MELLIN AND GREEN SYMBOLS FOR BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH EDGES

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We introduce the algebra of smoothing Mellin and Green symbols in a pseudodifferential calculus for manifolds with edges. In addition, we define scales of weighted Sobolev spaces with asymptotics based on the Mellin transform and analyze the mapping properties of the operators on these spaces. This will allow us to obtain complete information on the regularity and asymptotics of solutions to elliptic equations on these spaces.

Introduction

In this paper we develop a crucial part of the pseudodifferential calculus for boundary value problems on manifolds with edges, namely the smoothing Mellin and Green symbols along with the Sobolev spaces with asymptotics on which the associated operators act naturally. According to the basic principles in the design of pseudodifferential calculi on singular manifolds developed by the second author it is the interplay of the structure of the spaces with the structure of the symbols and their (pointwise) inverses which provides the specific information for the parametrix construction in the final calculus and the conclusions on regularity and asymptotics of solutions to elliptic boundary problems.

While the more technical details of the calculus are deferred to other publications [16, 17],..., we introduce here the concepts of meromorphic Mellin symbols and (non-branching, discrete) asymptotics. Both are new and differ from earlier versions in the boundaryless case.

A few details: Close to the edge, a manifold with edges and boundary has the structure of a wedge: It is the Cartesian product of q-dimensional Euclidean space with a cone whose base is a smooth compact manifold with boundary.

Following the general concept of iterated symbolic structures we devise the calculus on this wedge as a pseudodifferential calculus along the edge $\mathbf{R}^q$ with operator-valued symbols. They have values in the cone algebra developed by the authors in [14, 15]. Looking more closely, each of these pseudodifferential symbols will be a sum of three terms: An analytic edge symbol, a smoothing meromorphic Mellin symbol, and a Green symbol. Having dealt with the analytic edge symbols in [16, 17], the smoothing Mellin and Green symbols are the object of the present investigation.

Both symbol classes are subspaces of the parameter-dependent regularizing elements in Boutet de Monvel's calculus on the smooth open manifold obtained by deleting the edge of the wedge. However, they have a much finer interior structure. The smoothing Mellin symbols are pseudodifferential symbols along the edge with values in Mellin operators on the cone having regularizing meromorphic Mellin symbols with asymptotics; the Green symbols are described in terms of their mapping properties on weighted Mellin Sobolev spaces with asymptotics.

As a consequence, the Green symbols pointwise take values in compact operators. Hence the residual operators in the final edge calculus, namely the smoothing Green operators induced by the regularizing Green symbols, will be compact. A parametrix will yield a Fredholm inverse within the calculus. Moreover, the parametrix will act on the Mellin Sobolev spaces with asymptotics; the residual operators will map any Sobolev space into a corresponding Sobolev space of smooth functions with asymptotics. This will enable us to decribe precisely the structure of solutions to elliptic boundary value problems.

The construction of a pseudodifferential calculus on manifolds with edges is another step towards corresponding calculi for boundary value problems on manifolds with higher singularities and towards an index theory for these objects. More generally, the analysis on manifolds with singularities also is of considerable interest for concrete applications in mathematical physics and engineering. For pseudodifferential calculi in many different situations see Schulze [7, 18, 19, 21, 20].

1. Mellin Sobolev Spaces with Asymptotics

Throughout this article let X be an n-dimensional C^∞ manifold with boundary Y, embedded in an n-dimensional manifold $\tilde{X}$ without boundary. All of them are supposed to be compact. We write X

for the open interior of X, while $\overline{X}$ is its closure; $V_1, V_2, \dots$ are vector bundles over $\tilde{X}$ and $W_1, W_2, \dots$ vector bundles over Y. On $\tilde{X}$ we fix a Riemannian metric; moreover we endow the vector bundles with Hermitian structures so that we can speak of L^2-sections. By ∂_r we denote an operator which coincides with the normal derivative in a neighborhood of the boundary and vanishes outside a slightly larger neighborhood of the boundary.

Regularizing Operators in Boutet de Monvel's Calculus

In this paper we shall only need the notion of regularizing elements. For a short introduction to Boutet de Monvel's calculus see Section 2 of [14].

Definition 1.1. A *regularizing operator of type* 0 *in Boutet de Monvel's calculus* is an operator

$$R : L^2(X, V_1) \oplus L^2(Y, W_1) \to C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2)$$

whose formal adjoint with respect to the inner products on $L^2(X, V_1) \oplus L^2(Y, W_1)$ and $L^2(X, V_2) \oplus L^2(Y, W_2)$ respectively, R^*, induces a continuous operator

$$R^* : L^2(X, V_2) \oplus L^2(Y, W_2) \to C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1).$$

These mapping properties imply that R is an integral operator with a smooth kernel. A *regularizing operator of type* $d \in \mathbf{N}$ is a sum $R = \sum_{j=0}^d R_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}$ with all R_j regularizing of type zero. For $s > d - 1/2$, R defines a continuous operator

$$R : H^s(X, V_1) \oplus H^s(Y, W_1) \to C^\infty(X, V_2) \oplus C^\infty(Y, W_2).$$

Using integration by parts we may write $R = C + \sum_{j=1}^d \begin{bmatrix} K_j\gamma_{j-1} & 0 \\ S_j\gamma_{j-1} & 0 \end{bmatrix}$. Here, C is a regularizing operator of type zero, γ_j is defined by $\gamma_j u = \partial_r^j u|_Y$, while $K_j : L^2(Y, W_1) \to C^\infty(\overline{X}, V_2)$ and $S_j : L^2(Y, W_1) \to C^\infty(Y, W_2)$ are integral operators with smooth kernels; C as well as the K_j and S_j are uniquely determined.

We write $\mathcal{B}^{-\infty,d}(X)$ for the space of regularizing elements of type d and $\mathcal{B}^{-\infty,d}(X; \mathbf{R}^q)$ for the parameter-dependent regularizing elements, i.e., the Schwartz functions on $\mathbf{R}^q$ with values in $\mathcal{B}^{-\infty,d}(X)$.

Group Actions and Operator-Valued Symbols

1.2 Operator-valued symbols. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}$ and the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous for all $e \in E$.

We fix a smooth positive function $[\cdot] : \mathbf{R}^q \to \mathbf{R}_+$ with $[\eta] = |\eta|$ for large $|\eta|$. $H^s(\mathbf{R})$ is the usual Sobolev space on $\mathbf{R}$, while $H^s(\mathbf{R}_+) = \{u|_{\mathbf{R}_+} : u \in H^s(\mathbf{R})\}$ and $H_0^s(\mathbf{R}_+)$ is the set of all $u \in H^s(\mathbf{R})$ whose support is contained in $\overline{\mathbf{R}}_+$. Furthermore, $H^{s,t}(\mathbf{R}_+) = \{[r]^{-t}u : u \in H^s(\mathbf{R}_+)\}$, and $H_0^{s,t}(\mathbf{R}_+) = \{[r]^{-t}u : u \in H_0^s(\mathbf{R}_+)\}$; here r is the variable in $\mathbf{R}_+$. Finally, $\mathcal{S}(\mathbf{R}_+^q) = \{u|_{\mathbf{R}_+^q} : u \in \mathcal{S}(\mathbf{R}^q)\}$.

For all Sobolev spaces on $\mathbf{R}$ and $\mathbf{R}_+$, we will use the group action

$$(\kappa_\lambda f)(r) = \lambda^{\frac{1}{2}} f(\lambda r). \tag{1.1}$$

This action extends to distributions by $\kappa_\lambda u(\varphi) = u(\kappa_{\lambda^{-1}}\varphi)$. On $E = \mathbf{C}^l$ use the trivial group action $\kappa_\lambda = id$.

Let E, F be Banach spaces with strongly continuous group actions $\kappa, \tilde{\kappa}$, let $\Omega \subseteq \mathbf{R}^k$, $a \in C^\infty(\Omega \times \mathbf{R}^n, \mathcal{L}(E, F))$, and $\mu \in \mathbf{R}$. We shall write $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ provided that, for every $K \subset\subset \Omega$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta a(y, \eta) \kappa_{[\eta]}\|_{\mathcal{L}(E,F)} \le C[\eta]^{\mu - |\alpha|}. \tag{1.2}$$

The space $S^\mu(\Omega, \mathbf{R}^q; E, F)$ is Fréchet topologized by the choice of the best constants C.

The space $S^\mu(\Omega, \mathbf{R}^q; \mathbf{C}^k, \mathbf{C}^l)$ coincides with the ($l \times k$ matrix-valued) elements of Hörmander's class $S_{1,0}^\mu(\Omega \times \mathbf{R}^q)$. One has asymptotic summation: Given a sequence $\{a_j\}$ with $a_j \in S^{\mu_j}(\Omega, \mathbf{R}^q; E, F)$ and

$\mu_j \to -\infty$, there is an $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$, $\mu = \max\{\mu_j\}$ such that $a \sim \sum a_j$; a is unique modulo $S^{-\infty}(\Omega, \mathbf{R}^q; E, F)$. Note that $S^{-\infty}(\Omega, \mathbf{R}^q; E, F)$ is independent of the choice of κ and $\tilde{\kappa}$.

A symbol $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ is said to be *classical,* if it has an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ with $a_j \in S^{\mu-j}(\Omega, \mathbf{R}^q; E, F)$ satisfying the homogeneity relation

$$a_j(y, \lambda\eta) = \lambda^{\mu-j} \tilde{\kappa}_\lambda \, a_j(y, \eta) \, \kappa_{\lambda^{-1}} \tag{1.3}$$

for all $\lambda \geq 1, |\eta| \geq R$ with a suitable constant R. We write $a \in S^\mu_{cl}(\Omega, \mathbf{R}^q; E, F)$. For $E = \mathbf{C}^k,\ F = \mathbf{C}^l$ we recover the standard notion.

There is an extension to projective and inductive limits: Let $\tilde{E}, \tilde{F}$ be Banach spaces with group actions. If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ and $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ are sequences of Banach spaces with the same group action, and $F = \text{proj} - \lim F_k$, $E = \text{ind} - \lim E_k$, then let $S^\mu(\Omega, \mathbf{R}^q; \tilde{E}, F) = \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^q; \tilde{E}, F_k)$ and define $S^\mu(\Omega, \mathbf{R}^q; E, \tilde{F})$ as well as $S^\mu(\Omega, \mathbf{R}^q; E, F)$ similarly as projective limits.

Remark 1.3. Recall that $\mathcal{S}(\mathbf{R}_+) = \text{proj} - \lim_{\sigma,\tau\in\mathbf{N}} H^{\sigma,\tau}(\mathbf{R}_+)$, and $\mathcal{S}'(\mathbf{R}_+) = \text{ind} - \lim_{\sigma,\tau\in\mathbf{N}} H_0^{-\sigma,-\tau}(\mathbf{R}_+)$.

Example 1.4. Let $\gamma_j : \mathcal{S}(\mathbf{R}_+) \to \mathbf{C}$ be defined by $\gamma_j f = \lim_{r\to 0^+} \partial_r^j f(r)$. Then, for all $s > j + 1/2$, we can consider γ_j as a (y, η)-independent symbol in $S^{j+1/2}(\mathbf{R}^q \times \mathbf{R}^q; H^s(\mathbf{R}_+), \mathbf{C})$.

In fact, all we have to check is that $\|\tilde{\kappa}_{[\eta]^{-1}} \gamma_j \kappa_{[\eta]}\| = O([\eta]^{j+1/2})$ for the group actions $\tilde{\kappa}$ on $\mathbf{C}$ and κ on $H^s(\mathbf{R}_+)$. Since the group action on $\mathbf{C}$ is the identity, that on $H^s(\mathbf{R}_+)$ is given by (1.2), everything follows from the observation that

$$\partial_r^j \{[\eta]^{1/2} f([\eta] r)\}|_{r=0} = [\eta]^{j+1/2} \partial_r^j f(0).$$

The following lemma is obvious.

Lemma 1.5. *For $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ and $b \in S^\nu(\Omega, \mathbf{R}^q; F, G)$, the symbol c defined by $c(y, \eta) = b(y, \eta) a(y, \eta)$ (pointwise composition of operators) belongs to $S^{\mu+\nu}(\Omega, \mathbf{R}^q; E, G)$. Moreover, $D_\eta^\alpha D_y^\beta a \in S^{\mu-|\alpha|}(\Omega, \mathbf{R}^q; E, F)$ for all multi-indices α, β.*

Lemma 1.6. *Let $a = a(y, \eta) \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E, F))$, and suppose that $a(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda \, a(y, \eta) \, \kappa_{\lambda^{-1}}$ for all $\lambda \geq 1, |\eta| \geq R$. Then $a \in S^\mu_{cl}(\Omega, \mathbf{R}^n; E, F)$, and the symbol semi-norms for a can be estimated in terms of the semi-norms for a in $C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E, F))$.*

Proof. Without loss of generality let $R = 1$. We only have to consider the case of large $|\eta|$. For these, $D_\eta^\alpha D_y^\beta a(y, \eta) = \lambda^{-\mu+|\alpha|} \tilde{\kappa}_{\lambda^{-1}} \, (D_\eta^\alpha D_y^\beta a)(y, \lambda\eta) \, \kappa_\lambda$. Letting $\lambda = [\eta]$, we conclude that $\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta a(y, \eta) \kappa_{[\eta]} = [\eta]^{\mu-|\alpha|} (D_\eta^\alpha D_y^\beta a)(y, \eta/[\eta])$. The norm of the right hand side in $\mathcal{L}(E, F)$ clearly is $O([\eta]^{\mu-|\alpha|})$. Moreover, a is classical, since it is homogeneous of degree μ in the sense of (1.3). □

Mellin Sobolev Spaces

1.7 Parameter-dependent order reductions on $\tilde{X}$. Let V be a vector bundle over $\tilde{X}$. For each $\mu \in \mathbf{R}$ there is a parameter-dependent pseudodifferential operator $\Lambda^\mu = \{\Lambda^\mu(\tau) : \tau \in \mathbf{R}\}$ with local parameter-dependent elliptic symbols of order μ such that

$$\Lambda^\mu(\tau) : H^s(\tilde{X}, V) \to H^{s-\mu}(\tilde{X}, V)$$

is an isomorphism for all τ.

One way to construct such an operator is to start with symbols of the form $\langle \xi, (\tau, C) \rangle^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n_\xi; \mathbf{R}_\tau)$ with a large constant $C > 0$ and patch them together to an operator on the manifold $\tilde{X}$ with a partition of unity and cut-off functions.

Definition 1.8. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$. The Mellin transform Mu of $u \in C_0^\infty(\mathbf{R}_+)$ is given by

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt, \quad z \in \mathbf{C}. \tag{1.4}$$

M extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{1/2})$. Of course, (1.4) also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ motivates the definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+, E).$$

For a Hilbert space E, the inverse of M_γ is given by $(M_\gamma^{-1}h)(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz$.

1.9 Totally characteristic Sobolev spaces. Write $\tilde{X}^\wedge = \tilde{X}\times\mathbf{R}_+$, $X^\wedge = X\times\mathbf{R}_+$, $Y^\wedge = Y\times\mathbf{R}_+$.

(a) Let $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ be a family of parameter-dependent order reductions as in 1.7. For $s,\gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(\tilde{X}^\wedge)$ is the closure of $C_0^\infty(\tilde{X}^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\tilde{X}^\wedge)} = \left\{\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z)Mu(z)\|^2_{L^2(\tilde{X})}\,|dz|\right\}^{1/2}. \tag{1.5}$$

Recall that n is the dimension of X and $\tilde{X}$. The space $\mathcal{H}^{s,\gamma}(\tilde{X}^\wedge)$ is independent of the particular choice of the order reducing family.

(b) For $s \in \mathbf{N}$ we obtain the alternative description $u \in \mathcal{H}^{s,\gamma}(\tilde{X}^\wedge)$ iff $t^{n/2-\gamma}(t\partial_t)^k Du(x,t) \in L^2(X^\wedge)$ for all $k \le s$ and all differential operators D of order $\le s-k$ on $\tilde{X}$, cf. [20, Section 2.1.1, Proposition 2].

(c) Set $\mathcal{H}^{s,\gamma}(X^\wedge) = \{f|_{X^\wedge} : f \in \mathcal{H}^{s,\gamma}(\tilde{X}^\wedge)\}$, endowed with the quotient norm.

(d) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$, where the subscript '*loc*' refers to the t-variable only. Moreover, $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma\mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$ has a natural inner product

$$(u,v)_{\mathcal{H}^{0,0}(X^\wedge)} = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}\,dz.$$

(e) Let φ be the restriction to $X^\wedge$ of a function in $C_0^\infty(\tilde{X}\times\mathbf{R})$. Then the operator M_φ of multiplication by φ yields a bounded map $\mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$ for all $s,\gamma\in\mathbf{R}$. The mapping $\varphi \mapsto M_\varphi$ is continuous in the corresponding topology.

Definition 1.10. Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The exterior direct sum $E\oplus F$ is Fréchet and has the closed subspace $\mathcal{N} = \{(a,-a) : a \in E\cap F\}$. The non-direct sum of E and F then is the Fréchet space $E+F := E\oplus F/\mathcal{N}$.

1.11 The spaces H^s_{cone}. Let $\{\tilde{X}_j\}_{j=1}^J$ be a finite covering of $\tilde{X}$ by open sets, $\kappa_j : \tilde{X}_j \to U_j$ the coordinate maps onto bounded open sets in $\mathbf{R}^n$, and $\{\varphi_j\}_{j=1}^J$ a subordinate partition of unity. The maps κ_j induce a push-forward of functions and distributions: For a function u on $\tilde{X}_j$ for example,

$$(\kappa_{j*}u)(x) = u(\kappa_j^{-1}(x)), \quad x \in U_j. \tag{1.6}$$

For $j = 1,\dots,J$, consider the diffeomorphism

$$\chi_j : U_j\times\mathbf{R} \to \{(x[t],t) : x\in U_j, t\in\mathbf{R}\} \subset \mathbf{R}^{n+1}$$

given by $\chi_j(x,t) = (x[t],t)$. For $s\in\mathbf{R}$ we define $H^s_{cone}(\tilde{X}\times\mathbf{R})$ as the set of all $u\in H^s_{loc}(\tilde{X}\times\mathbf{R})$ such that, for $j=1,\dots,J$, the push-forward $(\chi_j\circ(\kappa_j\otimes\mathrm{id}))_*(\varphi_j u)$, which may be regarded as a distribution on $\mathbf{R}^{n+1}$ after extension by zero, is an element of $H^s(\mathbf{R}^{n+1})$. The space $H^s_{cone}(\tilde{X}\times\mathbf{R})$ is endowed with the natural Hilbert space topology. We let

$$H^s_{cone}(X^\wedge) = \{u|_{X\times\mathbf{R}_+} : u \in H^s_{cone}(\tilde{X}\times\mathbf{R})\}. \tag{1.7}$$

For more details see Schrohe&Schulze [15, Section 4.2]. The subscript "cone" is motivated by the fact that, away from zero, these are the Sobolev spaces for an infinite cone with center at the origin and cross-section X. In particular, the space $H^s_{cone}(S^n\times\mathbf{R}_+)$ coincides with $H^s(\mathbf{R}^{n+1}\setminus\{0\})$ outside any neighborhood of the origin.

Definition 1.12. For $s,\gamma\in\mathbf{R}$ and $\omega\in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(r)\equiv 1$ near $r=0$, let

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{u\in\mathcal{D}'(X^\wedge) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)u \in H^s_{cone}(X^\wedge)\}. \tag{1.8}$$

The definition is independent of the choice of ω by 1.9(e). We give $\mathcal{K}^{s,\gamma}(X^\wedge)$ the topology of the non-direct sum of the Hilbert spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $H^s_{cone}(X^\wedge)$.

Clearly, $\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

The lemma, below, can be deduced from the trace theorem for the usual Sobolev spaces. The shift in the weight $\gamma \mapsto \gamma - 1/2$ is due to the fact that $\dim Y = n-1$.

Lemma 1.13. *For $s > 1/2$ and $\gamma \in \mathbf{R}$ the restriction $\gamma_0 u = u|_{Y^\wedge}$ of u to $Y^\wedge$ induces a continuous operator $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)$.*

By r denote the normal coordinate in a neighborhood of Y. Then the operators $\gamma_j : u \mapsto \partial_r^j u|_{Y^\wedge}$ define continuous mappings $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-j-1/2,\gamma-1/2}(Y^\wedge)$.

Lemma 1.14. *A strongly continuous group action κ_λ is defined on $\mathcal{K}^{s,\gamma}(X^\wedge)$ by $(\kappa_\lambda f)(x,t) = \lambda^{(n+1)/2} f(x,\lambda t)$ provided $s \geq 0$. This action is unitary on $\mathcal{K}^{0,0}(X^\wedge)$ and naturally extends to distributions in $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s,\gamma \in \mathbf{R}$.*

Proof. It is lengthy but straightforward to see that κ is strongly continuous; it is unitary on $\mathcal{K}^{0,0}(X^\wedge)$ in view of 1.12. □

Remark 1.15. The definitions of the spaces $\mathcal{H}^{s,\gamma}$ and $\mathcal{K}^{s,\gamma}$ also make sense for functions and distributions taking values in a vector bundle V. We shall then write $\mathcal{H}^{s,\gamma}(X^\wedge, V)$ and $\mathcal{K}^{s,\gamma}(X^\wedge, V)$, respectively. In later constructions we will often have to deal with direct sums $\mathcal{K}^{s,\gamma}(X^\wedge, V) \oplus \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge, W)$ for vector bundles V and W over X and Y, respectively. On these spaces we use the natural group action $\kappa_\lambda(u,v) = (\lambda^{\frac{n+1}{2}} u(\cdot,\lambda\cdot), \lambda^{\frac{n}{2}} v(\cdot,\lambda\cdot))$.

Asymptotics

Convention: Whenever we shall write in the following $\omega, \tilde{\omega}, \omega_1, \ldots$, without further specification or refer to a function as a *cut-off function* we mean an element of $C_0^\infty(\overline{\mathbf{R}}_+)$ which is equal to one near the origin.

Definition 1.16. (a) A weight datum is a triple $\mathbf{g} = (\gamma, \delta, (-k, 0])$, where γ and δ are reals and $(-k, 0]$ is a finite interval with $0 \neq k \in \mathbf{N}$.

(b) An asymptotic type associated with the weight datum $\mathbf{g}$ is a pair $P = (P_1, P_2)$, where P_1 and P_2 are tuples with the following properties:

(i) $P_1 = \{(p_j, m_j) : j = 1, \ldots, J\}$, where $p_j \in \mathbf{C}$ with $\frac{n+1}{2} - \gamma - k < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma$, and $m_j \in \mathbf{N}$;

(ii) $P_2 = \{(q_j, n_j) : j = 1, \ldots, J'\}$, where $q_j \in \mathbf{C}$ with $\frac{n}{2} - \gamma - k < \operatorname{Re} q_j < \frac{n}{2} - \gamma$, and $n_j \in \mathbf{N}$.

The numbers J and J' may depend on P. We might have $J = 0$ or $J' = 0$. We then write $P_1 = O$ and $P_2 = O$, respectively. We let $\pi_{\mathbf{C}} P_1 = \{p_1, \ldots, p_J\}$, $\pi_{\mathbf{C}} P_2 = \{q_1, \ldots, q_{J'}\}$, and $\pi_{\mathbf{C}} P = \pi_{\mathbf{C}} P_1 \cup \pi_{\mathbf{C}} P_2$. Note that the conditions on P are independent of the second entry of the weight datum $\mathbf{g}$.

(c) A Mellin asymptotic type over an open set $U \subseteq \mathbf{R}^p$ is a sequence $Q = \{(q_j, n_j, L_j) : j \in \mathbf{Z}\}$; here $q_j \in \mathbf{C}$ with $\operatorname{Re} q_j \to \mp\infty$ as $j \to \pm\infty$, $n_j \in \mathbf{N}$, $L_j = \{L_j^\alpha(y) : y \in U, \alpha \in \mathbf{N}^p\}$, and each $L_j^\alpha(y)$ is a finite-dimensional subspace of finite rank operators in $\mathcal{B}^{-\infty,d}(X)$. As before, $\pi_{\mathbf{C}} Q = \{q_j : j \in \mathbf{Z}\}$, and we explicitly admit the case that $\{(q_j, n_j, L_j)\}$ has finitely many elements only or even no element at all. In this last case we speak of the trivial asymptotic type and use the notation $Q = O$.

(d) Let $P = (P_1, P_2)$ and γ be as above. For $s \in \mathbf{R}$ we define $\mathcal{H}^{s,\gamma}_{P_1}(X^\wedge)$ as the set of all distributions u in $\mathcal{H}^{s,\gamma}(X^\wedge)$ for which there are functions $c_{jl} \in C^\infty(\overline{X})$, $j = 1, \ldots, J$, $l = 0, \ldots, m_j$, and a cut-off function ω such that

$$\left(u - \sum_{j=1}^{J} \sum_{l=0}^{m_j} c_{jl}(x) t^{-p_j} \ln^l t\right) \omega(t) \in \mathcal{H}^{s,\gamma+l-\epsilon}(X^\wedge)$$

for every $\epsilon > 0$. Clearly, if one cut-off function has this property, then any other will also have it.

Similarly, we let $\mathcal{H}^{s,\gamma-1/2}_{P_2}(Y^\wedge)$ be the space of all distributions v in $\mathcal{H}^{s,\gamma-1/2}(Y^\wedge)$ for which there are functions $d_{jl} \in C^\infty(Y)$, $j = 1, \ldots, J'$, $l = 0, \ldots, n_j$, and a cut-off function ω such that, for every $\epsilon > 0$,

$$\left(v - \sum_{j=1}^{J'} \sum_{l=0}^{n_j} d_{jl}(x) t^{-q_j} \ln^l t\right)\omega(t) \in \mathcal{H}^{s,\gamma-1/2+k-\epsilon}(Y^\wedge).$$

This gives them natural Fréchet topologies.

(e) $\mathcal{K}^{s,\gamma}_{P_1}(X^\wedge)$ denotes those distributions u on $X^\wedge$ for which $\omega u \in \mathcal{H}^{s,\gamma}_{P_1}(X^\wedge)$ and to $(1-\omega)u \in H^s_{cone}(X^\wedge)$. Analogously, $\mathcal{K}^{s,\gamma-1/2}_{P_2}(Y^\wedge)$ is the space of all distributions u on $Y^\wedge$ with $\omega u \in \mathcal{H}^{s,\gamma-1/2}_{P_2}(Y^\wedge)$ and $(1-\omega)u \in H^s_{cone}(Y^\wedge)$.

(f) $\mathcal{S}^\gamma_{P_1}(X^\wedge)$ is the space of all smooth functions u on $X^\wedge$ with $\omega u \in \mathcal{H}^{\infty,\gamma}_{P_1}(X^\wedge)$ and $(1-\omega)u \in \mathcal{S}(X^\wedge)$. We define $\mathcal{S}^\gamma_{P_2}(Y^\wedge)$ correspondingly.

The spaces in (e) and (f) are topologized as non-direct sums of Fréchet spaces.

Our next goal is a description of the Mellin images of functions in the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$. We start with the following simple observation.

Lemma 1.17. (a) *Let ω be a cut-off function near 0. Then $M\omega(z) = z^{-1}M(-t\partial_t\omega)(z)$. Since $-t\partial_t\omega \in C_0^\infty(\mathbf{R}_+)$, its Mellin transform is rapidly decreasing on each line Γ_β. If χ is a smooth function on $\mathbf{C}$ which vanishes near zero and is equal to 1 near infinity, then $\chi M\omega$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.*

(b) *Given a cut-off function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(t) \equiv 1$ near zero, $p \in \mathbf{C}$, and $k \in \mathbf{N}$, let*

$$\psi_{p,k}(z) = M_{t\to z}(t^{-p}\ln^k t\,\omega(t))(z) = \frac{d^k}{dz^k}(-z^{-1}M(t\partial_t\omega)(z))(z-p).$$

Here we interpret $M_{t\to z}$ as the weighted Mellin transform M_γ with $\gamma < 1/2 - \mathrm{Re}\,p$. Then $\psi_{p,k}$ extends to a meromorphic function in $\mathbf{C}$ with a single pole of order $k+1$ at p. If χ is a smooth function on $\mathbf{C}$ which vanishes near p and is equal to 1 outside some compact set, then $\chi\psi_{p,k}$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.

Definition 1.18. Let $\mathbf{g} = (\gamma, \delta, (-k, 0])$ and $P = (P_1, P_2)$ be as in Definition 1.16.

(a) $\mathcal{A}^{s,\gamma}_{P_1}(X, V_1)$ is the space of all analytic functions

$$f : \{z \in \mathbf{C} : (n+1)/2 - \gamma - k < \mathrm{Re}\,z < (n+1)/2 - \gamma\} \setminus \pi_{\mathbf{C}}P_1 \to H^s(X, V_1)$$

with the following properties:

(i) In $p_j \in \pi_{\mathbf{C}}P_1$, the function f has a pole of order m_j and a Laurent expansion

$$f(z) = \sum_{l=0}^{m_j} c_{jl}(z - p_j)^{-l-1} + \tilde{f}(z),$$

with $c_{jl} \in C^\infty(\overline{X}, V_1)$ and $\tilde{f}$ analytic near p_j.

(ii) For $\epsilon > 0$ choose an excision function χ_ϵ, vanishing in an ϵ-neighborhood of $\pi_{\mathbf{C}}P_1$ and equal to 1 outside a 2ϵ-neighborhood of $\pi_{\mathbf{C}}P_1$. We then ask that, for each $\epsilon > 0$ and each β with $\gamma \le \beta < \gamma + k$,

$$\|M^{-1}_{\beta-n/2}(\chi_\epsilon f)\|_{\mathcal{H}^{s,\beta}(X^\wedge, V_1)} < \infty,$$

uniformly for β in compact subintervals of $[\gamma, \gamma + k)$.

(b) Similarly we define $\mathcal{A}^{s,\gamma}_{P_2}(Y, W_1)$ as the space of all analytic functions

$$f : \{z \in \mathbf{C} : n/2 - \gamma - k < \mathrm{Re}\,z < n/2 - \gamma\} \setminus \pi_{\mathbf{C}}P_2 \to H^s(Y, W_1)$$

with poles of order n_j at $q_j \in \pi_{\mathbf{C}}P_2$ and Laurent expansions

$$f(z) = \sum_{l=0}^{n_j} d_{jl}(z - q_j)^{-l-1} + \tilde{f}(z),$$

with $d_{jl} \in C^\infty(Y, W_1)$ and $\tilde{f}$ analytic near q_j. We also ask that, for each excision function χ_ϵ for $\pi_{\mathbf{C}} P_2$ and each β with $\gamma \le \beta < \gamma + k$, we have

$$\|M^{-1}_{\beta-(n-1)/2}(\chi_\epsilon f)\|_{\mathcal{H}^{s,\beta}(Y^\wedge, W_1)} < \infty,$$

uniformly in compact subintervals of $[\gamma, \gamma + k)$.

Proposition 1.19. *Let ω be a cut-off function, and let $P = (P_1, P_2)$, γ be as in Definition 1.16. Then the weighted Mellin transform together with the operator of multiplication by ω induces continuous maps*

(i) $M_{\gamma-n/2}\,\omega : \mathcal{K}^{s,\gamma}_{P_1}(X^\wedge, V_1) \to \mathcal{A}^{s,\gamma}_{P_1}(X, V_1)$,

(ii) $\omega M^{-1}_{\gamma-n/2} : \mathcal{A}^{s,\gamma}_{P_1}(X, V_1) \to \mathcal{K}^{s,\gamma}_{P_1}(X^\wedge, V_1)$,

(iii) $M_{\gamma-(n-1)/2}\,\omega : \mathcal{K}^{s,\gamma}_{P_2}(Y^\wedge, W_1) \to \mathcal{A}^{s,\gamma}_{P_2}(Y, W_1)$, *and*

(iv) $\omega M^{-1}_{\gamma-(n-1)/2} : \mathcal{A}^{s,\gamma}_{P_2}(Y, W_1) \to \mathcal{K}^{s,\gamma}_{P_2}(Y^\wedge, W_1)$.

Proof. This follows immediately from Lemma 1.17 and the definition above. □

Lemma 1.20. *The spaces $\mathcal{K}^{s,\gamma}_P$ can be written as projective limits of Hilbert spaces: $\mathcal{K}^{s,\gamma}_P(X^\wedge) = \text{proj} - \lim E_j$, where*

(i) $\mathcal{K}^{s,\gamma}(X) = E_0 \hookleftarrow E_1 \hookleftarrow \ldots \hookleftarrow \mathcal{K}^{s,\gamma}_P(X^\wedge)$, *and*

(ii) *the group action coincides on all spaces.*

The same statements hold for $\mathcal{S}^\gamma_P(X^\wedge)$ as well as for the corresponding spaces over $Y^\wedge$ or for distributional sections.

Proof. $C^\infty(\overline{X})$ is the projective limit of the Hilbert spaces $H^k(X), k \in \mathbf{N}$. Next, $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ is the non-direct sum of the intersection $\bigcap_{\epsilon>0} \mathcal{K}^{s,\gamma+k-\epsilon}(X^\wedge)$, which is a projective limit of Hilbert spaces, and the space of all linear combinations of functions of the form

$$d(x)t^{-p_j} \ln^l t\, \omega(t)$$

with $d \in C^\infty(\overline{X})$, $p_j \in \pi_{\mathbf{C}} P$, $0 \le l \le m_j$. This also is a projective limit in a natural way. For $\mathcal{S}^\gamma_P(X^\wedge)$ we use the representation as the non-direct sum in 1.16(f). We deal with $\mathcal{H}^{\infty,\gamma}_P(X^\wedge) = \bigcap_s \mathcal{H}^{s,\gamma}_P(X^\wedge)$ as before and use Remark 1.3 for $\mathcal{S}(X^\wedge)$. □

2. Operator-Valued Mellin and Green Edge Symbols

Green Symbols

2.1 Notation. In the following, Ω denotes an open set in $\mathbf{R}^q$. Given a weight datum $\mathbf{g} = (\gamma, \delta, (-k, 0])$, an asymptotic type $P = (P_1, P_2)$ associated with $\mathbf{g}$, and $s, \gamma \in \mathbf{R}$, $l = 1, 2, \ldots$, we introduce the abbreviations

$$\mathcal{K}^{s,\gamma}_l = \mathcal{K}^{s,\gamma}(X^\wedge, V_l) \oplus \mathcal{K}^{s,\gamma-1/2}(Y^\wedge, W_l), \tag{2.1}$$

$$\mathcal{S}^\gamma_{l,P} = \mathcal{S}^\gamma_{P_1}(X^\wedge, V_l) \oplus \mathcal{S}^{\gamma-1/2}_{P_2}(Y^\wedge, W_l), \quad \text{and} \tag{2.2}$$

$$\mathcal{A}^{s,\gamma}_{l,P} = \mathcal{A}^{s,\gamma}_{P_1}(X, V_l) \oplus \mathcal{A}^{s,\gamma-1/2}_{P_2}(Y, W_l). \tag{2.3}$$

Definition 2.2. Let $\mathbf{g} = (\gamma, \delta, (-k, 0])$ be a weight datum, $\mu \in \mathbf{Z}$, and $d \in \mathbf{N}$.
(a) $\mathcal{R}^{\mu,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ is the space of all operator-valued symbols

$$g \in \bigcap_{s>-1/2} S^\mu_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}^{s,\gamma}_1 \oplus \mathbf{C}^{N_1}, \mathcal{K}^{s-\mu,\delta}_2 \oplus \mathbf{C}^{N_2}) \tag{2.4}$$

with the following property: There is an asymptotic type $P = (P_1, P_2)$ associated with $\mathbf{g}$ such that, for each $s > -1/2$, the symbol g yields an element of

$$S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma} \oplus \mathbf{C}^{N_1}, \mathcal{S}^{\delta}_{2,P} \oplus \mathbf{C}^{N_2}), \tag{2.5}$$

while the pointwise formal adjoint g^*, defined by $g^*(y,\eta) = g(y,\eta)^*$, yields an element of

$$S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_2^{s,-\delta} \oplus \mathbf{C}^{N_2}, \mathcal{S}^{-\gamma}_{1,Q} \oplus \mathbf{C}^{N_1}) \tag{2.6}$$

for an asymptotic type $Q = (Q_1, Q_2)$ associated with the weight datum $(-\delta, -\gamma, (-k, 0])$. The last '$*$' denotes the formal adjoint with respect to the inner products on $\mathcal{K}_1^{0,0}$ and $\mathcal{K}_2^{0,0}$. The dimensions N_1, N_2 will not be indicated.

(b) $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ is the space of all operator-valued symbols

$$g \in \bigcap_{s > d-1/2} S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma} \oplus \mathbf{C}^{N_1}, \mathcal{K}_2^{s-\mu,\delta} \oplus \mathbf{C}^{N_2}), \tag{2.7}$$

which can be written in the form

$$g = g_0 + \sum_{j=1}^{d} g_j \begin{bmatrix} \begin{bmatrix} \partial_r^j & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.8}$$

with $g_j \in \mathcal{R}^{\mu-j,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. The matrix in the upper left corner refers to the decomposition (2.1). We call these elements Green edge symbols of order μ and type d. If we want to indicate the asymptotic types we will use the notation $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{P,Q}$.

For fixed asymptotic types P and Q the space $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{P,Q}$ is a Fréchet space topologized as a non-direct sum via (2.8). We write $\mathcal{R}^{\mu,d}_G(\Omega \times \mathbf{R}^q, \mathbf{g})$ for the space of all symbols that are independent of the variable $y' \in \Omega$. Since all symbols are classical, we naturally have the notion of a principal (operator-valued) symbol.

Remark 2.3. Similarly as in Boutet de Monvel's calculus, we are dealing with matrices of operators. Note, however, that we now have 3×3 block matrices. The additional entries correspond to trace and potential operators at the edge.

Proposition 2.4. *Let* $g_1 \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ *and* $g_2 \in \mathcal{R}^{\mu',d'}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. *Then*
(a) $D^{\alpha}_{\eta} D^{\beta}_{y} g_1 \in \mathcal{R}^{\mu-|\alpha|,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.
(b) *The pointwise composition* $g_1 g_2$ *is an element of* $\mathcal{R}^{\mu+\mu',d'}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.
(c) *If* $d = 0$, *then* g_1^*, *defined as the pointwise adjoint* $g_1^*(y,y',\eta) = g_1(y,y',\eta)^*$ *is an element of* $\mathcal{R}^{\mu,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.
(d) *Given a sequence* $g_j \in \mathcal{R}^{\mu-j,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{P,Q}$ *with constant asymptotic types* P, Q, *there is a* $g \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{P,Q}$ *with* $g \sim \sum_{j=0}^{\infty} g_j$.

Proof. (a), (b), and (c) are obvious. (d) Although we are dealing with operator-valued symbols, the usual asymptotic summation procedure with respect to the variable η furnishes the desired result. □

Proposition 2.5. *Let* $\nu_1, \nu_2 \in \mathbf{N}$ *and* $g \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. *Then*

$$t^{\nu_2} g \, t^{\nu_1} \in \mathcal{R}^{\mu-\nu_1-\nu_2,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}).$$

Here we understand t^{ν_1} as the operator of multiplication by the diagonal matrix diag $\{t^{\nu_1}, t^{\nu_1}, I\}$, acting a priori in $C_0^{\infty}(\overline{X}^{\wedge}, V_1) \oplus C_0^{\infty}(Y^{\wedge}, W_1) \oplus \mathbf{C}^{N_1}$ which is dense in $\mathcal{K}_1^{s,\gamma} \oplus \mathbf{C}^{N_1}$. A similar interpretation applies to t^{ν_2}. It is a part of the result that the composition extends to a symbol in

$$S^{\mu-\nu_1-\nu_2}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma} \oplus \mathbf{C}^{N_1}, \mathcal{K}_2^{s-\mu,\delta} \oplus \mathbf{C}^{N_2})$$

with the properties in Definition 2.2.

Proof. In view of the fact that ∂_r commutes with t we may assume that $d = 0$. For fixed y, y', η, the operator $g(y, y', \eta)$ has an integral kernel

$$k(x,y) = \begin{pmatrix} k_{11}(x,y) & k_{12}(x,y) \\ k_{21}(x,y) & k_{22}(x,y) \end{pmatrix},$$

with $k_{11} \in \mathcal{S}^{\delta}_{2,P} \hat{\otimes}_\pi \mathcal{S}^{-\gamma}_{1,\overline{Q}}, k_{12} \in \mathcal{S}^{\delta}_{2,P} \otimes \mathbf{C}^{N_1}, k_{21} \in \mathbf{C}^{N_2} \otimes \mathcal{S}^{-\gamma}_{1,\overline{Q}}$, and $k_{22} \in \mathbf{C}^{N_2} \otimes \mathbf{C}^{N_1}$. Here $\overline{Q}$ is the conjugate asymptotic type, i.e. $\overline{Q}_l = \{(\overline{q}^{(l)}_j, n_j)\}$ if $Q_l = \{(q^{(l)}_j, n_j)\}$. This can be deduced from [14, Theorem 3.3.2].

In particular, multiplication of $g(y, y', \eta)$ by powers of t from either side furnishes a continuous operator from $\mathcal{K}^{s,\gamma}_1 \oplus \mathbf{C}^{N_1}$ to $\mathcal{K}^{s-\mu,\delta}_2 \oplus \mathbf{C}^{N_2}$ for all $s > -1/2$. Moreover,

$$\kappa_{[\eta]^{-1}} t^{\nu_1} g(y,y',\eta) t^{\nu_2} \kappa_{[\eta]} = [\eta]^{-\nu_1-\nu_2} t^{\nu_1} \kappa_{[\eta]^{-1}} g(y,y',\eta) \kappa_{[\eta]} t^{\nu_2}.$$

Hence the order is lowered by $\nu_1 + \nu_2$. Properties (2.5) and (2.6) follow in the same way. □

Proposition 2.6. *Let* $\varphi \in \mathcal{S}(\mathbf{R}_+)$ *and* $g \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. *Then* φg, $g\varphi$, $\varphi(\cdot[\eta])g$, *and* $g\varphi(\cdot[\eta])$ *all are elements of* $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.

Again we interpret φ as the operator of multiplication by the diagonal matrix of functions $\mathrm{diag}(\varphi, \varphi)$, acting on $\mathcal{K}^{s,\gamma}_1$ and $\mathcal{K}^{s-\mu,\delta}_2$ for all $s \in \mathbf{R}$; furthermore, $\varphi(\cdot[\eta])$ is the corresponding η-dependent multiplier.

Proof. Since multiplication by $\varphi(t)$ and $\varphi(\cdot[\eta])$ commutes with ∂_r we may assume that $d = 0$. Next we note the identities

$$\begin{aligned} \kappa_{[\eta]^{-1}} \varphi g(y,y',\eta) \kappa_{[\eta]} &= \varphi([\eta]^{-1}\cdot) \kappa_{[\eta]^{-1}} g(y,y',\eta) \kappa_{[\eta]}, \\ \kappa_{[\eta]^{-1}} g(y,y',\eta) \varphi \kappa_{[\eta]} &= \kappa_{[\eta]^{-1}} g(y,y',\eta) \kappa_{[\eta]} \varphi([\eta]^{-1}\cdot). \end{aligned}$$

For fixed η, multiplication by $\varphi([\eta]^{-1}\cdot)$ furnishes a bounded operator on both $\mathcal{K}^{s,\gamma}_1$ and $\mathcal{K}^{s-\mu,\delta}_2$. Had we instead started with $\varphi(\cdot[\eta])$ then we would now only have to deal with multiplication by φ. Writing $\mathcal{S}^{\delta}_{2,P}$ and $\mathcal{S}^{-\gamma}_{1,\overline{Q}}$ as projective limits of Hilbert spaces, say $E^{\delta,j}_P$ and $E^{-\gamma,j}_{\overline{Q}}$, as indicated in Lemma 1.20, multiplication by $\varphi([\eta]\cdot)$ and φ in all cases induces continuous actions

$$E^{\delta,j}_P \to E^{\delta,j}_{\tilde{P}} \qquad \text{and} \qquad E^{-\gamma,j}_{\overline{Q}} \to E^{-\gamma,j}_{\tilde{Q}}$$

for suitable asymptotic types $\tilde{P}$ and $\tilde{Q}$ associated with $\mathbf{g}$. Moreover, the norm of the operator $\varphi([\eta]^{-1}\cdot)$ is uniformly bounded as η varies over $\mathbf{R}^q$. A corresponding argument applies to derivatives.

Hence we get symbols of the desired order. It remains to show that they are classical. So suppose g_j is homogeneous of degree j in the sense of (1.3). Then one has, for large $|\eta|$ and $\lambda \geq 1$,

$$\tilde{\kappa}_\lambda \varphi([\eta]\cdot) g_j(y,y',\eta) \kappa_{\lambda^{-1}} = \varphi(\lambda[\eta]\cdot) \tilde{\kappa}_\lambda \varphi g_j(y,y',\eta) \kappa_{\lambda^{-1}} = \varphi([\lambda\eta]\cdot) \lambda^{-j} g_j(y,y',\lambda\eta);$$

$$\tilde{\kappa}_\lambda \varphi g_j(y,y',\eta) \kappa_{\lambda^{-1}} = \varphi(\lambda\cdot) \tilde{\kappa}_\lambda \varphi g_j(y,y',\eta) \kappa_{\lambda^{-1}} = \varphi(\lambda\cdot) \lambda^{-j} g_j(y,y',\lambda\eta).$$

Similar equations hold for multiplication from the right. In the first case we therefore have homogeneity of degree j right away. For the second we use Taylor's formula to write

$$\varphi(t) = \sum_{l=0}^{N-1} \frac{\partial^l \varphi(0)}{l!} t^l + t^N \tilde{\varphi}$$

for some smooth bounded function $\tilde{\varphi}$. From the above and Proposition 2.5 we conclude that the operator induced via multiplication by the remainder term has order $j - N$ while the others are homogeneous for large $|\eta|$. This completes the proof. □

Smoothing Mellin Symbols

2.7 Smoothing Mellin symbols with asymptotics. Let $P = \{(p_j, m_j, L_j)\}$ be a Mellin asymptotic type over an open set U in Euclidean space. In applications, U will be Ω or $\Omega \times \Omega$. Recall that, for an open subset G of the complex plane and a Fréchet space $\mathcal{E}$, $\mathcal{A}(G, \mathcal{E})$ is the space of all analytic functions on G taking values in $\mathcal{E}$.

We define $C^\infty(U, M_P^{-\infty,d}(X))$ as the space of all smooth functions

$$h : U \to \mathcal{A}(\mathbf{C} \setminus \pi_{\mathbf{C}} P, \mathcal{B}^{-\infty,d}(X))$$

having the following additional properties:

(i) the points $p_j \in \pi_{\mathbf{C}} P$ are poles of order $\leq m_j$. For $l = 0, \ldots, m_j - 1$, the coefficient $h_{jl}(y)$ of $(z - p_j)^{-l-1}$ in the Laurent expansion of $h(y, \cdot)$ satisfies $\partial_y^\alpha h_{jl}(y) \in L_j^\alpha(y)$ for all multi-indices α.

(ii) For each finite strip $\{c_1 < \operatorname{Re} z < c_2\}$ we find functions $c_{jl} \in C^\infty(U, \mathcal{B}^{-\infty,d}(X))$ such that the difference

$$h(y,z) - \sum_{c_1 < \operatorname{Re} p_j < c_2} c_{jl}(y) M_{t \to z}(t^{-p_j} \ln^l t\, \omega(t))$$

is a smooth function on U with values in $\mathcal{A}(\{c_1 < \operatorname{Re} z < c_2\}, \mathcal{B}^{-\infty,d}(X))$; it is rapidly decreasing along Γ_β, uniformly for $y \in U$ and $c_1 < \beta < c_2$. Here ω is an arbitrary cut-off function.

These conditions furnish a natural Fréchet topology on $C^\infty(U, M_P^{-\infty,d}(X))$. The c_{jl} can be expressed in terms of the Laurent coefficients of the principal part; they are, however, more convenient to describe the decrease.

Proposition 2.8. *Let $h \in C^\infty(U, M_P^{-\infty,d}(X))$ and $f \in \mathcal{A}_{1,Q}^{s,\gamma}$ for a Mellin asymptotic type P over U, an asymptotic type Q associated with $\mathbf{g}$, and $s > d - 1/2$. Then $g(y,z) = h(y,z) f(z)$ defines an element g of $C^\infty(U, \mathcal{A}_{2,R}^{r,\gamma})$ for each $r \in \mathbf{R}$ and a suitable asymptotic type R; the induced mapping*

$$C^\infty(U, M_P^{-\infty,d}(X)) \times \mathcal{A}_{1,Q}^{s,\gamma} \to C^\infty(U, \mathcal{A}_{2,R}^{r,\gamma})$$

is continuous.

Proof. We are only interested in the strip $\{(n+1)/2 - \gamma - k < \operatorname{Re} z < (n+1)/2 - \gamma\}$. By linearity it therefore is no restriction to assume that $P = \{(p, m, L)\}$ and $Q = (Q_1, Q_2)$ with $Q_1 = \{(q, l)\}$ and $Q_2 = \emptyset$, i.e., both consist of a single element with p and q in the strip.

Choose a cut-off function ω. Employing Lemma 1.17 we may write

$$h(y,z) = \sum_{j=0}^{m-1} c_j(y) \psi_{p,j}(z) + h_0(y,z),$$

where $c_j \in C^\infty(U, \mathcal{B}^{-\infty,d}(X))$, $\psi_{p,j}(z) = M_{t \to z}(t^{-p} \ln^j t\, \omega)(z)$, and h_0 is a smooth function on U, taking values in the space $\mathcal{B}$ of all analytic functions on the strip with values in $\mathcal{B}^{-\infty,d}(X)$ that are (uniformly) rapidly decreasing along each vertical line in the strip. Since $C^\infty(U, \mathcal{B}) = C^\infty(U) \hat{\otimes}_\pi \mathcal{B}$ we may assume that $h_0(y,z) = v(y) e(z)$ with $v \in C^\infty(U)$ and $e \in \mathcal{B}$. Similarly,

$$f(z) = \sum_{k=0}^{l-1} d_k \psi_{q,k}(z) + f_0(z),$$

with $d_k \in C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1)$ and $f_0 \in \mathcal{A}_{1,O}^{s,\gamma}$; recall that O denotes the trivial asymptotic type.

Now we consider the terms separately: $\psi_{p,j} \psi_{q,k}$ is a meromorphic function in the strip with singularity set $\{p, q\}$, possibly $p = q$. Cutting out a neighborhood of the pole(s), this function is rapidly decreasing along each vertical line, uniformly in the strip. The continuity of the composition

$$\mathcal{B}^{-\infty,d}(X) \times C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1) \to C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2)$$

therefore shows that $c_j d_k \psi_{p,j} \psi_{q,k}$ is an element of $C^\infty(U, \mathcal{A}_{2,R}^{r,\gamma})$ for each $r \in \mathbf{R}$, provided the asymptotic type R takes care of the pole(s), say $R = (\{(p, m+l), (q, l)\}, \emptyset)$. Moreover, the semi-norms in $C^\infty(U, \mathcal{A}_{2,R}^{r,\gamma})$ depend continuously on those for c_j and d_k.

For fixed z in the strip, the composition $e(z) d_k$ defines an element of $C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2)$. It depends analytically on z in all the strip, and its semi-norms decay rapidly as z varies over a vertical line, uniformly in the strip. Hence also $ved_k\psi_{q,k} \in C^\infty(U, \mathcal{A}^{r,\gamma}_{2,R})$ for arbitrary r. The corresponding mapping is continuous. Finally one treats $c_j\psi_{p,j}f_0$ and vef_0 in the same way. □

2.9 Mellin operators. Given a subset U of Euclidean space, $h = h(y,z) \in C^\infty(U, M_P^{-\infty,d}(X))$, and $\gamma \in \mathbf{R}$ with $\Gamma_{1/2-\gamma} \cap \pi_{\mathbf{C}}P = \emptyset$, we introduce the operator family $\{\mathrm{op}_M^\gamma h(y) : y \in U\}$:

$$\mathrm{op}_M^\gamma h(y) : C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1) \to C^\infty(\overline{X}^\wedge, V_2) \oplus C^\infty(Y^\wedge, W_2)$$

is defined by

$$[\mathrm{op}_M^\gamma h(y)]u(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(y,z)M_\gamma u(z)\,dz = (M_\gamma^{-1}h(\cdot,y)M_\gamma u)(t).$$

Here, we identify u with a function in $C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$, and $M_\gamma u$ is the Fréchet space-valued weighted Mellin transform of this function. As an immediate consequence we have, for arbitrary $\beta \in \mathbf{R}$,

$$t^\beta \mathrm{op}_M^\gamma h(y) = \mathrm{op}_M^{\gamma+\beta} T^\beta h(y) t^\beta. \tag{2.9}$$

Here $T^\beta h(y,z) = h(y, z+\beta)$. For each $y \in U$ the operators $\mathrm{op}_M^\gamma h(y)$ extend to continuous maps

$$\mathrm{op}_M^\gamma h(y) : \begin{array}{c} \mathcal{H}^{s,\gamma+n/2}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+(n-1)/2}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma+n/2}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma+(n-1)/2}(Y^\wedge, W_2) \end{array}, \tag{2.10}$$

provided $s > d - 1/2$. It follows immediately that, for every choice of cut-off functions ω_1, ω_2, we have bounded operators

$$\omega_1 \mathrm{op}_M^\gamma h(y)\omega_2 : \mathcal{K}_1^{s,\gamma+n/2} \to \mathcal{K}_2^{\infty,\gamma+n/2}.$$

2.10 Convention. From now on we shall fix $\gamma \in \mathbf{R}$, $\mu, \nu \in \mathbf{Z}$, $\nu \le \mu$, $k \in \mathbf{N} \setminus \{0\}$, and the weight datum $\mathbf{g} = (\gamma, \gamma - \mu, (-k, 0])$.

2.11 The space $\mathcal{R}^{\nu,d}_{M+G}$. $\mathcal{R}^{\nu,d}_{M+G}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ is the space of all symbols $a = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} + g$, where $g \in \mathcal{R}^{\nu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$, and $m \in C^\infty(\Omega \times \Omega \times \mathbf{R}^q, \mathcal{B}^{-\infty,d}(X))$ is of the form

$$m(y,y',\eta) = \omega_1(t[\eta])\Big(\sum_{j=0}^{k-1}\sum_{|\alpha|\le j} t^{j-\nu}\mathrm{op}_M^{\gamma_j} h_{j\alpha}(y,y')\eta^\alpha\Big)\omega_2(t[\eta]) \tag{2.11}$$

with cut-off functions ω_1, ω_2,

(i) $h_{j\alpha} \in C^\infty(\Omega \times \Omega, M^{-\infty,d}_{P_{j\alpha}}(X))$,

(ii) $\gamma - (\mu - \nu) - j - n/2 \le \gamma_j \le \gamma - n/2$, and

(iii) Mellin asymptotic types $P_{j\alpha}$ with $\pi_{\mathbf{C}}P_{j\alpha} \cap \Gamma_{1/2-\gamma_j} = \emptyset$.

Note that m is a 2×2 matrix of operators defined e.g. on the space $C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1)$; the matrix $\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$ refers to the decompositions $[C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1)] \oplus \mathbf{C}^{N_1}$ and $[C^\infty(X^\wedge, V_2) \oplus C^\infty(Y^\wedge, W_2)] \oplus \mathbf{C}^{N_2}$.

Theorem 2.12. *Let m be as in Definition 2.11, $s > d - 1/2$, and let Q be an asymptotic type associated with $\mathbf{g}$. For fixed (y,y',η), the symbol $m(y,y',\eta)$ then induces continuous operators*

$$m(y,y',\eta) : \mathcal{K}_1^{s,\gamma} \to \mathcal{K}_2^{\infty,\gamma-\mu} \quad \textit{and} \quad m(y,y',\eta) : \mathcal{K}_{1,Q}^{s,\gamma} \to \mathcal{S}_{2,R}^{\gamma-\mu};$$

here R is a resulting asymptotic type. Furthermore,

$$m \in S_{cl}^{\nu,d}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{\infty,\gamma-\mu}) \cap S_{cl}^{\nu,d}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_{1,Q}^{s,\gamma}, \mathcal{S}_{2,R}^{\gamma-\mu}).$$

Proof. By 2.9, the first assertion is immediate from the fact that $m(y,y',\eta)$ takes values in $M_P^{-\infty,d}(X)$. For the second we apply Proposition 1.19 combined with Proposition 2.8. In order to see that it additionally defines a symbol in the asserted classes we focus on one summand as in (2.11); denote it by $m_{j\alpha}$, $j \geq |\alpha|$. Let us show that this is a homogeneous symbol in the sense of (1.3). Indeed, for large $|\eta|$ and $\lambda \geq 1$,

$$\begin{aligned}\kappa_\lambda m_{j\alpha}(y,y',\eta) &= \omega_1(\lambda t[\eta])\lambda^{j-\nu}t^{j-\nu}\kappa_\lambda \mathrm{op}_M^\gamma h_{j\alpha}(y,y')\eta^\alpha\omega_2(t[\eta]) \\ &= \omega_1(t[\lambda\eta])\lambda^{j-|\alpha|-\nu}t^{j-\nu}\mathrm{op}_M^\gamma h_{j\alpha}(y,y')(\lambda\eta)^\alpha\kappa_\lambda\omega_2(t[\eta]) \\ &= \lambda^{j-|\alpha|-\nu}m_{j\alpha}(y,y',\lambda\eta)\kappa_\lambda;\end{aligned}$$

here we have used the fact that, for $u \in C_0^\infty(\mathbf{R}_+, C^\infty(X,V_1)\oplus C^\infty(Y,W_1))$, we have $\kappa_\lambda[\mathrm{op}_M^\gamma h(y,y')u] = \mathrm{op}_M^\gamma h(y,y')\kappa_\lambda u$.

Hence $m_{j\alpha}(y,y',\lambda\eta) = \lambda^{\nu-j+|\alpha|}\kappa_\lambda m_{j\alpha}(y,y',\eta)\kappa_{\lambda^{-1}}$, and Lemma 1.6 yields the assertion. □

Proposition 2.13. *Suppose $R \in \mathcal{B}^{-\infty,d}(X)$ has finite rank and $r \in \mathbf{N}$. Let C be a contour in the half plane $\{\mathrm{Re}\, z > (n+1)/2-\gamma\}$ with winding number 1 with respect to the point $p \in \mathbf{C}$. Then the operator G defined by*

$$Gu(t) = \frac{1}{2\pi i}\int_C t^{-z}(z-p)^{-r-1}RM_{\gamma-n/2}(\omega u)(z)\,dz$$

maps $\mathcal{H}^{s,\gamma}(X^\wedge,V_1)\oplus\mathcal{H}^{s,\gamma-1/2}(Y^\wedge,W_1)$ to the finite-dimensional space of all functions of the form $\sum_{j=0}^r v_j(x)t^{-p}\ln^j t$, $v_j \in \mathrm{im}\, R$.

Proof. If $u \in \mathcal{H}^{s,\gamma}(X^\wedge,V_1)\oplus\mathcal{H}^{s,\gamma-1/2}(Y^\wedge,W_1)$, then $M_{\gamma-n/2}(\omega u)$ is analytic in $\{\mathrm{Re}\, z > (n+1)/2-\gamma\}$ with values in $H^s(X,V_1)\oplus H^s(Y,W_1)$. The assertion then follows from complex analysis in one variable: If the meromorphic function f has a pole in p of multiplicity $r+1$ then

$$\frac{1}{2\pi i}\int_C t^{-z}f(z)dz = \sum_{j=0}^r \frac{f_j}{j!}t^{-p}\ln^j t,$$

where f_j is the coefficient of $(z-p)^{-j-1}$ in the Laurent expansion of f near p. □

The following corollary shows that the $\mathcal{R}_{M+G}$-symbols commute with multiplication by t^β, $\beta \geq 0$, modulo elements of $\mathcal{R}_G$.

Corollary 2.14. *Let $h \in C^\infty(\Omega\times\Omega, M_P^{-\infty,d}(X))$ for some Mellin asymptotic type P over $\Omega\times\Omega$. Pick $\beta \geq 0$ as well as $\gamma_1 \in \mathbf{R}$ with $n/2-\gamma-j \leq \gamma_1 \leq n/2-\gamma$ and $\pi_{\mathbf{C}}P\cap\Gamma_{1/2-\gamma_1} = \pi_{\mathbf{C}}P\cap\Gamma_{1/2-\gamma_1+\beta} = \emptyset$. Then*

$$\omega_1(t[\eta])t^{j-\nu}\mathrm{op}_M^{\gamma_1}h(y,y')\omega_2(t[\eta])t^\beta - \omega_1(t[\eta])t^{j-\nu+\beta}\mathrm{op}_M^{\gamma_1}(T^{-\beta}h)(y,y')\omega_2(t[\eta])$$

is an element of $\mathcal{R}_G^{\nu,d}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$.

Proof. For $u \in C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X},V_1)\oplus C^\infty(Y,W_1))$,

$$\begin{aligned}&\left(\omega_1(t[\eta])\mathrm{op}_M^{\gamma_1}h(y,y')\omega_2(t[\eta])t^\beta\right)u(t) \\ =\;& \frac{1}{2\pi i}\omega_1(t[\eta])\int_{\Gamma_{1/2-\gamma_1}} t^{-z}h(y,y',z)(M_{\gamma_1}(\omega_2(\cdot[\eta])u)(z+\beta)dz \\ =\;& \frac{1}{2\pi i}\omega_1(t[\eta])\int_{\Gamma_{1/2-\gamma_1+\beta}} t^{-w+\beta}h(y,y',w-\beta)(M_{\gamma_1}(\omega_2(\cdot[\eta])u)(w)dw.\end{aligned}$$

Using Cauchy's theorem, we may change the line of integration; the error then is

$$g(y,y',\eta)u(t) = \frac{1}{2\pi i}\omega_1(t[\eta])\int_C t^{-w+\beta}h(y,y',w-\beta)(M_{\gamma_1}(\omega_2(\cdot[\eta])u)(w)dw,$$

where C is a countour which simply surrounds the poles of $h(y,y',\cdot)$ between the lines $\Gamma_{1/2-\gamma_1}$ and $\Gamma_{1/2-\gamma_1+\beta}$. Hence $g(y,y',\eta)\in\mathcal{L}(\mathcal{K}_1^{s,\gamma},\mathcal{K}_2^{s,\gamma})$. In addition it is homogeneous of degree zero in η for large $|\eta|$, and

$$\kappa_{[\eta]^{-1}}g(y,y',\eta)\kappa_{[\eta]}u(t)=\frac{1}{2\pi i}\omega_1\int_C t^{-w+\beta}h(y,y',w-\beta)M_{\gamma_1}(\omega_2 u)(w)dw$$

is independent of η. By definition, the coefficients of the principal part of the Laurent expansion of $h(y,y',\cdot)$ are finite rank operators in $\mathcal{B}^{-\infty,d}(X)$. By Proposition 2.13 we see that $\kappa_{[\eta]^{-1}}\tilde{g}(y,y',\eta)\kappa_{[\eta]}$ defines a continuous mapping from $\mathcal{K}_1^{s,\gamma}$ to $\mathcal{S}_{2,Q}^{\gamma}$ for suitable Q. Of course, the semi-norms are uniformly bounded in η. We may apply the same consideration to derivatives and the adjoint. Using the identity $\kappa_{[\eta]^{-1}}t^{j-\nu}\kappa_{[\eta]}=[\eta]^{\nu-j}t^{j-\nu}$ and Proposition 2.5, we see that $t^{\nu-j}g\in\mathcal{R}_G^{\nu-j}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$. □

In the same way one obtains the following result on invariance under changes in the weight:

Corollary 2.15. *Let h and γ_1 be as above. Choose γ_2 with $\gamma-n/2-j\le\gamma_2\le\gamma-n/2$ and $\Gamma_{1/2-\gamma_2}=\emptyset$. Then*

$$\omega_1(t[\eta])\,t^{j-\nu}\left\{\mathrm{op}_M^{\gamma_1}h(y,y')-\mathrm{op}_M^{\gamma_2}h(y,y')\right\}\omega_2(t[\eta])\in\mathcal{R}_G^{\nu-j,d}(\Omega\times\Omega\times\mathbf{R}^q).$$

Proposition 2.16. *For $l,r\in\mathbf{N}$ we interpret t^l as the operator of multiplication by* $\mathrm{diag}\{t^l,t^l,I\}$ *on $\mathcal{K}_1^{s,\gamma}\oplus\mathbf{C}^{N_1}$ and t^r as the operator of multiplication by* $\mathrm{diag}\{t^r,t^r,I\}$ *on $\mathcal{K}_2^{s-\mu,\gamma-\mu}\oplus\mathbf{C}^{N_2}$, recalling that $\mathcal{K}_1^{s,\gamma}$ and $\mathcal{K}_2^{s-\mu,\gamma-\mu}$ are direct sums of two spaces. Then*

$$t^l\,\mathcal{R}_{M+G}^{\nu,d}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})\,t^r\subseteq\mathcal{R}_G^{\nu-l-r,d}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$$

whenever $l+r\ge k$.

Proof. In view of Proposition 2.5 we only need to consider a symbol m as in Definition 2.11. Also it is sufficient to consider the case where $d=0$ and

$$m(y,y',\eta)=\omega_1(t[\eta])t^{-\nu+j}\mathrm{op}_M^{\gamma_0}h(y,y')\eta^\alpha\omega_2(t[\eta])$$

with $|\alpha|=j$. So let us estimate the norms of $\kappa_{[\eta]^{-1}}t^rD_y^\beta D_{y'}^\gamma D_\eta^\delta m(y,y',\eta)t^l\kappa_{[\eta]}$. Without loss of generality let $\beta=\gamma=\delta=0$.

$$\begin{aligned}
&\kappa_{[\eta]^{-1}}t^r m(y,y',\eta)t^l\kappa_{[\eta]}\\
=\;&[\eta]^{-l-r}t^r\kappa_{[\eta]^{-1}}m(y,y',\eta)\kappa_{[\eta]}t^l\\
=\;&[\eta]^{-l-r+\nu-j}\omega_1 t^{r+j-\nu}\mathrm{op}_M^{\gamma_0}h(y,y')\eta^\alpha t^l\omega_2\\
=\;&[\eta]^{-l-r+\nu-j}\omega_1 t^{r+l+j-\nu}\mathrm{op}_M^{\gamma_0-l}T^{-l}h(y,y')\eta^\alpha\omega_2,\\
=\;&[\eta]^{-l-r+\nu-j}\omega_1 t^{r+l+j-\nu}\mathrm{op}_M^{\gamma_1}T^{-l}h(y,y')\eta^\alpha\omega_2+g(y,y',\eta)
\end{aligned}$$

where we applied Corollary 2.15 for the last equality and (2.9) for that before; here γ_1 satisfies $\gamma-(\mu-\nu)-j-n/2\le\gamma_1\le\gamma-n/2$ and g is an element of $\mathcal{R}_G^{\nu-l-r}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$.

It remains to consider the first term on the right hand side of the above equation. We first employ the mapping properties of h stated in (2.10). In view of the factor $t^{l+r-\nu}\omega_1$ and the assumption $l+r\ge k$ we see that the operator maps into $\mathcal{S}_{2,O}^{\gamma-\mu}$. As before O denotes the trivial asymptotic type. In view of the special structure, the operator norm is $O([\eta]^{\nu-l-r})$ with respect to all spaces used to write $\mathcal{S}_{2,O}^{\gamma-\mu}$ as a projective limit of Hilbert spaces. We obtain a corresponding result for the adjoint by commuting the powers of t to the right. This completes the proof. □

Remark 2.17. In view of Proposition 2.16 we may confine the summation in (2.11) to $j=0,\ldots,k-(\mu-\nu)-1$.

Proposition 2.18. *A change in the choice of the cut-off functions ω_1,ω_2 or the weights γ_j in Definition 2.11 changes the symbol m in 2.11 only by a symbol in $\mathcal{R}_G^{\nu,d}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$.*

Proof. In Corollary 2.15 we saw already that a change of the weight for fixed cut-off functions results in a Green symbol. The lemma below, deals with the behavior under changes of the cut-off functions and therefore completes the proof of the above proposition. □

Lemma 2.19. *Let P be a Mellin asymptotic type and $h \in C^\infty(\Omega \times \Omega, M_P^{-\infty,d}(X))$. Fix $j \in \mathbf{N}$ and γ_1 with $\gamma-(\mu-\nu)-j-n/2 \le \gamma_1 \le \gamma - n/2$. Moreover, let ω be a cut-off function, $\varphi \in C_0^\infty(\mathbf{R}_+)$, and α a multi-index with $|\alpha| \le j$. Then both*

$$\varphi(t[\eta])\, t^{j-\nu} \mathrm{op}_M^{\gamma_1} h(y,y')\eta^\alpha\, \omega(t[\eta]) \text{ and } \omega(t[\eta])\, t^{j-\nu}\mathrm{op}_M^{\gamma_1} h(y,y')\eta^\alpha\, \varphi(t[\eta])$$

are elements of $\mathcal{R}_G^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.

Proof. Choose a cut-off function $\tilde\omega$ with $\tilde\omega\varphi = \varphi$. Write $\varphi(t[\eta]) = \varphi(t[\eta])(t[\eta])^{-k}\,[\eta]^k\tilde\omega(t[\eta])t^k$. In view of the fact that $t \mapsto \varphi(t)t^{-k}$ is a rapidly decreasing function on $\mathbf{R}_+$, the assertion follows from 2.6 and 2.16. □

Proposition 2.20. *The functions $h_{j\alpha} \in C^\infty(\Omega \times \Omega, M_P^{-\infty,d}(X))$ in Definition 2.11 are uniquely determined by the operator family*

$$\left\{ a(y,y',\eta) = \begin{bmatrix} m(y,y',\eta) & 0 \\ 0 & 0 \end{bmatrix} + g(y,y',\eta) : y,y' \in \Omega, \eta \in \mathbf{R}^q \right\}.$$

Proof. For each fixed choice of y, y', and η, we have a family of smoothing Mellin operators on the cone, as considered in [15, Proposition 3.1.27]. From this result we deduce that, for each j, the sum $\sum_{|\alpha|\le j} h_{j\alpha}(y,y',z)\eta^\alpha$ is uniquely determined. Since this is a polynomial in η, each of the terms $h_{j\alpha}(y,y',z)$ can be recovered. □

2.21 Conormal symbols. Let a, m, and g be as in Definition 2.11. For $j = 0,\dots,k-(\mu-\nu)-1$ we define the conormal symbol of order $\nu - j$ of a by

$$\sigma_M^{\nu-j}(a) = \sum_{|\alpha|\le j} h_{j\alpha}(y,y',z)\eta^\alpha.$$

This makes sense according to Proposition 2.20. Note that for $j \ge k-(\mu-\nu)$ the operator family $\{\omega_1(t[\eta])t^{j-\nu}\sum_{|\alpha|\le j}\mathrm{op}_M^{\gamma_j} h_{j\alpha}(y,y')\eta^\alpha\omega_2(t[\eta])\}$ defines an element in $\mathcal{R}_G^{\nu,d}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g})$ according to Proposition 2.18

2.22 Theorem: Adjoints. *Let m be as in Definition 2.11; assume additionally that $d=0$. Then the pointwise adjoint m^* defined by*

$$m^*(y,y',\eta) = m(y,y',\eta)^*$$

induces a symbol in $\mathcal{R}_{M+G}^{\nu,0}(\Omega\times\Omega\times\mathbf{R}^q,\mathbf{g}^*)$, where $\mathbf{g}^* = (-\gamma+\mu,-\gamma,(-k,0])$. In fact,

$$m^*(y,y',\eta) = \overline{\omega}_2(t[\eta])\sum_{j=0}^{k-1}\sum_{|\alpha|\le j} t^{j-\nu}\mathrm{op}_M^{-\gamma_j-n+\nu-j}\left(T^{\nu-j}h_{j\alpha}^{(*)}\right)(y,y')\eta^\alpha\overline{\omega}_1(t[\eta]),$$

where $h_{j\alpha}^{(*)}(y,y',z) = h_{j\alpha}(y,y',n+1-\overline{z})^*$ and the last asterisk denotes the adjoint in Boutet de Monvel's calculus.

Proof. It is well-known that, pointwise, the above formula furnishes the adjoint operator, see e.g. [14, Lemma 5.1.10]. In view of the fact that $T^{\nu-j}h_{j\alpha}(y,y',n+1-\overline{z})^*$ defines an element $C^\infty(\Omega \times \Omega, M_Q^{-\infty,d}(X))$ for a suitable (easily computable) Mellin asymptotic type, we get the desired result. □

The following proposition prepares the ground for treating compositions.

Proposition 2.23. *Let $h_1, h_2 \in C^\infty(\Omega\times\Omega, M_P^{-\infty,d}(X))$ for some Mellin asymptotic type P over $\Omega\times\Omega$. Choose γ_1,γ_2 with $\gamma - n/2 - j \le \gamma_1,\gamma_2 \le \gamma - n/2$. Then*

$$g(y,y',\eta) = \omega_1(t[\eta])\,t^{j-\nu}\mathrm{op}_M^{\gamma_1}h_1(y,y')(1-\omega_2(t[\eta]))\mathrm{op}_M^{\gamma_2}h_2(y,y')\omega_3(t[\eta])$$

is an element of $\mathcal{R}_G^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.

In order for the composition to make sense, we assume that the operators act between vector bundles in the following way: For fixed (y, y', η),

$$\begin{aligned} h_2(y,y',z) : C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1) &\to C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2), \quad \text{and} \\ h_1(y,y',\eta) : C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2) &\to C^\infty(\overline{X}, V_3) \oplus C^\infty(Y, W_3). \end{aligned}$$

Proof. In order to save space, we shall use, just for the moment, the notation $\omega_{j,\eta} = \omega_j(\cdot[\eta])$. Supposing that $\Gamma_{1/2-\gamma_1-k} \cap \pi_{\mathbf{C}} P = \emptyset$ we rewrite $g(y, y', \eta)$ as

$$\begin{aligned} &\omega_{1,\eta}\, t^{-\nu} \left\{ \mathrm{op}_M^{\gamma_1} h_1(y,y') - \mathrm{op}_M^{\gamma_1+k} h_1(y,y') \right\} (1-\omega_{2,\eta}) \mathrm{op}_M^{\gamma_1} h_2(y,y') \omega_{3,\eta} \\ &+ \omega_{1,\eta}\, t^{-\nu} \mathrm{op}_M^{\gamma_1+k} h_1(y,y')(1-\omega_{2,\eta}) \mathrm{op}_M^{\gamma_2} h_2(y,y') \omega_{3,\eta}. \end{aligned}$$

Due to the factor $1 - \omega_2$, the operator

$$(1-\omega_2)\mathrm{op}_M^{\gamma_2} h_2(y,y')\omega_3 : \mathcal{K}_1^{s,\gamma} \to \begin{array}{c} \mathcal{H}^{s,\gamma_1+k+n/2}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma_1+k+(n-1)/2}(Y^\wedge, W_2) \end{array}$$

is bounded, hence the last composition makes sense. Applying Corollary 2.15, the first term on the right hand side is a symbol in $\mathcal{R}_G^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. The second one we rewrite as

$$t^{k-\nu} \omega_{1,\eta} \mathrm{op}_M^{\gamma_1} T^{-k} h_1(y,y') t^{-k} (1-\omega_{2,\eta}) \mathrm{op}_M^{\gamma_2} h_2(y,y') \omega_{3,\eta};$$

then we apply Proposition 2.16 to see that it also is a Green symbol of the desired kind.

If h_1 has singularities on $\Gamma_{1/2-\gamma_1+k}$, then we may change γ_1 slightly without changing the operator; so we also get the assertion. Finally note that the adjoint is of the same type, so the proof is complete. □

2.24 Theorem: Compositions. *Let* $a \in \mathcal{R}_{M+G}^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ *and* $a' \in \mathcal{R}_{M+G}^{\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$. *Then the pointwise composition* $b(y, y', \eta) = a(y, y', \eta)a'(y, y', \eta)$ *defines an element* $b \in \mathcal{R}_{M+G}^{\nu+\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ *and therefore a mapping*

$$\mathcal{R}_{M+G}^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) \times \mathcal{R}_{M+G}^{\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) \to \mathcal{R}_{M+G}^{\nu+\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}).$$

This mapping is continuous. It restricts to continuous maps

$$\begin{aligned} \mathcal{R}_G^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) \times \mathcal{R}_{M+G}^{\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) &\to \mathcal{R}_G^{\nu+\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}), \\ \mathcal{R}_{M+G}^{\nu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) \times \mathcal{R}_G^{\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}) &\to \mathcal{R}_G^{\nu+\nu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}). \end{aligned}$$

The conormal symbols of b are given by

$$\sigma_M^{\nu+\nu'-j}(b) = \sum_{p+q=j} [T^{\nu'-q} \sigma_M^{\nu-p}(a)]\, \sigma_M^{\nu'-q}(a').$$

Proof. Consider symbols of the form

$$a = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} + g, \quad a' = \begin{bmatrix} m' & 0 \\ 0 & 0 \end{bmatrix} + g'$$

with m, m', g, and g' as in Definition 2.11. Applying Theorem 2.12, we see that the composition is a Green symbol whenever one of the factors is. So it only remains to consider the compositon of m and m'. By linearity we may focus on one summand of the type (2.11) for each of them, say

$$\begin{aligned} m &= \omega_1(t[\eta]) t^{\nu-j} \mathrm{op}_M^{\gamma_1} h_1(y,y') \eta^\alpha \omega_2(t[\eta]), \\ m' &= \tilde{\omega}_1(t[\eta]) t^{\nu'-j} \mathrm{op}_M^{\gamma_2} h_2(y,y') \eta^{\alpha'} \tilde{\omega}_2(t[\eta]). \end{aligned}$$

In the compostion we first remove the cut-off function in the middle at the expense of a Green term by applying Proposition 2.23. Then we commute the factor $t^{\nu'}$ to the left, taking advantage of Corollary 2.14. Next we move both weights γ_1, γ_2 to a single one, say γ_3; by Corollary 2.15 the error is a Green symbol. Finally, we apply the composition rule for Mellin symbols [15, Lemma 3.2.4] which implies that

$$\mathrm{op}_M^{\gamma_3} h_1(y,y')\mathrm{op}_M^{\gamma_3} h_2(y,y') = \mathrm{op}_M^{\gamma_3} h_3(y,y')$$

for $h_3 = h_1 h_2 \in C^\infty(\Omega \times \Omega, M_R^{\nu+\nu',d'}(X))$; here R is a resulting Mellin asymptotic type.

The rule for the conormal symbols is immediate from the standard case; it is not affected by the parameters y, y', η. The proof is complete. □

Operators of the considered type come up quite naturally:

Lemma 2.25. *Let $\pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$ and*

$$f = f(t,t',y,y',z,\eta) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\Omega \times \Omega, M_P^{-\infty,d}(X))\hat{\otimes}_\pi \mathcal{S}(\mathbf{R}^q).$$

For fixed (y,y',η) and cut-off functions ω_1, ω_2, we define the operator $m(y,y',\eta)$ on $C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$ by

$$\begin{aligned}(2.12)\qquad & m(y,y',\eta)u(t) \\ = \; & t^{-\mu}\frac{1}{2\pi i}\omega_1(t[\eta])\int_{\Gamma_{1/2-\gamma}}\int_0^\infty (t/t')^{-z} f(t,t',y,y',z,t\eta)\omega_2(t'[\eta])u(t')\frac{dt'}{t'}dz.\end{aligned}$$

Then m is an element of $\mathcal{R}_{M+G}^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$.

Proof. Employing Taylor's formula we write

$$\begin{aligned} & f(t,t',z,y,y',t\eta) \\ = \; & \sum_{j_1+|\alpha|=j\le k-1}\sum_{l\le k-1} c_{j_1 j_2 l}\partial_t^{j_1}\partial_\eta^\alpha \partial_{t'}^l f(0,0,y,y',z,0)\eta^\alpha t^j t'^l \\ & + \sum_{|\alpha|\le k} r_{1,\alpha}(t,t',y,y',z,t\eta)\eta^\alpha t^k + \sum_{|\alpha|\le j\le k-1} r_{2,j\alpha}(0,t',y,y',z,0)\eta^\alpha t^j t'^k \end{aligned}$$

with $r_{1,\alpha}, r_{2,j\alpha} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\Omega \times \Omega, M_P^{-\infty,d}(X))\hat{\otimes}_\pi \mathcal{S}(\mathbf{R}^q)$ and suitable constants $c_{j_1 j_2 l}$. The terms under the first sum on the right hand side are of the form (2.11).

Let us show that we obtain elements in $\mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})$ if we replace, on the right hand side of (2.12), the function $f(t,t',y,y',z,t\eta)$ by $r_{1,\alpha}(t,t',y,y',z,t\eta)\eta^\alpha t^k$ or $r_{2,j\alpha}(0,t',y,y',z,0)\eta^\alpha t^j t'^k$. Since $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+) = C^\infty(\overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\overline{\mathbf{R}}_+)$ we may assume that $r_{1,\alpha}(t,t',y,y',z,t\eta)$ is of the form $\varphi_1(t)\varphi_2(t')h(y,y',z)s(t[\eta])$ with $\varphi_1, \varphi_2 \in C^\infty(\overline{\mathbf{R}}_+)$, $h \in C^\infty(\Omega \times \Omega, M^{-\infty,d}(X))$, and $s \in \mathcal{S}(\mathbf{R}^q)$. We may even suppose that both φ_1 and φ_2 have compact support, since we will eventually multiply by $\omega_1(t[\eta])$ and $\omega_2(t[\eta])$ which both have compact support uniformly in η. Then, however, the assertion for $r_{1,\alpha}(t,t',y,y',z,t\eta)\eta^\alpha t^k$ follows by combining Propositions 2.6 and 2.16. With the terms for $r_{2,j\alpha}(0,t',y,y',z,0)\eta^\alpha t^j t'^k$ we can deal in the same way. □

2.26 Notation. Fix an element $h \in C^\infty(\Omega \times \Omega, M_P^{-\infty,d}(X))$ for some Mellin asymptotic type over $\Omega \times \Omega$. For given (y,y',z) the operator $h(y,y',z)$ is an element of $\mathcal{B}^{-\infty,d}(X;\mathbf{R}^q)$. We consider the operators $I + h(y,y',z)$ on the space

$$H^s = H^s(X,V_1) \oplus H^s(Y,W_1).$$

We will be interested in the structure of their inverses. In order for this to make sense we assume that $V_1 = V_2$ and $W_1 = W_2$.

2.27 Facts from Boutet de Monvel's calculus. If R in $\mathcal{B}^{-\infty,d}(X)$ and $I + R$ is invertible on H^s for some $s > d - 1/2$, then $(I+R)^{-1} = I + R'$ for suitable $R' \in \mathcal{B}^{-\infty,d}(X)$, see [14, Theorem 2.3.8].

For arbitrary $R \in \mathcal{B}^{-\infty,d}(X)$, the kernel of $I + R : H^s \to H^s$ is independent of s and consists of functions in $C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1)$ by elliptic regularity. The operators in $\mathcal{B}^{-\infty,d}(X)$ are compact on

H^s. Hence $I+R$ is a Fredholm operator of index zero; kernel and cokernel have the same dimension. As s increases the cokernel can only decrease. Therefore also the cokernel is independent of s and consists of smooth functions. Choosing orthonormal bases $\{\varphi_1,\ldots,\varphi_M\}$ and $\{\psi_1,\ldots,\psi_M\}$ of the kernel and cokernel, and defining $Fu=\sum_{j=1}^M \langle u,\varphi_j\rangle\,\psi_j$, we obtain a finite rank operator $F\in\mathcal{B}^{-\infty,0}(X)$ such that $I+R+F$ is invertible on H^s. The projection p onto its range is given by $pu=\sum_{j=1}^M\langle u,\psi_j\rangle\,\psi_j$, thus also is in $\mathcal{B}^{-\infty,0}(X)$.

2.28 Structure of the inverses. Let $y,y'\in\Omega$ and consider a finite strip $c_1\le \operatorname{Re} z\le c_2$ in $\mathbf{C}$. As $|\operatorname{Im} z|\to\infty$, the norm of $h(y,y',z)$ becomes small in the strip, uniformly for (y,y') in a compact subset of $\Omega\times\Omega$; therefore $I+h(y,y',z)$ will be invertible for large $|\operatorname{Im} z|$.

Next fix $(y_0,y_0',z_0)\in\Omega\times\Omega\times(\mathbf{C}\setminus\pi_{\mathbf{C}}P)$. According to 2.27 we find a finite rank operator $F\in\mathcal{B}^{-\infty,0}(X)$ such that $I+h(y_0,y_0',z_0)+F$ is invertible on H. Let $p\in\mathcal{B}^{-\infty,0}(X)$ be the projection onto its range and $q=I-p$.

For (y,y',z) in a small neighborhood of (y_0,y_0',z_0), $z\notin\pi_{\mathbf{C}}P$,

$$\begin{aligned} I+h(y,y',z) &= (I-F(I+h(y,y',z)+F)^{-1})(I+h(y,y',z)+F)\\ &(I-pF(y,y',z)q)(I-pF(y,y',z)p)(I+h(y,y',z)+F), \end{aligned} \tag{2.13}$$

where we used the abbreviation $F(y,y',z)=F(I+h(y,y',z)+F)^{-1})$. This decomposition has several consequences. First we note that $pF(y,y',z)q$ is an element of $\mathcal{B}^{-\infty,0}(X)$, and the first factor has the inverse $I+pF(y,y',z)q$. The last factor is invertible in a neighborhood of (y_0,y_0',z_0). Both the first and the last factor depend smoothly on (y,y') and analytically on z. The factor in the middle is a diagonal matrix; in the lower right corner we have the identity, in the upper left a finite rank matrix function, smooth in (y,y') and analytic in z. It is invertible whenever $I+h(y,y',z)$ is. The strip with $\pi_{\mathbf{C}}P$ removed is connected and $I+h(y,y',z)$ is known to be invertible for large $|\operatorname{Im} z|$. For fixed (y,y') it therefore is invertible outside a discrete set D contained in $\mathbf{C}\setminus\pi_{\mathbf{C}}P$. The singularities are poles of finite order; the coefficients of the principal part of the Laurent series are elements in $\mathcal{B}^{-\infty,0}(X)$. As (y,y') varies, the residue theorem shows that they vary smoothly with (y,y').

In general, the set D will depend on y and y'. There are, however, interesting situations when this is not the case. Under this assumption we shall see in Theorem 2.30 that the inverse $(I+h(y,y',z))^{-1}$ is of the form $I+h'(y,y',z)$ for some $h'\in C^\infty(\Omega\times\Omega,M_Q^{-\infty,d}(X))$, for a suitable Q.

The following lemma will be used in the proof of Theorem 2.30.

Lemma 2.29. ([14, Lemma 4.3.14]) *Let V be a neighborhood of zero in $\mathbf{C}$, E a Banach space, $N\in\mathbf{N}$, and $A_1,\ldots,A_N\in\mathcal{L}(E)$ operators of finite rank. Let H be an analytic function on V with values in $\mathcal{L}(E)$ with $H(z)e=0$ for all e in a finite-codimensional subspace E_0 of E. Then there is a $\delta>0$ such that the meromorphic E-valued function*

$$F(z)=I+H(z)+\sum_{k=1}^N A_k z^{-k} \tag{2.14}$$

is invertible for all $0<|z|<\delta$, unless it is nowhere invertible near 0.

Theorem 2.30. *Let $h\in C^\infty(\Omega\times\Omega,M_P^{-\infty,d}(X))$ with a Mellin asymptotic type P over $\Omega\times\Omega$. Assume that there is a discrete set D such that, for all $y,y'\in\Omega$ and all $z\in\mathbf{C}\setminus D$, the operator $I+h(y,y',z)$ is invertible. Let $\Omega_0\subset\subset\Omega$. Then the inverse $(I+h(y,y',z))^{-1}|_{\Omega_0\times\Omega_0}$ is of the form $I+h'(y,y',z)$ for some $h'\in C^\infty(\Omega_0\times\Omega_0,M_Q^{-\infty,d}(X))$, where Q is a Mellin asymptotic type over $\Omega_0\times\Omega_0$ with $\pi_{\mathbf{C}}Q\subseteq D$.*

Proof. Let us first show that the set D has no finite accumulation point. By 2.28, such a point necessarily would be an element $p_0\in\pi_{\mathbf{C}}P$. Near p_0 write $I+h(y,y',z)=I+h_0(y,y',z)+\sum_{k=0}^K F_j(y,y')(z-p_0)^{-k-1}=I+(h_0+h_s)(y,y',z)$ with some function $h_0\in C^\infty(\Omega\times\Omega,M_P^{-\infty,d}(X))$, which is analytic near p_0, a smooth family F_j of finite rank operators, and $h_s(y,y',z)=\sum_{k=0}^K F_j(y,y')(z-p_0)^{-k-1}$ for a suitable K.

Fix y and y'. The function h_0 also satisfies the assumptions of Lemma 2.26. Since it is analytic near p_0 we may conclude as in 2.28 that it is invertible in a neighborhood of p_0, execpt for (possibly) the point p_0, where it might have a pole of finite order. Hence

$$I + h(y,y',z) = (I + h_0(y,y',z))(I + (I + h_0(y,y',z))^{-1}h_s(y,y',z)). \tag{2.15}$$

We note that $I + (I + h_0(y,y',z))^{-1}h_s(y,y',z)$ is of the form (2.14) due to the fact that the operators F_j have finite rank. Applying Lemma 2.29, it is invertible near p_0, although possibly not in p_0. The same is true for $I + h(y,y',z)$.

Since h depends continuously on y and y' we may vary both variables a little and still have invertibility in a slightly smaller neighborhood. Since $\overline{\Omega}_0 \times \overline{\Omega}_0$ is compact, D cannot have accumulation points.

For $p_0 \in D$, the order of the pole of $(I + h(y,y',z))^{-1}$ might depend on y and y'. However, it will be uniformly bounded as y and y' vary over Ω_0: According to 2.28, the operator $I + h(y,y',z)$ is invertible whenever a scalar-valued function in $C^\infty(\Omega \times \Omega, \mathcal{A}(\mathbf{C} \setminus \pi_{\mathbf{C}}P))$ is, namely the determinant of the matrix operator in (2.13). For this scalar function, however, the order of the pole of its inverse at p_0 is the order of its zero at p_0; this in turn is the index of the first non-vanishing Taylor coefficient. By continuity, this number can locally decrease only. Since $\overline{\Omega}_0 \times \overline{\Omega}_0$ is compact, the order of the poles will be uniformly bounded.

So let Q be the Mellin asymptotic type with entries (p_j, m_j, L_j), where p_j are those elements of D where $I + h(y,y',z)|_{\Omega_0 \times \Omega_0}$ is not invertible, m_j is the bound on the order of the pole, and $L_j^\alpha(y,y')$, for $y,y' \in \Omega_0$, $\alpha = (\alpha_1, \alpha_2)$ an arbitrary multi-index, is the space of operators in $\mathcal{B}^{-\infty,d}(X)$ spanned by the coefficients of the principal part of the Laurent series of $\partial_y^{\alpha_1}\partial_{y'}^{\alpha_2}(I + h(y,y',z))^{-1}$ near $z = p_j$. Notice that $\mathrm{Re}\, p_j \to \mp\infty$ as $j \pm \infty$, since D has no finite accumulation point and $I + h(y,y',\cdot)$ is invertible for large $\mathrm{Im}\, z$ in any finite strip.

Supposing that the inverse to $I + h(y,y',z)$ exists for some fixed choice of y, y', and z, it is of the form $I + R$ with $R \in \mathcal{B}^{-\infty,d}(X)$ by 2.27. The continuity of the inversion implies that

$$(I + h(y,y',z))^{-1} = I - h(y,y',z)(I + h(y,y',z))^{-1} = I + h'(y,y',z) \tag{2.16}$$

with $h' \in C^\infty(\Omega_0 \times \Omega_0, \mathcal{A}(\mathbf{C} \setminus \pi_{\mathbf{C}}Q, \mathcal{B}^{-\infty,d}(X)))$. We saw already that the singularities are poles, the coefficients of the principal part of the Laurent series for $\partial_y^{\alpha_1}\partial_{y'}^{\alpha_2}(I + h(y,y',z))^{-1}$ near p_j being finite rank operators in the finite-dimensional subspace $L_j^{\alpha_1,\alpha_2}(y,y')$ of $\mathcal{B}^{-\infty,d}(X)$.

It remains to check the decay properties. Pick an arbitrary strip and a semi-norm $\|\cdot\|$ for $\mathcal{B}^{-\infty,d}(X)$. For large $|\mathrm{Im}\, z|$, say $|\mathrm{Im}\, z| > \rho$, we have $\|h(y,y',z)\| \le 1/2$, uniformly on $\Omega \times \Omega$. Therefore $\|(I + h(y,y',z))^{-1}\| \le 2$. Employing (2.16) we conclude that $\|h'(y,y',z)\| = O(\langle \mathrm{Im}\, z\rangle^{-N})$ for arbitrary N, uniformly for y, y' in compact subsets of $\Omega \times \Omega$. Since this estimate also holds on the compact set $\Omega_0 \times \Omega_0 \times \{z : |\mathrm{Im}\, z| \le \rho, \mathrm{dist}(z, D) \ge \epsilon\}$ the proof is complete. □

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