A simple remark on eigenvalues of Hecke operators on Siegel modular forms

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For an integer k let $S_k^{(n)}$ be the complex vector space of Siegel cusp forms of weight k on $\Gamma_n = Sp_{2n}(\mathbb{Z})$. For $m \in \mathbb{N}$ denote by $T_k^{(n)}(m)$ the Hecke operator which acts on $S_k^{(n)}$ according to

$$f|T_k^{(n)}(m) = m^{nk-n(n+1)/2} \sum_{M \in \Gamma_n \backslash \mathfrak{M}_m^{(n)}} f|_k M;$$

here $\mathfrak{M}_m^{(n)}$ is the set of integral $2n \times 2n$ matrices M satisfying $M' J_n M = m J_n$, where $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ and M' is the transpose of M, the summation is over a set of representatives for the action of Γ_n on $\mathfrak{M}_m^{(n)}$ by left-multiplication and $f|_k M$ is defined by

(1) $$(f|_k M)(Z) = \det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1})$$

$(Z \in \mathfrak{H}_n$ = Siegel upper half-space of degree n, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix})$.

R. Weissauer has proved the following

Theorem ([6], §6.). <u>Let</u> λ_m <u>be an eigenvalue of</u> $T_k^{(n)}(m)$, $m>1$. <u>Then</u>

(2) $$|\lambda_m| < m^{(nk-n(n+1))/2} d(m),$$

<u>where</u> d(m) <u>is the number of elements in</u> $\Gamma_n \backslash \mathfrak{M}_m^{(n)}$. <u>In particular, if</u> p <u>is prime, then</u>

$$|\lambda_p| < p^{(nk-n(n+1))/2} \prod_{\nu=1}^{n} (p^\nu + 1).$$

The above inequality is useful at several places (cf. [2],[6]; cf. also [1] for n=1, where it can be used to shorten some proofs in the theory of newforms).

The proof of (2) given in [6] is easy and essentially only uses the strong approximation theorem for the symplectic group.

The purpose of this note is to give a second (and almost trivial) proof of (2) based on the existence of the Petersson inner product (for n=1 this proof already appeared in [4], the case n>1 is a direct generalization).

We now recall some elementary properties of the Petersson product (cf. [5], pp.37-40 and [3], pp.270-273).

Let Γ be an admissible group, i.e. there is $\alpha\in\mathfrak{M}_m^{(n)}$ (for some $m\in\mathbb{N}$) such that $\alpha^{-1}\Gamma\alpha$ is of finite index in Γ_n. We put $\bar\Gamma=\Gamma\cdot\{\pm E_n\}/\{\pm E_n\}$. If f and g are Siegel cusp forms of weight k on Γ, we set

$$\langle f,g\rangle = \frac{1}{[\bar\Gamma_n:\bar\Gamma]}\int_{F_\Gamma} f(Z)\overline{g(Z)}(\det Y)^k\, d\omega,$$

where $X=\operatorname{Re}Z$, $Y=\operatorname{Im}Z$, $d\omega=dXdY/(\det Y)^{n+1}$ is the invariant symplectic volume element and F_Γ is a fundamental domain for the action of Γ on $\mathfrak{H}_n$. One easily shows that the integral is independent of the fundamental domain, converges absolutely and is also independent of the choice of Γ. Moreover, if for any symplectic similitude M with rational coefficients we define $f|_kM$ by (1), then

$$(3)\quad \langle f|_kM,g\rangle = \langle f,g|_kmM^{-1}\rangle \qquad (M\in\mathfrak{M}_m^{(n)}).$$

We put

$$\|f\| = \langle f,f\rangle^{1/2}.$$

Then from (3) we see that for $M\in\mathfrak{M}_m^{(n)}$

$$(4)\quad \|f|_kM\| = m^{-nk/2}\|f\|.$$

Now let $f\in S_k^{(n)}$ be a non-zero eigenform of $T_k^{(n)}(m)$ with eigenvalue λ_m. We have

$$\lambda_m \langle f,f\rangle = \langle f|T_k^{(n)}(m),f\rangle = m^{nk-n(n+1)/2} \sum_{M\in\Gamma_n\backslash\mathfrak{M}_m^{(n)}} \langle f|_k M,f\rangle .$$

From the Cauchy-Schwartz inequality and (4) we obtain

$$|\langle f|_k M,f\rangle| \le \|f|_k M\| \; \|f\|$$
$$= m^{-nk/2} \|f\|^2$$

with equality in the first line if and only if $f|_k M$ is proportional to f. Hence to prove (2) it is sufficient to show that if $f|_k M$ is proportional to f, for all $M\in\mathfrak{M}_m^{(n)}$, then f=0. This, however, is obvious; in fact, $\begin{pmatrix} mE_n & 0 \\ 0 & E_n \end{pmatrix}$ is in $\mathfrak{M}_m^{(n)}$, and from

$$f(Z) = cf(mZ) \qquad (c\in\mathbb{C})$$

it follows easily, e.g. by comparing Fourier coefficients, that f must be zero, if m>1.

References

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