho, \Sigma^B], [\Sigma^B, \mathrm{id}, \Sigma^B]\right) \geq \tau_Y\left([\Sigma^Y, \tilde{\rho}, \Sigma^Y], [\Sigma^Y, \mathrm{id}, \Sigma^Y]\right)$$

where τ_B and τ_Y are the Teichmüller metrics on $T^B(\Sigma^B)$ and $T^P(\Sigma^Y)$ respectively. To see this observe that the the equivalence class in the definition of τ_Y is larger than in the definition of τ_B.

Let $[\rho_n]$ be a sequence in $\mathrm{PModI}(\Sigma^B)$ such that

$$[\Sigma^B, \rho_n, \Sigma^B] \longrightarrow [\Sigma^B, \mathrm{id}, \Sigma^B]. \tag{5.2}$$

That is,

$$\tau_B([\Sigma^B, \rho_n, \Sigma^B], [\Sigma^B, \mathrm{id}, \Sigma^B]) \to 0.$$

The above inequality implies that

$$\tau_Y([\Sigma^B, \tilde{\rho_n}, \Sigma^B], [\Sigma^B, \mathrm{id}, \Sigma^B]) \to 0.$$

By Theorem 2.21 we know the action of $\mathrm{PMod}^P(\Sigma^Y)$ on $T^P(\Sigma^Y)$ is properly discontinuous. Thus there exists $N \in \mathbb{N}$ such that $[\tilde{\rho_n}]$ is in the stabilizer for $n \geq N$. From the definition of proper discontinuity we know that the stabilizer is finite.

By the injectivity of i_* we see that $[\rho_n]$ is in a finite set for $n \geq N$. Because $\mathrm{PModI}(\Sigma^B)$ acts fixed-point freely, the convergence in (5.2) implies that $[\rho_n] = [\mathrm{id}]$ for all $n \geq N$.

Therefore $\mathrm{PModI}(\Sigma^B)$ acts properly discontinuously on $T^B(\Sigma^B)$. $\square$

Let Σ^B be of type (g, n^-, n^+) and as in Section 3.2, we choose a boundary trivialization $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ of Σ^B. We define the map

$$P_{\Sigma^B} : T^B(\Sigma^B) \longrightarrow \widetilde{\mathcal{M}}^B(g, n^-, n^+) \tag{5.3}$$

by $[\Sigma^B, f, \Sigma_1^B] \mapsto [\Sigma_1^B, \boldsymbol{\tau} \circ f^{-1}]$. The ordering (and signs) of the boundary components of Σ_1^B is defined by pushing forward the ordering (and signs) on Σ^B by f.

Theorem 5.12. *The mapping P_{Σ^B} induces a bijection*

$$P^*_{\Sigma^B} : \frac{T^B(\Sigma^B)}{\mathrm{PModI}(\Sigma^B)} \longrightarrow \widetilde{\mathcal{M}}^B(g, n^-, n^+).$$

Proof.
Well defined: Assume there is an $[h] \in \mathrm{PModI}(\Sigma^B)$ such that $[\Sigma^B, f_1 \circ h^{-1}, \Sigma_1^B] = [\Sigma^B, f_2, \Sigma_2^B]$. Then there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1 \circ h^{-1}$ is isotopic to the identity rel $\partial\Sigma^B$. In particular $f_2^{-1} \circ \sigma = h \circ f_1^{-1} = f_1^{-1}$ when restricted to $\partial\Sigma_1$, so $\boldsymbol{\tau} \circ f_2^{-1} \circ \sigma = \boldsymbol{\tau} \circ f_1^{-1}$; that is, $[\Sigma_1^B, \boldsymbol{\tau} \circ f_1^{-1}] = [\Sigma_2^B, \boldsymbol{\tau} \circ f_2^{-1}]$.
Injective: Assume that $P_{\Sigma^B}([\Sigma^B, f_1, \Sigma_1^B]) = P_{\Sigma^B}([\Sigma^B, f_2, \Sigma_2^B])$. Then there exists a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $\boldsymbol{\tau} \circ f_2^{-1} \circ \sigma = \boldsymbol{\tau} \circ f_1^{-1}$ on $\partial\Sigma_1$; thus $f_1^{-1} \circ \sigma^{-1} \circ f_2$ is the identity on the boundary. Let $[h] = [f_1^{-1} \circ \sigma^{-1} \circ f_2] \in \mathrm{PModI}(\Sigma^B)$; thus

$$[h][\Sigma^B, f_1, \Sigma_1^B] = [\Sigma^B, f_1 \circ h, \Sigma_1^B] = [\Sigma^B, \sigma^{-1} \circ f_2, \Sigma_1^B] = [\Sigma^B, f_2, \Sigma_2^B].$$

Surjective: Let $[\Sigma_1, \boldsymbol{\psi}] \in \widetilde{\mathcal{M}}^B(g, n^-, n^+)$ and recall that such data includes an ordering of the boundary components. There exists an $f' : \Sigma^B \to \Sigma_1^B$ which is quasiconformal but we must modify it such that the ordering of the boundary components on Σ^B and Σ_1^B correspond under f'.

We claim that there exists a quasiconformal map $\gamma : \Sigma^B \to \Sigma^B$ that permutes the boundary components in any specified way. Consider the surface obtained by sewing caps onto the boundaries of Σ^B. For this punctured surface there exists a quasiconformal map which permutes two punctures. To see this, note that there exists a conformal mapping of a neighborhood of both punctures onto an open neighborhood of the closed unit disk. Clearly there exists a quasiconformal mapping which is the identity on $\partial\Delta$ and switches the punctures. Sewing this map back to Σ^B we obtain a quasiconformal map which interchanges the two punctures in question.

Such a map can be modified to preserve the boundaries by an application of Lemma 4.14. A sequence of such swaps can produce a map γ giving any desired permutation. We now choose γ so that $f = f' \circ \gamma : \Sigma^B \to \Sigma_1^B$ preserves the given orderings.

By Corollary 4.6, there exists a quasiconformal map $g : \Sigma_1^B \to \Sigma_1^B$ with boundary values $g = \boldsymbol{\psi}^{-1} \circ \boldsymbol{\tau} \circ f^{-1}$. Thus $P_{\Sigma^B}([\Sigma^B, g \circ f, \Sigma_1^B]) = [\Sigma_1^B, \boldsymbol{\psi}]$. $\square$

From Theorem 5.12, Proposition 2.22, and the fact that the group $\mathrm{PModI}(\Sigma^B)$ acts properly discontinuously and fixed-point freely as a group of biholomorphisms (see Lemmas 5.10, 5.11 and 2.19), we immediately have:

Theorem 5.13. *The rigged moduli space $\widetilde{\mathcal{M}}^B(g,n^-,n^+)$ is an infinite-dimensional complex manifold, with complex structure inherited from $T^B(\Sigma^B)$. That is, there exists a unique complex structure on $\widetilde{\mathcal{M}}^B(g,n^-,n^+)$ such that P_{Σ^B} is holomorphic. Moreover, P_{Σ^B} possesses local holomorphic sections.*

At first sight it appears that the complex structure on $\widetilde{\mathcal{M}}^B(g,n^-,n^+)$ depends on the choice of base surface Σ^B and its boundary parametrization $\boldsymbol{\tau}$. It is well known in Teichmüller theory that the complex structure on $T^B(\Sigma)$ is canonical in the sense that different choices of base surface give rise to biholomorphically equivalent spaces. This fact enables us to prove the following theorem.

Theorem 5.14. *The complex structure on $\widetilde{\mathcal{M}}^B(g,n^-,n^+)$ inherited from $T^B(\Sigma^B)$ is independent of choice of Σ^B and $\boldsymbol{\tau}$.*

Proof. Consider two base surface X and Y of type (g,n^-,n^+) with boundary trivializations $\boldsymbol{\tau}_X$ and $\boldsymbol{\tau}_Y$ respectively. Let $\widetilde{\mathcal{M}}^B_X$ and $\widetilde{\mathcal{M}}^B_Y$ be the underlying set $\widetilde{\mathcal{M}}^B(g,n^-,n^+)$ together with complex manifold structures inherited from $(X,\boldsymbol{\tau}_X)$ and $(Y,\boldsymbol{\tau}_Y)$ respectively. We need to show that the identity map $\mathrm{id}:\widetilde{\mathcal{M}}^B_X \to \widetilde{\mathcal{M}}^B_Y$ is biholomorphic.

Using Corollary 4.6 as in the proof of surjectivity in Theorem 5.12, we produce a quasiconformal map $g : Y \to X$ such that $\boldsymbol{\tau}_X \circ g|_{\partial Y} = \boldsymbol{\tau}_Y$. This quasiconformal map induces an *allowable bijection* $g_* : Y \to X$ defined by $g_*([X,f,\Sigma_1^B]) = ([Y,f\circ g,\Sigma_1^B])$. The map g_* is a biholomorphism (see for example [38, page 122 and 186]). It is straightforward to check that the diagram

$$\begin{tikzcd} T^B(X) \arrow[r, "g_*"] \arrow[d, "P_X"'] & T^B(Y) \arrow[d, "P_Y"] \\ \widetilde{\mathcal{M}}^B_X \arrow[r, "\mathrm{id}"] & \widetilde{\mathcal{M}}^B_Y \end{tikzcd}$$

commutes. The maps P_X and P_Y are defined in (5.3). By taking local holomorphic sections of P_X and P_Y we see that the bottom map, id, is biholomorphic. So $\widetilde{\mathcal{M}}^B_X$ and $\widetilde{\mathcal{M}}^B_Y$ have identical complex structures. $\square$

5.3. **Rigged Teichmüller spaces: puncture and border models.** We define the *rigged Teichmüller spaces* corresponding to the puncture and border models of the rigged moduli space.

In the puncture model, we construct a space of rigged, punctured Riemann surfaces as follows. As in Section 2, let Σ^P be a punctured Riemann surface of type (g,n^-,n^+) with oriented punctures $\mathbf{p} = (p_1,\ldots,p_n)$. Let

$$\overline{M}^P(\Sigma^P) = \{(\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi})\},$$

where Σ_1^P is another punctured Riemann surface, $f : \Sigma^P \to \Sigma_1^P$ is a quasiconformal mapping, and $\boldsymbol{\phi} \in \mathcal{O}^{\Delta}_{\mathrm{qc}}(\mathbf{p}_1)$ (see Definition 5.1). Here $\mathbf{p}_1$ denotes the punctures on Σ_1^P with order induced from Σ^P by f, that is, $p_1^i = f(p^i)$ for $i = 1,\ldots,n$.

We define an equivalence relation on $\overline{M}^P(\Sigma^P)$ by declaring

$$(\Sigma^P, f_1, \Sigma_1^P, \boldsymbol{\phi}_1) \sim^P (\Sigma^P, f_2, \Sigma_2^P, \boldsymbol{\phi}_2)$$

in the case that there exists a biholomorphism $\sigma : \Sigma_1^P \to \Sigma_2^P$ such that $\boldsymbol{\phi}_2 \circ \sigma = \boldsymbol{\phi}_1$ on $\boldsymbol{\phi}_1^{-1}(S^1)$ and $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity. Recall that for a punctured surface an isotopy necessarily fixes each puncture throughout.

Remark 5.15. As in Remark 5.4, the condition $\boldsymbol{\phi}_2 \circ \sigma = \boldsymbol{\phi}_1$ on $\boldsymbol{\phi}_1^{-1}(S^1)$ is equivalent to having equality on $\boldsymbol{\phi}_1^{-1}(\overline{\Delta})$. This follows from the fact that holomorphic maps are determined by their boundary values.

Definition 5.16. The *rigged Teichmüller space* (for punctured Riemann surfaces) is

$$\widetilde{T}^P(\Sigma^P) = \overline{M}(\Sigma^P)/ \sim^P .$$

The mapping class group $\mathrm{PMod}^P(\Sigma^P)$ acts on $\widetilde{T}^P(\Sigma^P)$ via

$$[\rho] \cdot [\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi}] = [\Sigma^P, f \circ \rho, \Sigma_1^P, \boldsymbol{\phi}],$$

just as in the usual Teichmüller space case (see Section 2.2). It can be easily shown, as in Lemma 5.10, that this action is fixed-point free. Later we will show that this action is also properly discontinuous.

Define

$$P_{\mathrm{mod}} : \widetilde{T}^P(\Sigma^P) \longrightarrow \widetilde{\mathcal{M}}^P(g, n^-, n^+) \tag{5.4}$$

by $P_{\mathrm{mod}}\left([\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi}_1]\right) = [\Sigma_1^P, \boldsymbol{\phi}_1]$. We immediately have:

Proposition 5.17. *The map P_{mod} induces a bijection*

$$P_{\mathrm{mod}}^* : \frac{\widetilde{T}^P(\Sigma^P)}{\mathrm{PMod}^P(\Sigma^P)} \longrightarrow \widetilde{\mathcal{M}}^P(g, n^-, n^+).$$

Proof. The map is surjective because given any ordering of the punctures on Σ_1^P, there exists a quasiconformal map $f : \Sigma^P \to \Sigma_1^P$ that induces the given ordering (as in the proof of surjectivity in Proposition 5.12.) The fact that the map is well-defined and injectivity are a simple consequence of the definitions. □

The border model of rigged Teichmüller space is given by a reduced Teichmüller space with boundary data. Fix a base Riemann surface Σ^B of a given type (g, n^-, n^+), which thus fixes an assignment of $\pm$ to each boundary component as well as an ordering of the set of components. Consider the space of quadruples $(\Sigma^B, f_1, \Sigma_1^B, \boldsymbol{\psi}_1)$ where Σ_1^B is another Riemann surface, $f_1 : \Sigma^B \to \Sigma_1^B$ is a quasiconformal map and $\boldsymbol{\psi}_1$ is a set of quasisymmetric parametrizations of $\partial \Sigma_1^B$ (see 'Border model' in Section 5.1 for details). The ordering of the boundary components on Σ_1^B is induced by f.

We say that

$$(\Sigma^B, f_1, \Sigma_1^B, \boldsymbol{\psi}_1) \sim_{\#} (\Sigma^B, f_2, \Sigma_2^B, \boldsymbol{\psi}_2)$$

if there exists a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity and $\boldsymbol{\psi}_2 \circ \sigma = \boldsymbol{\psi}_1$. It is important to observe that we do not require that the isotopy is 'rel boundary'.

Definition 5.18. The rigged reduced Teichmüller space of a bordered Riemann surface Σ^B is

$$\widetilde{T}^B_{\#}(\Sigma^B) \cong \{(\Sigma^B, f_1, \Sigma^B_1, \boldsymbol{\psi}_1)\}/\sim_{\#}.$$

The border and puncture models are equivalent. This can be seen by sewing 'caps' onto Σ^B and Σ^B_1 to produce an element in $\widetilde{T}^P(\Sigma^P)$. We describe this procedure carefully before giving the proof of equivalence. The analogous but simpler process for rigged moduli spaces was covered in Section 5.1. Certain technical points and notation will not be repeated.

As in Section 3.2, let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ be a trivialization of $\partial\Sigma^B$, let $\Sigma^P = \Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n$, and let $\mathbf{D} = (D_1 \cup \dots \cup D_n)$ be the images of the disks in Σ^P. Given $(\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1)$ we now describe a way to construct an element of $\widetilde{T}^P(\Sigma^P)$. Let $\Sigma^P_1 = \Sigma^B_1 \#_{\boldsymbol{\psi}_1} (\overline{\Delta}_0)^n$ be the punctured surface obtained by sewing on caps using the parametrization $\boldsymbol{\psi}_1$. Note that $\partial_i \Sigma^B_1$ and $\partial\overline{\Delta}_0$ are identified by $\mathrm{id} \circ J \circ \psi^i_1$. For $i = 1, \dots, n$, the map

$$h^i = J \circ \psi^i_1 \circ f \circ (\tau^i)^{-1} \circ J : S^1 \to S^1$$

is quasisymmetric and thus can be extended to a quasiconformal map $\tilde{h}^i : \overline{\Delta}_0 \to \overline{\Delta}_0$. Using these maps we extend f to a quasiconformal map $\tilde{f} : \Sigma^P \to \Sigma^P_1$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in \Sigma^B \\ \tilde{h}^i(x) & \text{for } x \in D_i \end{cases} \tag{5.5}$$

The boundary values h^i were chosen precisely to ensure $\tilde{f}$ is well defined. By construction, $\tilde{f}$ is quasiconformal everywhere except possibly on the seams $\partial_i \Sigma^B \subset \Sigma^P$, but Theorem 2.39 guarantees that $\tilde{f}$ is in fact quasiconformal everywhere.

Let $\mathbf{D}_1$ be the union of the caps on Σ^P_1 and let $\mathbb{A}_{\partial\Sigma^B_1}$ be the union of disjoint annular neighborhoods of the boundary components of Σ^B_1. As in (5.1), we use $\boldsymbol{\psi}_1$ to construct the local coordinates

$$\tilde{\boldsymbol{\psi}}_1 = \begin{cases} J \circ \boldsymbol{\psi}_{\mathrm{ext}} & \text{on } \mathbb{A}_{\partial\Sigma^B_1} \\ \mathrm{id} & \text{on } \mathbf{D}_1 \end{cases} \tag{5.6}$$

on Σ^P_1. That is, we first use Theorem 2.40 to extend ψ^i_1 to a quasiconformal map, ψ^i_{ext}, on an annular neighborhood $\mathbb{A}_{\partial_i \Sigma^B}$. Then $J \circ \psi^i_{\mathrm{ext}}$ is extended to the caps by the identity map.

We finally have an element $[\Sigma^P, \tilde{f}, \Sigma^P_1, \tilde{\boldsymbol{\psi}}_1] \in \widetilde{T}^P(\Sigma^P)$ as desired. The following theorem shows that this procedure defines a bijection between the border and puncture models of rigged Teichmüller space.

Theorem 5.19. *The map*

$$\begin{aligned} \mathcal{J} : \widetilde{T}^B_{\#}(\Sigma^B) &\longrightarrow \widetilde{T}^P(\Sigma^P) \\ [\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1] &\longmapsto [\Sigma^P, \tilde{f}, \Sigma^P_1, \tilde{\boldsymbol{\psi}}_1] \end{aligned}$$

is a bijection. Here, Σ^P and Σ^P_1 are as above and $\tilde{f} : \Sigma^P \to \Sigma^P_1$ is any quasiconformal extension of f that takes the punctures to the centres of the caps. For $i = 1, \dots, n$, $\tilde{\psi}^i_1$ is an extension of $J \circ \psi^i_1$ to a quasiconformal mapping that takes a neighborhood of the ith cap into a neighborhood of Δ and is conformal on $(\tilde{\psi}^i_1)^{-1}(\Delta)$.

Proof. By the above discussion we know that maps $\tilde{f}$ and $\tilde{\boldsymbol{\psi}}_1$ with the stated properties exist.

$\mathcal{J}$ *is well-defined*: It is immediate that the image of the map is independent of the choice of $\tilde{\boldsymbol{\psi}}_1$. By Lemma 4.9 and Remark 4.10, $[\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{\boldsymbol{\psi}}_1]$ is independent of the choice of extension $\tilde{f}$. It remains to check that if $[\Sigma^B, f_1, \Sigma_1^B, \boldsymbol{\psi}_1] = [\Sigma^B, f_2, \Sigma_2^B, \boldsymbol{\psi}_2]$ then $[\Sigma^P, \tilde{f}_1, \Sigma_1^P, \tilde{\boldsymbol{\psi}}_1] = [\Sigma^P, \tilde{f}_2, \Sigma_2^P, \tilde{\boldsymbol{\psi}}_2]$.

Assume that there is a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $f_2^{-1} \circ \sigma \circ f_1$ is isotopic to the identity and $\boldsymbol{\psi}_2 \circ \sigma = \boldsymbol{\psi}_1$ on $\partial \Sigma^B$. The second condition implies that σ extends to a biholomorphism $\tilde{\sigma} : \Sigma_1^P \to \Sigma_2^P$ by setting it to the identity on the caps. Applying Proposition 4.12 to the extension $\tilde{f}_2^{-1} \circ \tilde{\sigma} \circ \tilde{f}_1$ of $f_2^{-1} \circ \sigma \circ f_1$, we see that $\tilde{f}_2^{-1} \circ \tilde{\sigma} \circ \tilde{f}_1$ is isotopic to the identity. Thus $[\Sigma^P, \tilde{f}_1, \Sigma_1^P, \tilde{\boldsymbol{\psi}}_1] = [\Sigma^P, \tilde{f}_2, \Sigma_2^P, \tilde{\boldsymbol{\psi}}_2]$.

$\mathcal{J}$ *is injective*: Assume that $\mathcal{J}\left([\Sigma^B, f_1, \Sigma_1^B, \boldsymbol{\psi}_1]\right) = \mathcal{J}\left([\Sigma^B, f_2, \Sigma_2^B, \boldsymbol{\psi}_2]\right)$. Then there exists a biholomorphism $\sigma : \Sigma_1^P \to \Sigma_2^P$ such that $\tilde{\boldsymbol{\psi}}_2 \circ \sigma = \tilde{\boldsymbol{\psi}}_1$ on $\tilde{\boldsymbol{\psi}}_1^{-1}(S^1)$ (in fact on $\tilde{\boldsymbol{\psi}}_1^{-1}(\overline{\Delta})$ by Remark 5.15) and $\tilde{f}_2^{-1} \circ \sigma \circ \tilde{f}_1$ is isotopic to the identity on the punctured surface Σ^P. Proposition 4.12 applies to $\tilde{f}_2^{-1} \circ \sigma \circ \tilde{f}_1$ and we conclude that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity (in general not rel boundary) on Σ^B. Therefore $[\Sigma^B, f_1, \Sigma_1^B, \boldsymbol{\psi}_1] = [\Sigma^B, f_2, \Sigma_2^B, \boldsymbol{\psi}_2]$ and $\mathcal{J}$ is injective.

$\mathcal{J}$ *is surjective*: Let $[\Sigma^P, g, \Sigma_1^P, \boldsymbol{\phi}_1] \in \widetilde{T}^P(\Sigma^P)$ and $\Sigma_1^B = \Sigma_1^P \setminus \boldsymbol{\phi}_1^{-1}(\Delta_0)$. The essential step is to show that g can be replaced by a map $\hat{g}$ such that its restriction to Σ^B maps onto Σ_1^B without changing the equivalence class in $\widetilde{T}^P(\Sigma^P)$. One subtlety to keep in mind is that $\Sigma_1^B \# (\overline{\Delta}_0)^n$ is conformally equivalent, but not equal, to Σ_1^P.

Let $k : \Sigma^P \to \Sigma^P$ be a quasiconformal extension of $g^{-1} \circ \boldsymbol{\phi}_1^{-1} \circ \boldsymbol{\tau}|_{\partial \Sigma^B}$. The existence of k is guaranteed by Corollary 4.6. Let α be the correcting map obtained by applying Lemma 4.14 with$f = k$, and let $\hat{g} = g \circ \alpha$. From the properties of α it follows that $\hat{g}$ is homotopic to g and $\hat{g}(\partial_i \Sigma^B) = (\boldsymbol{\phi}_1^i)^{-1}(S^1)$.

The rest of the proof only involves keeping track of the details of the sewing operations. On $\boldsymbol{\phi}_1^{-1}(S^1)$ let $\boldsymbol{\psi} = 1/\boldsymbol{\phi}$. For each i let

$$\tilde{\psi}_i(x) = \begin{cases} \psi_1^i(x) & \text{for } x \in \phi_1^i|_{\Sigma_1^B} \\ x & \text{for } x \in \overline{\Delta}_0 \end{cases}$$

be the local coordinates on $\Sigma_1^B \#_{\boldsymbol{\psi}_1} (\overline{\Delta}_0)^n$. Let $f = \hat{g}|_{\Sigma^B}$ and $\tilde{f}$ be defined as in (5.5). We claim that with this $\tilde{f}$ and $\tilde{\boldsymbol{\psi}}_1$,

$$\mathcal{J}[\Sigma^B, f, \Sigma_1^B, \boldsymbol{\psi}_1] = [\Sigma^P, g, \Sigma_1^P, \boldsymbol{\phi}_1]$$

The equivalence is determined by the biholomorphism $\tilde{\sigma} : \Sigma_1^P \to \Sigma_1^B \#_{\boldsymbol{\psi}_1} (\overline{\Delta}_0)^n$ defined by

$$\tilde{\sigma} = \begin{cases} \mathrm{id} & \text{on } \Sigma_1^B \\ \boldsymbol{\phi}_1 & \text{on } \boldsymbol{\phi}_1^{-1}(\overline{\Delta}_0). \end{cases}$$

It is well defined because the sewing using $\boldsymbol{\psi}_1$ identifies $x \in \partial_i \Sigma_1^B$ with $1/\psi_i(x) = \phi_i(x)$. Holomorphicity on $\partial \Sigma_1^B$ follows from Theorem 2.39. Moreover, $\tilde{f}^{-1} \circ \tilde{\sigma} \circ g$ is homotopic to the identity because g is homotopic to $\hat{g}$ and $\tilde{f}^{-1} \circ \tilde{\sigma} \circ \hat{g}$ is the identity except on the caps. $\square$

5.4. **Relation between the Teichmüller spaces.** We can create a projection map from $T^B(\Sigma^B)$ onto $\widetilde{T}^P(\Sigma^P)$ from two equivalent points of view.

In the first method, we let $[\Sigma^B, f, \Sigma_1^B] \in T^B(\Sigma^B)$. Create the base space $\Sigma^P = \Sigma^B \#_{\boldsymbol{\tau}}(\overline{\Delta_0})^n$ as usual and let $\mathbf{D} = D_1 \cup \ldots \cup D_n$ be the union of the caps. Consider the local coordinates

$$\tilde{\tau}_i = \begin{cases} J \circ \tau_i & \text{on an annular neighborhood of } \partial_i \Sigma^B \\ \mathrm{id} & \text{on } D_i \end{cases}$$

on Σ^P as in equation (5.6). The map $\tau_i \circ f^{-1}|_{\partial_i \Sigma_1^B}$ is a quasisymmetric boundary parametrization of $\partial_i \Sigma_1^B$. Let $\Sigma_1^P = \Sigma_1^B \#_{\boldsymbol{\tau} \circ f^{-1}}(\overline{\Delta_0})^n$, and extend f to Σ^P according to

$$\tilde{f} = \begin{cases} f & \text{on } \Sigma^B \\ \mathrm{id} & \text{on } \mathbf{D}. \end{cases}$$

The projection is then given by $[\Sigma^B, f, \Sigma_1^B] \mapsto [\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ \tilde{f}^{-1}]$.

In the second method, let $\tilde{f}'$ be a quasiconformal map on Σ^P whose dilatation agrees with that of f on Σ^B and is 0 on $\mathbf{D}$. Let ${\Sigma'_1}^P = \tilde{f}'(\Sigma^P)$. The projection is given by $[\Sigma^B, f, \Sigma_1^B] \mapsto [\Sigma^P, \tilde{f}', {\Sigma'_1}^P, \tilde{\boldsymbol{\tau}} \circ (\tilde{f}')^{-1}]$.

It is not hard to check that both of these maps are well-defined. The biholomorphic map $\sigma = \tilde{f}' \circ \tilde{f}^{-1} : \Sigma_1^P \to {\Sigma'_1}^P$ establishes that $[\Sigma^P, \tilde{f}', {\Sigma'_1}^P, \tilde{\boldsymbol{\tau}} \circ (\tilde{f}')^{-1}] = [\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ \tilde{f}^{-1}]$.

In the following, we will adopt the first method. Define

$$P_{\mathrm{DB}} : T^B(\Sigma^B) \longrightarrow \widetilde{T}^P(\Sigma^P) \tag{5.7}$$

by $P_{\mathrm{DB}}\left([\Sigma^B, f, \Sigma_1^B]\right) = [\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ \tilde{f}^{-1}]$.

Theorem 5.20. *The map*

$$P^*_{\mathrm{DB}} : \frac{T^B(\Sigma^B)}{\mathrm{DB}(\Sigma^B)} \longrightarrow \widetilde{T}^P(\Sigma^P)$$

induced by P_{DB} is a bijection.

Proof.

P^*_{DB} *is well-defined*: The choice of quasiconformal extension in the definition of $\tilde{\boldsymbol{\tau}}$ is immaterial as the equivalence relation in $\widetilde{T}^P(\Sigma^P)$ only involves the local coordinates restricted to the caps.

If $[\Sigma^B, f_1, \Sigma_1^B] = [\Sigma^B, f_2, \Sigma_2^B]$ in $T^B(\Sigma^B)$, then the biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ extends by the identity to a map $\tilde{\sigma} : \Sigma_1^P \to \Sigma_2^P$. By using $\tilde{\sigma}$, a direct check shows that $[\Sigma^P, \tilde{f}_1, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ f_1^{-1}] = [\Sigma^P, \tilde{f}_2, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ f_2^{-1}]$.

Let $[h] \in \mathrm{DB}(\Sigma^B)$, so that $[h][\Sigma^B, f, \Sigma_1^B] = [\Sigma^B, f \circ h, \Sigma_1^B]$. Define $\tilde{h} : \Sigma^P \to \Sigma^P$ by

$$\tilde{h} = \begin{cases} h & \text{on } \Sigma^B \\ \mathrm{id} & \text{on } \mathbf{D}. \end{cases}$$

It follows from Proposition 4.19 that $\tilde{h}$ is isotopic to the identity. Clearly

$$P_{\mathrm{DB}}\left([\Sigma^B, f \circ h, \Sigma_1^B]\right) = [\Sigma^P, \tilde{f} \circ \tilde{h}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ \tilde{h}^{-1} \circ \tilde{f}^{-1}].$$

We need to show that $[\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ f^{-1}] = [\Sigma^P, \tilde{f} \circ \tilde{h}, \Sigma_1^P, \tilde{\boldsymbol{\tau}} \circ \tilde{h}^{-1} \circ \tilde{f}^{-1}]$ in $\widetilde{T}^P(\Sigma^P)$. Setting $\sigma : \Sigma_1^P \to \Sigma_1^P$ to be the identity, we see that $\tilde{f}^{-1} \circ \sigma \circ \tilde{f} \circ \tilde{h} = \tilde{h}$ is isotopic to the identity, and $\tilde{\boldsymbol{\tau}} \circ f^{-1} = \tilde{\boldsymbol{\tau}} \circ h^{-1} \circ f^{-1}$ on $\partial \Sigma_1^P$ since $h = \mathrm{id}$ on the boundary.

P^*_{DB} *is injective*: Assume that $P_{\mathrm{DB}}\left([\Sigma^B, f_1, \Sigma^B_1]\right) = P_{\mathrm{DB}}\left([\Sigma^B, f_2, \Sigma^B_2]\right)$; that is, there exists a biholomorphism $\tilde{\sigma} : \Sigma^P_1 \to \Sigma^P_2$, such that $\tilde{f}_2^{-1} \circ \tilde{\sigma} \circ \tilde{f}_1$ is isotopic to the identity, and $\tilde{\boldsymbol{\tau}} \circ \tilde{f}_2^{-1} \circ \sigma = \tilde{\boldsymbol{\tau}} \circ \tilde{f}_1^{-1}$ on $\partial\Sigma^B_1$. Therefore $\sigma = \tilde{\sigma}|_{\Sigma^B}$ maps Σ^B to itself and $f_2^{-1} \circ \sigma \circ f_1 = \mathrm{id}$ on $\partial\Sigma^B$. By Proposition 4.19, $\tilde{f}_2^{-1} \circ \sigma \circ \tilde{f}_1|_{\Sigma^B}$ represents an element $[h]$ of $\mathrm{DB}(\Sigma^B)$; thus

$$[h][\Sigma^B, f_2, \Sigma^B_2] = [\Sigma^B, f_2 \circ h, \Sigma^B_2] = [\Sigma^B, \sigma \circ f_1, \Sigma^B_2] = [\Sigma^B, f_1, \Sigma^B_1].$$

P^*_{DB} *is surjective*: Let $[\Sigma^P, g, \Sigma^P_1, \boldsymbol{\phi}_1] \in \widetilde{T}^P(\Sigma^P)$. We will produce $[\Sigma^B, f', \Sigma^B_1] \in T^B(\Sigma^B)$ that maps to the given element under P_{DB}. As in the proof of surjectivity in Theorem 5.19 we need to modify g such that it preserves the boundary curves. Moreover, to obtain $\boldsymbol{\phi}_1$ we need to specify the boundary values of modified map. We apply Theorem 5.19.

With $\mathcal{J}$ as in Theorem 5.19, it is routine to check that

$$P_{\mathrm{DB}}\left([\Sigma^B, f, \Sigma^B_1]\right) = \mathcal{J}\left([\Sigma^B, f, \Sigma^B_1, \boldsymbol{\tau} \circ f^{-1}]\right).$$

As $\mathcal{J}$ is onto, $\mathcal{J}\left([\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1]\right) = [\Sigma^P, g, \Sigma^P_1, \boldsymbol{\phi}_1]$ for some f and $\boldsymbol{\psi}_1$. Let $\tilde{f} : \Sigma^P \to \Sigma^P$ be an extension of f as in the definition of $\mathcal{J}$.

On Σ^P, we apply Lemma 4.16 with $g = \mathrm{id}$ and $h_i = f^{-1} \circ (\psi^i_1)^{-1} \circ \tau_i$ to produce $g' : \Sigma^P \to \Sigma^P$ which is homotopic to the identity and equals h_i on $\partial_i\Sigma^B$. The map $f' = \tilde{f} \circ g'$ is homotopic to $\tilde{f}$ and $f'|_{\partial_i\Sigma^B} = (\psi^i_1)^{-1} \circ \tau_i$. Thus $\boldsymbol{\tau} \circ (f')^{-1} = \boldsymbol{\psi}_1$ and

$$P_{\mathrm{DB}}\left([\Sigma^B, f'|_{\Sigma^B}, \Sigma^B_1]\right) = \mathcal{J}\left([\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1]\right) = [\Sigma^P, g, \Sigma^P_1, \boldsymbol{\phi}_1]$$

as required. The first equality follows from the fact that the extension of $f'|_{\Sigma^B}$, in the definition of P_{DB}, is homotopic to f' and thus is also homotopic to $\tilde{f}$. □

As a subgroup of $\mathrm{PModI}(\Sigma^B)$, we know that $\mathrm{DB}(\Sigma^B)$ acts properly discontinuously and fixed-point freely as a group of biholomorphisms on $T^B(\Sigma^B)$ (Lemmas 5.10, 5.11 and 2.19). As in Theorem 5.13, Proposition 2.22 immediately implies the following result.

Corollary 5.21. *The puncture model of rigged Teichmüller space, $\widetilde{T}^P(\Sigma^P)$, inherits an infinite-dimensional complex manifold structure from $T^B(\Sigma^B)$. That is, there is a unique complex structure on $\widetilde{T}^P(\Sigma^P)$ such that P_{DB} is holomorphic. Moreover, P_{DB} possesses local holomorphic sections.*

Lemma 5.22. *The group $\mathrm{PMod}^P(\Sigma^P)$ acts on $\widetilde{T}^P(\Sigma^P)$ by biholomorphisms.*

Proof. Let $[\rho]$ be an element of $\mathrm{PMod}^P(\Sigma^P)$ and recall that the action is defined by $[\rho] \cdot [\Sigma^P, f, \Sigma^P_1, \boldsymbol{\phi}_1] = [\Sigma^P, f \circ \rho, \Sigma^P_1, \boldsymbol{\phi}_1]$. This induces a bijection $\rho_* : \widetilde{T}^P(\Sigma^P) \to \widetilde{T}^P(\Sigma^P)$. The claim is that this map is a biholomorphism. By Corollary 4.17, ρ is isotopic to a map ρ' that is the identity on $\partial\Sigma$. In other words, $[\rho] = [\rho']$ and $\rho'|_{\Sigma^B}$ represents an element of $\mathrm{PModI}(\Sigma^B)$. From Lemma 2.19 we know that $(\rho'|_{\Sigma^B})_* : T^B(\Sigma^B) \to T^B(\Sigma^B)$ is a biholomorphism. Consider the diagram:

$$\begin{tikzcd}
T^B(\Sigma^B) \arrow[r, "(\rho'|_{\Sigma^B})_*"] \arrow[d, "P_{\mathrm{DB}}"'] & T^B(\Sigma^B) \arrow[d, "P_{\mathrm{DB}}"] \\
\widetilde{T}^P(\Sigma^P) \arrow[r, "\rho_*"] & \widetilde{T}^P(\Sigma^P)
\end{tikzcd}$$

The diagram commutes because $\widetilde{f \circ \rho'}$ is isotopic to $\tilde{f} \circ \rho$. Commutativity and the existence of local holomorphic sections of P_{DB} implies that ρ_* is a biholomorphism. □

Define

$$P^{\#}_{\mathrm{DB}} : T^B(\Sigma^B) \longrightarrow \widetilde{T}^B_{\#}(\Sigma^B) \tag{5.8}$$

by $P^{\#}_{\mathrm{DB}}\left([\Sigma^B, f, \Sigma^B_1]\right) = [\Sigma^B, f, \Sigma^B_1, \boldsymbol{\tau} \circ f^{-1}]$. From Theorems 5.19, 5.20 and the properties of the action of $\mathrm{DB}(\Sigma^B)$ we can immediately deduce:

Corollary 5.23. *The map $P^{\#}_{\mathrm{DB}}$ induces a bijection*

$$\frac{T^B(\Sigma^B)}{\mathrm{DB}(\Sigma^B)} \longrightarrow \widetilde{T}^B_{\#}(\Sigma^B)$$

and $\widetilde{T}^B_{\#}(\Sigma^B)$ has a unique complex structure such that $P^{\#}_{\mathrm{DB}}$ is holomorphic. Moreover, $P^{\#}_{\mathrm{DB}}$ possesses local holomorphic sections.

Recall that the subspace $\mathrm{DI}(\Sigma^B)$ of $\mathrm{PModI}(\Sigma^B)$ generated by "internal" Dehn twists (see Definition2.7) acts on $\widetilde{T}^B_{\#}(\Sigma^B)$ by

$$[\rho] \cdot [\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1] = [\Sigma^B, f \circ \rho, \Sigma^B_1, \boldsymbol{\psi}_1]$$

as usual.

Define

$$P_{\mathrm{DI}} : \widetilde{T}^B_{\#}(\Sigma^B) \longrightarrow \widetilde{M}^B(g, n^-, n^+) \tag{5.9}$$

by $P_{\mathrm{DI}}\left([\Sigma^B, f, \Sigma^B_1, \boldsymbol{\psi}_1]\right) = [\Sigma^B_1, \boldsymbol{\psi}_1]$.

Corollary 5.24. *The map P_{DI} induces a bijection*

$$\frac{\widetilde{T}^B_{\#}(\Sigma^B)}{\mathrm{DI}(\Sigma^B)} \longrightarrow \widetilde{M}^B(g, n^-, n^+)$$

and $\widetilde{M}^B(g, n^-, n^+)$ has a unique complex structure such that P_{DI} is holomorphic. Moreover, P_{DI} possesses local holomorphic sections.

Proof. We could directly prove the bijection along similar lines as Theorem 5.12. However we will make use of our previous work. First, observe that

$$\frac{T^B(\Sigma^B)}{\mathrm{PModI}(\Sigma^B)} \cong \frac{T^B(\Sigma^B)/\mathrm{DB}(\Sigma^B)}{\mathrm{PModI}(\Sigma^B)/\mathrm{DB}(\Sigma^B)} \cong \frac{T^B(\Sigma^B)/\mathrm{DB}(\Sigma^B)}{\mathrm{DI}(\Sigma^B)}$$

by Corollary 2.10. The required isomorphism now follows from Theorem 5.12 and Corollary 5.23. The action of $\mathrm{DI}(\Sigma^B)$ is proper discontinuous and fixed-point free by Lemmas 5.10 and 5.11, Theorem 5.12 and Corollary 5.23. The result now follows from Lemma 2.19 and Proposition 2.22. □

5.5. **Assembly of results: the big picture.** In this section we provide a conceptually satisfying commutative diagram and a slightly informal theorem which together summarize many of the results in this paper. Recall that Σ^B is a bordered surface and Σ^P is the corresponding punctured surface obtained by sewing on caps as described in "Border model", Section 5.1.

Consider the commutative diagram:

$$
\begin{tikzcd}
 & T^B(\Sigma^B) \arrow[dl, "P_{\mathrm{DB}}^{\#}"'] \arrow[dr, "P_{\mathrm{DB}}"] & \\
\widetilde{T}^B_{\#}(\Sigma^B) \arrow[rr, "\cong"] \arrow[d, "P_{\mathrm{DI}}"'] & & \widetilde{T}^P(\Sigma^P) \arrow[d, "P_{\mathrm{mod}}"] \\
\widetilde{\mathcal{M}}^B(g,n^-,n^+) \arrow[rr, "\cong"] & & \widetilde{\mathcal{M}}^P(g,n^-,n^+).
\end{tikzcd}
\tag{5.10}
$$

The horizontal isomorphisms are given in Theorems 5.9 and 5.19. The projection maps are defined in (5.3), (5.4), (5.7), (5.8) and (5.9). Checking commutativity is routine.

Producing such a diagram was one of the goals of this project. It gives a full relation between the puncture and and border picture at both the moduli space and Teichmüller space levels.

Before formulating a concluding theorem we need to show that P_{mod} is holomorphic.

Lemma 5.25. *The action of* $\mathrm{PMod}^P(\Sigma^P)$ *on* $\widetilde{T}^P(\Sigma^P)$ *is properly discontinuous and fixed-point free. The projection* P_{mod} *is holomorphic and possesses local holomorphic sections.*

Proof. The bijection $\mathcal{J} : \widetilde{T}^B_{\#}(\Sigma^B) \to \widetilde{T}^P(\Sigma^P)$ from Theorem 5.19 is a biholomorphism by the commutativity of the top triangle in Diagram (5.10) and the existence of local holomorphic sections of P_{DB} and $P_{\mathrm{DB}}^{\#}$ (Corollaries 5.21 and 5.23). The actions of DI on $\widetilde{T}^B_{\#}(\Sigma^B)$ and $\mathrm{PMod}^P(\Sigma^P)$ on $\widetilde{T}^P(\Sigma^P)$ can be seen to be equivalent by directly using the definitions of the actions and the isomorphism $\mathcal{J}$. The required properties of the action now follow from Corollary 5.24 and its proof. In Lemma 5.22 we showed that the action of $\mathrm{PMod}^P(\Sigma^P)$ is by biholomorphisms. Thus Proposition 2.22 guarantees the stated properties of P_{mod}. □

Theorem 5.26 (Summary of results)**.**

(1) *All the spaces in Diagram* (5.10) *are obtained from* $T^B(\Sigma^B)$ *by quotienting by the action of the mapping class group and certain subgroups. (Proposition 5.17, Theorem 5.20 and Corollaries 5.23 and 5.24.)*
(2) *These actions are by biholomorphisms and are properly discontinuous and fixed-point free. (Lemmas 2.19, 2.21, 5.10, 5.11 and 5.22 and Lemma 5.25.)*
(3) *With the complex structures inherited from* $T^B(\Sigma^B)$*, all the spaces in Diagram* (5.10) *become complex Banach manifolds. (Proposition 2.22, Theorem 5.13, Corollaries 5.21, 5.23 and 5.24 and Lemma 5.25.)*
(4) *These complex structures are the unique ones that make all the maps holomorphic. All the maps possess local holomorphic sections. (Proposition 2.22.)*
(5) *The horizontal bijections become biholomorphisms. (Commutativity of the diagram and existence of local holomorphic section)*
(6) *The complex structures on the moduli spaces are independent of the choice of base surface* Σ^B *and its boundary trivialization* $\boldsymbol{\tau}$*. (Theorem 5.14.)*

6. Holomorphicity of Sewing

As discussed in Section 1, the sewing operation is the fundamental geometric operation in conformal field theory. Holomorphicity of this operation is required in the case of chiral CFTs, or more formally, in the definition of a weakly conformal field theory. In Section 3 the sewing operation was defined for the case of quasisymmetric boundary parametrizations. We now express this operation as a map between both the rigged moduli and Teichmüller spaces. These sewing maps are shown to be holomorphic.

One advantage of working with quasisymmetric maps is the conceptually satisfying way in which the sewing maps can be defined and the holomorphicity proved.

Recall from Section 2.2 that for a bordered Riemann surface, Σ^B, the Teichmüller space $T^B(\Sigma^B)$ is endowed with the standard complex structure through the use of the Bers embedding (see [38, Chapter 3] or [32, V.5.]). With this structure the fundamental projection $\Phi_{\Sigma^B} : L^\infty_{(-1,1)}(\Sigma^B)_1 \to T^B(\Sigma^B)$ is holomorphic, and has a local holomorphic section in a neighborhood of every point.

Note that the projection $P_{\Sigma^B} : T^B(\Sigma^B) \to \widetilde{\mathcal{M}}(g, n^-.n^+)$ from (5.3) can be defined by first projecting

$$T^B(\Sigma^B) \to T^B(\Sigma^B)/\mathrm{PModI}(\Sigma^B)$$

and following with the isomorphism $[\Sigma^B, f, \Sigma_1^B] \mapsto [\Sigma_1^B, \boldsymbol{\tau} \circ f^{-1}]$ of Theorem 5.12. It can be checked directly that $P_{\Sigma^B} = P_{\mathrm{DI}} \circ P_{\mathrm{DB}\#}$. From Theorem 5.13 we know that the projection P_{Σ^B} is holomorphic and has local holomorphic sections near every point. Actually this projection was used to induce the complex structure on $\widetilde{\mathcal{M}}(g, n^-.n^+)$.

We now recall the sewing operation described in Section 3.1. To avoid excessive decorations, we change notation and let X and Y be bordered Riemann surfaces of type (g_X, n_X^-, n_X^+) and (g_Y, n_Y^-, n_Y^+) respectively where $n_X^+ > 0$ and $n_Y^- > 0$. Let $\boldsymbol{\tau}_X = (\tau_X^1, \ldots \tau_X^{n_X})$ and $\boldsymbol{\tau}_Y = (\tau_Y^1, \ldots, \tau_Y^{n_Y})$ be (quasisymmetric) boundary trivializations of X and Y respectively (see Section 3.2). Choose i and j such that $\partial_i X$ is an outgoing boundary component and $\partial_j Y$ is an incoming boundary component. Let $X \#_{ij} Y$ be the sewn surface obtained by identifying $\partial_i X$ with $\partial_j Y$ using $(\tau_Y^j)^{-1} \circ J \circ \tau_X^i$. Since the choice of i and j is fixed throughout we simply write $X \# Y$ for $X \#_{ij} Y$. Let $\iota_X : X \to X\#Y$ and $\iota_Y : Y \to X\#Y$ be the inclusion maps. Let $g_{X\#Y} = g_X + g_Y$, $n^-_{X\#Y} = n_X^- + n_Y^- - 1$ and $n^+_{X\#Y} = n_X^+ + n_Y^+ - 1$. The Riemann surface $X\#Y$ of type $(g_{X\#Y}, n^-_{X\#Y}, n^+_{X\#Y})$ with boundary trivialization

$$\boldsymbol{\tau}_{X\#Y} = (\tau_X^1, \ldots, \tau_X^{i-1}, \tau_Y^1, \ldots, \tau_Y^{j-1}, \tau_Y^{j+1}, \ldots, \tau_Y^{n_Y}, \tau_X^{i+1}, \ldots \tau_X^{n_X}) \tag{6.1}$$

will be considered as the base surface for the Teichmüller space $T^B(X\#Y)$. There are other ways of ordering the boundary components but this issue is not important for our purposes.

Remark 6.1. In conformal field theory the self-sewing operation must be considered. That is, the sewing of two boundary components of a single surface. Everything in this section can be altered without difficulty to cover this situation.

We describe three sewing maps: on the level of Beltrami differentials, the level of Teichmüller space, and the level of rigged moduli space.

- $\mathcal{S} : L^\infty_{(-1,1)}(X)_1 \times L^\infty_{(-1,1)}(Y)_1 \to L^\infty_{(-1,1)}(X\#Y)_1$ is defined by $(\mu, \nu) \mapsto \mu \cup \nu$ where

$$(\mu \cup \nu)(p) = \begin{cases} \mu(p) & \text{if } p \in \iota_X(X) \\ \nu(p) & \text{if } p \in \iota_Y(Y) \end{cases}$$

The values of $\mu \cup \nu$ on the seam of $X\#Y$ are not important as it is a set of measure zero.

- $\mathcal{S}_T : T^B(X) \times T^B(Y) \to T^B(X\#Y)$ is defined by
$$([X, f, X_1], [Y, g, Y_1]) \mapsto [X\#Y, f \cup g, X_1\#Y_1]$$
where
$$(f \cup g)(p) = \begin{cases} f(p) & \text{if } p \in \iota_X(X) \\ g(p) & \text{if } p \in \iota_Y(Y) \end{cases}$$
and X_1 and Y_1 are sewn using the boundary parametrizations $\tau_X^i \circ f^{-1}$ and $\tau_Y^j \circ g^{-1}$. By the definition of the sewing operation, the topologies on $X\#Y$ and $X_1\#Y_1$ are such that $f \cup g$ is automatically a homeomorphism. Since it is quasiconformal on $\iota_X(X)$ and $\iota_Y(Y)$, Theorem 2.39 guarantees it is quasiconformal on $X\#Y$, by an identical argument to the one preceding Remark 5.8. It is straightforward to check that $\mathcal{S}_T$ is well defined. For example, if $[X, f, X_1] = [X, f', X_1']$ via the biholomorphism $\sigma : X_1 \to X_1'$, then $\sigma_\# : X_1\#Y_1 \to X_1'\#Y_1$ defined by
$$\sigma_\# = \begin{cases} \sigma & \text{on } X_1 \\ \mathrm{id} & \text{on } Y_1 \end{cases}$$
gives the equivalence between $(X\#Y, f \cup g, X_1\#Y_1)$ and $(X\#Y, f' \cup g, X_1'\#Y_1)$.
- $\mathcal{S}_M : \widetilde{M}^B(g_X, n_X^-, n_X^+) \times \widetilde{M}^B(g_Y, n_Y^-, n_Y^+) \to \widetilde{M}^B(g_{X\#Y}, n_{X\#Y}^-, n_{X\#Y}^+)$ is defined by
$$\big([X_1, \boldsymbol{\psi}_{X_1}], [Y_1, \boldsymbol{\psi}_{X_2}]\big) \mapsto [X_1\#Y_1, \boldsymbol{\psi}]$$
where $\boldsymbol{\psi}$ is the parametrization of the remaining boundary components which are ordered in a way analogous to (6.1). (To be more precise we should write $\mathcal{S}_M^{ij}$ where i and j label the boundary components that are sewed.) It is easy to check that $\mathcal{S}_M$ is well defined by using maps such as $\sigma_\#$.

Remark 6.2. The spaces and maps $L^\infty_{(-1,1)}(X\#Y)$, $T^B(X\#Y)$, $\mathcal{S}$ and $\mathcal{S}_Y$ depend on the choice of boundary trivializations $\boldsymbol{\tau}_X$ and $\boldsymbol{\tau}_Y$. On the other hand $\mathcal{S}_M$ is independent of $\boldsymbol{\tau}_X$ and $\boldsymbol{\tau}_Y$.

Remark 6.3. Being able to sew with quasisymmetric boundary identification is crucial to defining $\mathcal{S}_T$. In the analytic case this is not possible, because even if $\boldsymbol{\tau}_X$ and $\boldsymbol{\tau}_Y$ are chosen to be analytic, there is no natural way to sew $[X, f, X_1]$ and $[Y, g, Y_1]$. This is because the maps $\boldsymbol{\tau}_X \circ f^{-1}$ and $\boldsymbol{\tau}_Y \circ g^{-1}$ are only quasisymmetric.

Consider the following diagram which relates the three sewing operations.

$$\begin{tikzcd} L^\infty_{(-1,1)}(X)_1 \times L^\infty_{(-1,1)}(Y)_1 \arrow[r, "\mathcal{S}"] \arrow[d, "(\Phi_X, \Phi_Y)"'] & L^\infty_{(-1,1)}(X\#Y) \arrow[d, "\Phi_{X\#Y}"] \\ T^B(X) \times T^B(Y) \arrow[r, "\mathcal{S}_T"] \arrow[d, "(P_X, P_Y)"'] & T^B(X\#Y) \arrow[d, "P_{X\#Y}"] \\ \widetilde{\mathcal{M}}^B(g_X, n_X^-, n_X^+) \times \widetilde{\mathcal{M}}^B(g_Y, n_Y^-, n_Y^+) \arrow[r, "\mathcal{S}_M"] & \widetilde{\mathcal{M}}^B(g_{X\#Y}, n_{X\#Y}^-, n_{X\#Y}^+) \end{tikzcd} \tag{6.2}$$

Lemma 6.4. *Diagram* (6.2) *commutes.*

Proof. The commutativity of the upper rectangle is immediate, since if $\mu = \mu(f)$ and $\nu = \mu(g)$ then $\mu \cup \nu = \mu(f \cup g)$.

For the lower rectangle, we take an element $([X, f, X_1], [Y, g, Y_1])$ of $T^B(X) \times T^B(Y)$ and let $\boldsymbol{\psi}_{X_1} = \boldsymbol{\tau}_X \circ f^{-1}$ and $\boldsymbol{\psi}_{X_2} = \boldsymbol{\tau}_X \circ f^{-1}$. Going clockwise, the image of $([X, f, X_1], [Y, g, Y_1])$ under $P_{X\#Y} \circ \mathcal{S}_T$ is $[X_1\#X_2, \boldsymbol{\tau}_{X\#Y} \circ (f \cup g)^{-1}]$ where the sewing is performed using the parametrizations $\tau^i_X \circ f^{-1}$ and $\tau^j_Y \circ g^{-1}$. Going anti-clockwise the image of $([X, f, X_1], [Y, g, Y_1])$ under $\mathcal{S}_M \circ (P_X, P_Y)$ is $[X_1\#X_2, \boldsymbol{\psi}]$, where $\boldsymbol{\psi}$ is formed from $\boldsymbol{\psi}_{X_1}$ and $\boldsymbol{\psi}_{X_2}$ with the appropriate ordering of the remaining boundary components. The sewing is performed using the parametrizations $\psi^i_{X_1}$ and $\psi^j_{X_2}$. A direct check using the definitions of $\boldsymbol{\tau}_{X\#Y}$, $f \cup g$ and $\boldsymbol{\psi}$ shows that the clockwise and anti-clockwise images are identical. □

To show holomorphicity of the sewing maps we need the following general result. See for example Lehto [32, page 206] or Nag [38, page 87].

Lemma 6.5. *Let E and F be complex Banach spaces and let U be an open subset of E. Let F^* be the (complex) dual space. A function $f : U \to F$ is holomorphic if it is continuous and for every $\alpha \in F^*$ and every $(x, e) \in U \times E$, the function $t \mapsto \alpha \circ f(x + te)$ is a holomorphic function in some neighborhood of the origin in $\mathbb{C}$.*

Lemma 6.6. *The sewing map $\mathcal{S}$ is holomorphic map.*

Proof. The directional derivatives can be computed directly but it is easier to use Lemma 6.5. Continuity is immediate because if $||\mu_1 - \mu_2||_\infty < \epsilon$, then $||\mu_1 \cup \nu - \mu_2 \cup \nu||_\infty < \epsilon$ and similarly for ν. For arbitrary (μ, ν) and (λ_1, λ_2) in $L^\infty_{(-1,1)}(X)_1 \times L^\infty_{(-1,1)}(Y)_1$,

$$(\alpha \circ \mathcal{S})((\mu, \nu) + t(\lambda_1, \lambda_2)) = \alpha(\mu \cup \nu) + t\alpha(\lambda_1 \cup \lambda_2).$$

So clearly $t \mapsto (\alpha \circ \mathcal{S})((\mu, \nu) + t(\lambda_1, \lambda_2))$ is holomorphic in t. □

Theorem 6.7. *The sewing maps $\mathcal{S}_T$ and $\mathcal{S}_M$ are holomorphic.*

Proof. Given any point (P, Q) in $T^B(X) \times T^B(Y)$ let σ_X and σ_Y be local holomorphic sections of Φ_X and Φ_Y near p and q respectively. Their existence is guaranteed by Theorem 2.17. It follows that $\sigma = (\sigma_X, \sigma_Y)$ is a local holomorphic section of $(\Phi_X, \Phi_Y) : L^\infty_{(-1,1)}(X)_1 \times L^\infty_{(-1,1)}(Y)_1 \longrightarrow T^B(X) \times T^B(Y)$. Since Diagram (6.2) commutes we have that $\mathcal{S}_T = \Phi_{X\#Y} \circ \mathcal{S} \circ \sigma$ and so, by Lemma 6.6, $\mathcal{S}_T$ is holomorphic.

Similarly let $\rho = (\rho_X, \rho_Y) : \widetilde{\mathcal{M}}^B(g_X, n^-_X, n^+_X) \times \widetilde{\mathcal{M}}^B(g_Y, n^-_Y, n^+_Y) \to T^B(X) \times T^B(Y)$ be a local holomorphic section in the neighborhood of any point in $\widetilde{\mathcal{M}}^B(g_X, n^-_X, n^+_X) \times \widetilde{\mathcal{M}}^B(g_Y, n^-_Y, n^+_Y)$. Then $\mathcal{S}_M = P_{X\#Y} \circ \mathcal{S}_T \circ \rho$ is holomorphic. □

Sewing on caps (as in Section 3.2) is a special case of the sewing operation. This results in a holomorphic map $T^B(\Sigma^B) \to T^P(\Sigma^P)$, and in particular we obtain the following.

Corollary 6.8. *The map*

$$\begin{aligned} \mathcal{C} : T^B(\Sigma^B) &\longrightarrow T^P(\Sigma^P) \\ [\Sigma^B, f, \Sigma^B_1] &\longmapsto [\Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n, f \cup id, \Sigma^B_1 \#_{\boldsymbol{\tau} \circ f^{-1}} (\overline{\Delta}_0)^n] \end{aligned}$$

is holomorphic.

Proof. The sewing map $\mathcal{S}_T : T^B(\Sigma^B) \times (T^B(\overline{\Delta}_0))^n \to T^B(\Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n)$ is holomorphic. Note that $T^B(\Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n)$ is the finite-dimensional Teichmüller space $T^P(\Sigma^P)$. To get the desired map, we fix the second entry of the sewing map to be the 'identity', $([\overline{\Delta}_0, \mathrm{id}, \overline{\Delta}_0])^n$. Note that the map depends on the choice of $\boldsymbol{\tau}$. $\square$

Not surprisingly this result enables us to show the compatibility of the complex structures on $\widetilde{T}^P(\Sigma^P)$ and the usual Teichmüller space $T^P(\Sigma^P)$. Consider the following diagram whose left-hand side is the right-hand side of Diagram (5.10).

(6.3)
$$\begin{tikzcd}
T^B(\Sigma^B) \arrow[dr, "P_{\mathrm{DB}}"'] \arrow[drr, bend left, "\mathcal{C}"] & & \\
& \widetilde{T}^P(\Sigma^P) \arrow[r, "\mathcal{F}_T"] \arrow[d, "P_{\mathrm{mod}}"'] & T^P(\Sigma^P) \arrow[d, "P_{\mathrm{mod}}"] \\
& \widetilde{\mathcal{M}}^P(g, n^-, n^+) \arrow[r, "\mathcal{F}_M"] & \mathcal{M}^P(g, n^-, n^+).
\end{tikzcd}$$

The horizontal maps just forget the rigging. That is,

$$\mathcal{F}_T([\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi}_1]) = [\Sigma^P, f, \Sigma_1^P] \quad \text{and} \quad \mathcal{F}_M([\Sigma_1^P, \boldsymbol{\phi}_1]) = [\Sigma_1^P].$$

Commutativity of the diagram can be checked directly.

Corollary 6.9. *The map $\mathcal{F}_T : \widetilde{T}^P(\Sigma^P) \to T^P(\Sigma^P)$ is holomorphic.*

Proof. We know that $\mathcal{C}$ is holomorphic and that P_{DB} possesses local holomorphic sections (Corollary 5.21). By commutativity of the diagram, $\mathcal{F}_T$ can locally be expressed as a composition of these maps, and is thus holomorphic. $\square$

Remark 6.10. The same arguments apply to $\mathcal{F}_M$, but one must be careful as $\mathcal{M}(g, n^-.n^+)$ is not a complex manifold. This is because the action of $\mathrm{PMod}^P(\Sigma^P)$ on $T^P(\Sigma^P)$ is not fixed-point free.

7. Local structure of rigged Teichmüller space

Although we have given a complex manifold structure to the rigged Teichmüller and Riemann Moduli spaces we have not described their local structure. We will focus on $T^B(\Sigma^B)$ and $\widetilde{T}^P(\Sigma^P)$.

From Corollary 6.9 we have the holomorphic map $\mathcal{F}_T : \widetilde{T}^P(\Sigma^P) \to T^P(\Sigma^P)$. The inverse image of a point is isomorphic to the space of local coordinates. That is

$$\mathcal{F}_T^{-1}([\Sigma^P, f, \Sigma_1^P]) = \left\{ [\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi}] \,|\, \boldsymbol{\phi} \in \mathcal{O}^{\Delta}_{\mathrm{qc}}(\mathbf{p}_1) \right\}$$

where $\mathbf{p}_1$ is the list of punctures on Σ_1^P. The goal is to show that every point $[\Sigma^P, f, \Sigma_1^P, \boldsymbol{\phi}_0] \in \widetilde{T}^P(\Sigma^P)$ has a neighborhood of the form $\Omega \times \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of $\boldsymbol{\phi}_{\mathbf{0}} \in \mathcal{O}^{\Delta}_{\mathrm{qc}}$, and Ω is a neighborhood of $[\Sigma^P, f, \Sigma_1^P]$ in the finite-dimensional Teichmüller space $T^P(\Sigma^P)$.

To reach this goal we must show the space $\mathcal{O}^{\Delta}_{\mathrm{qc}}$ is a complex manifold. How the fibers depend on the Riemann surface must also be understood. We intend to address these problems in a future publication.

However, a key result will be proved in Section 7.2: namely, if a family of riggings depends holomorphically on a parameter, then the corresponding family of Teichmüller space elements is holomorphic. This is the content of Corollary 7.7. The same result for $\widetilde{T}^P(\Sigma^P)$ then immediately follows, although for length considerations we have not presented all the details. This appears in Corollary 7.8.

Once these results are established we sketch, in Section 7.3, the relation of the present results to the standard approach to CFT using analytic riggings.

7.1. **Holomorphic motion of an annulus.** We first state a crucial lemma.

Let γ_1 and γ_2 be Jordan curves such that γ_1 is contained in the interior of γ_2. Consider a holomorphic motion β_t of γ_1 (see Definition 2.25), such that $\beta_t(\gamma_1) \cap \gamma_2 = \emptyset$ for all $t \in \Delta$.

Let $A = \gamma_1 \cup \gamma_2$ and define $h_t : A \to \mathbb{C}$ by

$$h_t(z) = \begin{cases} \beta_t(z) & \text{for } z \in \gamma_1 \\ z & \text{for } z \in \gamma_2. \end{cases}$$

It follows directly that h_t is a holomorphic motion of A.

Let $\mathbb{A}^{\gamma_2}_{\gamma_1}$ be the annulus bounded by γ_1 and γ_2. Applying the extended λ-lemma (Theorem 2.27) we obtain the following.

Lemma 7.1. *If γ_1, γ_2 and β_t are given as above, then there exists a holomorphic motion H_t of $\mathbb{A}^{\gamma_2}_{\gamma_1}$ having the properties guaranteed by Theorem 2.26. In particular, $H_t|_{\gamma_1} = \beta_t$ and $H_t|_{\gamma_2}$ is the identity.*

Remark 7.2. Although the proof of this lemma is simple it truly relies on the power of the extended λ-lemma. Moreover this lemma in one of the main technical results needed in the proof of the analyticity of the sewing operation in [46].

7.2. **Holomorphic family of surfaces.** We use Lemma 7.1 to produce a holomorphic family of surfaces obtained by cutting out holomorphically varying disks. The basic idea is to use the quasiconformal map H_t between annuli to produce a quasiconformal map between Riemann surfaces.

A family of surfaces is produced in the following way. Assume for simplicity that Σ^P has a single puncture p. Let t be a complex parameter and let ϕ_t be a family of local coordinates in $\mathcal{O}^{\Delta}_{\mathrm{qc}}(p)$ such that for fixed z, $t \mapsto \phi_t(z)$ is a holomorphic function of t. We say that ϕ_t is a holomorphic family of local coordinates.

Our aim is to show that $t \mapsto [\Sigma^P, \mathrm{id}, \Sigma^P, \phi_t]$ is a holomorphic map from a neighborhood of $0 \in \mathbb{C}$ to $\widetilde{T}^P(\Sigma^P)$. We do this by finding a suitable holomorphic family of elements in $T^B(\Sigma^B)$ where $\Sigma^B = \Sigma^P \setminus \phi_0^{-1}(\Delta)$.

Let $D_t = \phi_t^{-1}(\Delta)$, and $\gamma_t = \phi_t^{-1}(S^1)$. Consider the bordered Riemann surfaces $\Sigma^B_t = \Sigma^P \setminus D_t$ with (analytic) boundary parametrizations given by ϕ_t. Note that here we allow for a boundary parametrization to be orientation reversing.

By the definition of $\mathcal{O}^{\Delta}_{\mathrm{qc}}(p)$, there exists $r > 1$ such that ϕ_0^{-1} is quasiconformal on $B(0, r)$. Let $U = \phi^{-1}(B(0, r))$ and choose a biholomorphic map $G : U \to \mathbb{C}$. For $|t|$ sufficiently small, D_t in contained in U. Let A_t be the annular region on Σ^P bounded by ∂U and γ_t.

The map $\beta_t = G \circ \phi_t^{-1} \circ \phi_0 \circ G^{-1}|_{G(\gamma_0)}$ is a holomorphic motion of $G(\gamma_0)$. Applying Lemma 7.1 we get a holomorphic motion H_t of $G(A_0)$ such that $H_t|_{G(\gamma_0)} = \beta_t$ and $H_t|_{G(\partial U)}$ is the identity.

Proposition 7.3. *For $|t|$ sufficiently small, the map $F_t : \Sigma_0^B \to \Sigma_t^B$ defined by*

$$F_t = \begin{cases} \mathrm{id} & \text{on} \quad \Sigma^P \setminus A_0 \\ G^{-1} \circ H_t \circ G & \text{on} \quad A_0 \end{cases}$$

is quasiconformal and is holomorphic in t.

Remark 7.4. As Σ_0^B and Σ_t^B are subsets of Σ^P, it makes sense to talk about the identity map as well as holomorphicity in t.

Proof of Proposition 7.3. We first show that F_t is well defined. For $x \in \partial U$, the fact that H_t is the identity on $G(\partial U)$ implies that $G^{-1} \circ H_t \circ G(x) = x$. Because $H_t(z)$ is analytic in t for each fixed z, and the other maps are independent of t, we see that $F_t(z)$ is analytic in t. The map $F_t(z)$ is quasiconformal on $\Sigma_0^B \setminus \partial U$ because it is defined by a composition of conformal and quasiconformal maps. Theorem 2.39 guarantees that F_t is quasiconformal on ∂U. □

Some standard facts about the complex dilation of a composition of maps lead to the following.

Lemma 7.5. *Let $\{w_t\}_{t\in\Delta}$ be a family of quasiconformal homeomorphisms of $\mathbb{C}$. If $t \mapsto \mu(w_t(z))$ is holomorphic and $f : \mathbb{C} \to \mathbb{C}$ is quasiconformal then the map $\Delta \to L^\infty_{(-1,1)}(\mathbb{C})_1$ given by $t \mapsto \mu(w_t(f(z)))$ is holomorphic.*

Proposition 7.6. *For $|t|$ sufficiently small, the complex dilation $\mu(F_t)$ of F_t is holomorphic in t. That is, the map $t \mapsto \mu(F_t)$ is holomorphic.*

Proof. By Theorem 2.28, $\mu(H_t)$ is holomorphic in t. Apply Lemma 7.5 to $\mu(G^{-1} \circ H_t \circ G) = \mu(H_t \circ G)$. □

From the holomorphicity of the fundamental projection (see 2.16) we get the corresponding result for Teichmüller space.

Corollary 7.7. *For $|t|$ sufficiently small, the map $t \mapsto [\Sigma^B, F_t, \Sigma_t^B]$ is holomorphic.*

Consider the base surface $\Sigma^B = \Sigma_0^B$ whose single boundary component is parametrized by ϕ_0. With some work it can be checked directly that the holomorphic projection $P_{\mathrm{DB}} : T^B(\Sigma^B) \to \widetilde{T}^P(\Sigma^P)$ defined in (5.3) sends $[\Sigma^B, F_t, \Sigma_t^B]$ to $[\Sigma^P, \mathrm{id}, \Sigma^P, \phi_t]$. To see this, a change of base point must be used along with the fact that the extension of F_t to the punctured surface is homotopic to the identity. Corollary 7.7 now immediately gives the following.

Corollary 7.8. *For $|t|$ sufficiently small, the map $t \mapsto [\Sigma^P, \mathrm{id}, \Sigma^P, \phi_t]$ is holomorphic.*

We briefly recap the results of this section. Given a holomorphic family of local coordinate ϕ_t, define a family of surfaces $\Sigma_t^B = \Sigma^B \setminus \phi_t^{-1}(\Delta)$. Using the extended λ-lemma, quasiconformal maps $F_t : \Sigma^B \to \Sigma_t^B$ are obtained. The family of Teichmüller space elements $[\Sigma^B, F_t, \Sigma_t^B]$ is a holomorphic curve in $T^B(\Sigma^B)$. This holomorphic family projects to a holomorphic family $[\Sigma^P, \mathrm{id}, \Sigma^P, \phi_t]$ in $\widetilde{T}^P(\Sigma^P)$.

7.3. **Relation to analytic rigging.** In the standard approach to conformal field theory (as defined by Segal in [47]), the boundary components of the Riemann surfaces are parametrized with analytic maps, which extend to biholomorphic maps of a collared neighborhood of the boundary. In the puncture model, the equivalent picture is given by rigging the punctured surfaces with analytic local coordinates. We denote the corresponding rigged Teichmüller space by $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$. The complex structure of this space and the rigged moduli space are known. It was worked out in detail in the genus-zero case in [22] and in the higher-genus case in [46]. In this section we outline the compatibility between those complex structures and the one in the current paper. The precise statement is the following.

Claim: *The inclusion map*

$$\mathrm{inc} : \widetilde{T}^P_{\mathcal{O}}(\Sigma^P) \longrightarrow \widetilde{T}^P(\Sigma^P)$$

is holomorphic.

It would take significant preparation to properly define the complex structure on $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$, so this section is not as detailed as the previous ones. Moreover, the full details of the relationship we explore here will be included in a future publication.

The compatibility hinges on showing that cutting out varying disks using a holomorphic family of (analytic) local coordinates gives a holomorphic family in Teichmüller space. This is a special case of Corollary 7.8.

We now briefly describe (following [46]), the complex manifold structure on the Teichmüller space of analytically rigged Riemann surfaces.

Let $\mathcal{O}$ be the complex vector space of all formal series of the form $\sum_{i=1}^{\infty} a_n z^n$, which are absolutely convergent in some neighborhood of $0 \in \mathbb{C}$. Let H_Δ be the subspace of $\mathcal{O}$ consisting of series that have a radius of convergence strictly greater than one. Let L_Δ be the subspace of H_Δ consisting of functions that are one-to-one. These spaces of germs of holomorphic functions can be described as (LB)-spaces, that is, as inductive limits of complex Banach spaces. The theory of holomorphic maps on such spaces closely follows that of Banach spaces. See for example [30].

We refer the reader to Section 5.1 for notation and related ideas. In analogy with the local coordinates $\mathcal{O}^\Delta_{qc}(q)$ from Definition 5.1, we define

$$\mathcal{O}^\Delta(q) = \{\phi \in \mathcal{O}(q) \,|\, \Delta \subset \mathrm{Im}(\phi) \text{ , } \phi^{-1} \text{ biholomorphic on a neighborhood of } \Delta\}.$$

Let $\mathcal{O}^\Delta(\mathbf{p})$ be the space of local coordinates corresponding to $\mathcal{O}^\Delta_{qc}(\mathbf{p})$ in Definition 5.1. It can be shown that $\mathcal{O}^\Delta(q)$ and $\mathcal{O}^\Delta(\mathbf{p})$ are complex (LB)-manifolds modelled on L_Δ and $(L_\Delta)^n$ respectively.

An analytically rigged surface is a pair $(\Sigma^P, \boldsymbol{\phi})$ where Σ^P is of type (g, n^-, n^+), and $\boldsymbol{\phi} \in \mathcal{O}^\Delta(\mathbf{p})$ is the set of local coordinates at the punctures $\mathbf{p} = (p_1, \ldots, p_n)$. The corresponding rigged Moduli space and rigged Teichmuller space are defined exactly as in Definitions 5.3 and 5.16. We use the notation $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$ to distinguish from the earlier case.

In $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$, Schiffer variation of complex structure can be used to separate the "Teichmüller space part" from the "local coordinate part". More precisely, $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$ is a complex manifold which, in a neighborhood of $[\Sigma^P, f, \Sigma^P_1, \boldsymbol{\phi}]$, has charts of the form

$$S : \Omega \times \mathcal{U} \longrightarrow \widetilde{T}^P_{\mathcal{O}}(\Sigma^P) \tag{7.1}$$

where $\mathcal{U}$ is a neighborhood of $\boldsymbol{\phi} \in \mathcal{O}^\Delta(\mathbf{p}_1)$ and Ω is a neighborhood of $[\Sigma^P, f, \Sigma^P_1]$ in $T^P(\Sigma^P)$. Recall that $T^P(\Sigma^P)$ is a complex manifold of dimension $3g - 3 + n$.

The initial claim of compatibility of the complex structures on $\widetilde{T}^P_{\mathcal{O}}(\Sigma^P)$ and $\widetilde{T}^P(\Sigma^P)$ therefore reduces essentially to the compatibility of $\mathcal{O}^\Delta(\mathbf{p})$ and $\widetilde{T}^P(\Sigma^P)$. A key step in proving this compatibility is Corollary 7.8. In the current setting this corollary shows that a holomorphic curve in $\mathcal{O}^\Delta(\mathbf{p})$ maps to a holomorphic curve in $\widetilde{T}^P(\Sigma^P)$.

8. Concluding remarks

We conclude with some observations.

The first observation is regarding a possible further application of these results to conformal field theory. It is well known that the Teichmüller space $T^B(\Sigma^B)$ is contained in the universal Teichmüller space $T(1) = T(\Delta)$. Thus a representative of every possible topological type (g, n) is contained in $T(1)$. It was conjectured by Pekonen [43] (possibly following Nag) that the universal Teichmüller space is the proper arena for the path integral formulation of free bosonic string theory, and might be the basis of a non-perturbative formulation. In other words, the 'sum over topologies' could be accomplished by using the universal Teichmüller space (or perhaps some suitable subspace) as the space of all paths. The results of the present paper suggest that this may be correct.

On the other hand, the results of the present paper appear to be an application of Segal's definition of conformal field theory [47] to understanding the Teichmüller space of a bordered Riemann surface. First, we have shown that T^B is 'almost' $\tilde{T}^P$ (that is, up to the action of DB.) Second, we have provided two intermediate spaces between $\tilde{T}^P$ and $\mathcal{M}^P$, namely $\tilde{\mathcal{M}}^P$ and T^P (see Diagram (6.3)).

This can be interpreted in the following way. Given a bordered Riemann surface Σ^B, we want to understand its Teichmüller space by looking at the compactified surface Σ^P. Again ignoring the action of DB, we see that T^B contains the following extra information not contained in $\mathcal{M}^P$: the riggings, which we add in the horizontal direction of Diagram (6.3), and the markings, which we add in the vertical direction of Diagram (6.3). In some sense, the riggings can be regarded as 'external' information in that they specify how Σ^B sits inside a compact Riemann surface. The markings can be regarded then as containing 'internal' information.

Finally, we have shown that T^B/DB is fibred over T^P with fibres $\mathcal{O}^\Delta_{qc}$, and that the projection is holomorphic (Corollary 6.9). It is thus of interest to describe the complex structure of the fibres $\mathcal{O}^\Delta_{\mathrm{qc}}$ and to show that T^B is a holomorphic fiber space which is locally biholomorphic to an open subset of $T^P \times \mathcal{O}^\Delta_{\mathrm{qc}}$. We hope to pursue this point in a future publication.

9. Notation

Basic notation:

- Σ^B - Bordered Riemann surface of finite topological type.
- Σ^P - Riemann surface with punctures or marked points.
- S_r- circle or radius r (in $\mathbb{C}$).
- $B(0, r)$ - disk radius r (in $\mathbb{C}$)
- $\mathbb{A}^{r_2}_{r_1}$ - Standard annulus in $\mathbb{C}$ bounded by S_{r_1} and S_{r_2}.
- $\mathbb{A}_C$ - An annular neighborhood of a boundary component of Σ^B.
- S^1 - unit circle.
- Δ - Open unit disk.

- $\overline{\Delta}_0 = \overline{\Delta} \setminus \{0\}$ - punctured closed unit disk.
- $\hat{\mathbb{C}}$ - Riemann sphere.
- $J : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined by $J(z) = 1/z$.

General setup:

- Σ^B is the base surface with n boundary components.
- $\partial\Sigma^B = \bigcup_{i=1}^n \partial_i\Sigma^B$ where $\partial_i\Sigma^B$ are the *ordered* boundary components. Each component is specified as *incoming* or *outgoing*.
- $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ are fixed quasisymmetric parametrizations of the boundaries. A *base parametrization* for the *base surface*.
- $\Sigma^B \#_{\boldsymbol{\psi}} (\overline{\Delta}_0)^n$ - sewing in n disks using boundary parametrizations $(\psi_1, \ldots, \psi_n)$.
- $\Sigma^P = \Sigma^B \#_{\boldsymbol{\tau}} (\overline{\Delta}_0)^n$ is the punctured surface obtained by capping the boundaries.
- $\Sigma_1^B \# \Sigma_2^B$ - Result of sewing Σ_1^B and Σ_2^B along specified boundary components.
- $\mathcal{O}^{\Delta}_{\mathrm{qc}}(p)$ - Rigging data (puncture model). Space of holomorphic coordinates at p with quasiconformal extensions.
- $T^B(\Sigma^B)$ - Teichmüller space of a bordered surface.
- $\widetilde{T}^P(\Sigma^P)$ - Teichmüller space of rigged surfaces (puncture picture).
- $\widetilde{T}^B_{\#}(\Sigma^B)$ - reduced Teichmüller space of rigged surfaces (border pictue).
- $\widetilde{\mathcal{M}}^B$ - rigged moduli space (border picture).
- $\widetilde{\mathcal{M}}^P$ - rigged moduli space (puncture picture).
- $\mathrm{PMod}^B(\Sigma^B)$ - pure mapping class group.
- $\mathrm{PModI}(\Sigma^B) = \{[\rho] \in \mathrm{PMod}(\Sigma^B) \mid \rho|_{\partial\Sigma^B} = \mathrm{id}\}$.
- $\mathrm{DB}(\Sigma^B)$ - subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by "boundary" Dehn twists.
- $\mathrm{DI}(\Sigma^B)$ - subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by "internal" Dehn twists.

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