H.-J. Baues and F. Muro

THE CHARACTERISTIC COHOMOLOGY CLASS OF A TRIANGULATED CATEGORY

Introduction

This is a collection of five papers on the foundation of triangulated categories in the context of groupoid-enriched categories, termed track categories, and characteristic cohomology classes. As a main result it is shown that given an additive category $\mathbf{A}$ with a translation functor $t\colon \mathbf{A} \to \mathbf{A}$ and a class in translation cohomology

$$\nabla \in H^3(\mathbf{A}, t)$$

then two simple properties of ∇ imply that $(\mathbf{A}, t)$ is a triangulated category. The cohomology class ∇ yields an equivalence class $(\mathbf{B}, [s])$ where $\mathbf{B}$ is a track category with homotopy category $\mathbf{A}$ and $[s]$ is the homotopy class of a pseudofunctor $s\colon \mathbf{B} \rightsquigarrow \mathbf{B}$ inducing t. The two properties of ∇ correspond to natural axioms on $\mathbf{B}$ and s which again imply that $(\mathbf{A}, t)$ is a triangulated category.

The five papers of this volume depend on each other by cross references, but each paper can be read independently of the others so that the reader is free to choose one of the papers to start. Each paper has its own abstract, introduction and literature.

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Triangulated track categories

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Abstract

The concept of a triangulated homotopy category can be canonically lifted to the level of a groupoid-enriched category. This way two natural axioms on track triangles replace the four somewhat obscure axioms of a triangulated category.

Introduction

The axioms of a triangulated category are deduced from properties of cofiber sequences (or fiber sequences) in a stable homotopy category **A**. Here **A** is an additive category with a translation functor t as for example the homotopy category of chain complexes or the stable homotopy category of spectra. Cofibers or fibers, however, are defined in terms of homotopies and, as observed by Gabriel-Zisman [GZ], represent "cone functors" obtained by tracks. Here tracks are homotopy classes of homotopies. Therefore it is natural to consider properties of cofiber sequences in a *track category*, i.e. a groupoid enriched category or a 2-category for which all 2-cells are invertible. Gabriel-Zisman deal with the general unstable fiber sequences in a track category. In this paper, however, we consider stable fiber-cofiber sequences in a track category and we lift the classical concept of triangulation of a category (as introduced by Puppe and Verdier) to the new concept of triangulation of a track category. A triangulated category is an additive category with translation functor and a distinguished class of *exact triangles*. Similarly a triangulated track category is an additive track category with a translation track functor and a distinguished class of *track triangles* satisfying two natural axioms (TTr1) and (TTr2) which replace the rather obscure axioms of a triangulated category. As a main result we show that a triangulated track category always has a homotopy category which is a triangulated category in the classical sense. We also obtain this way a description of good morphisms between exact triangles in the sense of [Ne]. Further applications are obtained in [II] where we compute an algebraic model for the category of principal maps between 2-stage Postnikov spectra.

In [IV] one finds the cohomological interpretation of the results on track categories in this paper.

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1 Triangulated categories

To fix notation we repeat the following well known concepts. Let $\mathbf{A}$ be an additive category. A *translation functor* $t : \mathbf{A} \to \mathbf{A}$ is an additive automorphism of $\mathbf{A}$. Hence

$$t : \mathrm{Hom}_{\mathbf{A}}(X,Y) \cong \mathrm{Hom}_{\mathbf{A}}(tX,tY) \tag{1.1}$$

is an isomorphism of abelian groups and $t(X \oplus Y) = (tX) \oplus (tY)$. The inverse t^{-1} of t admits natural isomorphisms $tt^{-1}X \cong X$ and $t^{-1}tX \cong X$ in $\mathbf{A}$. We say that t is *strict* if these isomorphisms are the identity of X.

Now let $(\mathbf{A}, t)$ be an additive category with a translation functor. A *triangle* in $\mathbf{A}$ is a diagram in $\mathbf{A}$ of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} tA \tag{1.2}$$

Each such triangle yields an infinite sequence of maps $d_n : X_{n-1} \longrightarrow X_n$, $n \in \mathbb{Z}$, in $\mathbf{A}$ together with isomorphisms $\tau_X : X_n \cong tX_{n-3}$ satisfying $(td_{n-3})\tau_X = \tau_X d_n$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & X_{-3} \arrow[r] & X_{-2} \arrow[r, "d_{-1}"] \arrow[d, equal] & X_{-1} \arrow[r, "d_0"] \arrow[d, equal] & X_0 \arrow[r, "d_1"] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & X_2 \arrow[r] & \cdots \\
 & & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & tA & &
\end{tikzcd} \tag{1}$$

We define X_n for $n < -2$ by $X_n = t^{-1}X_{n+3}$ and for $n > 1$ by $X_n = tX_{n-3}$. Then $tt^{-1} \cong 1$ yields the isomorphism $X_n \cong tX_{n-3}$ for $n \leq 0$. If $X = (X_n, d_n)$ is a cochain complex, that is $d_n d_{n-1} = 0$ for all n, then X is a *candidate triangle* in $\mathbf{A}$, see [Ne]. A morphism between triangles is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & tA \arrow[d, "tf"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & tA'
\end{tikzcd} \tag{2}$$

This is an isomorphism if f, g, h are isomorphisms in $\mathbf{A}$. In the case of candidate triangles such a morphism induces a chain map $\alpha : X \longrightarrow X'$ between the associated cochain complexes with $\alpha_{-2} = f$, $\alpha_{-1} = g$, $\alpha_0 = h$ and $\alpha_n = t\alpha_{n-3}$ for $n \in \mathbb{Z}$. Following [Pu] and [Ve], see also [We] and [GM], we have the following notion of a triangulated category.

Definition 1.3. Let $\mathbf{A}$ be an additive category with a translation functor t. Then $\mathbf{A}$ is a *triangulated category* if $\mathbf{A}$ is equipped with a distinguished family $\mathcal{E}$ of triangles (called the *exact* triangles) which is subject to the following axioms:

(Tr0) Any triangle isomorphic to an exact triangle is exact. For any object A in $\mathbf{A}$ the identity 1_A of A yields the exact triangle:

$$A \xrightarrow{1} A \longrightarrow * \longrightarrow tA.$$

where $*$ is the zero object of $\mathbf{A}$.

(Tr1) Any morphism in $A \longrightarrow B$ in $\mathbf{A}$ is part of an exact triangle:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} tA.$$

Here the axiom (Tr3) below implies that the isomorphism type of C is well defined by u. We also write $C = C_u$ and we call $v = i_u : B \longrightarrow C_u$ the *inclusion* and $w = q_u : C_u \longrightarrow tA$ the *projection.*

(Tr2) If a triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} tA$ is exact then the triangles $B \xrightarrow{v} C \xrightarrow{w} tA \xrightarrow{-tu} tB$ and $t^{-1}C \xrightarrow{-t^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C$ are exact.

(Tr3) Any commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & tA \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & tA'
\end{tikzcd}$$

where the rows are exact extends to a morphism of triangles as defined in (2.1).

(Tr4) *The octahedral axiom*: For a composite $A \xrightarrow{u} B \xrightarrow{v} C$ let C_u, C_v and C_{vu} be chosen as in (Tr1). Then there exists an exact triangle

(#) $$C_{vu} \xrightarrow{\bar{u}} C_v \xrightarrow{\bar{v}} tC_u \xrightarrow{-t\bar{w}} tC_{vu}$$

with the following properties. The maps $\bar{u}$ and $\bar{w}$ are maps for which the diagram

(##) $$\begin{tikzcd}
B \arrow[r, "v"] \arrow[d, "i_u"'] & C \arrow[r, equal] \arrow[d, "i_{uv}"] & C \arrow[d, "i_v"] \\
C_u \arrow[r, "\bar{w}"] \arrow[d, "q_u"'] & C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "q_{vu}"] & C_v \arrow[d, "q_v"] \\
tA \arrow[r, equal] & tA \arrow[r, "tu"] & tB
\end{tikzcd}$$

commutes and $\bar{v}$ is the composite

$$\bar{v} : C_v \xrightarrow{q_v} tB \xrightarrow{ti_u} tC_u$$

Axiom (Tr4) is a reformulation of Verdier's octahedral axiom as described in Weibel [We]. This axiom does not say that all maps $\bar{u}$, $\bar{w}$ satisfying (##) yield an exact triangle (#). It is only required that there exist such maps. Neeman [Ne], [Ne1] raises the question to characterize maps $\bar{u}$, $\bar{w}$ for which (#) is an exact triangle, compare section 9 below.

It is easy to see that (Tr0), (Tr2) and (Tr3) imply that for all objects U in $\mathbf{A}$ the induced sequences of abelian groups

$$\leftarrow hom_{\mathbf{A}}(X_0, U) \longleftarrow hom_{\mathbf{A}}(X_1, U) \longleftarrow hom_{\mathbf{A}}(X_2, U) \leftarrow$$

$$\rightarrow hom_{\mathbf{A}}(U, X_0) \longrightarrow hom_{\mathbf{A}}(U, X_1) \longrightarrow hom_{\mathbf{A}}(U, X_2) \rightarrow$$

are long exact sequences.

2 Additive track categories

A *track category* is a category enriched in groupoids; in particular, for all of its objects X, Y their hom-groupoid $[\![X,Y]\!]$ is given, whose objects are maps $f : X \to Y$ and whose morphisms, denoted $\alpha : f \Rightarrow f'$, are called tracks. Equivalently, a track category is a 2-category all of whose 2-cells are invertible. For a track $\alpha : f \Rightarrow f'$ and maps $g : Y \to Y'$, $e : X' \to X$, the resulting composite tracks will be denoted by $g\alpha : gf \Rightarrow gf'$ and $\alpha e : fe \Rightarrow f'e$. Moreover there is a vertical composition of tracks, i.e. composition of morphisms in the groupoids $[\![X,Y]\!]$; for $\alpha : f \Rightarrow f'$ and $\beta : f' \Rightarrow f''$ it will be denoted by $\beta\square\alpha : f \Rightarrow f'$. An inverse of a track α with respect to this composition will be denoted $\alpha^{\boxminus}$. The identity track $f \Rightarrow f$ is denoted by $0^{\square}_f = 0^{\square}$ and is called the trivial track. A track category has the *homotopy category* which is an ordinary category obtained by identifying *homotopic* maps $f \cong f'$, i.e. maps f, f' for which there exists a track $f \Rightarrow f'$. If $\mathbf{A}$ is the homotopy category then we denote the associated track category by

$$\mathbf{B} = (\mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \to \mathbf{A}) \tag{2.1}$$

Here $\mathbf{B}_0$ is the underlying ordinary category of 1-cells. We have the quotient functor $\mathbf{B}_0 \to \mathbf{A}$ which is the identity on objects and which carries a 1-cell f to its homotopy class $\{f\}$. Moreover $\mathbf{B}_1$ denotes the category of 2-cells with the same objects as in $\mathbf{B}_0$ but with morphisms from X to Y being tracks $\alpha : f \Rightarrow f'$ with f, $f' : X \to Y$ in $\mathbf{B}_0$, composite of α and β in the diagram

$$\begin{tikzcd}
Z & Y \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow\alpha"] & X \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow\beta"]
\end{tikzcd}$$

being

$$\alpha\beta = \alpha g'\square f\beta = f'\beta\square\alpha g : fg \Rightarrow fg'. \tag{2.2}$$

There are thus two functors $\mathbf{B}_1 \to \mathbf{B}_0$ which are identity on objects and which send a morphism $\alpha : f \Rightarrow f'$ to f, resp. f'.

An object $X \times Y$ in $\mathbf{B}$ is a *strong product* if projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are given which induce for all objects U in $\mathbf{B}$ an isomorphism of groupoids

$$(p_{1*}, p_{2*}) : [\![U, X \times Y]\!] \to [\![U, X]\!] \times [\![U, Y]\!]$$

If this is an equivalence of groupoids then $X \times Y$ is a weak product.

We now assume given a track category $\mathbf{B}$ such that its homotopy category is an additive category like $\mathbf{A}$ from section 1,

$$\mathbf{B}_{\simeq} = \mathbf{A}$$

and that moreover $\mathbf{B}$ has a strong zero object, that is, an object $*$ such that for every object X of $\mathbf{B}$, $[\![X, *]\!]$ and $[\![*, X]\!]$ are trivial groupoids with a single morphism. It then follows that in each $[\![X,Y]\!]$ there is a distinguished map $0_{X,Y}$ obtained by composing the unique maps $X \to *$ and $* \to Y$. The identity track of this map will be denoted just by $0^{\square}$. Note that $0_{X,Y}$ may also admit non-identity self-tracks; one however has

$$0_{Y,Z}\beta = 0^{\square} = \alpha 0_{X,Y} \tag{2.3}$$

for any $\alpha : f \Rightarrow f', f, f' : Y \to Z, \beta : g \Rightarrow g', g, g' : X' \to Y$.

The secondary analogue of an additive category is an additive track category defined as follows.

Definition 2.4. A track category $\mathbf{B}$ is called *additive* if it has a strong zero object $*$, the homotopy category $\mathbf{A} = \mathbf{B}_{\simeq}$ is additive and moreover $\mathbf{B}$ is a linear track extension

$$D \to \mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \to \mathbf{A}$$

of $\mathbf{A}$ by a biadditive bifunctor

$$D : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}.$$

Explicitly, this means the following: a biadditive bifunctor D as above is given together with a system of isomorphisms

$$\sigma_f : D(X, Y) \to \mathrm{Aut}_{[\![X,Y]\!]}(f) \tag{2.5}$$

for each 1-arrow $f : X \to Y$ in $\mathbf{B}$, such that for any $f : X \to Y$, $g : Y \to Z$, $a \in D(X, Y)$, $b \in D(Y, Z)$, $\alpha : f \Rightarrow f'$ one has

$$\begin{aligned}\sigma_{gf}(ga) &= g\sigma_f(a);\\ \sigma_{gf}(bf) &= \sigma_g(b)f;\\ \alpha\square\sigma_f(a) &= \sigma_{f'}(a)\square\alpha.\end{aligned}$$

Remark 2.6. Using 2.5 we can identify the bifunctor D via the natural equation

$$D(X, Y) = \mathrm{Aut}(0_{X,Y}).$$

Here $\mathrm{Aut}(0_{X,Y})$ is easily seen to be a well defined bifunctor.

We use a translation functor t as in (1.1) for the definition of the *coefficient bifunctor*

$$\begin{aligned}&D : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}\\ &D(X, Y) = hom_{\mathbf{A}}(tX, Y)\end{aligned} \tag{2.7}$$

Then t yields a natural isomorphism

$$t_D : D \xrightarrow{\cong} D(t^{op} \times t) = t^*D$$

which on objects is given up to sign by the functor t, that is

$$t_D : D(X, Y) = hom_{\mathbf{A}}(tX, Y) \cong hom_{\mathbf{A}}(ttX, tY) = (t^*D)(X, Y)$$

is defined by $t_D(\alpha) = -t(\alpha)$.

Definition 2.8. Let $\mathbf{B}$ be an additive track category with homotopy category $\mathbf{A}$ and coefficient bifunctor $D(X,Y) = hom_{\mathbf{A}}(tX,Y)$ where t is a strict translation functor on $\mathbf{A}$. Then a *strict translation functor for* ***B*** is a track functor (i.e. a functor enriched in groupoids) as in the commutative diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "t_D"'] & \mathbf{B}_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "t^{(1)}"] & \mathbf{B}_0 \arrow[r] \arrow[d, "t^{(0)}"] & \mathbf{A} \arrow[d, "t"] \\
D \arrow[r] & \mathbf{B}_1 \arrow[r, shift left] \arrow[r, shift right] & \mathbf{B}_0 \arrow[r] & \mathbf{A}
\end{tikzcd}$$

such that $t^{(0)}$, $t^{(1)}$ are strict automorphisms of categories inducing t and t_D respectively. We also write $t = t^{(0)}$ and $t = t^{(1)}$.

Example 2.9. The category $Ch(\mathbf{A})$ of chain complexes in an additive category $\mathbf{A}$ is a track category with tracks given by homotopy classes of homotopies. In fact $Ch(\mathbf{A})$ is an additive track category with a strict translation functor given by the shift functor.

Example 2.10. The category of spectra *Spec* as introduced by Puppe [Pu] or Adams [Ma] is a track category with tracks defined by homotopy classes of homotopies. Moreover *Spec* is an additive track category with strict translation functor given by the shift functor.

Remark 2.11. The main results below are proved in the presence of a strict translation functor for $\mathbf{B}$. It is also possible to deal with with translation functors for $\mathbf{B}$ which are not strict. Such (non strict) translation functors are given by a pseudo functor $t : \mathbf{B} \to \mathbf{B}$ inducing t on $\mathbf{A}$ and $-t$ on D as in (2.8). In [IV] we study the more general case of non strict translation functors. It will be, however, more convenient for the reader to consider first the case of strict translation functors as this is done in this paper.

Example 2.12. Let $\mathbf{C}$ be a cofibration category with a zero object, for example given by a Quillen model category, see [Ba]. Assume that the suspension functor

$$\Sigma : \mathbf{C}_{cf}/\simeq \to \mathbf{C}_{cf}/\simeq$$

is an equivalence of categories. Here $\mathbf{C}_{cf}$ is the full subcategory of cofibrant and fibrant objects in $\mathbf{C}$. Then $\mathbf{C}_{cf}$ is an additive track category with a (non strict) translation functor Σ as in (2.11). The dual result holds for fibration categories. Tracks are defined as in [Ba](II.5).

3 Cone functors in additive track categories

In topology one defines the mapping cone C_f of a map $f : A \to B$ by the push out $B \cup_f CA$ of $CA \supset A \to B$. Here CA is the cone of A. Given a map $\alpha : B \to U$ and a homotopy $\hat{\alpha} : \alpha f \simeq 0$ one gets a map $\alpha \cup \hat{\alpha} : C_f \to U$. Therefore the set of homotopy classes of maps $C_f \to U$ denoted by $Cone_f(U)$ can be described only in terms of tracks which are homotopy classes of homotopies. This leads to the following definition of cone functors in track categories as introduced by Gabriel-Zisman, see the remark (3.7) below.

Let $\mathbf{B}$ be an additive track category as in (2.4) with homotopy category $\mathbf{A}$. Let $\mathbf{Ab}$ be the category of abelian groups. Given a morphism $f : X \to Y$ in $\mathbf{B}_0$ we define the *cone functor*

(3.1) $$Cone_f : \mathbf{A} \longrightarrow \mathbf{Ab}$$

and the *dual cone functor*

$$Cone^f : \mathbf{A}^{op} \longrightarrow \mathbf{Ab}$$

as follows. For an object U in $\mathbf{A}$ let $cone_f(U)$ be the set of all pairs $(\alpha, \hat{\alpha})$ where $\alpha : Y \to U$ is a map in $\mathbf{B}_0$ and

$$\hat{\alpha} : \alpha f \Rightarrow 0$$

is a track in $\mathbf{B}$. Two such pairs $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ in $cone_f(U)$ are equivalent if there exists a track $H : \alpha \Rightarrow \beta$ such that pasting of tracks in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend left=35, "\Uparrow\hat{\alpha}"] \arrow[rr, phantom, bend right=35, "\Downarrow\hat{\beta}"'] & Y \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Downarrow H"] & U
\end{tikzcd}$$

yields the trivial track $0^\square$. Now let

(1) $$Cone_f(U) = cone_f(U)/\sim$$

be the set of equivalence classes $\{\alpha, \hat{\alpha}\}$. Given a map $v : U \to V$ in $\mathbf{A}$ we choose v_0 in $\mathbf{B}_0$ representing v and we define the induced map

$$v_* : Cone_f(U) \longrightarrow Cone_f(V)$$

by $v_* : \{\alpha, \hat{\alpha}\} = \{v_0\alpha, v_0\hat{\alpha}\}$. One readily checks that v_* is well defined. Below we show that $Cone_f(U)$ is an abelian group.

Similarly let $cone^f(U)$ be the set of all pairs $(\alpha, \hat{\alpha})$ where $\alpha : U \to X$ is a map in $\mathbf{B}_0$ and $\hat{\alpha} : f\alpha \Rightarrow 0$ is a track in $\mathbf{B}$. Now two such pairs $(\alpha, \hat{\alpha})$, $(\beta, \hat{\beta})$ are equivalent if there exists a track $G : \alpha \Rightarrow \beta$ such that pasting of tracks in the diagram

$$\begin{tikzcd}
U \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Uparrow G"] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend left=35, "\Uparrow\hat{\alpha}"] \arrow[rr, phantom, bend right=35, "\Downarrow\hat{\beta}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

yields the trivial track $0^\square$. As above let

(2) $$Cone^f(U) = cone^f(U)/\sim$$

be the set of equivalence classes $\{\alpha, \hat{\alpha}\}$. Then a map $v : V \to U$ in $\mathbf{A}$ represented by v_0 in $\mathbf{B}_0$ yields the induced map

$$v^* : Cone^f(U) \longrightarrow Cone^f(V)$$

by $v^*\{\alpha, \hat{\alpha}\} = \{\alpha v_0, \hat{\alpha} v_0\}$.

A track $H : f' \Rightarrow f$ in **B** induces natural isomorphisms

$$(3) \qquad \begin{aligned} &H_* : Cone_f \cong Cone_{f'} \quad , resp. \\ &H_* : Cone^f \cong Cone^{f'} \end{aligned}$$

which carries $\{\alpha, \hat{\alpha}\}$ to $\{\alpha, \hat{\alpha} \square \alpha H\}$, resp. to $\{\alpha, \hat{\alpha} \square H\alpha\}$. Hence up to natural isomorphism cone functors depend only on the homotopy class of f.

Lemma 3.2. *Let $f : X \to Y$ be a map in $\boldsymbol{B}_0$ and let U, W be objects in $\boldsymbol{A}$ with direct sum $U \oplus W$ and projections $p_1 : U \oplus W \to U$, $p_2 : U \oplus W \to W$ in $\boldsymbol{A}$. Then*

$$(p_{1*}, p_{2*}) : Cone_f(U \oplus W) \to Cone_f(U) \times Cone_f(W)$$

is a bijection of sets.

Proof. Let $\{\alpha, \hat{\alpha}\} \in Cone_f(U), \{\beta, \hat{\beta}\} \in Cone_f(W)$. Then there exists γ together with tracks $Q_1 : p_1\gamma \Rightarrow \alpha$, $Q_2 : p_2\gamma \Rightarrow \beta$ since $U \oplus W$ is a direct sum in **A**. Moreover there exists $H : \gamma f \Rightarrow 0$ since $(\alpha, \beta)f = 0$ in **A**. Now $\hat{\alpha} \square Q_1 f$ and $p_1 H$ differ by $h_1 \in D(X, U)$ and $\hat{\beta} \square Q_2 f$ and $p_2 H$ differ by $H_2 \in D(X, W)$. Therefore we can alter H by $(H_1, H_2) \in D(X, U \oplus W)$ and get $\hat{\gamma}$ such that $(p_1)_*\{\gamma, \hat{\gamma}\} = \{\alpha, \hat{\alpha}\}$ and $(p_2)_*\{\gamma, \hat{\gamma}\} = \{\beta, \hat{\beta}\}$. Hence the map in (6.1) is surjective. Now let $\{\gamma, \hat{\gamma}\}$, $\{\varepsilon, \hat{\varepsilon}\} \in Cone_f(U \oplus W)$ be given with

$$(1) \qquad (p_i)_*\{\gamma, \hat{\gamma}\} = (p_i)_*\{\epsilon, \hat{\epsilon}\} \quad \text{for } i = 1, 2.$$

We have to show $\{\gamma, \hat{\gamma}\} = \{\varepsilon, \hat{\varepsilon}\}$, that is, there is a track $H : \gamma \Rightarrow \varepsilon$ with $\hat{\varepsilon} \square (Hf) = \hat{\gamma}$. By (1) we obtain tracks $H_i : p_i\gamma \Rightarrow p_i\varepsilon$ with $(p_i\hat{\varepsilon}) \square (H_i f) = p_i\hat{\gamma}$. Hence γ and ε represent the same homotopy class so that there exists a track $G : \gamma \Rightarrow \varepsilon$. Now H_i and $p_i G$ differ by an element D_i with $D_1 \in D(X, U)$, $D_2 \in D(X, W)$. We can alter G by $D = (D_1, D_2) \in D(X, U \oplus W)$ and get H with $p_i H = H_i$. Moreover

$$(2) \qquad p_i(\hat{\varepsilon} \square Hf) = p_i\hat{\gamma}$$

Now $\hat{\varepsilon} \square Hd$ differs from $\hat{\gamma}$ by an element $A \in D(Y, U \oplus W)$. Then (2) shows $A = 0$. Hence $\hat{\varepsilon} \square Hf = \hat{\gamma}$ is satisfied.

□

Corollary 3.3. *The sets $Cone_f(U)$ and $Cone^f(U)$ have the structure of abelian groups and the functors (3.1) are well defined additive functors*

Proof. Using the isomorphism (3.2) we define addition by the composite

$$Cone_f(U) \times Cone_f(U) \cong Cone_f(U \oplus U) \xrightarrow{(1,1)_*} Cone_f(U)$$

where $(1, 1) : U \oplus U \to U$ is the folding map. We obtain for $Cone^f$ a result similar to (3.2) and define the addition in $Cone^f(U)$ in a similar way.

□

Proposition 3.4. *Let $\boldsymbol{B}$ be an additive category with $\boldsymbol{B}_\simeq = \boldsymbol{A}$ and bifunctor $D : \boldsymbol{A}^{op} \times \boldsymbol{A} \to \boldsymbol{Ab}$. Then the cone functors are embedded in natural exact sequences in $\boldsymbol{Ab}$*

$$D(Y,U) \xrightarrow{f^*} D(X,U) \xrightarrow{i} Cone_f(U) \xrightarrow{q} hom_{\boldsymbol{A}}(Y,U) \xrightarrow{f^*} hom_{\boldsymbol{A}}(X,U)$$

$$D(U,X) \xrightarrow{f_*} D(U,Y) \xrightarrow{\bar{i}} Cone^f(U) \xrightarrow{\bar{q}} hom_{\boldsymbol{A}}(U,X) \xrightarrow{f_*} hom_{\boldsymbol{A}}(U,Y)$$

where $f : X \to Y$ is a map in $\boldsymbol{B}_0$. Moreover i, q and $\bar{i}, \bar{q}$ are compatible with H_* in (3.1)(3), that is, $H_* i = i$, $qH_* = q$.

Proof. We define $q\{\alpha, \hat{\alpha}\} = \{\alpha\}$. Then $f^* q = 0$ since $\hat{\alpha} : \alpha f \Rightarrow 0$. On the other hand any element $\{\alpha\} \in hom_{\mathbf{A}}(Y,U)$ with $f^*\{\alpha\} = 0$ admits a track $\hat{\alpha} : \alpha f \Rightarrow 0$ representing an element $\{\alpha, \hat{\alpha}\} \in Cone_f(U)$ with $q\{\alpha, \hat{\alpha}\} = \{\alpha\}$. We define $i(\delta)$ by the track $\sigma\delta : 0 \Rightarrow 0$ in $\mathbf{B}_1$, see (2.5), so that we get the element $i(\delta) = (0, \sigma\delta)$ by the diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left=25, "0"] \arrow[rr, bend left=75, "0"] \arrow[rr, phantom, bend left=50, "\Uparrow\sigma\delta"] & Y \arrow[r, "0"'] & U
\end{tikzcd}$$

with $0f = 0$ since $*$ is a strict zero object of $\mathbf{B}$. Cleary $iq = 0$. Moreover assume $q\{\alpha, \hat{\alpha}\} = 0$. Then there exists a track $A : 0 \Rightarrow \alpha$ and $\{\alpha, \hat{\alpha}\} = \{0, \hat{\alpha} \square Af\}$. Moreover by (2.5) there is δ with $\hat{\alpha} \square Af : 0 \Rightarrow 0$ being $\sigma(\delta)$. Next assume that $\delta = f^*\delta'$ with $\delta' \in D(Y,U)$. Then $\{0, \sigma\delta\} = \{0, 0^{\square}\}$ is the trivial element since $\sigma\delta$ coincides with

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend left=70] \arrow[rr, phantom, bend left=55, "\Uparrow 0^{\square}"] & Y \arrow[r, "0"'] \arrow[r, bend left=60, "0"] \arrow[r, phantom, bend left=25, "\Uparrow\sigma\delta'"] & U
\end{tikzcd}$$

This shows $if^* = 0$. Moreover let δ be given with $i\delta = \{0, \sigma\delta\} = 0$. Then there exixts a track $A : 0 \Rightarrow 0$ with $\sigma\delta = Af$. Here $A = \sigma\delta'$ by (2.5). Hence $\delta = f^*\delta'$.

In a similar way one proves exactness of the second sequence in (3.4) with

$$\begin{aligned} \bar{q}\{\alpha, \hat{\alpha}\} &= \{\alpha\} \quad \text{for } \{\alpha, \hat{\alpha}\} \in Cone^f(U), \\ \bar{i}\delta &= \{0, \sigma\delta\} \quad \text{for } \delta \in D(U,Y). \end{aligned}$$

□

Corollary 3.5. *Let $\boldsymbol{B}$ be an additive track category with $\boldsymbol{B}_{\simeq} = \boldsymbol{A}$ and with a bifunctor $D(X,Y) = hom_{\boldsymbol{A}}(tX,Y)$ as in (2.7). Let $f : X \to Y$ be a map in $\boldsymbol{A}$ and let $f(n) : t^n X \to t^n Y$ be a map in $\boldsymbol{B}_0$ representing the homotopy class $t^n\{f\}$. Then one has long exact sequences in $\boldsymbol{Ab}$ as follows, $n \in \mathbb{Z}$.*

$$hom_{\boldsymbol{A}}(t^{n+1}Y,U) \xrightarrow{f^*} hom_{\boldsymbol{A}}(t^{n+1}X,U) \xrightarrow{i} Cone_{f(n)}(U) \xrightarrow{q} hom_{\boldsymbol{A}}(t^n Y,U) \xrightarrow{f^*} hom_{\boldsymbol{A}}(t^n X,U)$$

$$hom_{\boldsymbol{A}}(U,t^{n+1}X)\xrightarrow{f^*}hom_{\boldsymbol{A}}(U,t^{n+1}Y)\xrightarrow{\bar{i}}Cone^{f(n)}(U)\xrightarrow{\bar{q}}hom_{\boldsymbol{A}}(U,t^nX)\xrightarrow{f_*}hom_{\boldsymbol{A}}(U,t^nY)$$

Proof.

$$D(X,Y)=hom_{\mathbf{A}}(tX,Y)\cong hom_{\mathbf{A}}(X,t^{-1}Y)$$

and we can apply (3.4).

□

Proposition 3.6. *Let $\boldsymbol{B}$ be an additive track category with $\boldsymbol{B}_{\simeq}=\boldsymbol{A}$ and with a bifunctor $D(X,Y)=hom_{\boldsymbol{A}}(tX,Y)$ as in (2.7). Let*

$$A\xrightarrow{u}B\xrightarrow{v}C$$

be maps in $\boldsymbol{B}_0$ and let be X an object in $\boldsymbol{A}$. Then we have a long exact sequence in $\boldsymbol{Ab}$.

$$Cone_{vu}(t^{-1}X)\xrightarrow{v^*}Cone_u(t^{-1}X)\xrightarrow{\delta}Cone_v(X)\xrightarrow{u^*}Cone_{vu}(X)\xrightarrow{v^*}Cone_u(X)$$

Here we set $v^*\{\alpha,\hat{\alpha}\}=\{\alpha v,\hat{\alpha}\}$ and $u^*\{\beta,\hat{\beta}\}=\{\beta,\hat{\beta}u\}$. Moreover δ is defined by $\delta\{\gamma,\hat{\gamma}\}=it\{\gamma\}$ where $\gamma:B\to t^{-1}X$ and $t\{\gamma\}\in hom_{\boldsymbol{A}}(tB,X)=D(B,X)$ and i is given as in (3.4).

We point out that the exact sequence (3.6) is the analogue of the exact triangle (#) in the octahedral axiom (1.3). The operators v^*,δ,u^* in (3.6), however, are canonically given while in (1.3)(#) the maps $\bar{u},\bar{w}$ are certain choices.

Proof of (3.6). For $\{\beta,\hat{\beta}\}\in Cone_v(X)$ with $\hat{\beta}:\beta v\Rightarrow 0$ we get $\hat{\beta}u:\beta vu\Rightarrow 0$ so that $v^*u^*\{\beta,\hat{\beta}\}=\{\beta vu,\hat{\beta}u\}=0$ in $Cone_u(X)$.

Now let $\{\alpha,\hat{\alpha}\}\in Cone_{vu}(X)$ with $\hat{\alpha}:\alpha vu\Rightarrow 0$ and assume $u^*\{\alpha,\hat{\alpha}\}=\{\alpha v,\hat{\alpha}\}=0$ in $Cone_u(X)$. Then there exists $z:\alpha v\Rightarrow 0$ with $zu=\hat{\alpha}$. Hence $\{\alpha,\hat{\alpha}\}=\{\alpha,zu\}=u^*\{\alpha,z\}$. Hence we proved that $im(u^*)=ker(v^*)$.

Next let $\{\gamma,\hat{\gamma}\}\in Cone_u(t^{-1}X)$ with $\hat{\gamma}:\gamma u\Rightarrow 0$. Let $z:0\Rightarrow 0$ be given by $\sigma t\{\gamma\}$ where $0:B\to *\to X$. Then $u^*\delta\{\gamma,\hat{\gamma}\}=u^*\{0,z\}=\{0,zu\}=\{0,\sigma(t\{\gamma u\})\}=0$ since $\gamma u=0$ in $\mathbf{A}$ by the track $\hat{\gamma}$. Now let $\{\beta,\hat{\beta}\}\in Cone_v(X)$ with $u^*\{\beta,\hat{\beta}\}=\{\beta,\hat{\beta}u\}=0$ in $Cone_{vu}(X)$. Then there exixts $z:\beta\Rightarrow 0$ with $\hat{\beta}u=zvu$. Hence we get an element

$$\sigma(y)=\hat{\beta}\square(zv)^{\boxminus}:0\Rightarrow 0$$

with $y\in D(B,X)$ and $u^*y=0$. Thus we can find an element $\bar{y}\in Cone_u(t^{-1}X)$ with $\delta\bar{y}=\{\beta,\hat{\beta}\}$. This completes the proof that $im(\delta)=ker(u^*)$.

Finally let $\{\rho,\hat{\rho}\}\in Cone_{vu}(t^{-1}X)$ with $\hat{\rho}:\rho vu\Rightarrow 0$ and $\rho:C\to t^{-1}X$. Then $\delta v^*\{\rho,\hat{\rho}\}=\delta\{\rho v,\hat{\rho}\}=it\{\rho v\}=\{0,\sigma(t(\rho))v\}=0$ in $Cone_v(X)$.

Moreover let $\{\gamma,\hat{\gamma}\}\in Cone_u(t^{-1}X)$ with $\hat{\gamma}:\gamma u\Rightarrow 0$ and $\delta\{\gamma,\hat{\gamma}\}=0$ in $Cone_v(X)$. Then there exists $z\in D(C,X)$ such that in the diagram

$$\begin{tikzcd}
X & C \arrow[l, "0"{name=M}] \arrow[l, bend left=50, "0"{name=L}] \arrow[from=M, to=L, phantom, "\Downarrow\sigma z"] & B \arrow[l, "v"] \arrow[ll, bend right=60, "0"'{name=U}] \arrow[from=M, to=U, phantom, "\Uparrow\sigma t\{\gamma\}"]
\end{tikzcd}$$

we have $\sigma(z)v = \sigma t\{\gamma\}$. Let $\rho : C \to t^{-1}X$ be a map in $\mathbf{B}_0$ which represents $t^{-1}z$. Then there exixts $\hat{\rho} : \rho vu \Rightarrow 0$ since $\gamma u = 0$ in $\mathbf{A}$. This completes the proof that $im(v^*) = ker(\delta)$.

□

Remark 3.7. Gabriel-Zisman consider cone functors $Cone^f$ in a track category $\mathbf{B}$ with zero object $*$ as follows. Given an object U and a map $f : X \longrightarrow Y$ in $\mathbf{B}$ one gets the morphism

$$f^U : [\![U, X]\!] \longrightarrow [\![U, Y]\!]$$

between pointed groupoids induced by f. Then the groupoid $\Gamma(f^U)$ is defined by the objects $(\alpha, \hat{\alpha})$ and by the tracks $G : (\beta, \hat{\beta}) \Longrightarrow (\alpha, \hat{\alpha})$ satisfying the condition in (3.1)(2). Hence

$$Cone^f(U) = \pi_0(\Gamma(f^U))$$

is the set of path components of $\Gamma(f^U)$. Here the groupoid $\Gamma(f^U)$ coincides with the construction V.3.1 [GZ] and axiom C in V.3.2 [GZ] implies that $Cone^f$ is representable as in (4.1) below. Gabriel-Zisman use the axiom C for the proof of the "exact fiber sequence", see V.4.2 [GZ], which in the context of this paper is easily achieved by the dual of (3.5) and (4.3).

4 Representable cone functors

Let $\mathbf{B}$ be an additive track category with homotopy category $\mathbf{A}$ and with bifunctor $D(X, Y) = hom_{\mathbf{A}}(tX, Y)$ as in (2.7). An additive functor $F : \mathbf{A} \to \mathbf{Ab}$ is *representable* if there exists an object X in $\mathbf{A}$ and a natural isomorphism of functors

(4.1) $$\chi : F \cong hom_{\mathbf{A}}(X, -)$$

in $\mathbf{Ab}$. Then X is well defined up to isomorphism in $\mathbf{A}$ and is called a *representation* of F. We now consider representations of cone functors.

If $C = C_f$ represents $Cone_f$ with $f : A \to B$ in $\mathbf{B}_0$ then we obtain the triangle

(4.2) $$A \xrightarrow{f} B \xrightarrow{i_f} C_f \xrightarrow{q_f} tA$$

in $\mathbf{A}$ as follows. For $U = C_f$ we have the operator

$$hom_{\mathbf{A}}(C_f, C_f) \xrightarrow{\chi_f^{-1}} Cone_f(C_f) \xrightarrow{q} hom_{\mathbf{A}}(B, C_f)$$

by (3.4) which carries the identity of C_f to i_f. Moreover for $U = tA$ we have the operator

$$hom_{\mathbf{A}}(tA, tA) \xrightarrow{i} Cone_f(tA) \xrightarrow{\chi_f} hom_{\mathbf{A}}(C_f, tA)$$

by (3.4) which carries the identity of tA to q_f.

Of course i_f and q_f depend on the representation (C_f, χ_f) of $Cone_f$. Using the exact sequence (3.4) we get by a representation the following diagram in $\mathbf{Ab}$

(4.3)

$$\begin{tikzcd}
hom_{\mathbf{A}}(tB, U) \arrow[r, "f^*"] \arrow[d, equal] & hom_{\mathbf{A}}(tA, U) \arrow[r, "i"] \arrow[d, equal] & Cone_f(U) \arrow[r, "q"] \arrow[d, "\chi_f", "\cong"'] & hom_{\mathbf{A}}(B, U) \arrow[r, "f^*"] \arrow[d, equal] & hom_{\mathbf{A}}(A, U) \arrow[d, equal] \\
hom_{\mathbf{A}}(tB, U) \arrow[r, "f^*"] & hom_{\mathbf{A}}(tA, U) \arrow[r, "q_f^*"] & hom_{\mathbf{A}}(C_f, U) \arrow[r, "{i_f}^*"] & hom_{\mathbf{A}}(B, U) \arrow[r, "f^*"] & hom_{\mathbf{A}}(A, U)
\end{tikzcd}$$

Here we have $i_f^*\chi_f = q$ since

$$\begin{aligned} q\chi_f^{-1}(\alpha) &= q\chi_f^{-1}(\alpha 1_{C_f}) = \alpha_* q\chi_f^{-1}(1_{C_f}) \\ &= \alpha_* i_f = i_f^*(\alpha). \end{aligned}$$

Similarly we get $\chi_f i = q_f^*$ by

$$\begin{aligned} \chi_f i(\beta) &= \chi_f i(\beta 1_{t_A}) = \beta_* \chi_f i(1_{tA}) \\ &= \beta_* q_f = q_f^* \beta. \end{aligned}$$

Hence diagram (4.3) commutes and therefore the bottom of (4.3) is an exact sequence.Compare also (1.5).

Proposition 4.4. *If $Cone_f$ is representable by (C_f, χ_f) then there exixts a diagram in $\boldsymbol{B}$*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left=50, "0"] \arrow[r, phantom, shift left=18, "\Uparrow \hat{i}_f"] & B \arrow[r, "i_f"] \arrow[rr, bend right=50] \arrow[rr, phantom, shift right=18, "\Downarrow \hat{q}_f"] & C_f \arrow[r, "q_f"] & tA \end{tikzcd}$$

with the properties

$$\chi_f\{i_f, \hat{i}_f\} = 1_{C_f},$$

$$\sigma^{-1}(q_f \hat{i}_f \square (\hat{q}_f^{\boxminus}) f) = 1_{tA} \in D(A, tA).$$

Proof. We have

$$i(1_{tA}) = \chi_f^{-1} q^*(1_{tA}) = \chi_f^{-1}(q_f 1_{C_f}) = q_{f*}\{i_f, \hat{i}_f\}$$

Hence by definition of i and the equivalence relations in $Cone_f$ there exists $\hat{q}_f$ such that the following diagram represents the trivial track.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rrr, bend left=40, "0"] \arrow[rrd, bend right=40] & B \arrow[rr, "0"] \arrow[dr, "i_f"] \arrow[rr, phantom, shift left=20, "\Uparrow \sigma 1_{tA}"] \arrow[d, phantom, "\Downarrow \hat{i}_f"] & & tA \\ & {} & C_f \arrow[ur, "q_f"'] \arrow[u, phantom, "\Uparrow \hat{q}_f"] & \end{tikzcd}$$

This yields the second formula in (4.4).

□

The converse of (4.4) is also true. We get the following criterion for representability of cone functors.

Theorem 4.5. *Let $f : A \to B$ be a map in $\boldsymbol{B}_0$. Then $Cone_f$ is representable if and only if there exists a diagram in $\boldsymbol{B}$*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\hat{i}"] & B \arrow[r, "i_f"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Uparrow\hat{q}^{\boxminus}"] & C_f \arrow[r, "q_f"] & tA
\end{tikzcd}$$

which by pasting yields $\varepsilon 1_{tA} \in D(A, tA)$, with $\varepsilon = 1$ or $\varepsilon = -1$ and for which the bottom row in (4.3) is exact for all U.

Proof. If $Cone_f$ is representable then a diagram as in (4.5) exists by (4.4). Conversely given a diagram as in (4.5) we define the function

$$\chi_f^{-1} : hom_{\mathbf{A}}(C_f, U) \to Cone_f(U)$$

which carries α to $\alpha_*\{i_f, \hat{i}\}$. Now we obtain a diagram as in (4.3) satisfying $i_f^* \chi_f = q$. Moreover $\chi_f^{-1} q_f^* = \varepsilon i$ since the diagram in (4.5) represents $\varepsilon 1_{tA}$, compare the proof of (4.4). Hence diagram (4.3) commutes up to sign and therefore χ_f is a bijection by the five lemma.

□

5 Toda complexes

As in (2.4) let **B** an additive track category with homotopy category **A** and bifunctor D. A *Toda pair* (H, G) is a diagram in **B** of the form

(5.1)
$$\begin{tikzcd}
Z \arrow[r, "\gamma"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H"] & Y \arrow[r, "\beta"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow G"] & X \arrow[r, "\alpha"] & W
\end{tikzcd}$$

We say that (H, G) is *associated* to the triple (a, b, c) where $a = \{\alpha\}$, $b = \{\beta\}$ and $c = \{\gamma\}$ are homotopy classes in **A**. Moreover we say that (H, G) represents $\xi \in D(Z, W)$ or $[H|G] \in \mathrm{Aut}_{[\![z,w]\!]}(\alpha\beta\gamma)$ if the equation

$$\sigma(\xi) = [H|G] = \alpha H \Box G^{\boxminus} \gamma \tag{1}$$

holds where we use σ in (2.4). We point out that the convention in (1) uses the positive direction of H to define ξ. Given a triple (a, b, c) in **A** with $ab = 0$ and $bc = 0$ the *Toda bracket*

$$< a, b, c > \subset D(Z, W) \tag{2}$$

consists of all elements ξ represented by Toda pairs associated to (a,b,c). This is a coset of

$$\gamma^* D(Y,W) + \alpha_* D(Z,X) \subset D(Z,W).$$

We now define a *Toda complex* by a sequence of maps $d = d_n : X_{n-1} \to X_n$, $n \in \mathbb{Z}$, in $\mathbf{B}_0$ together with a sequence of tracks H_n as in the diagram

(5.2)
$$\begin{tikzcd}
X_{n-3} \arrow[r, "d_{n-2}"'] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_{n-1}"] & X_{n-2} \arrow[r, "d_{n-1}"'] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow H_n"] & X_{n-1} \arrow[r, "d_n"'] & X_n
\end{tikzcd}$$

Hence (H_{n-1}, H_n) is a Toda pair representing $\alpha_n \in D(X_{n-3}, X_n)$ for $n \in \mathbb{Z}$.

Let $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ and $X' = (X'_n, d'_n, H'_n, n \in \mathbb{Z})$ be Toda complexes. A *morphism between Toda complexes* $f : X \to Y$ is a sequence $f = (f_n, F_n, n \in \mathbb{Z})$ of maps f_n in $\mathbf{B}_0$ and tracks F_n in $\mathbf{B}_1$ as in the diagram

(5.3)
$$\begin{tikzcd}
X_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, phantom, "\Downarrow F_{n-1}"] & X_n \arrow[d, "f_n"] \arrow[dl, phantom, "\Downarrow F_n"] \\
X'_{n-2} \arrow[r] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow H'_n"] & X'_{n-1} \arrow[r] & X'_n
\end{tikzcd}$$

such that this diagram represents the trivial track for all n, that is

$$f_n H_n = H'_n f_{n-2} \square d'_n F_{n-1} \square F_n d_{n-1}.$$

Let **Tod** be the category of Toda complexes and morphisms as in (5.3). We have the *shift functor*

(5.4) $$sh : \mathbf{Tod} \to \mathbf{Tod}$$

which is a strict isomorphism of categories defined as follows. For $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ in **Tod** let $sh(X) = (X'_n, d'_n, H'_n, n \in \mathbb{Z})$ be given by $X'_n = X_{n+1}, d'_n = d_{n+1}, H'_n = H_{n+1}$.

We also define the category of rows denoted by **row** as follows. A row in $\mathbf{B}_0$ is a sequence of maps $(X_n, d_n : X_{n-1} \to X_n, n \in \mathbb{Z})$. A map $f : (X_n, d_n) \to (X'_n, d'_n)$ between rows is a sequence $f = (f_n, F_n, n \in \mathbb{Z})$ of diagrams in $\mathbf{B}$

(5.5)
$$\begin{tikzcd}
X_{n-1} \arrow[r, "d_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow F_n"] & X_n \arrow[d, "f_n"] \\
X'_{n-1} \arrow[r, "d'_n"'] & X'_n
\end{tikzcd}$$

Two such maps $f, f' : X \to X'$ are *homotopic* if there exists tracks $G_n : f_n \Rightarrow f'_n$ such that for $n \in \mathbb{Z}$

$$(d'_n G_{n-1}) \Box F_n = F'_n \Box (G_n d_n).$$

A *weak equivalence* in **row** is a map f for which all f_n are isomorphisms in the homotopy category **A**.

We have the shift functor also on the category **row** and the faithful forgetful functor

(5.6)
$$\mathbf{Tod} \longrightarrow \mathbf{row}$$

carries $X = (X_n, d_n, H_n)$ to (X_n, d_n). Then we say that the Toda complex X *extends* the row (X_n, d_n).

6 Triangulated track categories

Let **B** be an additive track category with a strict translation functor $t : \mathbf{B} \to \mathbf{B}$ as in (2.8). Let **A** be the homotopy category of **B** and $t : \mathbf{A} \to \mathbf{A}$ the induced translation functor on **A**.

A *track triangle* is a diagram in **B**

(6.1)
$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=50, "0", "\Uparrow H_0"'] & B \arrow[r, "i_f"'] \arrow[rr, bend right=50, "0"', "\Downarrow H_1"] & C \arrow[r, "q_f"'] \arrow[rr, bend left=50, "0", "\Uparrow H_2"'] & tA \arrow[r, "tf"'] & tB
\end{tikzcd}$$

for which the Toda pairs (H_0, H_1), (H_1, H_2) and (H_2, tH_0) represent $1_{tA}, -1_{tB}$ and 1_{tC} respectively, see (5.1). As in (1.2)(1) a track triangle can be equivalently described as follows, compare the notation in section 5.

Definition 6.2. A *track triangle* X is a Toda complex in **B** satisfying the following properties:

(1)
$$sh^3(X) = tX$$

that is, for $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ we have $X_{n+3} = tX_n, d_{n+3} = td_n, H_{n+3} = tH_n$. Moreover the Toda pairs (H_{n-1}, H_n) represent for $n \in \mathbb{Z}$ the element

$$\sigma[H_{n-1}|H_n] = (-1)^{n+1} 1_{X_n} \in D(X_{n-3}, X_n) = hom_{\mathbf{A}}(tX_{n-3}, X_n)$$

$$(2) \qquad = hom_{\mathbf{A}}(X_n, X_n)$$

given by the identity 1_{X_n} of X_n. A *map* between track triangles X, X' is map $f : X \to X'$ between Toda complexes such that

$$(3) \qquad sh^3(f) = tf,$$

that is, for $f = (f_n, F_n, n \in \mathbb{Z})$ we have $f_{n+3} = tf_n,\ F_{n+3} = tF_n$.

Remark 6.3. If $\mathbf{B}$ has a (non strict) translation functor we define a track triangle X as in (6.1) but we replace the track tH_0 by the obvious track $(ti_f)(tf) \Rightarrow 0$ given by tH_0 and the pseudofunctor track $t_{i_f,f}$. Similarly we define maps between such track triangles. For details see [IV]. It is easy to generalize the results and proofs in this paper to the non strict case.

The definition of a track triangle in $\mathbf{B}$ is motivated by the representability result for cone functors in (4.5) which implies:

Proposition 6.4. *Given a track triangle X one gets for $n \in \mathbb{Z}$ representations of cone functors*

$$hom_{\mathbf{A}}(X_n, U) \cong Cone_{d_{n-1}}(U)$$

which carry α to $\alpha_*\{d_n, H_n\}$. Moreover the exact sequences (4.3) for these representations coincide up to the sign with the long exact sequence

$$... \to hom_{\mathbf{A}}(X_{n+1}, U) \overset{d^*}{\to} hom_{\mathbf{A}}(X_n, U) \overset{d^*}{\to} hom_{\mathbf{A}}(X_{n-1}, U)$$

induced by X with $n \in \mathbb{Z}$.

Proof. Using (4.5) we only have to show that the sequence is exact. We do this for $n = 1$ so that by the convention (1.2)(1) we have $X_1 = tA$. The track H_2 implies that $d^*d^* = 0$. Hence it remains to show $ker(d^*) \subset image(d^*)$. Let $\alpha : tA \to U$ be a map with $G : \alpha q_f \Rightarrow 0$. We then construct a map $\beta : tB \to U$ in $\mathbf{A}$ with $\beta(tf) = \pm t\alpha$. Let β be given by the Toda pair (H_1, G), that is, $\sigma(\beta) = [H_1|G]$ as in (5.1). Then we get

$$\sigma((tf)^*\beta) = f^*[H_1|G] = \pm\alpha_*[H_0|H_1] = \pm \quad \alpha_*\sigma(1_{tA}) = \sigma(\pm\alpha).$$

This completes the proof □

Definition 6.5. Let $\mathbf{B}$ be an additive track category with a (strict) translation functor $t : \mathbf{B} \to \mathbf{B}$ as in (2.8) or (2.11). Then $\mathbf{B}$ is a *triangulated track category* if $\mathbf{B}$ is equipped with a distinguished family of track triangles which is subject to the following axioms:

(TTr1) For each morphism $f : A \to B$ in $\mathbf{B}_0$ there exists a distinguished track triangle X extending f, that is, $d_{-1} = f$.

(TTr2) Let X and X' be distinguished track triangles extending $f : A \to B$ and $f' : A' \to B'$ respectively and let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow H"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a diagram in $\mathbf{B}$. Then there exists a morphism $h = (h_n, H_n, n \in \mathbb{Z}) : X \to X'$ between track triangles extending (a, b, H), that is $h_{-2} = a$, $h_{-1} = b$, $H_{-1} = H$.

The category $\mathbf{Ch}(\mathbf{A})$ of chain complexes in (2.9) and the category **Spec** in (2.10) are examples of triangulated track categories with a strict translation functor. Moreover a stable cofibration category as in (2.12) is an example of a triangulated track category with a non strict translation functor, see (2.11).

The author does not know any example of a triangulated category which is not derived from a triangulated track category. We shall use triangulated track categories in [II].

We are ready to describe the main result of this paper.

Theorem 6.6. *Let* $(\mathbf{B}, t)$ *be a triangulated track category and let* $\mathbf{A}$ *be the homotopy category of* $\mathbf{B}$ *and* $t\colon \mathbf{A} \to \mathbf{A}$ *be induced by* $t\colon \mathbf{B} \to \mathbf{B}$*. Let* $\mathcal{E}$ *be the class of triangles in* $(\mathbf{A}, t)$ *which are induced by track triangles in* $(\mathbf{B}, t)$ *via the quotient functor* $\mathbf{B}_0 \to \mathbf{A}$*. Then* $(\mathbf{A}, t, \mathcal{E})$ *is a triangulated category.*

More precisely we prove the following addendum.

Addendum 6.7. *Let* $\mathbf{B}$ *be an additive track category with a strict translation functor* $t\colon \mathbf{B} \to \mathbf{B}$ *inducing* t *on the homotopy category* $\mathbf{A}$ *of* $\mathbf{B}$*. Let* $\mathcal{E}$ *be the class of triangles in* $(\mathbf{A}, t)$ *which are induced by track triangles in* $(\mathbf{B}, t)$*. Then* $(\mathbf{A}, t, \mathcal{E})$ *satisfies axioms (Tr0), (Tr2) and (Tr3). Moreover, if* $(\mathbf{B}, t)$ *is a triangulated track category, that is,* $\mathbf{B}, t)$ *satisfies (TTr1) and (TTr2), then* $(\mathbf{A}, t, \mathcal{E})$ *is a triangulated category, that is, (TTr1) obviously implies (Tr1) and (TTr1), (TTr2) both imply (Tr4).*

The theorem has a generalization for an additive track category with a non-strict translation functor $t\colon \mathbf{B} \to \mathbf{B}$, see [IV]. In fact in [IV] we show that $(\mathbf{B}, t)$ determines a characteristic cohomology class

$$\nabla = \langle \mathbf{B}, t \rangle \in H^3(\mathbf{A}, t)$$

in the translation cohomology of $(\mathbf{A}, t)$ defined in [III]. Moreover, the triangulated structure $\mathcal{E}$ of $(\mathbf{A}.t)$ depends only on this class, that is $\mathcal{E} = \mathcal{E}_\nabla$.

We point out that the family of exact triangles in $\mathbf{A}$ is well defined only by the additive track category $\mathbf{B}$ and the translation functor t on $\mathbf{B}$, that is, the distinguished track triangles in (6.5) are not used in the definition of exact triangles in $\mathbf{A}$. The next result shows that each exact triangle in $\mathbf{A}$ is isomorphic to an exact triangle induced by a distinguished track triangle. For this we use (TTr1).

Proposition 6.8. *Let X and X' be track triangles both extending $f : A \to B$. Then there exists a commutative diagram in* $\boldsymbol{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "h"] & tA \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r] & C' \arrow[r] & tA
\end{tikzcd}$$

where the rows are the exact triangles induced by X and X' and where h is an isomorphism.

Proof. We have a commutative diagram of natural transformations, see (4.3) and (4.4),

$$\begin{tikzcd}
 & hom_{\mathbf{A}}(C, U) \arrow[dr] \arrow[d, phantom, "\wr\|"] & \\
hom_{\mathbf{A}}(tA, U) \arrow[ur] \arrow[r, "i"] \arrow[dr] & Cone_f(U) \arrow[r, "q"] \arrow[d, phantom, "\wr\|"] & hom_{\mathbf{A}}(B, U) \\
 & hom_{\mathbf{A}}(C', U) \arrow[ur] &
\end{tikzcd}$$

Now the isomorphism in the middle carries the identity of C' to h and carries the identity of C to the inverse of h.

□

Proof of (6.6). We show (Tr0), (Tr2) and (Tr4) in the following sections. We here only consider (Tr1) and (Tr3). It is clear that (TTr1) implies (Tr1). Moreover given track triangles X and X' extending f and f' we find by (TTr1) distinguished track triangles Y and Y' extending f and f' and we can choose (a, b) as in (TTr2) inducing the square in (Tr3). Then (TTr2) yields a map $Y \to Y'$ between distinguished track triangles extending (a, b). Using this map and (6.8) we get (Tr3).

□

Remark 6.9. Grandis [G] describes triangulated categories derived from an abstract setting of homotopical algebra. This setting axiomatizes the behaviour of continuous maps and homotopies between them and does not use tracks as above which are homotopy classes of homotopies.

7 Induced Toda complexes and proof of (Tr0)

Lemma 7.1. *Let X be a Toda complex extending the row (X_n, d_n) and let $f : (X_n, d_n) \to (X'_n, d'_n)$ be weak equivalence between rows. Then there exists a unique Toda complex X' extending (X'_n, d'_n) such that $f : X \to X'$ is a morphism between Toda complexes.*

We denote $X' = f_* X$ the *Toda complex induced by the weak equivalence f.*

Proof. Since f_n is an isomorphism in $\mathbf{A}$ we can choose a track $G_n : d'_n d'_{n-1} \Rightarrow 0$. Then the tracks (H_n, F_{n-1}, F_n, G_n) represent an element $\xi_n \in D(X_{n-2}, X'_n)$ and hence we can alter G_n by $(f^*_{n-2})^{-1}(\xi) \in D(X'_{n-2}, X'_n)$ so that we get H'_n. Then $(H_n, F_{n-1}, F_n, H'_n)$ represents the trivial track and hence $f : X \to X'$ is a morphism between Toda complexes with $X' = (X'_n, d'_n, H'_n, n \in \mathbb{Z})$.

□

Lemma 7.2. *Let X be a track triangle in $\boldsymbol{B}$ with row (X_n, d_n) and let $f : (X_n, d_n) \to (X'_n, d'_n)$ be a weak equivalence of rows with $sh^3 f = tf$. Then the induced Toda complex f_*X is a track triangle.*

Proof. We only show that the Toda pair H'_0, H'_1 represents $1_{tA'}$. Let $F = F_{-1},\ G = F_0,\ H = F_1$ and $a = f_{-2},\ b = f_{-1},\ c = f_0,\ e = f_1$ be given by $f = (f_n, H_n, n \in \mathbb{Z})$. Moreover we write $d = d_i$ and $d = d'_i$. Assume the Toda pair (H'_0, H'_1) represents $\xi \in D(A', tA')$. Then we get the equations:

$$\begin{aligned}
\sigma(\alpha^*\xi) &= (dH'_0\Box(H'_1 d)^{\boxminus})a \\
&= dH'_0 a\Box(H'_1)^{\boxminus} da \\
&= d(cH_0\Box G^{\boxminus} d\Box dF^{\boxminus})\Box((H'_1)^{\boxminus} bd) \\
&= d(cH_0\Box G^{\boxminus} d\Box dF^{\boxminus})\Box(dG\Box Hd\Box eH_1^{\boxminus})d \\
&= dcH_0\Box dG^{\boxminus} d\Box ddF^{\boxminus}\Box dGd\Box Hdd\Box eH_1^{\boxminus})d \\
&= edH_0\Box dG^{\boxminus} d\Box dGd\Box eH_1^{\boxminus} d \\
&= e(dH_0\Box H_1^{\boxminus} d) \\
&= \sigma((t\alpha)_* 1_{tA}) \\
&= \sigma(1_{tA'} t\alpha) \\
&= \sigma(\alpha^* 1_{tA'})
\end{aligned}$$

Here $\alpha : A \to A'$ is the isomorphism induced by a. We know that e induces $t\alpha$. The equations above show that $\xi = 1_{tA'}$.

□

Proof of (Tr0) in (6.6). The identity 1_A yields a track triangle by considering

$$\begin{tikzcd}
A \arrow[r, "1"'] \arrow[rr, bend left=70, "\Uparrow H_0"'] & A \arrow[r] \arrow[rr, bend right=70, "0"', "\Downarrow H_1"] & * \arrow[r] \arrow[rr, bend left=70, "\Uparrow H_2"'] & tA \arrow[r, "1"'] & tA
\end{tikzcd}$$

Here $H_0 = 0,\ H_2 = 0$ since $*$ is a strict zero object and $H_1 = \sigma(-1_{tA})$. Therefore

$$A \xrightarrow{1} A \longrightarrow * \longrightarrow tA$$

is exact. Next let $\alpha : T \cong T'$ be an isomorphism of triangles where T is induced by the track triangles X. We obtain for T' a cochain complex (X'_n, d'_n) in $\mathbf{A}$ as in (2.1) and we choose a row (X'_n, d'_n) in $\mathbf{B}$ representing this cochain complex. Then α shows that there is a weak equivalence of rows $f : (X_n, d_n) \to (X'_n, d'_n)$ with $sh^3 f = tf$

inducing α so that the track triangle f_*X is defined by (7.2). Hence T' is induced by f_*X and therefore T' is exact.

$\square$

8 Negative Toda complexes and the proof of (Tr2)

Let $\mathbf{B}$ be an additive track category with $\mathbf{B}_\simeq = \mathbf{A}$. In this section we use the existence of strong products $X \times Y$ in $\mathbf{B}$. The case of strong coproducts is treated in a dual way. We can choose for each object X maps in $\mathbf{B}_0$

$$\mu : X \times X \longrightarrow X$$

$$-1_X : X \longrightarrow X$$

which induce $(1_X, 1_X)$ and -1_X in the additive category $\mathbf{A}$. Moreover we choose tracks

$$\begin{aligned} \tau_X &: \mu_X(1_X, -1_X) \Rightarrow 0 \\ \tau'_X &: \mu_X(0_X, 1_X) \Rightarrow 1_X \\ \tau''_X &: \mu_X(1_X, 0_X) \Rightarrow 1_X \end{aligned}$$

Here $0_X : X \to * \to X$ is given by the strict zero object $*$ in $\mathbf{B}$. We now define for each morphism $f : X \to Y$ in $\mathbf{B}_0$ tracks Γ_f and Γ_f^+ as in the diagrams (compare [Ba1]):

(8.1)
$$\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y \times Y \arrow[d, "\mu_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "-1_X"'] \arrow[dr, phantom, "\overset{\Gamma_f}{\Rightarrow}"] & Y \arrow[d, "-1_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Here Γ_f^+ is the unique track for which the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_\varepsilon"] \arrow[dd, bend right=60, "1"'] & Y \arrow[d, "i_\varepsilon"'] \arrow[dd, bend left=60, "1"] \\
\Leftarrow X \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y \times Y \Rightarrow \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

represents the trivial track $0^\square : f \Rightarrow f$ for $\varepsilon = 1$ and $\varepsilon = 2$ with $i_1 = (0,1)$ and $i_2 = (1,0)$. Moreover Γ_f is the unique track for which the following diagram represents the trivial track $0^\square : 0 \Rightarrow 0$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "{(1,-1)}"'] \arrow[dd, bend right=60] \arrow[dr, phantom, "\overset{(0^\square,\Gamma_f)}{\Rightarrow}"] & Y \arrow[d, "{(1,-1)}"] \arrow[dd, bend left=60, "0"] \\
\Leftarrow X \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y \times Y \Rightarrow \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

The uniqueness of Γ_f^+ and Γ_f follows from the biadditivity of the coefficient functor D in the additive track category $\mathbf{B}$.

Lemma 8.2. *Pasting in the diagram*

(1)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "-1"'] \arrow[dr, phantom, "\overset{\Gamma_f}{\Rightarrow}"] & Y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\overset{\Gamma_g}{\Rightarrow}"] & Z \arrow[d, "-1"] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

yields the track Γ_{gf}. *Moreover for each track* $H : f \Rightarrow g$ *pasting in the diagram*

(2)
$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[r, bend left=70, "f"] \arrow[r, phantom, bend left=35, "\Uparrow H^\boxminus"] \arrow[d, "-1"'] \arrow[dr, phantom, "\overset{\Gamma_g}{\Rightarrow}"] & Y \arrow[d, "-1"] \\
X \arrow[r, "g"] \arrow[r, bend right=70, "f"'] \arrow[r, phantom, bend right=35, "\Uparrow H"] & Y
\end{tikzcd}$$

yields Γ_f.

Using Γ-tracks (8.1) we can define the *negative of a Toda complex*

(8.3) $$neg(X) = X^- = (X_n^-, d_n^-, H_n^-, n \in \mathbb{Z})$$

with $X_n^- = X_n$ and $d_n^- = d_n$ for $n \equiv 0, 1 \; mod \; 3$ and $d_n^- = (-1_{X_n})d_n$ for $n \equiv 2 \; mod \; 3$. Moreover H_n^- is obtained by $H_n^- = H_n$ for $n \equiv 1 \; mod \; 3$ and $H_n^- = (-1_{X_n})H_n$ for $n \equiv 2 \; mod \; 3$ and for $n \equiv 0 \; mod \; 3$ the track H_n^- is given by pasting the diagram

(8.4)
$$\begin{tikzcd}
 & X_{n-1} \arrow[r, "d"] \arrow[d, "-1"] \arrow[dr, phantom, "\overset{\Gamma_d}{\Rightarrow}"] & X_n \arrow[d, "-1"] \\
X_{n-2} \arrow[ur, "d"] \arrow[urr, bend left=60, "0"] \arrow[urr, phantom, bend left=35, "\Uparrow H_n"] \arrow[r, "d_{n-1}^-"'] & X_{n-1} \arrow[r, "d_n^-"'] & X_n
\end{tikzcd}$$

Here we have $d_{n-1}^- = (-1)d_{n-1}$ and $d_n^- = d_n$.

Proposition 8.5. *Toda pairs* (H_{n-1}^-, H_n^-) *in the negative Toda complex* $neg(X) = X^-$ *satisfy the formula*

$$\sigma^{-1}[H_{n-1}^- | H_n^-] = -\sigma^{-1}[H_{n-1} | H_n] \in D(X_{n-3}, X_n)$$

Proof of (8.5). We check three cases

$$\begin{aligned}
&[H_1^- | H_2^-] \\
&= [H_1 | (-1)H_2] \\
&= (-1)d_2 H_1 \Box (-1) H_2^{\boxminus} d_0 \\
&= (-1)(d_2 H_1 \Box H_2^{\boxminus} d_0) = (-1)[H_1, H_2]
\end{aligned}$$

$$\begin{aligned}
&[H_2^- | H_3^-] \\
&= d_3(-1)H_2 \Box \Gamma_d^{\boxminus} d_2 d_1 \Box (-1) H_3^{\boxminus} d_1 \\
&= (-1) d_3 H_2 \Box (-1) H_3^{\boxminus} d_1 \\
&= (-1)(d_3 H_2 \Box H_3^{\boxminus} d_1) = (-1)[H_2 | H_3]
\end{aligned}$$

$$\begin{aligned}
&[H_3^- | H_4^-] \\
&= d_4((-1)H_3 \Box \Gamma_{d_3} d_2) \Box H_4^{\boxminus}(-1) d_2 \\
&= (-1) d_4 H_3 \Box (\Gamma_{d_4} d_3 d_2 \Box d_4 \Gamma_{d_3} d_2) \Box H_4^{\boxminus}(-1) d_2 \\
&= (-1) d_4 H_3 \Box \Gamma_{d_4 d_3} d_2 \Box H_4^{\boxminus}(-1) d_2 \quad \text{, see (8.2)(1),} \\
&= (-1) d_4 H_3 \Box (\Gamma_{d_4 d_3} \Box H_4^{\boxminus}(-1)) d_2 \\
&= (-1) d_4 H_3 \Box ((-1) H_4^{\boxminus} \Box \Gamma_0) d_2 \quad \text{, see (8.2)(1),} \\
&= (-1)(d_4 H_3 \Box H_4^{\boxminus} d_2) \Box \Gamma_0 d_2 \\
&= (-1)[H_3 | H_4]
\end{aligned}$$

Here Γ_0 for $0 : X \to * \to Z$ is trivial by (8.2)(1).

□

The shift functor *sh* in (5.4) does not carry track triangles X to track triangles since $sh(X)$ does not satisfy condition (6.2)(2) on Toda pairs. We get, however, by (8.5) and (8.2)(3) the next result.

Proposition 8.6. *If $X = (X_n, d_n, H_n, i \in \mathbb{Z})$ is a track triangle then (H_1^-, H_2^-, H_3^-) and (H_{-1}^-, H_0^-, H_2^-) are track triangles in the sense of (6.1).*

Proof of (Tr2). Let X be a track triangle in $\mathbf{B}$ inducing (u, v, w) in (Tr2). Then by (8.6) we get track triangles (H_1^-, H_2^-, H_3^-) and (H_{-1}^-, H_0^-, H_2^-) inducing $(v, w, -tu)$ and $(-t^{-1}w, u, v)$ respectively. Here we assume that $\mathbf{B}$ has strong products. If this is not satisfied we can choose by [BJP] a track category $\mathbf{B}'$ with strong products and a pseudofunctor $\lambda\colon \mathbf{B} \to \mathbf{B}'$ with homotopy inverse $\rho\colon \mathbf{B}' \to \mathbf{B}$ inducing the identity on $\mathbf{A}$ and coefficients D. For X in $\mathbf{B}$ we get the Toda complex λX in $\mathbf{B}'$ and $neg(\lambda X) = (X_n^-, d_n^-, H_n^-, n \in \mathbb{Z})$ is defined as above. Then $(\rho H_1^-, \rho H_2^-, \rho H_3^-)$ and $(\rho H_{-1}^-, \rho H_0^-, \rho H_2^-)$ are track triangles in $\mathbf{B}$ inducing $(v, w, -tu)$ and $(-t^{-1}w, u, v)$ respectively. ☐

Proof of (6.6). Let $\mathbf{B}$ be an additive track category with (strict) translation functor t. Then track triangles in $\mathbf{B}$ yield exact triangles in $\mathbf{A}$ which satisfy (Tr0) and (Tr2) by the proof above. Moreover, by an argument as in the proof of (6.7) we see that (Tr3) holds, compare [IV] 7.5. If $\mathbf{B}$ is a triangulated track category then (TTr1) shows that $(\mathbf{A}, t, \mathcal{E})$ satisfies (Tr1). Moreover $(\mathbf{A}, t, \mathcal{E})$ satisfies (Tr4) by the proof in the next section. Hence $(\mathbf{A}, t, \mathcal{E})$ is a triangulated category. ☐

9 Proof of (Tr4)

Using (6.8) we see that (Tr4)(#) is isomorphic to a sequence of the form (Tr4)(#) where, however, C_u, C_v and C_{uv} are chosen by distinguished track triangles. Then (Tr0) already proved above shows that it suffices to prove (Tr4) for such distinguished choices. We consider the following diagram in $\mathbf{B}$.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "vu"] & B \arrow[d, "v"] \\
B \arrow[r, "v"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & C \arrow[r, equal] \arrow[d, "d'"] \arrow[dr, phantom, "\overset{F'}{\Rightarrow}"] & C \arrow[d, "d''"] \\
C_u \arrow[r, "\bar{w}"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{G}{\Rightarrow}"] & C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "d'"] \arrow[dr, phantom, "\overset{G'}{\Rightarrow}"] & C_v \arrow[d, "d''"] \arrow[dd, bend left=60, "0", "\overset{H_2''}{\Rightarrow}"'] \\
tA \arrow[r, equal] \arrow[d, "tu"] \arrow[dd, bend right=60, "0"', "\overset{tH_0}{\Leftarrow}"] & tA \arrow[r, "tu"] \arrow[d, "tvu"] & tB \arrow[d, "tv"] \\
tB \arrow[r, "tv"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{tF}{\Rightarrow}"] & tC \arrow[r, equal] \arrow[d, "d'"] & tC \\
tC_u \arrow[r, "t\bar{w}"] & tC_{vu} &
\end{tikzcd}$$

The subdiagrams without tracks are commutative. Let $\bar{C}_u = (H_0, H_1, H_2)$ be a distinguished track triangle extending the left column, let $\bar{C}_{vu} = (H_0', H_1', H_2')$ be a distinguished track triangle extending the column in the middle and let $\bar{C}_v = (H_0'', H_1'', H_2'')$ be a distinguished track triangle extending the right hand column. Then (TTr2) shows that we can find tracks F, G and F', G' which determine maps between track triangles $\bar{C}_u \to \bar{C}_{vu}$ and $\bar{C}_{vu} \to \bar{C}_v$ respectively. This implies the equations

$$(1) \qquad (t\bar{w})(tH_0) = (tH_0')\square(tF)(tu)$$

$$(2) \qquad H_2' = (H_2''\bar{u})\square(tv)G'$$

Moreover we know (H_2', tH_0') represents

$$(3) \qquad \sigma(1_{tC_{vu}}) = d'H_2'\square(tH_0')^{\boxminus}d'$$

Therefore we get the equation

(4) $$\sigma(1_{tC_{vu}}) = d'H_2''\bar{u}\square d'(tv)G'\square(tF)(tu)d'\square(t\bar{w})(tH_0)^{\boxminus}d'$$

We now choose a distinguished track triangle for the map $\bar{u} : C_{vu} \to C_v$ which exists by (TTr1). This yields $C_{\bar{u}}$ and the following diagram in the homotopy category $\mathbf{A}$ where the bottom row corresponds to (2.3)(#).

(5) $$\begin{tikzcd}
 & & C_{\bar{u}} \arrow[d, "\varphi"] \arrow[dr, "-q_{\bar{u}}"] & \\
C_{vu} \arrow[r, "\bar{u}"'] & C_v \arrow[ur, "i_{\bar{u}}"] \arrow[r, "dd''"'] & tC_u \arrow[r, "t\bar{w}"'] & tC_{vu}
\end{tikzcd}$$

We shall construct a map φ for which this diagram commutes. For this we use the representation $\chi : Cone_{\bar{u}}(U) \to hom_{\mathbf{A}}(C_{\bar{u}}, U)$ defining $i_{\bar{u}}$, $q_{\bar{u}}$ as in (8.5) with $U = tC_u$. Moreover we use the following diagram of tracks:

(6) $$\begin{tikzcd}
C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "d'"'] \arrow[dr, phantom, "\overset{G'}{\Rightarrow}"] & C_v \arrow[d, "d''"'] \arrow[dr, "0"] \arrow[dr, phantom, "\overset{H_2''}{\Rightarrow}", near end, swap] & \\
tA \arrow[r, "tu"] \arrow[dr, "0"'] \arrow[dr, phantom, "\overset{(tH_0)^{\boxminus}}{\Rightarrow}"] & tB \arrow[r, "tv"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{tF}{\Rightarrow}"] & tC \arrow[d, "d'"] \\
 & tC_u \arrow[r, "t\bar{w}"'] & tC_{vu}
\end{tikzcd}$$

According to (4) this diagram represents $\sigma(1_{tC_{vu}})$. Now let

(7) $$\varphi = \chi\{\alpha, \hat{\alpha}\} \text{ with } \alpha = dd'' \ \textit{and} \ \hat{\alpha} = (tH_0)d'\square d(G')^{\boxminus}$$

Then we have $\varphi i_{\bar{u}} = dd'$ and we get $(t\bar{w})\varphi = -\bar{q}_v$ since using $R = (d'H_2''\square(tF)d'')^{\boxminus}$ yields the equation

(8) $$\chi((t\bar{w})\varphi) = (t\bar{w})_*\{\alpha, \hat{\alpha}\} = \{(t\bar{w})\alpha, (t\bar{w})\hat{\alpha}\} = \{0, \sigma(-1_{tC_{vu}})\} = \chi(-q_{\bar{u}})$$

Hence the proof of the commutativity of (5) is complete.

Finally we observe that there is a commutative diagram

(9)

$$\begin{tikzcd}
hom_{\mathbf{A}}(tC_{vu},X) \arrow[r, "(t\bar{w})^*"] & hom_{\mathbf{A}}(tC_u,X) \arrow[r, "(dd'')^*"] & hom_{\mathbf{A}}(C_v,X) \arrow[r, "\bar{u}^*"] & hom_{\mathbf{A}}(C_{vu},X) \\
Cone_{vu}(t^{-1}X) \arrow[r, "v^*"] \arrow[u, "t\chi"'] & Cone_u(t^{-1}X) \arrow[r, "\delta"] \arrow[u, "t\chi"'] & Cone_v(X) \arrow[r, "u^*"] \arrow[u, "\chi"'] & Cone_{vu}(X) \arrow[u, "\chi"']
\end{tikzcd}$$

Here the bottom row is the exact sequence in (3.6). The vertical arrows are isomorphisms as in (4.4) by the track triangles $\bar{C}_v$, $\bar{C}_u$ and $\bar{C}_{vu}$ respectively. One readily checks by the properties of maps between track triangles that the diagram commutes. Hence the top row is exact by (3.6) and therefore φ in (5) is an isomorphism. Now (Tr0) shows that the bottom row of (5) is an exact triengle. This completes the proof of (Tr4).

10 The dual of a triangulated track category

We have seen in (1.5) that an exact triangle $A \xrightarrow{f} B \to C \to tA$ in a triangulated category $\mathbf{A}$ induces for each object U of $\mathbf{A}$ the long exact sequence

(10.1)

$$hom_{\mathbf{A}}(U,t^{-1}A) \to hom_{\mathbf{A}}(U,t^{-1}B) \to hom_{\mathbf{A}}(U,t^{-1}C) \to hom_{\mathbf{A}}(U,A) \xrightarrow{f_*} hom_{\mathbf{A}}(U,B)$$

which resembles the exact sequence for the dual cone functor $Cone^f$ in (3.4). In fact we get:

Proposition 10.2. *Let* $\boldsymbol{B}$ *be a triangulated track category. Then for each* $f : A \to B$ *in* $\boldsymbol{B}_0$ *the dual cone functor* $Cone^f$ *has a canonical representation*

$$\chi : Cone^f(U) \cong hom_{\boldsymbol{A}}(U,t^{-1}C_f)$$

where $A \xrightarrow{f} B \to C_f \to tA$ *is induced by a track triangle extending* f. *Moreover* χ *yields an isomorphism of exact sequences (3.4) and (10.1).*

Proof. We have the tracks

$$\begin{tikzcd}
t^{-1}A \arrow[r, "t^{-1}f"] \arrow[rr, bend right=60, "\Downarrow H_{-3}"] & t^{-1}B \arrow[r, "u"] \arrow[rr, bend left=60, "\Uparrow H_{-2}"'] & t^{-1}C_f \arrow[r, "w"] \arrow[rr, bend right=60, "\Downarrow H_{-1}"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

which define the element

$$\{w, H_{-1}\} \in Cone^f(t^{-1}C_f).$$

Now we define the inverse of χ in (10.1) by

$$\chi^{-1}(\alpha) = \alpha^*\{w, H_{-1}\}.$$

We claim that the following diagram commutes up to sign

$$\begin{tikzcd}
 & Cone^f(U) \arrow[dr, "\bar{q}"] & \\
hom_{\mathbf{A}}(U, t^{-1}B) \arrow[ur, "\bar{i}"] \arrow[dr, "u_*"'] & & hom_{\mathbf{A}}(U, A) \\
 & hom_{\mathbf{A}}(U, t^{-1}C_f) \arrow[uu, "\chi^{-1}"'] \arrow[ur, "w_*"'] &
\end{tikzcd}$$

where $\bar{i}$ and $\bar{q}$ are defined as in (3.4). In fact $\bar{q}\chi^{-1} = w_*$ since

$$\bar{q}\chi^{-1}(\alpha) = \bar{q}\{w\alpha, H_{n-1}\alpha\} = w\alpha = w_*(\alpha).$$

Moreover, for $\delta \in hom_{\mathbf{A}}(U, t^{-1}B)$ we have to show that $\bar{i}\delta = \chi^{-1}u_*\delta = \chi^{-1}(u\delta)$ where

$$\bar{i}\delta = \{0, \sigma\delta\} = \delta^*\{0, \sigma 1_{t^{-1}B}\}$$
$$\chi^{-1}(u\delta) = \delta^*\chi^{-1}(u) = \delta^*\{wu, H_{-1}u\}$$

Hence it suffices to prove that in $Cone^f(t^{-1}B)$ we have $\{0, \sigma 1_{t^{-1}B}\} = -\{wu, H_{-1}u\}$. But this follows since the pair (H_{-2}, H_{-1}) represents $\sigma 1_B$. Hence we proved $\chi^{-1}u_x = -\bar{i}$. Using the exact sequences (3.4) and (10.1) we see that χ^{-1} is an isomorphism and the proposition (10.2) is proved.

□

Proposition (10.1) corresponds to the following result:

Theorem 10.3. *The categorical dual $\boldsymbol{B}^{op}$ of a triangulated track category $\boldsymbol{B}$ is again a triangulated track category with translation functor given by the inverse t^{-1} of the strict translation functor t for $\boldsymbol{B}$. The distinguished track triangles in $\boldsymbol{B}^{op}$ are the duals X^{op} of distinguished track triangles X in $\boldsymbol{B}$ with $X^{op}_n = X_{-n}$.*

11 The natural notion of good morphism

This section should be compared with section 3 in [Ne] which has the same title. Let $\mathbf{B}$ be a triangulated track category with homotopy category $\mathbf{A}$ and quotient functor $p : \mathbf{B}_0 \to \mathbf{A}$. Then p induces the functor

(11.1) $$p : \mathbf{TTr_B} \to \mathbf{Tri_A}$$

Here $\mathbf{Tri_A}$ is the category of triangles in $\mathbf{A}$ and morphisms as in (2.1). Moreover $\mathbf{TTr_B}$ is the category of distinguished track triangles in $\mathbf{B}$ and maps as in (6.2).

A *special* isomorphism in $\mathbf{Tri_A}$ is an isomorphism of the form

(11.2)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & tA \arrow[d, equal] \\
A \arrow[r] & B \arrow[r] & C' \arrow[r] & tA
\end{tikzcd}$$

For example (6.8) yields such special isomorphisms. We use the functor p in (11.1) and special isomorphisms for the following definition:

Definition 11.3. A *good triangle* is a triangle T in $\mathbf{Tri_A}$ together with an equivalence class of special isomorphisms

$$T \cong p(X)$$

where X is an object in $\mathbf{TTr_B}$. Two such isomorphisms $T \cong p(X)$, $T \cong p(Y)$ are equivalent if there exists a map $X \to Y$ in $\mathbf{TTr_B}$ inducing the composite $p(X) \cong T \cong p(Y)$. A *good morphism* between good triangles T, T' is a morphism $f : T \to T'$ in $\mathbf{Tri_A}$ for which there exists a commutative diagram in $\mathbf{Tri_A}$

$$\begin{tikzcd}
p(X) \arrow[r, "p(\varphi)"] & p(X') \\
T \arrow[u, "\alpha"] \arrow[r, "f"'] & T' \arrow[u, "\alpha'"']
\end{tikzcd}$$

where α and α' are representing the equivalence classes of isomorphisms for T and T' respectively and where $p(\varphi)$ is induced by a map $\varphi : X \to X'$ in $\mathbf{TTr_B}$. Let $\mathbf{GTr}$ be the category of good triangles and good morphisms. We have the faithful forgetful functor

$$F : \mathbf{GTr} \longrightarrow \mathbf{Tri_A}.$$

Theorem 11.4. *For a triangulated track category $\boldsymbol{B}$ the homotopy category $\boldsymbol{A}$ of $\boldsymbol{B}$ together with the functor $F : \mathbf{GTr} \longrightarrow \mathbf{Tri}_{\boldsymbol{A}}$ satisfy the axioms (GTR1)...(GTR10) of a good triangulated category defined in [Ne](3.4).*

We do not want to recall the 10 axioms of Neeman and leave the proof of the theorem to the reader. The author does not know any examples of triangulated categories or good triangulated categories which are not given by triangulated track categories. Therefore it seems reasonable to replace the complicated set of axioms (Trn) for $n = 0, ..., 4$ or (GTRn) for $n = 0, ..., 10$ by two very natural axioms (TTr1) and (TTr2) of a triangulated track category. Many of the numerous results on triangulated categories in the literature have an analogue in a triangulated track category yielding a natural and interesting extension of such results.

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[II]

Principal maps in triangulated categories and maps between 2-stage spectra

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Abstract

Using the two natural axioms on track triangles in a triangulated track category we show that the mapping cone construction is functorial, that is, principal maps between mapping cones in a triangulated category can be identified with the homotopy classes of homotopy pairs. This is applied to obtain an algebraic model of the category of principal maps between 2-stage spectra.

Introduction

Consider the following commutative diagram in a triangulated category with translation functor t

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & t(A) \arrow[d, "t(f)"] \\
A^{'} \arrow[r, "d'"] & B^{'} \arrow[r] & C^{'} \arrow[r] & t(A^{'})
\end{tikzcd}$$

Here the rows are exact (i.e. distinguished) triangles and C is called a *cone* of d. Then one of the axioms of a triangulated category says: *Given f and g with $gd = d'f$, then there exists h such that the diagram commutes.* Conversely we define that *a* map $h : C \longrightarrow C'$ is *principal* with respect to the cone structure of C and C' if there exists (f, g) such that the diagram commutes.

It is an old mystery in triangulated categories how pairs (f, g) and principal maps h are related. For this compare Gelfand-Manin [GM] IV.7 who remark: *"This non-functoriality of a cone is the first symptom that something is going wrong in the axioms of a triangulated category. Unfortunately, at the moment, we don't have a more satisfactory version."* We think that triangulated track categories as introduced in [I] are in this respect more satisfactory since we show that in a triangulated track category the category of principal maps is equivalent to the homotopy category of the category of homotopy pairs. This leads (in the case of the stable homotopy category of spectra) to the algebraic computation of all principal maps $C \longrightarrow C'$ where A, B, A' and B' are finite direct sums of Eilenberg-Mac Lane spectra over a prime field $\mathbb{F}$. In this case we call C and C' 2-*stage spectra*, see (1.5) and section 6.

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1 Principal maps

Let **A** be a triangulated category with translation functor $t : \mathbf{A} \longrightarrow \mathbf{A}$ which is an additive equivalence. Then a distinguished family of triangles

(1.1)
$$X \quad = \quad (A \xrightarrow{d} B \xrightarrow{d_0} C \xrightarrow{d_1} tA)\ ,$$

termed *exact triangles*, is given satisfying axioms (Tr0)...(Tr4), see [Ne1], [We]. We also write $C = C_d$. A morphism between exact triangles $X \longrightarrow X'$ is a commutative diagram in **A** of the form

(1.2)
$$\begin{tikzcd}
X \arrow[r, phantom, "="] & (A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & tA) \arrow[d, "t(f)"] \\
X' \arrow[r, phantom, "="] & (A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & tA')
\end{tikzcd}$$

We say that the morphism $(f, g, h) : X \longrightarrow X'$ extends the map $h : C \longrightarrow C'$ and we say that a map $h : C \longrightarrow C'$ in **A** is *principal* with respect to X and X' if there exists a morphism $X \longrightarrow X'$ extending h. Let **ET** be the category of exact triangles (1.1) and morphisms (1.2). Then we have the forgetful functor

(1.3)
$$\mathbf{ET} \longrightarrow \mathbf{A}$$

which carries X to C and (f, g, h) to h. The image category of this functor is the category **Prin** of principal maps. Objects in **Prin** are exact triangles X, X' and morphisms $X \longrightarrow X'$ in **Prin** are the maps $C \longrightarrow C'$ in **A** which are principal, i.e. in the image of the functor (1.3). We have the faithful functor **Prin** $\longrightarrow$ **A**.

Lemma 1.4. *A map $h : C \longrightarrow C'$ is principal with respect to X and X' if there exists a commutative diagram in **A** as in* (1) *or* (2) *respectively.*

(1.5)
$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & C \arrow[d, "h"] & C \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & tA \arrow[d] \\
B' \arrow[r] & C' & C' \arrow[r] & tA'
\end{tikzcd}$$

Axiom $(Tr3)$ says that given (f, g) as above with $gd = df$ there exists $(f, g, h) : X \longrightarrow X'$ extending (f, g). Here, however, the map h is not well defined by the pairs (f, g). Therefore the axiom yields a misterious relationship between the pairs (f, g) and the principal maps $h : C \longrightarrow C'$ in a triangulated category. In this paper we clarify this relationship in the context of a triangulated track category [I]. This leads in section 6 below to a computation as in the following example.

Example 1.6. Let $\mathbf{A} = \mathbf{Spec}$ be the stable homotopy category of spectra. Then **A** is a triangulated category. For a prime field $\mathbb{F}$ we have the full subcategory

$$2\text{-}\ \mathbf{stage} \subset \mathbf{Prin}.$$

The objects of 2-**stage** are triangles X for which A and B are finite direct sums of Eilenberg-Mac Lane spectra over $\mathbb{F}$. Hence for an object $X = C_d$ in 2-**stage** the map

$d : A \longrightarrow B$ in $\mathbf{A}$ is algebraicly determined by a map between free modules over the Steenrod algebra. Also for objects $X \longrightarrow X'$ in 2-**stage** the map $(f, g) : d \longrightarrow d'$ is easily expressed algebraicly. It is, however, an old problem to compute the principal maps $C_d \longrightarrow C_{d'}$ in the category 2-**stage**, see [KM]. As an application of the main result in this paper we obtain an algebraic description of the category 2-**stage** in terms of the pair algebra $\mathcal{B}$ of secondary cohomology operations computed in [Ba3], see section 6.

2 The category of homotopy pairs

Let $\mathbf{B}$ be a triangulated track category (see [I]) with strict translation functor $t : \mathbf{B} \longrightarrow \mathbf{B}$ and homotopy category $\mathbf{A}$. Then we have a linear extension of categories

(2.1)
$$D \longrightarrow \mathbf{B_1} \rightrightarrows \mathbf{B_0} \longrightarrow \mathbf{A}$$

with coefficient $D(A, B) = Hom_{\mathbf{A}}(tA, B)$. Here $\mathbf{B} = (\mathbf{B_1} \rightrightarrows \mathbf{B_0})$ is a track category (i.e. a groupoid enriched category) with hom-groupoids $[\![A, B]\!]$ for all objects A, B in $\mathbf{B}$. The objects of the groupoid $[\![A, B]\!]$ are maps $f : A \longrightarrow B$ (or 1-cells) and morphisms, denoted $F : f \Longrightarrow f'$, are called tracks (or 2-cells). The 1-cells are the morphisms of the category $\mathbf{B_0}$. Two 1-cells f, f' are *homotopic* if there exists a track $f \Longrightarrow f'$. The homotopy classes of 1-cells in $\mathbf{B_0}$ are the morphisms in $\mathbf{A}$ and $\mathbf{B_0} \longrightarrow \mathbf{A}$ is the quotient functor. The track category $\mathbf{B}$ has a strong zero object $*$ with $[\![A, *]\!] = 0 = [\![*, A]\!]$ which yields the zero maps $0 : A \longrightarrow * \longrightarrow B$ in $\mathbf{B_0}$. Moreover, we assume that $\mathbf{B}$ has strong coproducts $A \vee B$ (which yield the direct sum $A \oplus B$ in $\mathbf{A}$) with $[\![A \vee B, C]\!] = [\![A, C]\!] \times [\![B, C]\!]$.

A *track-triangle* $X = (H_0, H_1, H_2)$ in $\mathbf{B}$ is a diagram of the form

(2.2)
$$\begin{tikzcd}
A \arrow[r, "d"'] \arrow[rr, bend left=60, "0", "\Uparrow H_0"'] & B \arrow[r] \arrow[rr, bend right=60, "0"', "\Downarrow H_1"] & C \arrow[r] \arrow[rr, bend left=60, "0", "\Uparrow H_2"'] & tA \arrow[r, "td"'] & tB
\end{tikzcd}$$

satisfying certain properties [I]. In a triangulated track category $\mathbf{B}$ one has a distinguished class of track triangles subject to two axioms (TTr1) and (TTr2)'. We show in [I] that the two axioms imply that $\mathbf{A}$ is a triangulated category in which exact triangles are induced by track triangles via the quotient functor $\mathbf{B_0} \longrightarrow \mathbf{A}$.

Definition 2.3. Let $\mathbf{B}$ be a track category. Then the *category* **HP** *of homotopy pairs in* $\mathbf{B}$ is defined as follows. Objects are the morphisms $d : A \longrightarrow B$ in $\mathbf{B_0}$ and morphisms $F : d \longrightarrow d'$ in **HP** are diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & A' \arrow[d, "d'"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

in **B**. Composition in **HP** of the morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & A' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\overset{F'}{\Rightarrow}"] & A'' \arrow[d, "d''"] \\
B \arrow[r, "g"'] & B' \arrow[r, "g'"'] & B''
\end{tikzcd}$$

is defined by ff', gg' and

$$F' \bar{\Box} F \;=\; F'f \Box g'F.$$

Here $F \Box G$ denotes the composition of tracks. Let $F^{\boxminus}$ be the inverse of the track F.

Proposition 2.4. *Let **B** be a triangulated track category. Then there is a functor*

$$\psi : \mathbf{HP} \longrightarrow \mathbf{Prin}$$

*from the category of homotopy pairs in **B** to the category of principal maps in **A** such that this functor is full and representative.*

This result shows that there is a natural equivalence relation $\sim$ on **HP** such that quotient category **HP**$/\sim$ is equivalent to the category **Prin**. We shall compute the equivalence relation in section 4. The functor ψ in (2.4) corresponds to the construction of principal maps between mapping cones in [Ba2].

Addendum 2.5. *If **B** has strong coproducts then there is a functor ψ in such a way that ψ carries coproducts in* **HP** *to coproducts in **A**.*

Proof of (2.4). For each object $d : A \longrightarrow B$ we choose a distinguished track triangle $X = X_d$ as in (2.2) extending d, this corresponds to axiom (TTr1) in **B**. We define the functor ψ on the object d by $\psi(d) = C_d$ where $C = C_d$ is given by X as in (2.2), that is $\psi(d)$ is the triangle induced by X_d in **A**. Let $(f, g, F) : d \longrightarrow d'$ be a morphism in **HP**. Then we can choose by the axiom (TTr2)' in **B** a map $\varphi : X_d \longrightarrow X_{d'}$ between track triangles extending (f, g, F). The map φ defines a map $h : C_d \longrightarrow C_{d'}$ and a track G as in the diagram

(1)
$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left=45, "0", "\Uparrow H_0"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow F"] & B \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow G"] & C \arrow[d, "h"] \\
A' \arrow[r] \arrow[rr, bend right=45, "0"', "\Downarrow H_0'"] & B' \arrow[r] & C'
\end{tikzcd}$$

which represents the trivial track $0 \Longrightarrow 0$. The homotopy class {h} of h in **A** corresponds via the representation

(2)
$$\{h\} \in hom_{\mathbf{A}}(C, C') \cong Cone_d(C')$$

in [I] (4.4) to the element $\{hd, hH_0\}$ in $Cone_d(C')$. By the equivalence relation defining the cone functor in [I] (3.1) (1) we see that for any other map $\varphi' : X_d \longrightarrow X_{d'}$ extending (f, g, F) we have

$$\{hd\,,\, hH_0\} \;=\; \{d'g\,,\, hH_0 \,\square\, G^{\boxminus} d\} \;\overset{(*)}{=}\; \{d'g\,,\, H_0' f \,\square\, d'F\}$$

$$\overset{(*)}{=}\; \{d'g\,,\, h'H_0 \,\square\, G'^{\boxminus} d\} \;=\; \{h'd\,,\, h'H_0\} \tag{3}$$

Here (h', G') is given by φ' and in (*) we use the fact that diagram (1) represents the trivial track. This shows that the homotopy class $\{h\}$ of h does not depend on the choice of φ. We now define the functor ψ on morphisms by $\psi(f, g, F) = \{h\}$. It is clear that ψ is a well defined functor. By [I] (6.6) we see that ψ is representative, that is, for each exact triangle (1.1) in **Prin** there is an object d in **HP** with $\psi(d) \cong C$ in **Prin**. Finally we check that the functor ψ is full. For this let $h : \psi(d) \longrightarrow \psi(d')$ be a map in **Prin**. Then we can choose (g, G) as in (1) such that $(g, h, G) : d_0 \longrightarrow d_0'$ is a map in **HP** with $d_0 = d : B \longrightarrow C$. We can find a map $\varphi' : X_{d_0} \longrightarrow X_{d_0'}$ extending (g, h, G) and we define (f, F) in such a way that the strict translation functor t on **B** yields (tf, tg, tF) as part of φ_0. Then we get $\psi(f, g, F) = \{h\}$. This completes the proof of (2.4). □

For the proof of the addendum we observe that the coproduct of track triangles is a track triangle.

3 Cogroups and Γ-tracks

Let **gr** be the category of finitely generated free groups $< E >$.

Definition 3.1. Let C be a category with coproducts $X \vee Y$. Then a *cogroup* X in **C** is given by a coproduct preserving functor

$$\mathbf{gr} \longrightarrow \mathbf{C}$$

which carries $\mathbb{Z}$ to X. If **B** is a track category with strong coproducts $X \vee Y$ then a *pseudo cogroup* X in **B** is a pseudo functor

$$\mathbf{gr} \longrightarrow \mathbf{B}$$

which preserves coproducts. Hence we have for $\alpha :< E > \longrightarrow < E' >$ in **gr** a map $\alpha_* : \vee_E X \longrightarrow \vee_{E'} X$ and for $\beta :< E > \longrightarrow < E'' >$ we have a track

$$\begin{tikzcd} \vee_E X \arrow[r, "\alpha_*"] \arrow[rr, bend right=40, "(\beta\alpha)_*"', ""{name=B}] & \vee_{E'} X \arrow[r, "\beta_*"] \arrow[phantom, to=B, "\Downarrow\tau"] & \vee_{E''} X \end{tikzcd}$$

We call τ a *linearity track* for X.

For $-1 : \mathbb{Z} \to \mathbb{Z}$ we write $-1_X = (-1)_* : X \to X$ and for $i_1 + i_2 : \mathbb{Z} \to \mathbb{Z} \vee \mathbb{Z}$ in **gr** we write $(i_1 + i_2)_* = i_1 + i_2 : X \to X \vee X$. This is the *comultiplication* of X.

Now let **B** be an additive track category with strong coproducts and a strong zero object $*$. Then we obtain as in [I] (8.1) by use of the linearity tracks above well defined Γ-Tracks for each $f : X \longrightarrow Y$ in $\mathbf{B}_0$.

(3.2)
$$\begin{tikzcd}
X \vee X \arrow[r, "f \vee f"] & Y \vee Y \\
X \arrow[u, "i_1+i_2"] \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\Gamma}{\Longrightarrow}"] & Y \arrow[u, "i_1+i_2"']
\end{tikzcd}$$

(1)
$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X \arrow[u, "-1"] \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\Gamma}{\Longrightarrow}"] & X \arrow[u, "-1"']
\end{tikzcd}$$

We use these tracks to define for $-i_1 + i_2 : X \longrightarrow X \vee X$ also the Γ-track

(2)
$$\Gamma : (-i_1 + i_2)f \Longrightarrow -i_1 f + i_2 f$$

given by pasting in the diagram

(3)
$$\begin{tikzcd}
X \vee X \arrow[r, "f \vee f"] & Y \vee Y \\
X \vee X \arrow[u, "-1 \vee 1"] \arrow[r, "f \vee f"] \arrow[ur, phantom, "\Gamma \vee 0^{\square}_{f}"] & Y \vee Y \arrow[u, "-1 \vee 1"'] \\
X \arrow[u, "i_1+i_2"] \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\Gamma}{\Longrightarrow}"] & Y \arrow[u, "i_1+i_2"']
\end{tikzcd}$$

Here $0^{\square}_{f} : f \Longrightarrow f$ is the trivial track. In [BP] we prove the following result.

Lemma 3.3. *Let* **B** *be an additive track category with strong coproducts. Then each object* X *in* **B** *admits the structure of a pseudo cogroup.*

In some of the proofs below we assume that objects X in **B** are actually cogroups so that we omit this way the linearity tracks for X. We leave it to the reader to generalize the argument in case the linearity tracks are non trivial.

Example 3.4. Let $\mathbf{S}_{\mathbf{P}_\mathbf{F}}$ be the category of simplicial group spectra FX where X is a simplicial spectrum, see [Ka] (5.2). Then $\mathbf{S}_{\mathbf{P}_\mathbf{F}}$ has a cylinder functor and the homotopy category $\mathbf{S}_{\mathbf{P}_\mathbf{F}}/\simeq\; = \mathbf{Spec}$ is the stable homotopy category of the spectra. Moreover the associated track category for $\mathbf{S}_{\mathbf{P}_\mathbf{F}}$ is a triangulated track category **B** with a strict translation functor t in which each object is a cogroup in $\mathbf{B}_\mathbf{0}$. Hence the results of this paper hold in particular in stable homotopy theory.

4 The homotopy relation for homotopy pairs

Let **B** be an additive track category with strong coproducts and a strong zero object. For each object X in **B** we choose the structure of a pseudo cogroup as in Lemma 3.3.

Definition 4.1. For a morphism $d : A \longrightarrow B$ in $\mathbf{B_0}$ we define the *cylinder object* $I(d)$ by the diagram in **B**

$$\begin{tikzcd}
A \arrow[r, "\partial_2"] \arrow[rr, bend right=40, "0"', ""{name=Z}] & A \vee B \vee A \arrow[r, "\partial_1"] \arrow[to=Z, phantom, "\Downarrow\partial_{12}"] & B \vee B
\end{tikzcd}$$

Here we set $\partial_1 = (i_1 d, -i_1 + i_2, i_2 d)$ and $\partial_2 = i_1 + i_2 d - i_3$. Moreover the track ∂_{12} is given by Γ in(3.2)(2) and by linearity tracks, that is, for $\partial_1\partial_2 = i_1 d + (-i_1 + i_2)d - i_2 d$ and $\delta = i_1 d + d(-i_1 + i_2) - i_2 d = (d \vee d)(i_1 + (-i_1 + i_2) - i_2)$ let $\partial_1\partial_2$ be the composite

$$\partial_1\partial_2 \xRightarrow{i_1 d + \Gamma - i_2 d} \delta \xRightarrow{(d\vee d)\tau} 0.$$

Here τ is the appropriate linearity track.

Definition 4.2. Let $d : A \longrightarrow B$ and $d' : A' \longrightarrow B'$ be objects in the category of homotopy pairs. Then two morphisms

$$(d, g, F)\ ,\ (f', g', F') :\ d \longrightarrow d'$$

in **HP** are *homotopic* if and only if there exists a map $a : B \longrightarrow A$ and there exist tracks

$$\alpha : (g, g')(-i_1 + i_2) \Longrightarrow d'a$$

$$\beta : (f, a, f')\partial_2 \Longrightarrow 0$$

such that pasting in the following diagram yields the trivial track $0 \Longrightarrow 0$.

$$\begin{tikzcd}
& A \arrow[d, "\partial_2"'] \arrow[dr, bend left, "0"] \arrow[dr, phantom, "\overset{\beta}{\Longrightarrow}"] \arrow[dd, bend right=80, "0"'] & \\
{} & A \vee B \vee A \arrow[l, Rightarrow] \arrow[r, "{(f,a,f')}"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\overset{(F,\alpha,F')}{\Longrightarrow}"] & A' \arrow[d, "d'"] \\
& B \vee B \arrow[r, "{(g,g')}"'] & B'
\end{tikzcd}$$

Here the left hand side of the diagram is given by the cylinder $I(d)$ in (4.1).

Remark 4.3. A Toda complex $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ is a sequence of maps $d_n : X_{n+1} \longrightarrow X_n$ and tracks $H_n : d_{n+1}d_n \Longrightarrow 0$ in **B**. This is a *secondary chain complex* if all Toda pairs (H_{n+1}, H_n) represent the zero element, see [I] (5.1). Let **SCh** be the category of secondary chain complexes with morphisms defined as maps between Toda complexes,see [I] (5.3). We consider an object $d : A \longrightarrow B$ in **HP** as an object X in **SCh** concentrated in degrees 0 and 1, that is, $X_n = *$ for $n \neq 0, 1$. This way we have the full inclusion of categories $\mathbf{HP} \subset \mathbf{SCh}$. In [BG] we shall describe a cylinder $I(X)$ in the category **SCh** which generalizes the cylinder $I(d)$ in (4.1). The cylinder $I(X)$ yields a homotopy relation $\simeq$ on **SCh** such that one has the full inclusion of homotopy categories $\mathbf{HP}/\simeq\ \subset \mathbf{SCh}/\simeq$.

Theorem 4.4. *Let* $\mathbf{B}$ *be a triangulated track category with strict coproducts* $X \vee Y$*. Then the functor* ψ *in (2.4) induces an equivalence of categories*

$$\psi : \mathbf{HP}/\simeq\ \xrightarrow{\sim} \mathbf{Prin}.$$

Hence the homotopy category of the homotopy pairs in **B** is equivalent to the category of principal maps in $\mathbf{A} = \mathbf{B}_0/\simeq$. Results of this kind under connectivity and dimensional restrictions are described in [Ba2]. We prove the theorem in the next section.

5 Proof of the equivalence (4.4)

Let $d : A \longrightarrow B$ be a map of $\mathbf{B_0}$ and let $C = C_d$ be given by a distinguished track triangle extending d, see (2.2). We consider the following diagram in **B**

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d] \arrow[dr, phantom, "\overset{H_0^{\boxminus}}{\Longrightarrow}"] & B \arrow[r, equal] \arrow[d, "d_0"] \arrow[dr, phantom, "\overset{\Gamma}{\Longrightarrow}"] & B \arrow[d, "\partial"] & \\
* \arrow[r, "0"'] \arrow[d] & C \arrow[r, "-i_1+i_2"'] \arrow[d, "j"] & C \vee C \arrow[r, "{(\varphi,\varphi')}"] \arrow[d, "i"] & X \\
tA \arrow[r, "v"] & C_{d_0} \arrow[r, "w"] & C_\partial \arrow[ur, "u"] &
\end{tikzcd}$$

Here $\partial = i_1 d_0 + i_2 d_0$ and Γ is defined by (3.2)(2). Moreover H_0 is given by (2.2). The tracks in the diagram describe homotopy pairs in **HP** and wv is a map in $\mathbf{B_0}$ representing $\psi(\Gamma \bar{\Box} H_0^{\boxminus})$ where ψ is the functor in (2.4) and $w = \psi(\Gamma)$ and $v = \psi(H_0^{\boxminus})$. Here v is an isomorphism in **A** by (2.2).

Lemma 5.1. *Let* $\varphi, \varphi' : C \longrightarrow X$ *in* $\mathbf{B_0}$*. Then* $\varphi \simeq \varphi'$ *if and only if there exists* $u : C_\partial \longrightarrow X$ *in* $\mathbf{B_0}$ *with* $ui = (\varphi, \varphi')$ *and* $uw = 0$ *in* $\boldsymbol{A}$*.*

Proof. We have $\varphi \simeq \varphi'$ if and only if $(\varphi, \varphi')(-i_1 + i_2) \simeq 0$. Hence $\varphi \simeq \varphi'$ implies that $(\varphi, \varphi')\partial = 0$ in **A** and hence there is u' with $u'i = (\varphi, \varphi')$ in **A**. Then $u'wj = 0$ so that $u'w$ factors in **A** through the cofiber tB of j. This cofiber is isomorphic to the cofiber of i. Therefore we can alter u' by a map $tB \longrightarrow X$ in such a way that we get u with $uw = 0$. Conversely assume u is given. Then $uv = 0$ and hence $(\varphi, \varphi')(-i_1 + i_2) = uwj = 0$ in **A**. $\square$

Next we consider the following diagram in **B** where we use the cylinder object

$I(d)$ in (4.1)

$$\begin{tikzcd}
A \arrow[r, "\partial_2"] \arrow[d] \arrow[dr, phantom, "\overset{\partial_{12}^{\boxminus}}{\Longrightarrow}"] & A \vee B \vee A \arrow[r, "{(0,1,0)}"] \arrow[d, "\partial_1"] \arrow[dr, phantom, "\overset{V}{\Longrightarrow}"] & B \arrow[d, "\partial"] \\
* \arrow[r] \arrow[d] & B \vee B \arrow[r, "d_0 \vee d_0"'] \arrow[d] & C \vee C \arrow[d] \\
tA \arrow[r, "w'"] & C_{\partial_1} \arrow[r, "v'"] & C_{\partial}
\end{tikzcd}$$

Here V is given by $(i_1H_0, 0^{\Box}, i_2H_0)$. We consider V and $\partial_{12}^{\boxminus}$ as morphisms in **HP**. Let v' and w' be maps which represent $\psi(V)$ and $\psi(\partial_{12}^{\boxminus})$ respectively. We observe v' is an isomorphism in **A**.

Lemma 5.2. *We have $v'w' = wv$ in* ***A****.*

Proof. We have $wv = \psi(\Gamma \,\bar{\Box}\, H_0^{\boxminus})$ and $v'w' = \psi(V \,\bar{\Box}\, \partial_{12}^{\boxminus})$. Therefore it suffices to show that in **HP** we have the equation

$$\Gamma \,\bar{\Box}\, H_0^{\boxminus} = V \,\bar{\Box}\, \partial_{12}^{\boxminus}. \tag{1}$$

Let $p_1 = (1,0) : C \vee C \longrightarrow C$ and $p_2 = (0,1) : C \vee C \longrightarrow C$ be the projections. Then (1) holds if and only if (2) holds.

$$p_i(\Gamma \,\bar{\Box}\, H_0^{\boxminus}) = p_i(V \,\bar{\Box}\, \partial_{12}^{\boxminus}) \quad \text{for } i = 1, 2. \tag{2}$$

This follows from the bilinearity of the coefficients D in (2.1). For convenience we now assume that additive tracks τ in (3.1) are trivial tracks. Then we get $p_2V = (0^{\Box}, H_0)$, $p_2\partial_{12}^{\boxminus} = 0^{\Box}$ and $p_2\Gamma = 0^{\Box}$. Hence (2) for $i = 2$ is equivalent to

$$H_0^{\boxminus} = d_0d - H_0. \tag{3}$$

This holds since for any track $H : f \Rightarrow 0$ one has $0^{\Box} = H - H = H \,\Box\, (f - H)$.

Next we consider the case $i = 1$ in (2). Then we have $p_1V = (H_0, 0^{\Box})$, $p_1\partial_{12}^{\boxminus} = d + \bar{\Gamma}$ with $\bar{\Gamma} : d(-1) \Rightarrow (-1)d$, $p_1\Gamma = \bar{\Gamma}_0^{\boxminus} : (-1)d_0 \Rightarrow d_0(-1)$. Therefore (2) for $i = 1$ is equivalent to

$$\bar{\Gamma}_0^{\boxminus} \,\bar{\Box}\, H_0^{\boxminus} = (H_0^{\boxminus}, 0^{\Box}) \,\bar{\Box}\, (d + \bar{\Gamma}). \tag{4}$$

In fact, we have

$$\begin{aligned}
(H_0, 0^{\Box}) \,\bar{\Box}\, (d + \bar{\Gamma}) &= (H_0, 0^{\Box})(i_1 + i_2d) \,\Box\, d_0(d + \bar{\Gamma}) \\
&= (H_0 + d_0(-1)d) \,\Box\, (d_0d + d_0\bar{\Gamma}) \\
&= H_0 + d_0\bar{\Gamma} : 0 \Longrightarrow d_0(-1)d
\end{aligned} \tag{5}$$

On the other hand we get:

$$\begin{aligned}
\bar{\Gamma}_0^{\boxminus} \,\bar{\Box}\, H_0^{\boxminus} &= \bar{\Gamma}_0^{\boxminus}d \,\Box\, (-1)H_0^{\boxminus} \\
&= d_0\bar{\Gamma} \,\Box\, (-H_0^{\boxminus}) && \text{, see (7),} \\
&= d_0\bar{\Gamma} \,\Box\, (H_0 - d_0d) && \text{, see (3),} \\
&= H_0 + d_0\bar{\Gamma}.
\end{aligned} \tag{6}$$

Here we use the fact that by the rules for Γ-tracks the following diagram represents the trivial track.

(7)

$$\begin{tikzcd}
\bullet \arrow[r, "d"] \arrow[d, "-1"'] \arrow[rr, bend left=40, "0", "\Uparrow H_0"'] \arrow[dr, phantom, "\overset{\bar{\Gamma}}{\Longrightarrow}"] & \bullet \arrow[r, "d_0"] \arrow[d, "-1"] \arrow[dr, phantom, "\overset{\bar{\Gamma}_0}{\Longrightarrow}"] & \bullet \arrow[d, "-1"] \\
\bullet \arrow[r, "d"'] \arrow[rr, bend right=40, "0"', "\Downarrow H_0"] & \bullet \arrow[r, "d_0"'] & \bullet
\end{tikzcd}$$

This completes the proof of (5.2). $\square$

Proof of (4.4). Let $d : A \longrightarrow B$ and $d' : A' \longrightarrow B'$ be objects in **HP** and let $F = (f, g, F)$ and $F' = (f', g', F')$ be morphisms $d \longrightarrow d'$ in **HP**. We have to show that F is homotopic to F' as in (4.2) if and only if $\psi(F) = \psi(F')$. By (4.2) we obtain the following diagram in **B**.

$$\begin{tikzcd}
A \arrow[r, "\partial_2"] \arrow[d] \arrow[rr, bend left=30, "0", "\Uparrow \beta"'] \arrow[dr, phantom, "\overset{\partial_{12}^{\boxminus}}{\Longrightarrow}"] & A \vee B \vee A \arrow[r, "{(f,a,f')}"] \arrow[d, "\partial_1"] \arrow[dr, phantom, "\overset{(F,\alpha,F')}{\Longrightarrow}"] & A' \arrow[d, "d'"] \arrow[dd, bend left=60, "0", "\overset{H_0'}{\Rightarrow}"'] \\
* \arrow[r] \arrow[d] & B \vee B \arrow[r, "{(g,g')}"'] \arrow[d] & B' \arrow[d] \\
tA \arrow[r, "w'"] & C_{\partial_1} \arrow[r, "W"] & C_{d'} \\
& C_d \vee C_d \arrow[u, "i"] \arrow[ur, "{(\psi,\psi')}"'] &
\end{tikzcd}$$

Here w', W and i are maps representing $\psi(\partial_{12}^{\boxminus})$, $\psi(F, \alpha, F')$ and $d \vee d \subset \partial_1$ respectively and ψ and ψ' are maps representing $\psi(F)$, resp. $\psi(F')$. Since ψ is a functor we have $Wi = (\psi, \psi')$ in **A**.

Now assume $F \simeq F'$ in **HP** so that a, α, β are given. Then we can choose W as above and we observe that Ww' in **A** is represented by pasting tracks $\partial_{12}^{\boxminus}$, (F, α, F') and H_0'. Compare the proof of (2.4). This pasting yields the same element as the pasting of $\partial_{12}^{\boxminus}$, (F, α, F') and β which by definition of $F \simeq F'$ in (4.2) represents 0. Hence $Ww' = 0$ and therefore (5.1) and (5.2) shows $\psi \simeq \psi'$ since v and v' are isomorphisms in **A**.

On the other hand assume now $\psi \simeq \psi'$. Then by (5.1) and (5.2) there exists W with $Wi = (\psi, \psi')$ in **A** and $Ww' = 0$ in **A**. Since the functor ψ in (2.4) is full we find a, α as above with $\psi(F, \alpha, F') = W$ and $\psi(F) = \psi$, $\psi(F') = \psi'$ in **A**. Moreover there exists $\beta' : (f, a, f')\partial_2 \Rightarrow 0$ since $Ww' = 0$ induces $t((f, a, f')\partial_2) = 0$. Now pasting $\partial_{12}^{\boxminus}$, (F, α, F') and β yields an element $\xi \in Hom_{\mathbf{A}}(tA, B')$. This element maps to 0 in $Hom_{\mathbf{A}}(tA, C_{d'})$ since $Ww' = 0$. Hence there exists $\xi' \in Hom(tA, A')$ which maps to ξ, that is, $d'\xi' = \xi$. Now we alter β' by ξ' and get β such that pasting of $\partial_{12}^{\boxminus}$, (F, α, F') and β represents the zero element. This shows $F \simeq F'$ in **HP**.

$\square$

6 The category of 2-stage spectra

Let k be a commutative ring with unit and let **Mod** be the category of k-modules and k-linear maps. This is a symmetric monoidal category via the tensor product $A \otimes B$ over k. A *pair* of modules is a morphism

$$X = \left(X_1 \xrightarrow{\partial} X_0 \right)$$

in **Mod**. A *morphism* $f : X \longrightarrow Y$ of pairs is a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\partial"'] & Y_1 \arrow[d, "\partial"] \\
X_0 \arrow[r, "f_0"] & Y_0
\end{tikzcd}$$

Clearly a pair in **Mod** coincides with a chain complex concentrated in degrees 0 and 1. For two pairs X and Y the tensor product of the chain complexes corresponding to them is concentrated in degrees 0, 1 and 2 and is given by

$$X_1 \otimes Y_1 \xrightarrow{\partial_1} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 \xrightarrow{\partial_0} X_0 \otimes Y_0$$

with $\partial_0 = (\partial \otimes 1, 1 \otimes \partial)$ and $\partial_1 = (-1 \otimes \partial, \partial \otimes 1)$. Truncating $X \otimes Y$ we get the pair

$$X \bar{\otimes} Y = \left((X \bar{\otimes} Y)_1 = \mathit{coker}(\partial_1) \xrightarrow{\partial} X_0 \otimes Y_0 = (X \bar{\otimes} Y)_0 \right)$$

with ∂ induced by ∂_0.

We define the tensor product $A \otimes B$ of two graded modules in the usual way, i.e. by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j .$$

A *(graded) pair module* is a sequence $X^n = (\partial : X_1^n \to X_0^n)$ with $n \in \mathbb{Z}$ of pairs in **Mod**. We identify such a graded pair module X with the underlying morphism ∂ of degree 0 between graded modules

$$X = \left(X_1 \xrightarrow{\partial} X_0 \right) .$$

Now the tensor product $X \bar{\otimes} Y$ of graded pair modules X, Y is defined by

$$(X \bar{\otimes} Y)^n = \bigoplus_{i+j=n} X^i \bar{\otimes} Y^j .$$

This defines a monoidal structure on the category of graded pair modules. Morphisms in this category are of degree 0. For two morphisms f, $g : X \longrightarrow Y$ between graded pairs modules, a *homotopy* $H : f \Longrightarrow g$ is a morphism $H : X_0 \longrightarrow Y_1$ of degree 0 as in the diagram

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, "\partial"'] & Y_1 \arrow[d, "\partial"] \\
X_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[ur, "H"] & Y_0
\end{tikzcd}$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$. A *pair algebra* B is a monoid in the monoidal category of graded pair modules, with multiplication

$$\mu : B\bar{\otimes}B \longrightarrow B.$$

We assume that B is concentrated in nonnegative degrees, that is $B^n = 0$ for $n < 0$. A *left B-Module* is a graded pair module M together with a left action

$$\mu : B\bar{\otimes}M \longrightarrow M$$

of the monoid B on M. More explicitly pair algebras and modules over them can be described as follows.

Definition 6.1. A *pair algebra* B is a graded pair

$$\partial : B_1 \longrightarrow B_0$$

in **Mod** with $B_1^n = B_0^n = 0$ for $n < 0$ such that B_0 is a graded algebra in **Mod**, B_1 is a graded B_0-B_0-bimodule, and ∂ is a bimodule homomorphism. Moreover for x, y $\in B_1$ the equality

$$\partial(x)y = x\partial(y)$$

holds in B_1.

Definition 6.2. A *(left) module* over a pair algebra B is a graded pair $M = (\partial : M_1 \longrightarrow M_0)$ in **Mod** such that M_1 and M_0 are left B_0-modules and ∂ is B_0-linear. Moreover a B_0-linear map

$$\bar{\mu} : B_1 \otimes_{B_0} M_0 \longrightarrow M_1$$

is given fitting in the commutative diagram

$$\begin{tikzcd}
B_1 \otimes_{B_0} M_1 \arrow[r, "1\otimes\partial"] \arrow[d, "\mu"'] & B_1 \otimes_{B_0} M_0 \arrow[d, "\mu"] \arrow[dl, "\bar{\mu}"] \\
M_1 \arrow[r, "\partial"'] & M_0
\end{tikzcd}$$

where $\mu(b \otimes m) = \partial(b)m$ for $b \in B_1$ and $m \in M_1 \cup M_0$.

For an indeterminate element x of degree $n = |x|$ let $B[x]$ denote the B-module with $B[x]_i$ consisting of expressions bx with $b \in B_i$, i =0,1, with bx having degree $|b|+n$, and structure maps given by $\partial(bx) = \partial(b)x$, $\mu(b'\otimes bx) = (b'b)x$ and $\bar{\mu}(b'\otimes bx) = (b'b)x$. A *free* B-module is direct sum of several copies of modules of the form $B[x]$, with $x \in I$ for some set I of indeterminates of possibly different degrees. It will be denoted

$$B[I] = \bigoplus_{x\in I} B[x].$$

Let B-**Mod** be the category of left modules over the pair algebra B. Morphisms $f = (f_0, f_1) : M \longrightarrow N$ are pair morphisms which are B-equivariant, that is, f_0 and f_1 are B_0-equivariant and compatible with $\bar{\mu}$ above, i.e. the diagram

$$\begin{tikzcd}
B_1 \otimes_{B_0} M_0 \arrow[r, "\bar{\mu}"] \arrow[d, "1\otimes f_0"'] & M_1 \arrow[d, "f_1"] \\
B_1 \otimes_{B_0} N_0 \arrow[r, "\bar{\mu}"] & N_1
\end{tikzcd}$$

commutes.

For two such maps $f,\, g : M \longrightarrow N$ a track $H : f \Longrightarrow g$ is a degree zero map

$$H : M_0 \longrightarrow N_1$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$ such that H is B_0-equivariant. For tracks $H : f \Longrightarrow g$, $K : g \Longrightarrow h$ their composition $K \square H : f \Longrightarrow h$ is $K + H$.

The category B-**Mod** with these tracks is a well defined additive track category with strong direct sums $M \oplus N$. Let

(6.3) $$B\text{-}\mathbf{mod} \subset B\text{-}\mathbf{Mod}$$

be the full subcategory consisting of finitely generated free B-modules. The category of homotopy pairs $\mathbf{HP}(B\text{-}\mathbf{mod})$ is defined dually to (2.3). Morphisms $d' \longrightarrow d$ are diagrams

(6.4) $$\begin{tikzcd} A & A' \arrow[l, "f"'] \\ B \arrow[u, "d"] \arrow[ur, phantom, "\overset{F}{\Leftarrow}"] & B' \arrow[l, "g"] \arrow[u, "d'"'] \end{tikzcd}$$

in B-**mod**. Two such morphisms (f,g,F), $(f',g',F) : d' \longrightarrow d$ are *homotopic* if there exists (a,α,β) as in the diagram

$$\begin{tikzcd} A & \\ A \oplus B \oplus A \arrow[u, "\partial_2"] \arrow[ur, phantom, "\overset{\beta}{\Leftarrow}"] & A' \arrow[l, "{(f,a,f')}"'] \arrow[ul, bend right, ""] \\ B \oplus B \arrow[u, "\partial_1"] \arrow[ur, phantom, "\overset{(F,\alpha,F')}{\Longleftarrow}"] & B' \arrow[l, "{(g,g')}"] \arrow[u, "d'"'] \end{tikzcd}$$

with $\partial_1 = (dp_1, -p_1 + p_2, dp_2)$ and $\partial_2 = p_1 + dp_2 - p_3$. Here p_1, p_2, p_3 denote the projections. This defines a natural equivalence relation $\simeq$ on $\mathbf{HP}(B\text{-}\mathbf{mod})$.

Theorem 6.5. *Let B be the pair algebra of secondary cohomology operations computed in [Ba3]. Then one has an equivalence of categories*

$$(\mathbf{HP}(B\text{- }\mathbf{mod})/\simeq)^{op} \xrightarrow{\sim} 2\text{- }\mathbf{stage}.$$

Here 2-**stage** *is the category of* 2*-stage spectra and principal maps in (1.5).*

Proof. It is proved in [Ba3] (5.5.6) that the track category of finite direct sums of Eilenberg-Mac Lane spectra is equivalent to the track category $(B\text{-}\mathbf{mod})^{op}$. This equivalence is compatible with strong coproducts and hence induces the equivalence in the theorem by use of theorem (4.4). Since composition in B-**mod** is bilinear Γ-tracks are trivial in the notion of homotopy in (6.4). ☐

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[III]

The homotopy category of pseudofunctors and translation cohomology

H.-J. Baues and F. Muro

Abstract

We develop the obstruction theory of the 2-category of abelian track categories, pseudofunctors and pseudonatural transformations by using the cohomology of categories. We introduce translation cohomology to classify endomorphisms in this category.

Introduction

A homotopy category $\mathbf{A}$ is derived from a groupoid enriched category $\mathbf{B}$ in which the morphism groupoids $[\![X,Y]\!]$ are given by maps $f,g\colon X\to Y$ and tracks $\alpha\colon f\Rightarrow g$ which are homotopy classes of homotopies. The set of connected components

$$[X,Y]=\pi_0[\![X,Y]\!]$$

is the set of morphisms in $\mathbf{A}=\pi_0\mathbf{B}$. Now let $\mathbf{A}$ be the homotopy category of a pointed stable Quillen model category. The suspension functor Σ is an endofunctor of $\mathbf{A}$ which on $\mathbf{B}$, however, is only a pseudofunctor $\tilde{\Sigma}\colon\mathbf{B}\rightsquigarrow\mathbf{B}$. If $\mathbf{B}$ is enriched in abelian groupoids with $\pi_0\mathbf{B}=\mathbf{A}$ and $\pi_1\mathbf{A}=D$ then the equivalence class of $\mathbf{B}$ is determined by an element in the cohomology group

$$\langle\mathbf{B}\rangle\in H^3(\mathbf{A},D), \tag{0.1}$$

as proved in [BD89], [Pir88]. In this paper we consider the equivalence class of the pair $(\mathbf{B},\tilde{\Sigma})$. For this we introduce the *translation cohomology* $H^*(\mathbf{A},\Sigma)$ and we show as a main result that the equivalence class of $(\mathbf{B},\tilde{\Sigma})$ is determined by the cohomology class

$$\langle\mathbf{B},\tilde{\Sigma}\rangle\in H^3(\mathbf{A},\Sigma), \tag{0.2}$$

generalizing the result (0.1).

In order to achieve the result (0.2) we study the homotopy category of pseudofunctors $\mathbf{Pseudo}^{ab}_{\simeq}$ which is a 2-category via pseudonatural transformations. We achieve a useful obstruction theory for this category with obstructions in the cohomology of

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categories. As the motivating application we show in [IV] that for a stable homotopy category $(\mathbf{A}, \Sigma)$ the cohomology class $\langle \mathbf{B}, \tilde{\Sigma} \rangle$ determines the triangulation of $(\mathbf{A}, \Sigma)$, so that triangulated categories have a cohomological description. On the other hand there is an application concerning the category of topological 2-types, which can be considered as a full subcategory of $\mathbf{Pseudo}_{\simeq}^{ab}$. This leads to the *translation cohomology of groups* which classifies pairs (X, h) where X is a 2-type and $h \colon X \to X$ is an endomorphism in the homotopy category of spaces, see [BM01] and [BM].

Some proofs in this paper are somewhat technical, for this reason we present the results in the first sections, 2, ..., 9, and concentrate most of the proofs in the final ones, 10, ..., 13. Section 1 contains some background material about track categories which is completed by the Appendix, where we recall the classical concepts of pseudofunctor and pseudonatural transformation. In Section 2 we define the homotopy category of pseudofunctors $\mathbf{Pseudo}_{\simeq}$ as a quotient category, which is the main object of study in this paper. The full subcategory $\mathbf{Pseudo}_{\simeq}^{ab}$ given by abelian track categories is the localization of the category of pseudofunctors and of track functors with respect to weak equivalences, as we prove in Section 3 (Theorem 3.9). In Section 4 we recall the definition of the cohomology of small categories in the sense of [BW85] as well as some of its 2-functorial structure developed in [Mur04]. Translation cohomology of categories is defined in Section 5, and the classification result (0.2) is proved in Section 6 (Theorem 6.3).

Section 7 is the core of the paper, it is devoted to the obstruction theory in the homotopy category of pseudofunctors $\mathbf{Pseudo}_{\simeq}^{ab}$. In Section 8 we give an interpretation of this obstruction theory in terms of exact sequences for functors and linear extensions of categories in the sense of [Bau89]. As a consequence we show in Section 9 that the homotopy category of pseudofunctors $\mathbf{Pseudo}_{\simeq}^{ab}$ is equivalent to a category defined only in terms of cocycles in the cohomology of categories.

1 Categories and track categories

Categories, functors and natural transformations

The category **Cat** of all categories is a well-known 2-category. The objects or 0-cells of **Cat** are the categories which are small with respect to a fixed universe. The 1-cells are functors, and the 2-cells are natural transformations. The category **Cat** has products denoted by $\mathbf{A} \times \mathbf{B}$. A natural transformation can be regarded as a functor from the *cylinder category* $\mathbf{A} \times \mathbb{I} \to \mathbf{B}$. Here $\mathbb{I}$ is the *interval category*, it has two objects 0, 1 and a unique non-trivial morphism $0 \to 1$. There are two obvious functors from the trivial category $*$, which has only one morphism, $i_0, i_1 \colon * \to \mathbb{I}$, and a functor $p \colon \mathbb{I} \to *$. These functors induce the structural functors of a cylinder

(1.1)
$$\begin{tikzcd} \mathbf{A} \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & \mathbf{A} \times \mathbb{I} \arrow[r, "p"] & \mathbf{A} . \end{tikzcd}$$

The *double interval category* $\mathbb{J}$ has three objects 0, 1, 2 and two essential morphisms $0 \to 1 \to 2$. It fits into a push-out diagram in **Cat**

(1.2)
$$\begin{tikzcd} * \arrow[r, "i_1"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\text{push}"] & \mathbb{I} \arrow[d] \\ \mathbb{I} \arrow[r] & \mathbb{J} \end{tikzcd}$$

There is a unique functor $j\colon \mathbb{I} \to \mathbb{J}$ with $j(0) = 0$ and $j(1) = 2$. It can be used to define vertical composition of natural transformations regarded as functors from cylinders since the product functor $\mathbf{A} \times -$ preserves colimits in **Cat**.

In this paper a single arrow $\to$ will denote a morphism, a functor, or a 1-cell in a 2-category, while double arrows $\Rightarrow$ will be kept for natural transformations and 2-cells in 2-categories. The composite of morphisms f, g will be indicated by juxtaposition fg, as well as horizontal compositions in 2-categories. The symbol $\square$ will be used for vertical composition of 2-cells $\alpha\square\beta$, and the inverse of an invertible 2-cell α will be denoted by $\alpha^{\boxminus}$.

Track categories, track functors and track natural transformations

A *track category* $\mathbf{A}$ is a category enriched in groupoids. Hence for objects X, Y in $\mathbf{A}$ one has the hom-groupoid $[\![X,Y]\!]_{\mathbf{A}}$ in $\mathbf{A}$. The objects $f\colon X \to Y$ of this groupoid are called *maps*, and the morphisms $\alpha\colon f \Rightarrow g$ are called *tracks*. Let $[\![X,Y]\!]_{\mathbf{A}}(f,g)$ be the set of all tracks $f \Rightarrow g$. The symbol for the identity track in f is $0^{\square}_{f}$. Equivalently, a track category is a 2-category all of whose 2-cells are invertible. Ordinary categories can be regarded as track categories with only the identity tracks.

A track category $\mathbf{A}$ will also be depicted as $\mathbf{A}_1 \rightrightarrows \mathbf{A}_0$. Here $\mathbf{A}_0$ is the underlying ordinary category; $\mathbf{A}_1$ has the same objects as $\mathbf{A}$, morphisms form X to Y in $\mathbf{A}_1$ are tracks $\alpha\colon f \Rightarrow g$ between maps $f, g\colon X \to Y$ in $\mathbf{A}$, and the composition of α and β as in the following diagram

(1.3)
$$\begin{tikzcd} Z & Y \arrow[l, bend right, "f"', ""{name=A, below}] \arrow[l, bend left, "f'", ""{name=B, above}] \arrow[Rightarrow, from=A, to=B, "\alpha"] & X \arrow[l, bend right, "g"', ""{name=C, below}] \arrow[l, bend left, "g'", ""{name=D, above}] \arrow[Rightarrow, from=C, to=D, "\beta"] \end{tikzcd}$$

is horizontal composition

$$\alpha\beta = (\alpha g')\square(f\beta) = (f'\beta)\square(\alpha g).$$

The arrows $\rightrightarrows$ from $\mathbf{A}_1$ to $\mathbf{A}_0$ denote the obvious "source" and "target" functors.

A *track functor* is an enriched functor between track categories, that is a 2-functor, or equivalently, an assignment of objects, maps and tracks which preserves all composition laws and identities. The category **Track** of track categories and track functors also has colimit-preserving products.

A *track natural transformation* $\alpha\colon \varphi \Rightarrow \psi$ between track functors $\varphi, \psi\colon \mathbf{A} \to \mathbf{B}$. is a collection of maps $\alpha_X\colon \varphi(X) \to \psi(X)$ in $\mathbf{B}$ indexed by the objects of $\mathbf{A}$ satisfying certain axioms described in the Appendix. One can alternatively define such a track natural transformation α as a track functor $\alpha\colon \mathbf{A} \times \mathbb{I} \to \mathbf{B}$. Track natural transformations endow **Track** with a 2-category structure. Moreover, (1.2) can be used to define vertical composition of track natural transformations.

Pseudofunctors and pseudonatural transformations

A *pseudofunctor* between track categories $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is an assignment of objects, maps and tracks which preserves composition and identities only up to certain given tracks

$$\varphi_{f,g}\colon \varphi(f)\varphi(g) \Rightarrow \varphi(fg) \text{ and } \varphi_X\colon \varphi(1_X) \Rightarrow 1_{\varphi(X)}.$$

These tracks must satisfy well-known coherence and naturality properties which are recalled in the Appendix. Moreover, the category **Pseudo** of small track categories and pseudofunctors is defined.

Track functors can be regarded as a special case of pseudofunctors. The inclusion functor **Track** → **Pseudo** has a left adjoint

$$\mathcal{P}\colon \mathbf{Pseudo} \longrightarrow \mathbf{Track}. \tag{1.4}$$

This adjoint is defined in [Gra74] I,4.23 where $\mathcal{P}(\mathbf{A}) = \tilde{\mathbf{A}}$. A more handy description that we now recall can be found in [BJP03] 2.4. The category $\mathcal{P}(\mathbf{A})_0$ is the free category on $\mathbf{A}_0$ regarded as a graph, that is objects are the same as in $\mathbf{A}_0$ and a morphism $(\sigma_1, \cdots, \sigma_n)\colon X \to Y$ in $\mathcal{P}(\mathbf{A})_0$ is a sequence of composable morphisms $Y \overset{\sigma_1}{\leftarrow} \cdots \overset{\sigma_n}{\leftarrow} X$ in $\mathbf{A}$, the identity morphism is the empty sequence $()\colon X \to X$ and composition is defined by concatenation. Track sets in $\mathcal{P}(\mathbf{A})$ are defined by the formula

$$[\![X, Y]\!]_{\mathcal{P}(\mathbf{A})}((\sigma_1, \dots, \sigma_n), (\tau_1, \dots, \tau_m)) = [\![X, Y]\!]_{\mathbf{A}}(\sigma_1 \cdots \sigma_n, \tau_1 \cdots \tau_m),$$

and their composition laws are induced by the laws in $\mathbf{A}$. The adjointness yields natural bijections

$$ad\colon \mathbf{Pseudo}(\mathbf{A}, \mathbf{B}) \cong \mathbf{Track}(\mathcal{P}(\mathbf{A}), \mathbf{B}). \tag{1.5}$$

Here **Track** is a 2-category so that the right hand side of (1.5) is a category consisting of track functors as objects and track natural transformations as morphisms. If we endow **Pseudo** with the 2-category structure given by pseudonatural transformations then (1.5) is an isomorphism of categories.

Here a *pseudonatural transformation* $\alpha\colon \varphi \Rightarrow \psi$ between two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is a collection of maps $\alpha_X : \varphi(X) \to \psi(X)$ in $\mathbf{B}$ indexed by the objects of $\mathbf{A}$ such that the usual squares only commute up to given tracks $\alpha_f\colon \alpha_Y \varphi(f) \Rightarrow \psi(f)\alpha_X$ indexed by the maps $f\colon X \to Y$ in $\mathbf{A}$. These tracks must satisfy certain axioms described in the Appendix. By using the isomorphisms of categories (1.5) one can also regard the pseudonatural transformation $\alpha\colon \varphi \Rightarrow \psi$ as a track functor $\alpha\colon \mathcal{P}(\mathbf{A}) \times \mathbb{I} \to \mathbf{B}$.

There is a 2-functor

$$\pi_0\colon \mathbf{Pseudo} \longrightarrow \mathbf{Cat} \tag{1.6}$$

defined on objects by sending a track category $\mathbf{A}$ to its *homotopy category* $\pi_0\mathbf{A} = \mathbf{A}_{\simeq}$ obtained from $\mathbf{A}_0$ by identifying those maps which are connected by tracks. The equivalence class in $\pi_0\mathbf{A}$ of a map f in $\mathbf{A}$ is denoted by $\{f\}$. The quotient category $\pi_0\mathbf{A}$ of $\mathbf{A}_0$ is a coequalizer in **Cat** of the form

$$\mathbf{A}_1 \rightrightarrows \mathbf{A}_0 \to \pi_0\mathbf{A}.$$

2 The homotopy category of pseudofunctors

A *homotopy* $\xi\colon \varphi \Rightarrow \psi$ between two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ in **Pseudo** is a pseudonatural transformation ξ such that the maps $\xi_X : \varphi(X) \to \psi(X)$ are identities, that is $\varphi(X) = \psi(X)$ and $\xi_X = 1_{\varphi(X)} = 1_{\psi(X)}$, in particular homotopic pseudofunctors induce the same functor on π_0.

The homotopy relation $\simeq$ is a natural equivalence relation since homotopies are invertible 2-cells in this 2-category. Therefore the homotopy category of pseudofunctors is well defined

$$\mathbf{Pseudo}_{\simeq}.$$

This category, which is of main interest in this paper, can be endowed with a 2-category structure in the following way: let us write $[\varphi]$ for the homotopy class of a pseudofunctor φ, a 2-cell $[\alpha]\colon [\varphi] \Rightarrow [\psi]$ is represented by a pseudonatural transformation $\alpha\colon \varphi \Rightarrow \psi$ between two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$, which is equivalent to $\alpha'\colon \varphi' \Rightarrow \psi'$, $[\alpha] = [\alpha']$, if $\varphi \simeq \varphi'$, $\psi \simeq \psi'$, and there exist tracks $\alpha_X \Rightarrow \alpha'_X$ for all objects X in $\mathbf{A}$. The horizontal composition of 2-cells is the composition of representatives

$$\begin{tikzcd}
\mathbf{A} \arrow[r, bend left=40, "{[\varphi]}"] \arrow[r, bend right=40, "{[\varphi']}"'] \arrow[r, phantom, "\Downarrow{[\alpha]}"] & \mathbf{B} \arrow[r, bend left=40, "{[\psi]}"] \arrow[r, bend right=40, "{[\psi']}"'] \arrow[r, phantom, "\Downarrow{[\beta]}"] & \mathbf{C} \arrow[r, phantom, "="] & \mathbf{A} \arrow[r, bend left=40, "{[\psi\varphi]}"] \arrow[r, bend right=40, "{[\psi'\varphi']}"'] \arrow[r, phantom, "\Downarrow{[\beta\alpha]}"] & \mathbf{C}\,.
\end{tikzcd}$$

For the vertical composition of 2-cells represented by pseudonatural transformations $\alpha\colon \varphi \Rightarrow \psi$ and $\beta\colon \psi' \Rightarrow \phi$ we have to choose a homotopy $\xi\colon \psi \Rightarrow \psi'$

$$\begin{tikzcd}[column sep=huge]
\mathbf{A} \arrow[r, bend left=70, "{[\varphi]}"] \arrow[r, "{[\psi]=[\psi']}" description] \arrow[r, bend right=70, "{[\phi]}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow{[\alpha]}"] \arrow[r, phantom, shift right=4ex, "\Downarrow{[\beta]}"] & \mathbf{C} \arrow[r, phantom, "="] & \mathbf{A} \arrow[r, bend left=40, "{[\varphi]}"] \arrow[r, bend right=40, "{[\phi]}"'] \arrow[r, phantom, "\Downarrow{[\beta\xi\alpha]}"] & \mathbf{C}\,.
\end{tikzcd}$$

The 2-functor π_0 in (1.6) clearly factors through the homotopy 2-category of pseudo functors

$$\pi_0\colon \mathbf{Pseudo}_{\simeq} \longrightarrow \mathbf{Cat}. \tag{2.1}$$

Moreover, the following result is an immediate consequence of the equivalence relation defining 2-cells in $\mathbf{Pseudo}_{\simeq}$.

Proposition 2.2. *The 2-functor π_0 in (2.1) is faithful on 2-cells.*

3 Abelian track categories

In this paper we will concentrate on *abelian track categories*, which are the track categories $\mathbf{A}$ for which the automorphism group $\mathrm{Aut}_{[\![X,Y]\!]_{\mathbf{A}}}(f)$ of any map $f\colon X \to Y$ in the groupoid $[\![X,Y]\!]_{\mathbf{A}}$ is an abelian group. Equivalently an abelian track category is a category enriched in abelian groupoids. We write $\mathbf{Pseudo}^{ab}$ for the 2-category of small abelian track categories, pseudofunctors and pseudonatural transformations, and $\mathbf{Track}^{ab}$ for the 2-category of small abelian track categories, track functors and track natural transformations.

The π_0 of a track category was defined in Section 1. In order to define the π_1 of an abelian track category we need to recall some concepts from [BW85].

For any category $\mathbf{C}$ the *factorization category* $\mathcal{F}\mathbf{C}$ is given as follows. Objects are the morphisms in $\mathbf{C}$, and a morphism $(h,k)\colon f \to g$ is a pair of morphisms in $\mathbf{C}$ with $kfh = g$. This gives rise to a functor

$$\mathcal{F}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}. \tag{3.1}$$

A *natural system* D (of abelian groups) on $\mathbf{C}$ is just a functor $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$ from the factorization category to the category of abelian groups. We usually denote $D_f = D(f)$, $h^* = D(h,1)$ and $k_* = D(1,k)$.

For any abelian track category $\mathbf{A}$ there is a well-defined natural system $\pi_1\mathbf{A}$ on $\pi_0\mathbf{A}$, see [BJ02] 2.4. This natural system is up to isomorphism determined by the existence of isomorphisms in $\mathbf{Ab}$

$$\sigma_f\colon (\pi_1\mathbf{A})_{\{f\}} \cong \mathrm{Aut}_{[\![X,Y]\!]_{\mathbf{A}}}(f) \tag{3.2}$$

such that given a track $\alpha\colon f \Rightarrow g$ in $\mathbf{A}$,

$$\alpha\Box\sigma_f(x) = \sigma_g(x)\Box\alpha; \tag{3.3}$$

and given composable maps $\bullet \xrightarrow{h} \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ in $\mathbf{A}$

$$f\sigma_g(x) = \sigma_{fg}(\{f\}_* x) \text{ and } \sigma_g(x)h = \sigma_{gh}(\{h\}^* x). \tag{3.4}$$

This means that any abelian track category $\mathbf{A}$ is endowed with the structure of a *linear track extension* denoted by

$$\pi_1\mathbf{A} \to \mathbf{A}_1 \rightrightarrows \mathbf{A}_0 \to \pi_0\mathbf{A},$$

compare [BD89].

The functorial properties of π_1 are described in terms of the following 2-category **nat**. The 0-cells of **nat** are pairs $(\mathbf{C}, D)$ where D is a natural system on a small category $\mathbf{C}$, 1-cells $(\varphi,\lambda)\colon (\mathbf{C},D) \to (\mathbf{D},E)$ are given by a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ together with a natural transformation $\lambda\colon D \Rightarrow E\mathcal{F}(\varphi)$, and 2-cells $\alpha\colon (\varphi,\lambda) \Rightarrow (\psi,\zeta)$ are natural transformations $\alpha\colon \varphi \Rightarrow \psi$ such that for any morphism $f\colon X \to Y$ in the source of φ and ψ the following diagram of abelian groups commutes

$$\begin{tikzcd} D_f \arrow[r, "\lambda_f"] \arrow[d, "\zeta_f"'] & E_{\varphi(f)} \arrow[d, "\alpha_{Y*}"] \\ E_{\psi(f)} \arrow[r, "\alpha_X^*"'] & E_{\alpha_Y\varphi(f)} = E_{\psi(f)\alpha_X} \end{tikzcd} \tag{3.5}$$

Composition of 1-cells is defined as

$$(\varphi,\lambda)(\psi,\zeta) = (\varphi\psi, (\lambda\mathcal{F}(\psi))\Box\zeta),$$

and the composition laws involving 2-cells in **nat** are induced by those in **Cat**.

There is a 2-functor

$$\pi\colon \mathbf{Pseudo}^{ab} \longrightarrow \mathbf{nat} \tag{3.6}$$

defined on cells of dimensions 0, 1 and 2 as

$$\pi\mathbf{A} = (\pi_0\mathbf{A}, \pi_1\mathbf{A}), \quad \pi\varphi = (\pi_0\varphi, \pi_1\varphi), \text{ and } \pi\tau = \pi_0\tau,$$

respectively. Here given a pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ between abelian track categories the natural transformation

$$\pi_1\varphi\colon \pi_1\mathbf{A} \Rightarrow (\pi_1\mathbf{B})\mathcal{F}(\pi_0\varphi)$$

is determined by the formula

$$(3.7) \qquad (\pi_1\varphi)_{\{f\}}\sigma_f^{-1}(\alpha) = \sigma_{\varphi(f)}^{-1}(\varphi(\alpha)),$$

where $\alpha\colon f \Rightarrow f$ is any self-track in $\mathbf{A}$. In fact this 2-functor factors through the homotopy 2-category of small abelian track categories

$$(3.8) \qquad \pi\colon \mathbf{Pseudo}^{ab}_{\simeq} \longrightarrow \mathbf{nat}.$$

A pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ between abelian track categories is said to be a *homotopy equivalence* if $[\varphi]$ is an isomorphism in the homotopy category of pseudofunctors, and it is a *weak equivalence* if $\pi\varphi$ is an isomorphism in $\mathbf{nat}$. Obviously homotopy equivalences are weak equivalences. We write $\mathrm{Ho}(\mathbf{Pseudo}^{ab})$ and $\mathrm{Ho}(\mathbf{Track}^{ab})$ for the corresponding localized categories with respect to weak equivalences.

Theorem 3.9. *Weak equivalences are homotopy equivalences. Moreover, the quotient functor induces a functor*

$$\mathrm{Ho}(\mathbf{Pseudo}^{ab}) \cong \mathbf{Pseudo}^{ab}_{\simeq}$$

which is an isomorphism of categories. Furthermore, the inclusion functor induces an isomorphism of categories

$$\mathrm{Ho}(\mathbf{Track}^{ab}) \cong \mathrm{Ho}(\mathbf{Pseudo}^{ab}).$$

This result underlines the natural significance of he homotopy category $\mathbf{Pseudo}^{ab}_{\simeq}$ as the localization of $\mathbf{Track}^{ab}$.

Proof of Theorem 3.9. Weak equivalences in $\mathbf{Pseudo}^{ab}$ are homotopy equivalences because π in (3.8) fits into an exact sequence for functors, as we show in (8.1) below, see [Bau89] IV.4.11 and IV.1.3 (a).

Let us check that the quotient functor $\varrho\colon \mathbf{Pseudo}^{ab} \to \mathbf{Pseudo}^{ab}_{\simeq}$ satisfies the universal property of the localization $\mathrm{Ho}(\mathbf{Pseudo}^{ab})$. We just have to show that a functor $\varsigma\colon \mathbf{Pseudo}^{ab} \to \mathbf{C}$ carrying weak equivalences to isomorphisms does not distinguish between homotopic pseudofunctors.

In the Appendix we consider the reduced cylinder track category $\mathbb{D}$. This is an abelian track category together with a diagram of weak equivalences

$$\begin{tikzcd} * \arrow[r, shift left, "j^0"] \arrow[r, shift right, "j^1"'] & \mathbb{D} \arrow[r, "q"] & * \end{tikzcd},$$

where $*$ is the final track category. We can multiply this diagram by any other abelian track category $\mathbf{A}$ to obtain a new diagram of weak equivalences

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift left, "j^0"] \arrow[r, shift right, "j^1"'] & \mathbf{A} \times \mathbb{D} \arrow[r, "q"] & \mathbf{A} \end{tikzcd}.$$

Suppose that we have a homotopy $\xi\colon \varphi \Rightarrow \psi$ between pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ in $\mathbf{Pseudo}^{ab}$. By Proposition A.10 there is a pseudofunctor $\bar{\xi}\colon \mathbf{A} \times \mathbb{D} \rightsquigarrow \mathbf{B}$ with $\varphi = \bar{\xi}j^0$ and $\psi = \bar{\xi}j^1$. The equalities $qj^0 = 1 = qj^1$ hold in $\mathbf{Pseudo}^{ab}$, therefore $\varsigma(j^0) = \varsigma(q)^{-1} = \varsigma(j^1)$ in $\mathbf{C}$, and so $\varsigma(\varphi) = \varsigma(\bar{\xi})\varsigma(q)^{-1} = \varsigma(\psi)$.

Let us now check that the functor $\bar{\varrho}\colon \mathbf{Track}^{ab} \to \mathrm{Ho}(\mathbf{Pseudo}^{ab})$ given by the inclusion $\mathbf{Track}^{ab} \subset \mathbf{Pseudo}^{ab}$ and the quotient functor ϱ above satisfies the universal property of the localization Ho($\mathbf{Track}^{ab}$). Obviously $\bar{\varrho}$ sends weak equivalences to isomorphisms.

Recall from (1.4) and (1.5) that there is a natural track functor $ad(1_{\mathbf{A}})\colon \mathcal{P}(\mathbf{A}) \to \mathbf{A}$, namely the adjoint of the identity. The track category $\mathcal{P}(\mathbf{A})$ is abelian provided $\mathbf{A}$ is, and in this case $ad(1_{\mathbf{A}})$ is a week equivalence, see [BJP03] 2.4. The track functor $ad(1_{\mathbf{A}})$ is a retraction in $\mathbf{Pseudo}^{ab}$ with splitting section $\epsilon\colon \mathbf{A} \rightsquigarrow \mathcal{P}(\mathbf{A})$. Here ϵ is the unit of the adjunction between the inclusion $\mathbf{Track} \subset \mathbf{Pseudo}$ and $\mathcal{P}$. In particular the section ϵ is necessarily a weak equivalence. Actually any pseudofunctor φ in $\mathbf{Pseudo}^{ab}$ factors as $\varphi = ad(\varphi)\epsilon$

Suppose now that we have a functor $\varsigma\colon \mathbf{Track}^{ab} \to \mathbf{C}$ sending weak equivalences to isomorphisms, then we define a factorization $\bar{\varsigma}\colon \mathbf{Pseudo}^{ab} \to \mathbf{C}$ through the inclusion as follows, given a pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ in $\mathbf{Pseudo}^{ab}$,

$$\bar{\varsigma}\varrho(\varphi) = \varsigma(ad(\varphi))\varsigma(ad(1_{\mathbf{A}}))^{-1}.$$

This actually defines an factorization because we know that ϱ is full, since we have already established the first isomorphism of the statement. This factorization is the unique possible one because of the equalities $ad(1_{\mathbf{A}})\epsilon = 1_{\mathbf{A}}$ and $\varphi = ad(\varphi)\epsilon$, therefore we will be done if we manage to prove that $\bar{\varsigma}$ also sends weak equivalences to isomorphisms. But if φ is a weak equivalence then $ad(\varphi)$ is also a weak equivalence because the equality $\varphi = ad(\varphi)\epsilon$ holds and in this case $\bar{\varsigma}\varrho(\varphi)$ is an isomorphism. The proof is now finished. □

4 Cohomology of categories

The cohomology $H^*(\mathbf{C}, D)$ of a small category $\mathbf{C}$ with coefficients in a natural system D is the cohomology of the Baues-Wirsching complex defined in [BW85]. This complex $F^*(\mathbf{C}, D)$ is a cochain complex of abelian groups concentrated in dimensions ≥ 0. Its group in dimension n is the following product indexed by all sequences of morphisms of length n in $\mathbf{C}$

$$F^n(\mathbf{C}, D) = \prod_{\bullet \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} \bullet} D_{\sigma_1 \cdots \sigma_n}.$$

Here we assume that a sequence of length 0 is an object X in $\mathbf{C}$ which we also identify with the identity morphism 1_X. The coordinate of $c \in F^n(\mathbf{C}, D)$ in $\bullet \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} \bullet$ will be denoted by $c(\sigma_1, \dots, \sigma_n) \in D_{\sigma_1 \cdots \sigma_n}$. The differential δ is defined as

$$\begin{aligned}\delta(c)(\sigma_1, \dots, \sigma_{n+1}) \quad = \quad & {\sigma_1}_* c(\sigma_2, \dots, \sigma_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i c(\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_{n+1}) \\ & + (-1)^{n+1} \sigma_{n+1}^* c(\sigma_1, \dots, \sigma_n)\end{aligned}$$

over an n-cochain c for $n \geq 1$, and $\delta(c)(\sigma) = \sigma_* c(X) - \sigma^* c(Y)$ for $n = 0$ and $\sigma\colon X \to Y$.

A functor $\varphi\colon \mathbf{D} \to \mathbf{C}$ and a natural transformation $\lambda\colon D \Rightarrow E$ induce cochain homomorphisms

$$\varphi^*\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D\mathcal{F}(\varphi)), \;\; \varphi^*(c)(\sigma_1, \dots, \sigma_n) = c(\varphi(\sigma_1), \dots, \varphi(\sigma_n));$$

$$\lambda_*\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{C}, E), \;\; \lambda_*(c)(\sigma_1, \dots, \sigma_n) = \lambda_{\sigma_1 \cdots \sigma_n}(c(\sigma_1, \dots, \sigma_n));$$

and the corresponding homomorphisms in cohomology

$$\varphi^*\colon H^*(\mathbf{C}, D) \longrightarrow H^*(\mathbf{D}, D\mathcal{F}(\varphi)),$$

$$\lambda_*\colon H^*(\mathbf{C}, D) \longrightarrow H^*(\mathbf{C}, E).$$

See [BW85] for further details on the functoriality of cohomology of categories.

A cochain $c \in F^n(\mathbf{C}, D)$ is said to be *normalized* (at identities) if $c(\sigma_1, \dots, \sigma_n) = 0$ provided σ_i is an identity morphism for some $1 \leq i \leq n$. The inclusion of the subcomplex $\bar{F}^*(\mathbf{C}, D) \subset F^*(\mathbf{C}, D)$ of normalized cochains induces an isomorphism in cohomology, see [BD89]. Moreover, the induced cochain homomorphisms above restrict to normalized cochains.

The factorization functor $\mathcal{F}$ in (3.1) can be extended to a 2-functor if we consider the 2-category $\mathbf{Cat}_{\mathcal{F}}$, see [Mur04] 3.2. This 2-category is obtained by applying the product-preserving functor $\mathcal{F}$ to morphism categories in the 2-category $\mathbf{Cat}$. This 2-functor will be used in the next section to describe the functorial properties of translation cohomology as well as from Section 7 on in the context of the obstruction theory for pseudonatural transformations. More precisely, 0-cells in $\mathbf{Cat}_{\mathcal{F}}$ are small categories, a 1-cell $\alpha\colon \mathbf{C} \to \mathbf{D}$ is a natural transformation $\alpha\colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi\colon \mathbf{C} \to \mathbf{D}$, composition is horizontal composition, so that composition is defined in the same way as in (1.3). A 2-cell $(\varepsilon, \gamma)\colon \alpha \Rightarrow \beta$ between morphisms $\alpha, \beta\colon \mathbf{C} \to \mathbf{D}$ is a pair of natural transformations such that $\gamma \Box \alpha \Box \varepsilon = \beta$. We can place $\mathbf{Cat}$ as an ordinary category inside $\mathbf{Cat}_{\mathcal{F}}$ by using the ordinary functor $\mathbf{Cat} \to \mathbf{Cat}_{\mathcal{F}}$ which is the identity on objects and sends a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ to the identity natural transformation $0^{\Box}_{\varphi}\colon \varphi \Rightarrow \varphi$ regarded as a morphism $0^{\Box}_{\varphi}\colon \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{\mathcal{F}}$.

The 2-functor

$$\mathcal{F}\colon \mathbf{Cat}_{\mathcal{F}} \longrightarrow \mathbf{Cat} \tag{4.1}$$

extending (3.1) is defined in the same way on 0-cells, on a 1-cell as above $\mathcal{F}(\alpha)\colon \mathbf{C} \to \mathbf{D}$ is the functor with $\mathcal{F}(\alpha)(f) = \alpha_Y \varphi(f) = \psi(f)\alpha_X$ for any morphism $f\colon X \to Y$ in $\mathbf{C}$ regarded as an object in $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\alpha)(h, k) = (\varphi(h), \psi(k))$ for any morphism (h, k) in $\mathcal{F}(\mathbf{C})$, and on a 2-cell as above $\mathcal{F}(\varepsilon, \gamma)\colon \mathcal{F}(\alpha) \Rightarrow \mathcal{F}(\beta)$ is the natural transformation given by $\mathcal{F}(\varepsilon, \gamma)_f = (\varepsilon_X, \gamma_Y)$.

This formalism can be used to restate the condition defining 2-cells in **nat**, more precisely, (3.5) commutes for all f if and only if the following equality holds

$$(E\mathcal{F}(0^{\Box}_{\varphi}, \alpha)) \Box \lambda = (E\mathcal{F}(\alpha, 0^{\Box}_{\psi})) \Box \zeta. \tag{4.2}$$

A natural transformation $\alpha\colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi\colon \mathbf{D} \to \mathbf{C}$ induces a cochain homomorphism

$$\alpha^*\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D\mathcal{F}(\alpha))$$

defined as

$$\alpha^*(c)(\sigma_1, \cdots, \sigma_n) = (D\mathcal{F}(0^{\Box}_{\varphi}, \alpha))_{\sigma_1 \cdots \sigma_n}(c(\varphi(\sigma_1), \dots, \varphi(\sigma_n))).$$

This cochain homomorphism is called $F^*(\alpha, 0^{\Box}_{D\mathcal{F}(\alpha)})$ in [Mur04] 4.1. The induced homomorphism in cohomology is denoted in the same way

$$\alpha^*\colon H^*(\mathbf{C}, D) \longrightarrow H^*(\mathbf{D}, D\mathcal{F}(\alpha)).$$

Notice that the cochain homomorphism defined by an identity natural transformation coincides with the cochain homomorphism defined by the corresponding functor

$$(0^{\square}_{\varphi})^* = \varphi^*.$$

The natural transformation $\alpha\colon \varphi \Rightarrow \psi$ also induces cochain homotopies given by degree -1 homomorphisms

$$\alpha_{\#}, \alpha^{\#}\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D\mathcal{F}(\alpha)),$$

with

$$\delta\alpha_{\#} + \alpha_{\#}\delta = -(D\mathcal{F}(0^{\square}_{\varphi}, \alpha))_*\varphi^* + \alpha^*, \tag{4.3}$$

$$\delta\alpha^{\#} + \alpha^{\#}\delta = -(D\mathcal{F}(\alpha, 0^{\square}_{\psi}))_*\psi^* + \alpha^*. \tag{4.4}$$

The cochain homotopies $\alpha_{\#}$ and $\alpha^{\#}$ are $h_{0^{\square}_{\varphi},\alpha}$ and $h_{\alpha,0^{\square}_{\psi}}$ in [Mur04] 4.1, respectively. They are defined by the following formulas

$$\alpha_{\#}(c)(\sigma_1, \dots, \sigma_n) = \sum_{i=0}^{n}(-1)^i(D\mathcal{F}(0^{\square}_{\varphi}, \alpha))_{\sigma_1\cdots\sigma_n}(c(\varphi(\sigma_1), \dots, \varphi(\sigma_i), 1_{\varphi(X_i)}, \varphi(\sigma_{i+1}), \dots, \varphi(\sigma_n))),$$

$$\alpha^{\#}(c)(\sigma_1, \dots, \sigma_n) = \sum_{i=0}^{n}(-1)^i c(\psi(\sigma_1), \dots, \psi(\sigma_i), \alpha_{X_i}, \varphi(\sigma_{i+1}), \dots, \varphi(\sigma_n)). \tag{4.5}$$

Here X_i is the source of σ_i and/or the target of σ_{i+1}. These homotopies are part of the 2-functorial properties of the Baues-Wirsching complex studied in [Mur04]. Notice that these induced cochain homomorphisms and homotopies restrict to the subcomplex of normalized cochains. Moreover, as one can readily notice on the normalized cochain complex the following formulas hold

$$\alpha_{\#} = 0, \tag{4.6}$$

$$(0^{\square}_{\varphi})^{\#} = 0, \tag{4.7}$$

$$\alpha^* = (D\mathcal{F}(0^{\square}_{\varphi}, \alpha))_*\varphi^*, \tag{4.8}$$

in particular by (4.3) and (4.4)

$$\delta\alpha^{\#} + \alpha^{\#}\delta = -(D\mathcal{F}(\alpha, 0^{\square}_{\psi}))_*\psi^* + (D\mathcal{F}(0^{\square}_{\varphi}, \alpha))_*\varphi^*. \tag{4.9}$$

Observe that for any natural transformation $\lambda\colon D \to E$ the following equality holds,

$$\alpha^{\#}\lambda_* = (\lambda\mathcal{F}(\alpha))_*\alpha^{\#}. \tag{4.10}$$

We finally recall the behaviour of these cochain homotopies with respect to horizontal and vertical composition, see [Mur04] 5.1. Given natural transformations $\phi \overset{\beta}{\Rightarrow} \varphi \overset{\alpha}{\Rightarrow} \psi$ there is a degree -2 homomorphism

$$(\alpha, \beta)^{\%}\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D\mathcal{F}(\alpha\square\beta))$$

such that

$$\delta(\alpha,\beta)^{\%}-(\alpha,\beta)^{\%}\delta \;=\; -(D\mathcal{F}(\beta,0^{\square}_{\psi}))_*\alpha^{\#} \quad -(D\mathcal{F}(0^{\square}_{\phi},\alpha))_*\beta^{\#}+(\alpha\square\beta)^{\#}. \tag{4.11}$$

This order 2 cochain homotopy coincides with $r_{(\beta,0^{\square}_{\varphi});(\alpha,0^{\square}_{\psi})}$ in [Mur04] 4.1. If we have functors $\varphi,\varphi'\colon \mathbf{D}\to\mathbf{C}$, $\psi,\psi'\colon\mathbf{E}\to\mathbf{D}$ and natural transformations $\alpha\colon\varphi\Rightarrow\varphi'$, $\beta\colon\psi\Rightarrow\psi'$, there is a degree -2 homomorphism

$$(\alpha,\beta)^{\$}\colon F^*(\mathbf{C},D)\longrightarrow F^*(\mathbf{E},D\mathcal{F}(\alpha\beta)),$$

such that

$$\delta(\alpha,\beta)^{\$}-(\alpha,\beta)^{\$}\delta \;=\; -\beta^{\#}\alpha^*-\beta^*\alpha^{\#}+(\alpha\beta)^{\#}. \tag{4.12}$$

The order 2 cochain homotopy $(\alpha,\beta)^{\$}$ is $r'_{(\beta,0^{\square}_{\psi'});(\alpha,0^{\square}_{\varphi'})}$ in [Mur04] 4.1. We shall not give here the explicit formulas of $(\alpha,\beta)^{\%}$ and $(\alpha,\beta)^{\$}$ since they are not needed in this paper, however we point out that they restrict to the subcomplex of normalized cochains. Moreover, on normalized cochains the following equalities holds

$$(\alpha,0^{\square}_{\psi})^{\$}=0, \tag{4.13}$$

$$(0^{\square}_{\varphi},\beta)^{\$}=0, \tag{4.14}$$

and therefore by (4.12) the equalities

$$\psi^*\alpha^{\#}=(\alpha\psi)^{\#}, \tag{4.15}$$

$$\beta^{\#}\varphi^*=(\varphi\beta)^{\#}, \tag{4.16}$$

are also satisfied on the normalized cochain complex.

5 Translation cohomology

Let $\mathbf{A}$ be a category, D a natural system on $\mathbf{A}$, and $(t,\bar{t})$ an endomorphism of $(\mathbf{A},D)$ in **nat**. Such an endomorphism is termed a *translation* of $(\mathbf{A},D)$. The *translation cohomology*

$$H^*(t,\bar{t})$$

is the cohomology of the homotopy fiber of the cochain homomorphism

$$\bar{t}_*-t^*\colon \bar{F}^*(\mathbf{A},D)\longrightarrow \bar{F}^*(\mathbf{A},D\mathcal{F}(t)),$$

which will be denoted by $\bar{F}^*(t,\bar{t})$. In dimension n this cochain complex is

$$\bar{F}^n(t,\bar{t})=\bar{F}^n(\mathbf{A},D)\oplus\bar{F}^{n-1}(\mathbf{A},D\mathcal{F}(t))$$

and the differential is the matrix

$$\left(\begin{array}{cc}\delta & 0\\ \bar{t}_*-t^* & -\delta\end{array}\right).$$

In particular there is a long exact sequence ($n\in\mathbb{Z}$)

$$\cdots\to H^n(t,\bar{t})\xrightarrow{j}H^n(\mathbf{A},D)\xrightarrow{\bar{t}_*-t^*}H^n(\mathbf{A},D\mathcal{F}(t))\xrightarrow{\partial}H^{n+1}(t,\bar{t})\to\cdots. \tag{5.1}$$

Remark 5.2. This is a general concept of *cohomology of diagrams* as for example described in [MS95] or [Bau99] A.2. Then translation cohomology corresponds to the cohomology of the diagram

with one vertex and one arrow.

Given functors $s\colon \mathbf{B} \to \mathbf{B}$ and $\varphi\colon \mathbf{B} \to \mathbf{A}$ such that the square

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "s"] \arrow[d, "\varphi"'] & \mathbf{B} \arrow[d, "\varphi"] \\
\mathbf{A} \arrow[r, "t"'] & \mathbf{A}
\end{tikzcd}$$

commutes up to a given natural isomorphism $\alpha\colon t\varphi \Rightarrow \varphi s$ there is an induced translation

$$(s, (D\mathcal{F}(\alpha^{\boxminus}, \alpha))\square(\bar{t}\mathcal{F}(\varphi)))\colon (\mathbf{B}, D\mathcal{F}(\varphi)) \longrightarrow (\mathbf{B}, D\mathcal{F}(\varphi))$$

and an induced cochain homomorphism

$$(\varphi, \alpha)^*\colon \bar{F}^*(t, \bar{t}) \longrightarrow \bar{F}^*(s, (D\mathcal{F}(\alpha^{\boxminus}, \alpha))\square(\bar{t}\mathcal{F}(\varphi)))$$

given by the matrix

$$(\varphi, \alpha)^* = \left(\begin{array}{cc} \varphi^* & 0 \\ (D\mathcal{F}(\alpha^{\boxminus}, 0^{\square}_{\varphi s}))_*\alpha^{\#} & (D\mathcal{F}(\alpha^{\boxminus}, \alpha))_*\varphi^* \end{array} \right).$$

Here we use (4.9). The induced homomorphisms in translation cohomology will be denoted in the same way

$$(\varphi, \alpha)^*\colon H^*(t, \bar{t}) \longrightarrow H^*(s, (D\mathcal{F}(\alpha^{\boxminus}, \alpha))\square(\bar{t}\mathcal{F}(\varphi))).$$

Similarly two natural transformations $\lambda\colon D \Rightarrow E$ and $\bar{s}\colon E \Rightarrow E\mathcal{F}(t)$ between natural systems on $\mathbf{A}$ such that $(\lambda\mathcal{F}(t))\square\bar{t} = \bar{s}\square\lambda$ give rise to a translation

$$(t, \bar{s})\colon (\mathbf{A}, E) \longrightarrow (\mathbf{A}, E)$$

and a cochain homomorphism

$$\lambda_*\colon \bar{F}^*(t, \bar{t}) \longrightarrow \bar{F}^*(t, \bar{s})$$

given by the matrix

$$\lambda_* = \left(\begin{array}{cc} \lambda_* & 0 \\ 0 & (\lambda\mathcal{F}(t))_* \end{array} \right).$$

The corresponding homomorphism in translation cohomology will also be denoted by

$$\lambda_*\colon H^*(t, \bar{t}) \longrightarrow H^*(t, \bar{s}).$$

These homomorphisms in translation cohomology can be used to describe the functorial properties of this cohomology. More precisely, let **Natt** be the category whose objects are all translations. A morphism $(\varphi, \alpha, \lambda)\colon (t, \bar{t}) \to (s, \bar{s})$ between two translations $(t, \bar{t})\colon (\mathbf{A}, D) \to (\mathbf{A}, D)$ and $(s, \bar{s})\colon (\mathbf{B}, E) \to (\mathbf{B}, E)$ is given by a functor $\varphi\colon \mathbf{B} \to \mathbf{A}$, a natural isomorphism $\alpha\colon t\varphi \Rightarrow \varphi s$, and a natural transformation $\lambda\colon D\mathcal{F}(\varphi) \Rightarrow E$ such that

$$(\lambda\mathcal{F}(s))\square(D\mathcal{F}(\alpha^{\boxminus}, \alpha))\square(\bar{t}\mathcal{F}(\varphi)) = \bar{s}\square\lambda.$$

Composition of morphisms

$$(t,\bar{t}) \xrightarrow{(\varphi,\alpha,\lambda)} (s,\bar{s}) \xrightarrow{(\psi,\beta,\zeta)} (r,\bar{r})$$

in **Natt** is defined by the formula

$$(\psi,\beta,\zeta)(\varphi,\alpha,\lambda) = (\varphi\psi, (\varphi\beta)\Box(\alpha\psi), \zeta\Box(\lambda\mathcal{F}(\psi))).$$

Proposition 5.3. *Translation cohomology groups are functors*

$$H^*\colon \mathbf{Natt} \longrightarrow \mathbf{Ab}$$

given on morphisms by $H^*(\varphi,\alpha,\lambda) = \lambda_*(\varphi,\alpha)^*$.

This is just a simplified version of a more extended functoriality of translation cohomology that we now describe. Suppose that we have a translation $(t,\bar{t})\colon (\mathbf{A},D) \to (\mathbf{A},D)$, a functor $s\colon \mathbf{B} \to \mathbf{B}$, a natural transformation $\gamma\colon \varphi \Rightarrow \psi$ between functors $\varphi,\psi\colon \mathbf{B} \to \mathbf{A}$, and two natural isomorphisms $\alpha\colon t\varphi \Rightarrow \varphi s$ and $\beta\colon t\psi \Rightarrow \psi s$ such that

$$\beta\Box(t\gamma) = (\gamma s)\Box\alpha. \tag{5.4}$$

In this situation there is an induced translation

$$(s, (D\mathcal{F}(\alpha^{\boxminus},\beta))\Box(\bar{t}\mathcal{F}(\gamma)))\colon (\mathbf{B}, D\mathcal{F}(\gamma)) \longrightarrow (\mathbf{B}, D\mathcal{F}(\gamma))$$

and an induced cochain homomorphism

$$(\gamma,\alpha,\beta)^*\colon \bar{F}^*(t,\bar{t}) \longrightarrow \bar{F}^*(s, (D\mathcal{F}(\alpha^{\boxminus},\beta))\Box(\bar{t}\mathcal{F}(\gamma)))$$

given by the matrix

$$(\gamma,\alpha,\beta)^* = \begin{pmatrix} \gamma^* & 0 \\ (D\mathcal{F}(\alpha^{\boxminus},\gamma s))_*\alpha^{\#} & (D\mathcal{F}(\alpha^{\boxminus},\beta))_*\gamma^* \end{pmatrix}.$$

The reader can check that this is indeed a cochain homomorphism by using (4.8), (4.9) and (5.4).

Let $\mathbf{Natt}_1$ be the category whose objects are all translations. A morphism $(\gamma,\alpha,\beta,\lambda)\colon (t,\bar{t}) \to (s,\bar{s})$ between two translations $(t,\bar{t})\colon (\mathbf{A},D) \to (\mathbf{A},D)$ and $(s,\bar{s})\colon (\mathbf{B},E) \to (\mathbf{B},E)$ is given by a natural transformation $\gamma\colon \varphi \Rightarrow \psi$ between functors $\varphi,\psi\colon \mathbf{B} \to \mathbf{A}$, two natural isomorphisms $\alpha\colon t\varphi \Rightarrow \varphi s$ and $\beta\colon t\psi \Rightarrow \psi s$ satisfying (5.4), and a natural transformation $\lambda\colon D\mathcal{F}(\gamma) \Rightarrow E$ such that

$$(\lambda\mathcal{F}(s))\Box(D\mathcal{F}(\alpha^{\boxminus},\beta))\Box(\bar{t}\mathcal{F}(\gamma)) = \bar{s}\Box\lambda.$$

Composition of morphisms

$$(t,\bar{t}) \xrightarrow{(\gamma,\alpha,\beta,\lambda)} (s,\bar{s}) \xrightarrow{(\varepsilon,\mu,\nu,\zeta)} (r,\bar{r})$$

in $\mathbf{Natt}_1$ with $\gamma\colon \varphi \Rightarrow \psi$ and $\varepsilon\colon \bar{\varphi} \Rightarrow \bar{\psi}$ is defined by the formula

$$(\varepsilon,\mu,\nu,\zeta)(\gamma,\alpha,\beta,\lambda) = (\gamma\varepsilon, (\varphi\mu)\Box(\alpha\bar{\varphi}), (\psi\nu)\Box(\beta\bar{\psi}), \zeta\Box(\lambda\mathcal{F}(\varepsilon))).$$

Compare with the definition of $\mathbf{Cat}_{\mathcal{F}}$ in Section 4.

Proposition 5.5. *Translation cohomology groups are functors*

$$H^*\colon \mathbf{Natt}_1 \longrightarrow \mathbf{Ab}$$

given on morphisms by $H^*(\gamma,\alpha,\beta,\lambda)=\lambda_*(\gamma,\alpha,\beta)^*$.

Proof. All we have to check is that composition is preserved. For this we are going to prove that the cochain homomorphisms

$$\zeta_*(\varepsilon,\mu,\nu)^*\lambda_*(\gamma,\alpha,\beta)^*, (\zeta\Box(\lambda\mathcal{F}(\varepsilon)))_*(\gamma\varepsilon,(\varphi\mu)\Box(\alpha\bar{\varphi}),(\psi\nu)\Box(\beta\bar{\psi}))^*\colon \bar{F}^*(t,\bar{t})\to\bar{F}^*(s,\bar{s})$$

are homotopic. The first of these cochain homomorphisms is given by the matrix

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

with entries

$$\begin{aligned}
a_{11} &= \zeta_*\varepsilon^*\lambda_*\gamma^*,\\
a_{12} &= 0,\\
a_{21} &= (\zeta\mathcal{F}(r))_*(E\mathcal{F}(\mu^{\boxminus},\varepsilon r))_*\mu^{\#}\lambda_*\gamma^*\\
&\quad +(\zeta\mathcal{F}(r))_*(E\mathcal{F}(\mu^{\boxminus},\nu))_*\varepsilon^*(\lambda\mathcal{F}(s))_*(D\mathcal{F}(\alpha^{\boxminus},\gamma s))_*\alpha^{\#},\\
a_{22} &= (\zeta\mathcal{F}(r))_*(E\mathcal{F}(\mu^{\boxminus},\nu))_*\varepsilon^*(\lambda\mathcal{F}(s))_*(D\mathcal{F}(\alpha^{\boxminus},\beta))_*\gamma^*,
\end{aligned}$$

and the second one by the matrix

$$\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$$

with

$$\begin{aligned}
b_{11} &= (\zeta\Box(\lambda\mathcal{F}(\varepsilon)))_*(\gamma\varepsilon)^*,\\
b_{12} &= 0,\\
b_{21} &= ((\zeta\Box(\lambda\mathcal{F}(\varepsilon)))\mathcal{F}(r))_*(D\mathcal{F}((\varphi\mu^{\boxminus})\Box(\alpha^{\boxminus}\bar{\varphi}),0^{\Box}_{\gamma\varepsilon r}))_*((\varphi\mu)\Box(\alpha\bar{\varphi}))^{\#},\\
b_{22} &= ((\zeta\Box(\lambda\mathcal{F}(\varepsilon)))\mathcal{F}(r))_*(D\mathcal{F}((\varphi\mu^{\boxminus})\Box(\alpha^{\boxminus}\bar{\varphi}),(\psi\nu)\Box(\beta\bar{\psi})))_*(\gamma\varepsilon)^*.
\end{aligned}$$

One can easily check that the equalities $a_{ij}=b_{ij}$ hold for $(i,j)\neq(2,1)$. Moreover, by using (4.8), (4.10) and (5.4) the reader can check that

$$\begin{aligned}
a_{21} &= ((\zeta\Box(\lambda\mathcal{F}(\varepsilon)))\mathcal{F}(r))_*(D\mathcal{F}(\varphi\mu^{\boxminus},\gamma\varepsilon r))\mu^{\#}\varphi^*\\
&\quad +((\zeta\Box(\lambda\mathcal{F}(\varepsilon)))\mathcal{F}(r))_*(D((\alpha^{\boxminus}\bar{\varphi})\Box(\varphi\mu^{\boxminus}),(\gamma\varepsilon r)\Box(\varphi\mu))_*\bar{\varphi}^*\alpha^{\#}
\end{aligned}$$

Furthermore, by using (4.15), (4.16), and (4.9) one can now check that the matrix

$$\begin{pmatrix} 0 & 0\\ (\varphi\mu,\alpha\bar{\varphi})^{\%} & 0\end{pmatrix}$$

yields the desired cochain homotopy. □

Now Proposition 5.3 follows from Proposition 5.5 since there is a commutative diagram of functors

$$\begin{tikzcd}
\mathbf{Natt} \arrow[d] \arrow[dr, "H^*"] & \\
 & \mathbf{Ab} \\
\mathbf{Natt}_1 \arrow[ur, "H^*"'] &
\end{tikzcd}$$

where the vertical arrow is the identity on objects and sends a morphism (φ, α) to $(0^{\square}_{\varphi}, \alpha, \alpha)$. Compare with the definition of the functor $\mathbf{Cat} \to \mathbf{Cat}_{\mathcal{F}}$ in Section 4.

The following identities between induced homomorphisms in translation cohomology will be useful for future applications.

Proposition 5.6. *Let be a morphism* $(\gamma, \alpha, \beta, \lambda)\colon (t, \bar{t}) \to (s, \bar{s})$ *between two translations* $(t, \bar{t})\colon (\mathbf{A}, D) \to (\mathbf{A}, D)$ *and* $(s, \bar{s})\colon (\mathbf{B}, E) \to (\mathbf{B}, E)$ *in* $\mathbf{Natt}_1$*. The following homomorphisms* $H^*(t, \bar{t}) \to H^*(s, \bar{s})$ *coincide*

$$\lambda_*(\gamma, \alpha, \beta)^* = (\lambda \square D(0^{\square}_{\varphi}, \gamma))_*(\varphi, \alpha)^* = (\lambda \square D(\gamma, 0^{\square}_{\psi}))_*(\psi, \beta)^*$$

Proof. The first equality can be easily checked at the level of cochains by using (4.8) and (5.4). For the second one we only need to use the cochain homotopy in the translation cochain complex given by the matrix

$$\left(\begin{array}{cc} \lambda_* \gamma^{\#} & 0 \\ 0 & ((\lambda \mathcal{F}(s)) \square D\mathcal{F}(\alpha^{-1}, \beta))_* \gamma^{\#} \end{array} \right),$$

see (4.9). □

Remark 5.7. As the reader may suspect Proposition 5.6 is a consequence of a higher functoriality of the translation cochain complex. More precisely, the category $\mathbf{Natt}_1$ can be endowed with a 2-category structure by using factorizations, compare with the definition of the 2-category $\mathbf{Cat}_{\mathcal{F}}$ in Section 4. The translation cochain complex turns out to be a pseudofunctor on the 2-category $\mathbf{Natt}_1$. This pseudofunctor can be described by using the 2-functorial properties of the Baues-Wirsching cochain complex described in [Mur04].

If $\mathbf{A}$ is additive and $t\colon \mathbf{A} \to \mathbf{A}$ is an additive endofunctor we can consider the natural system D given by the $\mathbf{A}$-bimodule $\mathrm{Hom}_{\mathbf{A}}(t, -)$ and the natural transformation $\bar{t}\colon D = \mathrm{Hom}_{\mathbf{A}}(t, -) \Rightarrow \mathrm{Hom}_{\mathbf{A}}(t^2, t) = D\mathcal{F}(t)$ which is given by $(-1)t$. In this particular case we denote the translation cohomology as follows

$$H^*(\mathbf{A}, t) = H^*(t, \bar{t}). \tag{5.8}$$

This cohomology is of particular interest in case $\mathbf{A}$ is a triangulated category and t its translation functor. This example, in fact, is the motivation of the theory in this paper. We shall use translation cohomology in order to study the "characteristic cohomology class" in $H^3(\mathbf{A}, t)$ of a triangulated category $(\mathbf{A}, t)$, see [IV] and [V].

6 Classification of abelian track categories and translation track categories

Let D be a natural system over $\mathbf{C}$. We introduce the categories $\mathbf{Track}(\mathbf{C}, D)$ and $\mathbf{Pseudo}_{\simeq}(\mathbf{C}, D)$. Both categories have the same objects $(\mathbf{A}, \chi)$ which are given by an abelian track category $\mathbf{A}$ together with an isomorphism $\chi = (\chi_0, \chi_1)\colon \pi\mathbf{A} \cong (\mathbf{C}, D)$ in $\mathbf{nat}$. We often use χ as an identification and drop it from notation. Morphisms $\varphi\colon (\mathbf{A}, \chi) \to (\mathbf{B}, \kappa)$ in $\mathbf{Track}(\mathbf{C}, D)$ are track functors $\varphi\colon \mathbf{A} \to \mathbf{B}$ such that $\chi = \kappa(\pi\varphi)$, and morphisms $[\psi]\colon (\mathbf{A}, \chi) \to (\mathbf{B}, \kappa)$ in $\mathbf{Pseudo}_{\simeq}(\mathbf{C}, D)$ are homotopy classes of pseudofunctors $[\psi]\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ such that $\chi = \kappa(\pi[\psi])$.

Given an ordinary category $\mathbf{G}$ we write $\pi_0\mathbf{G}$ for the set of connected components in $\mathbf{G}$. Two objects X, Y belong to the same connected component iff there is a chain of morphisms in $\mathbf{G}$ as follows

$$X \leftarrow \bullet \rightarrow \cdots \leftarrow \bullet \rightarrow Y.$$

This equivalence relation simplifies when $\mathbf{G}$ is a groupoid, in this case two objects belong to the same connected component if and only if they are isomorphic.

The next result classifies connected components in $\mathbf{Track}(\mathbf{C}, D)$ by the cohomology in Section 4.

Proposition 6.1 ([BD89] 4.6, [Pir88]). *There is a bijection*

$$\pi_0\mathbf{Track}(\mathbf{C}, D) \cong H^3(\mathbf{C}, D)$$

which carries $(\mathbf{A}, \chi)$ *to the characteristic class* $\langle \mathbf{A} \rangle \in H^3(\mathbf{C}, D)$ *given by the universal Toda bracket of the linear track extension* $(\mathbf{A}, \chi)$.

A 3-cocycle $c_{\mathbf{A}}$ representative of the cohomology class $\langle \mathbf{A} \rangle$ is defined in Section 13.

The category $\mathbf{Track}(\mathbf{C}, D)$ is far from being a groupoid, however $\mathbf{Pseudo}^{ab}_{\simeq}(\mathbf{C}, D)$ is indeed a groupoid as a consequence of Theorem 3.9, therefore its set of connected components can be described in a much easier way. Moreover, we have the following result.

Proposition 6.2. *There is a bijection*

$$\pi_0\mathbf{Pseudo}^{ab}_{\simeq}(\mathbf{C}, D) \cong H^3(\mathbf{C}, D)$$

which carries $(\mathbf{A}, \chi)$ *to* $\langle \mathbf{A} \rangle$.

Proof. This follows from the existence of the exact sequence for functors (8.1), the definition of its obstruction operator θ in (7.1), and [Bau89] IV.4.12. ☐

A *translation track category* $(\mathbf{A}, [\varphi])$ is an abelian track category equipped with a homotopy class of pseudofunctors $[\varphi]\colon \mathbf{A} \rightsquigarrow \mathbf{A}$. Notice that $\pi[\varphi]$ is a translation in the sense of Section 5.

Let $(t, \bar{t})\colon (\mathbf{C}, D) \to (\mathbf{C}, D)$ be a translation. We now introduce the category $\mathbf{Trans}(t, \bar{t})$ whose objects are triples $(\mathbf{A}, [\varphi], \chi)$ where $(\mathbf{A}, [\varphi])$ is a translation track category and $\chi\colon \pi\mathbf{A} \cong (\mathbf{C}, D)$ is an isomorphism with $(t, \bar{t}) = \chi(\pi[\varphi])\chi^{-1}$. A morphism $[\phi]\colon (\mathbf{A}, [\varphi], \chi) \to (\mathbf{B}, [\psi], \kappa)$ is a homotopy class of pseudofunctors $[\phi]\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ such that $[\phi][\varphi] = [\psi][\phi]$ and $\chi = \kappa(\pi[\phi])$. Again by Theorem 3.9 we see that $\mathbf{Trans}(t, \bar{t})$ is in fact a groupoid. The abelian group $H^2(\mathbf{C}, D\mathcal{F}(t))$ acts on this groupoid by using the action defined in Theorem 7.12. More precisely, this action is given by $(\mathbf{A}, [\varphi], \xi) + \omega = (\mathbf{A}, [\varphi] + \omega, \xi)$ on objects and the identity on morphisms. Moreover, there is a functor

$$\bar{\jmath}\colon \mathbf{Trans}(t, \bar{t}) \longrightarrow \mathbf{Pseudo}_{\simeq}(\mathbf{C}, D), \;\; \bar{\jmath}(\mathbf{A}, [\varphi], \chi) = (\mathbf{A}, \chi).$$

Theorem 6.3. *There is a bijection*

$$\pi_0\mathbf{Trans}(t, \bar{t}) \cong H^3(t, \bar{t})$$

where we use the translation cohomology in Section 5. This bijection is equivariant with respect to the action of $H^2(\mathbf{C}, D\mathcal{F}(t))$ *on* $H^3(t, \bar{t})$ *given by* ∂ *in (5.1). Moreover, if we use this bijection and the bijection in Proposition 6.2 as identifications then the following diagram commutes*

$$\begin{tikzcd}
\pi_0\mathbf{Trans}(t, \bar{t}) \arrow[r, "\pi_0\bar{\jmath}"] \arrow[d, equal] & \pi_0\mathbf{Pseudo}^{ab}_{\simeq}(\mathbf{C}, D) \arrow[d, equal] \\
H^3(t, \bar{t}) \arrow[r, "j"'] & H^3(\mathbf{C}, D)
\end{tikzcd}$$

Here j is the homomorphism in the exact sequence (5.1).

Proof. The characteristic class of a translation track category $(\mathbf{A},[\varphi])$ is the translation cohomology class

$$\langle \mathbf{A},[\varphi]\rangle \in H^3(\pi\varphi)$$

represented by the cocycle

$$(c_{\mathbf{A}}, b_\varphi).$$

Here we use the cochains defined in Section 13 obtained from a fixed global section. This is indeed a cocycle by Proposition 13.7 (1). Moreover, its cohomology class does not depend on the choice of the global section by Proposition 13.9. Notice also that the equality $j\langle \mathbf{A},[\varphi]\rangle = \langle \mathbf{A}\rangle$ holds from the very definition of j and these characteristic classes.

Let us check that the map

$$\mathbf{Trans}(t,\bar{t}) \longrightarrow H^3(t,\bar{t}),\ \ (\mathbf{A},[\varphi],\chi)\mapsto \langle \mathbf{A},[\varphi]\rangle$$

induces the desired bijection. This map is equivariant with respect to the action of $H^2(\mathbf{C},D\mathcal{F}(t))$, see Proposition 12.1 and the proof of Theorem 7.12 in Section 12. Let us check that it factors through the set of connected components $\pi_0\mathbf{Trans}(t,\bar{t})$. Let $[\phi]\colon (\mathbf{A},[\varphi],\chi)\to(\mathbf{B},[\psi],\kappa)$ be an isomorphism in $\mathbf{Trans}(t,\bar{t})$ and $\xi\colon \phi\varphi\Rightarrow\psi\phi$ a homotopy. By Proposition 13.7 the coboundary of $(b_\phi, o_{\phi,\varphi}-o_{\psi,\phi}+e_\xi)$ in the translation cochain complex is $(c_{\mathbf{A}}-c_{\mathbf{B}}, b_\varphi - b_\psi)$, therefore $\langle \mathbf{A},[\varphi]\rangle = \langle \mathbf{B},[\psi]\rangle$.

We claim that the isotropy groups of the action of $H^2(\mathbf{C},D\mathcal{F}(t))$ on $\pi_0\mathbf{Trans}(t,\bar{t})$ are the image of $\bar{t}_*-t^*\colon H^2(\mathbf{C},D)\to H^2(\mathbf{C},D\mathcal{F}(t))$ in the exact sequence (5.1). By Theorem 7.12 two translation track categories $(\mathbf{A},[\varphi],\chi)$ and $(\mathbf{A},[\varphi]+\omega,\chi)$ are isomorphic in $\mathbf{Trans}(t,\bar{t})$ if and only if there exists $\varpi\in H^2(\mathbf{C},D)$ such that

$$(1_{(\mathbf{A},\chi)}+\varpi)[\varphi] = ([\varphi]+\omega)(1_{(\mathbf{A},\chi)}+\varpi).$$

Now by the effectivity of the action in Theorem 7.12 and the linear distributivity law in Proposition 7.13 we obtain that

$$\omega = (\bar{t}_*-t^*)(-\varpi).$$

In order to complete the proof it is enough to check that the image of the map $\pi_0\bar{\jmath}$ coincides with the image of j under the identification in Proposition 6.2. By Proposition 6.2 any cohomology class $j(c)\in H^3(\mathbf{C},D)$ is the characteristic class of some $(\mathbf{A},\chi)$ in $\mathbf{Pseudo}^{ab}_{\simeq}(\mathbf{C},D)$. We claim that there exists a homotopy class $[\varphi]\colon \mathbf{A}\rightsquigarrow\mathbf{A}$ such that $(\mathbf{A},[\varphi],\chi)$ is a translation. By Theorem 7.2 the existence of such a $[\varphi]$ is equivalent to the vanishing of $\theta_{\mathbf{A},\mathbf{A}}(t,\bar{t})$, and by applying (7.1) and the exactness of (5.1) we obtain

$$\theta_{\mathbf{A},\mathbf{A}}(t,\bar{t}) = \bar{t}_*j(c) - t^*j(c) = 0.$$

□

7 Obstruction theory for the 2-functor π

This section is devoted to the realizability of 0-, 1- and 2-cells through the 2-functor

$$\pi\colon \mathbf{Pseudo}^{ab}_{\simeq} \longrightarrow \mathbf{nat}$$

in (3.8). All 0-cells are known to be realizable since there is always the trivial linear track extension for $(\mathbf{C},D)$, see [BD89]. We here define classes in cohomology of categories which represent obstructions to the realizability of 1- and 2-cells. We also consider their behaviour with respect to the various composition laws.

Given two abelian track categories **A**, **B** and a morphism $(\varphi, \lambda)\colon \pi\mathbf{A} \to \pi\mathbf{B}$ in **nat** we define the cohomology class

$$\theta_{\mathbf{A},\mathbf{B}}(\varphi, \lambda) = \lambda_*\langle\mathbf{A}\rangle - \varphi^*\langle\mathbf{B}\rangle \in H^3(\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\varphi)). \tag{7.1}$$

Theorem 7.2. *The element $\theta_{\mathbf{A},\mathbf{B}}(\varphi, \lambda)$ vanishes if and only if there exists a pseudofunctor $\psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ such that $\pi\psi = (\varphi, \lambda)$.*

The proof of this result will be given in Section 10.

Proposition 7.3. *Given abelian track categories **A**, **B**, **C** and morphisms $\pi\mathbf{A} \overset{(\varphi,\lambda)}{\to} \pi\mathbf{B} \overset{(\psi,\zeta)}{\to} \pi\mathbf{C}$ in **nat** the following formula holds*

$$\theta_{\mathbf{A},\mathbf{C}}((\psi,\zeta)(\varphi,\lambda)) = (\zeta\mathcal{F}(\varphi))_*\theta_{\mathbf{A},\mathbf{B}}(\varphi,\lambda) + \varphi^*\theta_{\mathbf{B},\mathbf{C}}(\psi,\zeta).$$

This proposition follows from (7.1) and the composition law for 1-cells in **nat**.

Proposition 7.4. *Given abelian track categories **A**, **B** and a 2-cell $\alpha\colon (\varphi,\lambda) \Rightarrow (\psi,\zeta)$ in **nat** between 1-cells $(\varphi,\lambda), (\psi,\zeta)\colon \pi\mathbf{A} \to \pi\mathbf{B}$ the following equality holds*

$$((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi}, \alpha))_*\theta_{\mathbf{A},\mathbf{B}}(\varphi,\lambda) = ((\pi_1\mathbf{B})\mathcal{F}(\alpha, 0^{\square}_{\pi_0\psi}))_*\theta_{\mathbf{A},\mathbf{B}}(\psi,\zeta).$$

This proposition is a consequence of (7.1), (4.2) and (4.9).

Proposition 7.5. *Given an abelian track category **A** and $\omega \in H^3(\pi_0\mathbf{A}, \pi_1\mathbf{A})$ there exists another abelian track category **B** with $\pi\mathbf{A} = \pi\mathbf{B}$ such that $\theta_{\mathbf{A},\mathbf{B}}(1_{\pi\mathbf{A}}) = \omega$. This abelian track category is unique up to isomorphism in $\mathbf{Pseudo}^{ab}_{\simeq}$ and we write $\mathbf{A} = \mathbf{B} + \omega$.*

This result is a consequence of the definition of θ and [BJ02] 3.3.

Let $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ be pseudofunctors between small abelian track categories and $\alpha\colon \pi\varphi \Rightarrow \pi\psi$ a 2-cell in **nat**. We define the element

$$\vartheta_{\varphi,\psi}(\alpha) \in H^2(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\alpha))$$

as the cohomology class represented by

$$-((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi}, \alpha))_*b_\varphi - \alpha^{\#}c_{\mathbf{B}} + ((\pi_1\mathbf{B})\mathcal{F}(\alpha, 0^{\square}_{\pi_0\psi}))_*b_\psi. \tag{7.6}$$

Here we use the cochains defined in Section 13 for a fixed global section. This is indeed a cocycle by Proposition 13.8. Moreover, by using Proposition 13.10 we see that its cohomology class does not depend on the global section chosen for its definition.

Theorem 7.7. *The element $\vartheta_{\varphi,\psi}(\alpha)$ vanishes if and only if there exists a pseudonatural transformation $\tau\colon \varphi \Rightarrow \psi$ such that $\pi_0\tau = \alpha$.*

This proposition will be proved in Section 11.

Proposition 7.8. *Two pseudofunctors between abelian track categories $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ are homotopic if and only if $\pi\varphi = \pi\psi$ and $\vartheta_{\varphi,\psi}(0^{\square}_{\pi_0\varphi}) = 0$.*

Proof. Obviously if $\xi\colon \varphi \Rightarrow \psi$ is a homotopy then $\pi\varphi = \pi\psi$ and $\pi_0\xi = 0^{\square}_{\pi_0\varphi}$, hence by Theorem 7.7 $\vartheta_{\varphi,\psi}(0^{\square}_{\pi_0\varphi}) = 0$. On the other hand if we suppose that $\pi\varphi = \pi\psi$ and $\vartheta_{\varphi,\psi}(0^{\square}_{\pi_0\varphi}) = 0$ the pseudonatural transformation $\tau\colon \varphi \Rightarrow \psi$ with $\pi_0\tau = 0^{\square}_{\pi_0\varphi}$ constructed in the proof of Theorem 7.7, given in Section 11, satisfies $\tau_X = t(0^{\square}_{\pi_0\varphi})_X = t\left\{1_{\varphi(X)}\right\} = 1_{\varphi(X)}$, here we use (13.1), therefore τ is a homotopy. □

This result proves that, if we restrict to the category of abelian track categories, we could have defined homotopies as pseudonatural transformations $\xi\colon \varphi \Rightarrow \psi$ such that $\pi_0\xi$ is and identity natural transformation, obtaining in this way the same homotopy category $\mathbf{Pseudo}^{ab}_{\simeq}$. Therefore homotopies in the sense of Section 2 would be a reduced version of these more general ones, however the existence of $\mathbf{Pseudo}^{ab}_{\simeq}$ is not so easy to establish if we try to use non-reduced homotopies from the beginning.

Proposition 7.9. *Given pseudofunctors* $\phi, \varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *and* 2*-cells* $\pi\phi \overset{\beta}{\Rightarrow} \pi\varphi \overset{\alpha}{\Rightarrow} \pi\psi$ *in* **nat** *the following equality holds*

$$\vartheta_{\phi,\psi}(\alpha\Box\beta) = ((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\alpha))_*\vartheta_{\phi,\varphi}(\beta) + ((\pi_1\mathbf{B})\mathcal{F}(\beta,0^{\Box}_{\pi_0\psi}))_*\vartheta_{\varphi,\psi}(\alpha).$$

Proof. The result follows from the equations:

$$\begin{aligned}
((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\alpha))_*(-((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\beta))_*b_\phi \\
-\beta^{\#}c_{\mathbf{B}} + ((\pi_1\mathbf{B})\mathcal{F}(\beta,0^{\Box}_{\pi_0\varphi}))_*b_\varphi) \\
+((\pi_1\mathbf{B})\mathcal{F}(\beta,0^{\Box}_{\pi_0\psi}))_*(-((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\varphi},\alpha))_*b_\varphi \\
-\alpha^{\#}c_{\mathbf{B}} + ((\pi_1\mathbf{B})\mathcal{F}(\alpha,0^{\Box}_{\pi_0\psi}))_*b_\psi) &= -((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\alpha\Box\beta))_*b_\phi \\
&\quad -((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\alpha))_*\beta^{\#}c_{\mathbf{B}} \\
&\quad +((\pi_1\mathbf{B})\mathcal{F}(\beta,\alpha))_*b_\varphi \\
&\quad -((\pi_1\mathbf{B})\mathcal{F}(\beta,\alpha))_*b_\varphi \\
&\quad -((\pi_1\mathbf{B})\mathcal{F}(\beta,0^{\Box}_{\pi_0\psi}))_*\alpha^{\#}c_{\mathbf{B}} \\
&\quad +((\pi_1\mathbf{B})\mathcal{F}(\alpha\Box\beta,0^{\Box}_{\pi_0\psi}))_*b_\psi \\
&= -((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\phi},\alpha\Box\beta))_*b_\phi \\
&\quad -(\alpha\Box\beta)^{\#}c_{\mathbf{B}} \\
&\quad +((\pi_1\mathbf{B})\mathcal{F}(\alpha\Box\beta,0^{\Box}_{\pi_0\psi}))_*b_\psi \\
&\quad +\delta(\alpha,\beta)^{\%}c_{\mathbf{B}}.
\end{aligned}$$

Here we use (4.11) and the fact that $c_{\mathbf{B}}$ is a cocycle. □

Proposition 7.10. *Given pseudofunctors* $\varphi, \varphi'\colon \mathbf{B} \rightsquigarrow \mathbf{A}$*,* $\psi, \psi'\colon \mathbf{C} \rightsquigarrow \mathbf{B}$ *and* 2*-cells* $\alpha\colon \pi\varphi \Rightarrow \pi\varphi'$*,* $\beta\colon \pi\psi \Rightarrow \pi\psi'$ *in* **nat** *the following equalities hold*

$$\begin{aligned}
\vartheta_{\varphi\psi,\varphi'\psi'}(\alpha\beta) &= ((((\pi_1\mathbf{A})\mathcal{F}(0^{\Box}_{\pi_0\varphi},\alpha))\Box\pi_1\varphi)\mathcal{F}(\beta))_*\vartheta_{\psi,\psi'}(\beta) + \beta^*\vartheta_{\varphi,\varphi'}(\alpha) \\
&= ((((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\Box}_{\pi_0\varphi'}))\Box\pi_1\varphi')\mathcal{F}(\beta))_*\vartheta_{\psi,\psi'}(\beta) + \beta^*\vartheta_{\varphi,\varphi'}(\alpha).
\end{aligned}$$

Proof. The result follows from the next equalities

$$\begin{aligned}
&((((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))\square\pi_1\varphi)\mathcal{F}(\beta))_*(\\
&\qquad -((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\psi},\beta))_*b_\psi \\
&-\beta^{\#}c_{\mathbf{B}}+((\pi_1\mathbf{B})\mathcal{F}(\beta,0^{\square}_{\pi_0\psi'}))_*b_{\psi'}) \\
&\quad +\beta^*(-((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_*b_\varphi \\
&-\alpha^{\#}c_{\mathbf{A}}+((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*b_{\varphi'}) \overset{(a)}{=} -((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi\psi},\alpha\beta))_*((\pi_1\varphi)\mathcal{F}(\pi_0\psi))_*b_\psi \\
&\qquad -\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_*(\pi_1\varphi)_*c_{\mathbf{B}} \\
&\qquad +((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*((\pi_1\varphi')\mathcal{F}(\pi_0\psi'))_*b_{\psi'} \\
&\qquad -((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi\psi},\alpha\beta))_*(\pi_0\varphi)^*b_\varphi \\
&\qquad -\beta^*\alpha^{\#}c_{\mathbf{A}} \\
&\qquad +((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*(\pi_0\psi')^*b_{\varphi'} \\
&\qquad +\delta\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*b_{\varphi'} \\
&\qquad +\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*\delta b_{\varphi'} \\
&\overset{(b)}{=} -(\alpha\beta)^*(b_{\varphi\psi}-\delta(o_{\varphi,\psi})) \\
&\qquad -\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_*(\pi_1\varphi)_*c_{\mathbf{B}} \\
&\qquad -\beta^*\alpha^{\#}c_{\mathbf{A}} \\
&\qquad +\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*((\pi_1\varphi')_*c_{\mathbf{B}} \\
&\qquad -(\pi_0\varphi')^*c_{\mathbf{A}}) \\
&\qquad +((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*(b_{\varphi'\psi'}-\delta(o_{\varphi',\psi'})) \\
&\qquad +\delta\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*b_{\varphi'} \\
&\overset{(c)}{=} -(\alpha\beta)^*b_{\varphi\psi}-\beta^*\alpha^{\#}c_{\mathbf{A}}-\beta^{\#}\alpha^*c_{\mathbf{A}} \\
&\qquad +((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*b_{\varphi'\psi'} \\
&\qquad +\delta((\alpha\beta)^*o_{\varphi,\psi}-((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*o_{\varphi',\psi'} \\
&\qquad +\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*b_{\varphi'}) \\
&\overset{(d)}{=} -(\alpha\beta)^*b_{\varphi\psi}-(\alpha\beta)^{\#}c_{\mathbf{A}} \\
&\qquad +((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*b_{\varphi'\psi'} \\
&\qquad +\delta((\alpha\beta)^*o_{\varphi,\psi}-((\pi_1\mathbf{A})\mathcal{F}(\alpha\beta,0^{\square}_{\pi_0\varphi'\psi'}))_*o_{\varphi',\psi'} \\
&\qquad +\beta^{\#}((\pi_1\mathbf{A})\mathcal{F}(\alpha,0^{\square}_{\pi_0\varphi'}))_*b_{\varphi'}+(\alpha,\beta)^{\$}c_{\mathbf{A}})
\end{aligned}$$

Here for (a) we use (4.2), (4.8), (4.9) and (4.10); in (b) we apply (4.8) as well as (13.7) (1) and (3); for (c) we use (4.2) and (4.8); and finally for (d) we apply (4.12) and the fact that $c_{\mathbf{A}}$ is a cocycle. □

Proposition 7.11. *Given pseudofunctors $\varphi,\varphi',\psi,\psi'\colon \mathbf{A}\rightsquigarrow\mathbf{B}$ between abelian track categories such that $\varphi\simeq\varphi'$, $\psi\simeq\psi'$ and a 2-cell $\alpha\colon\pi\varphi=\pi\varphi'\Rightarrow\pi\psi=\pi\psi'$ in* **nat** *then $\vartheta_{\varphi,\psi}(\alpha)=\vartheta_{\varphi',\psi'}(\alpha)$.*

Proof. This is a consequence of the following equalities

$$\begin{aligned}
\vartheta_{\varphi,\psi}(\alpha) &= ((\pi_1\mathbf{B})\mathcal{F}(\alpha,0^{\square}_{\pi_0\psi}))_*\vartheta_{\psi',\psi}(0^{\square}_{\pi_0\psi}) \\
&\quad +\vartheta_{\varphi',\psi'}(\alpha)+((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_*\vartheta_{\varphi,\varphi'}(0^{\square}_{\pi_0\varphi}) \\
&= \vartheta_{\varphi',\psi'}(\alpha).
\end{aligned}$$

In the first equality we use Proposition 7.9 applied to the vertical composite $\alpha = 0^{\square}_{\pi_0\psi}\square\alpha\square 0^{\square}_{\pi_0\varphi}$, for the second one we use Proposition 7.8. $\square$

This proposition proves that the cohomology class $\vartheta_{\varphi,\psi}(\alpha)$ only depends on the homotopy classes of φ and ψ, therefore we can denote it by

$$\vartheta_{[\varphi],[\psi]}(\alpha).$$

Theorem 7.12. *Given a homotopy class of pseudofunctors between abelian track categories* $[\varphi]\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *and* $\omega \in H^2(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0[\varphi]))$ *there exists a unique homotopy class* $[\varphi] + \omega\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *such that* $\vartheta_{[\varphi]+\omega,[\varphi]}(0^{\square}_{\pi_0[\varphi]}) = \omega$*. This induces a transitive and effective action of the abelian group* $H^2(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0[\varphi]))$ *on the morphism set* $\pi^{-1}(\psi,\lambda) \subset \mathbf{Pseudo}^{ab}_{\simeq}(\mathbf{A},\mathbf{B})$ *for any* 1*-cell in* $(\psi,\lambda)\colon \pi\mathbf{A} \to \pi\mathbf{B}$ *in* **nat** *such that* $\theta_{\mathbf{A},\mathbf{B}}(\psi,\lambda) = 0$*.*

This proposition will be proved in Section 12.

Proposition 7.13. *The action in Theorem 7.12 satisfies the following linear distributivity law: given composable morphisms* $\mathbf{C} \overset{[\psi]}{\to} \mathbf{B} \overset{[\varphi]}{\to} \mathbf{A}$ *in* $\mathbf{Pseudo}^{ab}_{\simeq}$ *and cohomology classes* $\varpi \in H^2(\pi_0\mathbf{C}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0[\psi]))$ *and* $\omega \in H^2(\pi_0\mathbf{B}, (\pi_1\mathbf{A})\mathcal{F}(\pi_0[\varphi]))$ *the equality*

$$([\varphi] + \omega)([\psi] + \varpi) = [\varphi][\psi] + ((\pi_1[\varphi])\mathcal{F}(\pi_0[\psi]))_*\varpi + (\pi_0[\psi])^*\omega$$

is satisfied.

Proof. The formula is a consequence of the following equalities,

$$\begin{aligned}
\vartheta_{([\varphi]+\omega)([\psi]+\varpi),[\varphi][\psi]}(0^{\square}_{\pi_0[\varphi][\psi]}) &= ((\pi_1[\varphi])\mathcal{F}(\pi_0[\psi]))_*\vartheta_{[\psi]+\varpi,[\psi]}(0^{\square}_{\pi_0[\psi]}) \\
&\quad +(\pi_0[\psi])^*\vartheta_{[\varphi]+\omega,[\varphi]}(0^{\square}_{\pi_0[\varphi]}) \\
&= ((\pi_1[\varphi])\mathcal{F}(\pi_0[\psi]))_*\varpi + (\pi_0[\psi])^*\omega.
\end{aligned}$$

Here we apply Proposition 7.10 to the composite $0^{\square}_{\pi_0[\varphi][\psi]} = 0^{\square}_{\pi_0[\varphi]}0^{\square}_{\pi_0[\psi]}$. $\square$

8 The homotopy category of pseudofunctors as a linear extension

An *exact sequence for a functor* λ

$$D \xrightarrow{+} \mathbf{A} \xrightarrow{\lambda} \mathbf{B} \xrightarrow{\theta} H$$

is given by two natural systems D, H on $\mathbf{B}$ together with an obstruction operator θ which determines whether a morphism in $\mathbf{B}$ is realizable through λ or not, and an action + counting all possible realizations. We refer the reader to [Bau89] IV.4.10 for an explicit definition. *Linear extensions of categories*, in the sense of [Bau89] IV.3.2,

$$D \overset{+}{\rightarrowtail} \mathbf{A} \xrightarrow{\lambda} \mathbf{B}$$

can be regarded as special cases of exact sequences for functors with $H = 0$ and an effective action + of D. A weak linear extension of categories occurs when we change $\mathbf{B}$ by an equivalent category in a linear extension as above.

As a consequence of the results in Section 7 the 2-functor π in (3.8) regarded as an ordinary functor embeds in an exact sequence. More precisely, we can consider the natural system H^n $(n \geq 0)$ on **nat** defined by the n^{th}-cohomology of categories $H^n(\mathbf{C}, D)$. The homomorphisms induced by a 1-cell (φ, λ) in **nat** are $(\varphi, \lambda)_* = \lambda_*$ and $(\varphi, \lambda)^* = \varphi^*$. This is indeed a natural system because of the functorial properties of the cohomology of categories. The exact sequence for π

$$H^2 \xrightarrow{+} \mathbf{Pseudo}^{ab}_{\simeq} \xrightarrow{\pi} \mathbf{nat} \xrightarrow{\theta} H^3 \tag{8.1}$$

is given by the obstruction operator θ in (7.1) and the action $+$ in Theorem 7.12. Theorems 7.2 and 7.12 and Propositions 7.3, 7.11 and 7.13 prove that the axioms of an exact sequence for a functor are satisfied in this case.

In fact the theory developed in Section 7 is much richer, because it also involves an obstruction theory for the realization of 2-cells through the 2-functor π. However we have not developed general axioms for this sort of exact sequence for a 2-functor because in this paper we concentrate only in the particular case of π.

The obstruction operator in (8.1) is an inner derivation in the sense of [Bau89] IV.7.3, the action $+$ is effective by Theorem 7.12, and any object in **nat** is isomorphic to an object in the image of π by Proposition 6.2, therefore we can obtain a weak linear extension from the exact sequence (8.1) in the following canonical way: consider the category **CoH** whose objects are triples $(\mathbf{C}, D, c)$ where $\mathbf{C}$ is a category, D is a natural system on $\mathbf{C}$ and $c \in H^3(\mathbf{C}, D)$. A morphism $(\varphi, \lambda)\colon (\mathbf{C}, D, c) \to (\mathbf{D}, E, d)$ in **CoH** is a morphism $(\varphi, \lambda)\colon (\mathbf{C}, D) \to (\mathbf{D}, E)$ in **nat** which is compatible with the cohomology classes, that is $\lambda_* c = \varphi^* d$. By Theorem 7.2 the functor π extends to a functor

$$\bar{\pi}\colon \mathbf{Pseudo}^{ab}_{\simeq} \longrightarrow \mathbf{CoH}, \ \ \bar{\pi}(\mathbf{A}) = (\pi_0\mathbf{A}, \pi_1\mathbf{A}, \langle\mathbf{A}\rangle).$$

This functor fits into a weak linear extension

$$H^2 \overset{+}{\rightarrowtail} \mathbf{Pseudo}^{ab}_{\simeq} \xrightarrow{\bar{\pi}} \mathbf{CoH}. \tag{8.2}$$

Here $+$ is the same action as in (8.1), and H^2 is regarded as a natural system on **CoH** through the obvious forgetful functor $\mathbf{CoH} \to \mathbf{nat}$.

9 A 2-category in terms of cocycles

In this section we describe an alternative and more algebraic model of the 2-category $\mathbf{Pseudo}^{ab}_{\simeq}$ only in terms of the Baues-Wirsching cochain complex as a 2-functor. For this we will use the exact sequence for functors (8.1) constructed in Section 8.

We define the 2-category **CoC** as follows: 0-cells $(\mathbf{C}, D, c)$ are triples where $\mathbf{C}$ is a category D is a natural system on $\mathbf{C}$ and $c \in \bar{F}^3(\mathbf{C}, D)$ is a 3-cocycle, that is $\delta c = 0$. A 1-cell $(\varphi, \lambda, e)\colon (\mathbf{C}, D, c) \to (\mathbf{D}, E, d)$ is given by a morphism $(\varphi, \lambda)\colon (\mathbf{C}, D) \to (\mathbf{D}, E)$ in **nat** together with a 2-cochain modulo coboundaries

$$e \in \frac{\bar{F}^2(\mathbf{C}, E\mathcal{F}(\varphi))}{\operatorname{Im}\delta}$$

satisfying $\delta e = \lambda_* c - \varphi^* d$. A 2-cell $\alpha\colon (\varphi, \lambda, e) \Rightarrow (\psi, \zeta, b)$ between 1-cells

$$(\varphi, \lambda, e), (\psi, \zeta, b)\colon (\mathbf{C}, D, c) \to (\mathbf{D}, E, d)$$

is just a 2-cell $\alpha\colon (\varphi, \lambda) \Rightarrow (\psi, \zeta)$ in **nat** such that

$$0 = -(E\mathcal{F}(0^{\square}_{\varphi}, \alpha))_* e - \alpha^{\#} d + (E\mathcal{F}(\alpha, 0^{\square}_{\psi}))_* b \in H^2(\mathbf{C}, E\mathcal{F}(\alpha)) \subset \frac{\bar{F}^2(\mathbf{C}, E\mathcal{F}(\alpha))}{\operatorname{Im}\delta}.$$

One can check as in Proposition 13.8 that this linear combination of 2-cochains modulo coboundaries lies in the cohomology group. Composition of 1-cells is determined by the formula

$$(\varphi, \lambda, e)(\psi, \zeta, b) = (\varphi\psi, (\lambda\mathcal{F}(\psi))\square\zeta, (\lambda\mathcal{F}(\psi))_* b + \psi^* e).$$

Composition laws for 2-cells are as in **nat**. It is not trivial at all to check that the horizontal or vertical composition of two 2-cells in **CoC** is indeed in this 2-category, however the necessary computations are similar to those in the proofs of Propositions 7.9 and 7.10, and for this reason we do not include them here.

There is an obvious 2-functor

$$\wp\colon \mathbf{CoC} \longrightarrow \mathbf{nat}, \;\; \wp(\mathbf{C}, D, c) = (\mathbf{C}, D),$$

which is faithful in dimension 2, as π is, see Proposition 2.2. This functor also embeds in an exact sequence

$$H^2 \xrightarrow{+} \mathbf{CoC} \xrightarrow{\wp} \mathbf{nat} \xrightarrow{\theta} H^3. \tag{9.1}$$

The obstruction operator is defined as follows

$$\theta_{(\mathbf{C},D,c),(\mathbf{D},E,d)}(\varphi, \lambda) = \lambda_* \{c\} - \varphi^* \{d\}.$$

Here $\{\cdot\}$ denotes the cohomology class of the cocycle inside. Moreover, the action $+$ is defined as follows, given a 1-cell $(\varphi, \lambda, e)\colon (\mathbf{C}, D, c) \to (\mathbf{D}, E, d)$ in **CoC** and $\{s\} \in H^2(\mathbf{C}, E\mathcal{F}(\varphi))$

$$(\varphi, \lambda, e) + \{s\} = (\varphi, \lambda, e + s).$$

Notice that this action is effective. The 2-functor $\wp$ lifts to **CoH** through the forgetful functor $\mathbf{CoH} \to \mathbf{nat}$ giving rise to a weak linear extension of categories

$$H^2 \overset{+}{\rightarrowtail} \mathbf{CoC} \xrightarrow{\bar{\wp}} \mathbf{CoH}, \tag{9.2}$$

where $\bar{\wp}(\mathbf{C}, D, c) = (\mathbf{C}, D, \{c\})$.

Proposition 13.7 proves that there is a 2-functor

$$\mathcal{C}\colon \mathbf{Pseudo}^{ab}_{\simeq} \longrightarrow \mathbf{CoC},$$

such that the image of an abelian track category $\mathbf{A}$ is $\mathcal{C}(\mathbf{A}) = (\pi_0\mathbf{A}, \pi_1\mathbf{A}, c_{\mathbf{A}})$, and given a pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ between abelian track categories $\mathcal{C}(\varphi) = (\pi_0\varphi, \pi_1\varphi, b_\varphi)$. Here we use the cochains defined in Section 13 for a fixed global section, but notice that Proposition 13.9 proves that up to 2-natural isomorphism this 2-functor does not depend on that choice.

By Proposition 12.1 the 2-functor $\mathcal{C}$ induces a map of exact sequences from (8.1) to (9.1) in the sense of [Bau89] 4.13

$$\begin{tikzcd}
H^2 \arrow[r, "+"] \arrow[d, equal] & \mathbf{Pseudo}^{ab}_{\simeq} \arrow[r, "\pi"] \arrow[d, "\mathcal{C}"] & \mathbf{nat} \arrow[r, "\theta"] \arrow[d, equal] & H^3 \arrow[d, equal] \\
H^2 \arrow[r, "+"] & \mathbf{CoC} \arrow[r, "\wp"] & \mathbf{nat} \arrow[r, "\theta"] & H^3
\end{tikzcd}$$

hence by [Bau89] 4.14 $\mathcal{C}$ is an equivalence of categories regarded as an ordinary functor. Furthermore, the definition of 2-cells in **CoC** together with (7.6), Proposition 2.2 and Theorem 7.7 prove that $\mathcal{C}$ is fully faithful in dimension 2, hence it is not only an ordinary equivalence but an equivalence of enriched categories.

10 Proof of Theorem 7.2

The following result is a consequence of Proposition 13.7 (1).

Proposition 10.1. *If* $\psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *is a pseudofunctor between abelian track categories then* $\theta_{\mathbf{A},\mathbf{B}}(\pi\psi) = 0$.

In order to prove the converse we introduce the pull-back and push-forward of an abelian track category.

Let $\mathbf{A}$ be an abelian track category and $\varphi\colon \mathbf{C} \to \pi_0\mathbf{A}$ an ordinary functor, the *pull-back* of $\mathbf{A}$ along φ is the abelian track category $\varphi^*\mathbf{A}$ defined as follows. The category $(\varphi^*\mathbf{A})_0$ is the following pull-back in $\mathbf{Cat}$

(10.2)
$$\begin{tikzcd} (\varphi^*\mathbf{A})_0 \arrow[r, "p'"] \arrow[d, "\varphi'"'] \arrow[dr, phantom, "\text{pull}"] & \mathbf{C} \arrow[d, "\varphi"] \\ \mathbf{A}_0 \arrow[r, "p"'] & \pi_0\mathbf{A} \end{tikzcd}$$

where p is the quotient functor. Track sets in $\varphi^*\mathbf{A}$ are given by the formula

(10.3)
$$[\![X,Y]\!]_{\varphi^*\mathbf{A}}(f,g) = [\![\varphi'(X),\varphi'(Y)]\!]_{\mathbf{A}}(\varphi'(f),\varphi'(g)).$$

This pull-back comes equipped with a track functor

$$\bar{\varphi}\colon \varphi^*\mathbf{A} \longrightarrow \mathbf{A}$$

which is given by φ' on objects and maps, and the equality (10.3) on tracks. This track functor determines composition laws in $\varphi^*\mathbf{A}$ involving tracks.

Proposition 10.4. *The pull back* $\varphi^*\mathbf{A}$ *satisfies the equalities* $\pi_0\varphi^*\mathbf{A} = \mathbf{C}$ *and* $\pi_1\varphi^*\mathbf{A} = (\pi_1\mathbf{A})\mathcal{F}(\varphi)$. *Moreover, the track functor* $\bar{\varphi}$ *induces* $\pi_0\bar{\varphi} = \varphi$ *and* $\pi_1\bar{\varphi} = 0^{\square}_{(\pi_1\mathbf{A})\mathcal{F}(\varphi)}$.

Proof. The natural projection $(\varphi^*\mathbf{A})_0 \to \mathbf{C}$ is p' in (10.2), the equality $\pi_0\bar{\varphi} = \varphi$ follows from the commutativity of (10.2), and the identities $\pi_1\varphi^*\mathbf{A} = (\pi_1\mathbf{A})\mathcal{F}(\varphi)$ and $\pi_1\bar{\varphi} = 0^{\square}_{(\pi_1\mathbf{A})\mathcal{F}(\varphi)}$ are direct consequences of (10.3). ☐

If $\mathbf{A}$ is an abelian track category and $\lambda\colon \pi_1\mathbf{A} \Rightarrow D$ is a natural transformation between natural systems on $\pi_0\mathbf{A}$ the *push-forward* of $\mathbf{A}$ along λ is an abelian track category $\lambda_*\mathbf{A}$ with the same objects and maps as $\mathbf{A}$. A track $(\alpha|a)\colon f \Rightarrow g$ in $\lambda_*\mathbf{A}$ is represented by a track $\alpha\colon f \Rightarrow g$ in $\mathbf{A}$ together with an element $a \in D_{\{f\}}$. Two of these pairs represent the same track

$$(\alpha|a) = (\beta|b)$$

if and only if

$$b = a + \lambda_{\{f\}}(\sigma_f^{-1}(\beta^{\boxminus}\square\alpha)).$$

The vertical composition of tracks $\bullet \overset{(\beta|b)}{\Rightarrow} \bullet \overset{(\alpha|a)}{\Rightarrow} \bullet$ in $\lambda_*\mathbf{A}$ is given by

$$(\alpha|a)\square(\beta|b) = (\alpha\square\beta|a+b).$$

Moreover, given a diagram in $\lambda_*\mathbf{A}$ as follows

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow(\alpha|a)"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

the two horizontal compositions in $\lambda_*\mathbf{A}$ are given by

$$(\alpha|a)f = (\alpha f|\{f\}^*(a)) \quad \text{and} \quad g(\alpha|a) = (g\alpha|\{g\}_*(a)).$$

Notice that there is a track functor

$$\bar{\lambda}\colon \mathbf{A} \longrightarrow \lambda_*\mathbf{A}$$

defined as the identity on objects and maps, and $\bar{\lambda}(\alpha) = (\alpha|0)$ on tracks.

Proposition 10.5. *The push-forward $\lambda_*\mathbf{A}$ satisfies $\pi_0\lambda_*\mathbf{A} = \pi_0\mathbf{A}$ and $\pi_1\lambda_*\mathbf{A} = D$. Moreover, the track functor $\bar{\lambda}$ induces $\pi_0\bar{\lambda} = 1_{\pi_0\mathbf{A}}$ and $\pi_1\bar{\lambda} = \lambda$.*

Proof. It is enough to notice that for any map $f\colon X \to Y$ in $\mathbf{A}_0 = (\lambda_*\mathbf{A})_0$ the homomorphism

$$D_{\{f\}} \longrightarrow [\![X,Y]\!]_{\lambda_*\mathbf{A}}(f,f), \;\; a \mapsto (0_f^\square|a),$$

is an isomorphism with inverse

$$[\![X,Y]\!]_{\lambda_*\mathbf{A}}(f,f) \longrightarrow D_{\{f\}}, \;\; (\alpha|a) \mapsto \lambda_{\{f\}}(\sigma_f^{-1}(\alpha)) + a.$$

□

Now we are ready to prove Theorem 7.2.

Proof of Theorem 7.2. The "if" part is Proposition 10.1. Now suppose that we have a 1-cell $(\varphi,\lambda)\colon \pi\mathbf{A} \to \pi\mathbf{B}$ in **nat** such that $\theta_{\mathbf{A},\mathbf{B}}(\varphi,\lambda) = \lambda_*\langle\mathbf{A}\rangle - \varphi^*\langle\mathbf{B}\rangle = 0$. By Propositions 10.1, 10.4 and 10.5 we have that $\langle\lambda_*\mathbf{A}\rangle = \lambda_*\langle\mathbf{A}\rangle$ and $\langle\varphi^*\mathbf{B}\rangle = \varphi^*\langle\mathbf{B}\rangle$, therefore $\langle\lambda_*\mathbf{A}\rangle = \langle\varphi^*\mathbf{A}\rangle$. By [BJP03] 2.4.2 and [BJ02] 2.4 there exists a pseudofunctor $\phi\colon \lambda_*\mathbf{A} \rightsquigarrow \varphi^*\mathbf{B}$ with $\pi\phi = (1_{\pi_0\mathbf{A}}, 0^\square_{(\pi_1\mathbf{B})\mathcal{F}(\varphi)})$, hence again by Propositions 10.4 and 10.5 we see that $\psi = \bar{\varphi}\phi\bar{\lambda}$ satisfies the requirements of Theorem 7.2. □

11 Proof of Theorem 7.7

The "if" part follows from Proposition 13.7 (2).

Suppose now that a 2-cell α in **nat** is given with $\vartheta_{\varphi,\psi}(\alpha) = 0$. We choose a 1-cochain

$$e \in \bar{F}^1(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\alpha))$$

with

$$\delta(e) = -((\pi_1\mathbf{B})\mathcal{F}(0^\square_{\pi_0\varphi},\alpha))_*b_\varphi - \alpha^\# c_\mathbf{B} + ((\pi_1\mathbf{B})\mathcal{F}(\alpha, 0^\square_{\pi_0\psi}))_*b_\psi. \tag{11.1}$$

Notice that implicitly we have chosen a global section (t,μ,ν,η) as in Section 13. It will remain fixed throughout this proof.

We claim that there exists a unique pseudonatural transformation $\tau\colon \varphi \Rightarrow \psi$ such that for all objects X and morphisms $\{f\}: X \to Y$ in $\pi_0\mathbf{A}$, $\tau_X = t\alpha_X$ and $\tau_{t\{f\}}$ is the following composite track

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\varphi(X) \arrow[r, "t\alpha_X", ""'{name=T}] \arrow[d, bend right=60, "\varphi(t\{f\})"', ""{name=L1}] \arrow[d, "t\{\varphi(f)\}", ""'{name=L2}] \arrow[dr, bend left=25, "t\{\psi(f)\alpha_X\}", ""'{name=U}] \arrow[dr, bend right=25, "t\{\alpha_Y\varphi(f)\}"', ""{name=D}] & \psi(X) \arrow[d, "t\{\psi(f)\}"', ""{name=R1}] \arrow[d, bend left=60, "\psi(t\{f\})", ""'{name=R2}] \\
\varphi(Y) \arrow[r, "t\alpha_Y"', ""{name=B}] & \psi(Y)
\arrow[Rightarrow, from=L1, to=L2, "\nu_{\{f\},\varphi}"]
\arrow[Rightarrow, from=R1, to=R2, "\nu^{\boxminus}_{\{f\},\psi}"]
\arrow[Rightarrow, from=U, to=T, "\mu^{\boxminus}_{\{\psi(f)\},\alpha_X}"]
\arrow[Rightarrow, from=D, to=U, "\sigma_{t\{\alpha_Y\varphi(f)\}}(e(\{f\}))"]
\arrow[Rightarrow, from=B, to=D, "\mu_{\alpha_Y,\{\varphi(f)\}}"]
\end{tikzcd}$$

that is

$$\tau_{t\{f\}} = (\nu^{\boxminus}_{\{f\},\psi}t\alpha_X)\square\mu^{\boxminus}_{\{\psi(f)\},\alpha_X}\square\sigma_{t\{\alpha_Y\varphi(f)\}}(e(\{f\}))\square\mu_{\alpha_Y,\{\varphi(f)\}}\square(t\alpha_Y\nu_{\{f\},\varphi}).$$

If such a τ exists then by (A.7) for any map f in $\mathbf{A}$ if $\xi\colon f \Rightarrow t\{f\}$ is any track we have

$$\tau_f = (\psi(\xi)^{\boxminus}t\alpha_X)\square\tau_{t\{f\}}\square(t\alpha_Y\varphi(\xi)),$$

In fact we can take this formula as a definition for τ_f. In order to check that τ_f does not depend on the track ξ chosen for its definition it is enough to see that the equality holds when ξ is a self-track of $t\{f\}$, or equivalently that

$$\tau_{t\{f\}} = (\psi(\sigma_{t\{f\}}(a))^{\boxminus}t\alpha_X)\square\tau_{t\{f\}}\square(t\alpha_Y\varphi(\sigma_{t\{f\}}(a)))$$

for all $a \in (\pi_1\mathbf{A})_{\{f\}}$. This follows from the next equalities, where we use (3.3), (3.4) and (3.7) in the first one, and the commutativity of (3.5) in the second one

$$\begin{aligned}
(\psi(\sigma_{t\{f\}}(a))^{\boxminus}t\alpha_X)\square\tau_{t\{f\}}\square(t\alpha_Y\varphi(\sigma_{t\{f\}}(a))) &= (\nu^{\boxminus}_{\{f\},\psi}t\alpha_X)\square\mu^{\boxminus}_{\{\psi(f)\},\alpha_X} \\
&\quad \square\sigma_{t\{\alpha_Y\varphi(f)\}}(-\alpha_X^*(\pi_1\psi)_*(a) \\
&\quad +e(\{f\})+\alpha_{Y*}(\pi_1\varphi)_*(a)) \\
&\quad \square\mu_{\alpha_Y,\{\varphi(f)\}}\square(t\alpha_Y\nu_{\{f\},\varphi}) \\
&= \tau_{t\{f\}}.
\end{aligned}$$

This can also be used to check that τ satisfies (A.7). Moreover, (A.9) follows easily from (13.1), ..., (13.4), so it is only left to prove (A.8). For this given composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{A}$ we choose fixed tracks $\xi_f\colon f \Rightarrow t\{f\}$ and $\xi_g\colon g \Rightarrow t\{g\}$ for the definition of τ_f and τ_g, respectively. We will use the track $\mu_{\{g\},\{f\}}\square(\xi_g\xi_f)\colon gf \Rightarrow t\{gf\}$ for the definition of τ_{gf}.

$$(\psi_{g,f}t\alpha_X)\square(\psi(g)\tau_f)\square(\tau_g\varphi(f))\square(t\alpha_Z\varphi^{\boxminus}_{g,f}) \quad =$$

$$\begin{aligned}
= \quad & (\psi_{g,f} t\alpha_X)\square(\psi(g)\psi(\xi_f)^{\boxminus} t\alpha_X)\square(\psi(g)\nu^{\boxminus}_{\{f\},\psi} t\alpha_X)\square(\psi(g)\mu^{\boxminus}_{\{\psi(f)\},\alpha_X}) \\
& \square(\psi(g)\sigma_{t\{\alpha_Y\varphi(f)\}}(e(\{f\})))\square(\psi(g)\mu_{\alpha_Y,\{\varphi(f)\}})\square(\psi(g)t\alpha_Y\nu_{\{f\},\varphi}) \\
& \square(\psi(g)t\alpha_Y\varphi(\xi_f))\square(\psi(\xi_g)^{\boxminus} t\alpha_Y\varphi(f))\square(\nu^{\boxminus}_{\{g\},\psi} t\alpha_Y\varphi(f))\square(\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y}\varphi(f)) \\
& \square(\sigma_{t\{\alpha_Z\varphi(f)\}}(e(\{g\}))\varphi(f))\square(\mu_{\alpha_Z,\{\varphi(g)\}}\varphi(f))\square(t\alpha_Z\nu_{\{g\},\varphi}\varphi(f)) \\
& \square(t\alpha_Z\varphi(\xi_g)\varphi(f))\square(t\alpha_Z\varphi^{\boxminus}_{g,f}) \\
\overset{(a)}{=} \quad & (\psi_{g,f} t\alpha_X)\square(\psi(\xi_g)^{\boxminus}\psi(\xi_f)^{\boxminus} t\alpha_X)\square(\nu^{\boxminus}_{\{g\},\psi}\nu^{\boxminus}_{\{f\},\psi} t\alpha_X)\square(t\left\{\psi(g)\right\}\mu^{\boxminus}_{\{\psi(f)\},\alpha_X}) \\
& \square\sigma_{t\{\psi(g)\}t\{\alpha_Y\varphi(f)\}}(\left\{\psi(g)\right\}_* e(\{f\}))\square(t\left\{\psi(g)\right\}\mu_{\alpha_Y,\{\varphi(f)\}}) \\
& \square(\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y} t\left\{\varphi(f)\right\})\square\sigma_{t\{\alpha_Z\varphi(f)\}t\{\varphi(f)\}}(\left\{\varphi(f)\right\}^* e(\{g\})) \\
& \square(\mu_{\alpha_Z,\{\varphi(g)\}} t\left\{\varphi(f)\right\})\square(t\alpha_Z\nu_{\{g\},\varphi}\nu_{\{f\},\varphi})\square(t\alpha_Z\varphi(\xi_g)\varphi(\xi_f))\square(t\alpha_Z\varphi^{\boxminus}_{g,f}) \\
\overset{(b)}{=} \quad & (\psi(\xi_g\xi_f)^{\boxminus} t\alpha_X)\square(\psi_{t\{g\},t\{f\}} t\alpha_X)\square(\nu^{\boxminus}_{\{g\},\psi}\nu^{\boxminus}_{\{f\},\psi} t\alpha_X) \\
& \square\sigma_{t\{\psi(g)\}t\{\psi(f)\}t\alpha_Y}(\left\{\psi(g)\right\}_* e(\{f\}) + \left\{\varphi(f)\right\}^* e(\{g\})) \\
& \square(t\left\{\psi(g)\right\}\mu^{\boxminus}_{\{\psi(f)\},\alpha_X})\square(t\left\{\psi(g)\right\}\mu_{\alpha_Y,\{\varphi(f)\}})\square(\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y} t\left\{\varphi(f)\right\}) \\
& \square(\mu_{\alpha_Z,\{\varphi(g)\}} t\left\{\varphi(f)\right\})\square(t\alpha_Z\nu_{\{g\},\varphi}\nu_{\{f\},\varphi})\square(t\alpha_Z\varphi^{\boxminus}_{t\{g\},t\{f\}})\square(t\alpha_Z\varphi(\xi_g\xi_f)) \\
\overset{(c)}{=} \quad & (\psi(\xi_g\xi_f)^{\boxminus} t\alpha_X)\square(\psi_{t\{g\},t\{f\}} t\alpha_X)\square(\nu^{\boxminus}_{\{g\},\psi}\nu^{\boxminus}_{\{f\},\psi} t\alpha_X) \\
& \square\sigma_{t\{\psi(g)\}t\{\psi(f)\}t\alpha_Y}(\alpha_X^* b_\psi(\left\{f\right\},\left\{g\right\}) - (\alpha^{\#} c_{\mathbf{B}})(\left\{f\right\},\left\{g\right\}) + e(\{gf\})) \\
& \square(t\left\{\psi(g)\right\}\mu^{\boxminus}_{\{\psi(f)\},\alpha_X})\square(t\left\{\psi(g)\right\}\mu_{\alpha_Y,\{\varphi(f)\}})\square(\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y} t\left\{\varphi(f)\right\}) \\
& \square(\mu_{\alpha_Z,\{\varphi(g)\}} t\left\{\varphi(f)\right\})\square(t\alpha_Z\nu_{\{g\},\varphi}\nu_{\{f\},\varphi})\square(t\alpha_Z\varphi^{\boxminus}_{t\{g\},t\{f\}}) \\
& \square\sigma_{t\alpha_Z t\{\varphi(g)\}t\{\varphi(f)\}\alpha_Y}(-\alpha_{Z*} b_\varphi(\left\{f\right\},\left\{g\right\}))\square(t\alpha_Z\varphi(\xi_g\xi_f)) \\
\overset{(d)}{=} \quad & (\psi(\xi_g\xi_f)^{\boxminus} t\alpha_X)\square(\psi(\mu^{\boxminus}_{\{g\},\{f\}}) t\alpha_X)\square(\nu^{\boxminus}_{\{gf\},\psi} t\alpha_X)\square(\mu_{\{\psi(g)\},\{\psi(f)\}} t\alpha_X) \\
& \square\sigma_{t\{\psi(g)\}t\{\psi(f)\}t\alpha_Y}(-c_{\mathbf{B}}(\left\{\psi(g)\right\},\left\{\psi(f)\right\},\alpha_X))\square(t\left\{\psi(g)\right\}\mu^{\boxminus}_{\{\psi(f)\},\alpha_X}) \\
& \square\sigma_{t\{\psi(g)\}t\{\alpha_Y\varphi(f)\}}(e(\{gf\}) + c_{\mathbf{B}}(\left\{\psi(g)\right\},\alpha_Y,\left\{\varphi(f)\right\})) \\
& \square(t\left\{\psi(g)\right\}\mu_{\alpha_Y,\{\varphi(f)\}})\square(\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y} t\left\{\varphi(f)\right\}) \\
& \square\sigma_{t\{\alpha_Z\varphi(g)\}t\{\varphi(f)\}}(-c_{\mathbf{B}}(\alpha_Z,\left\{\varphi(g)\right\},\left\{\varphi(f)\right\}))\square(\mu_{\alpha_Z,\{\varphi(g)\}} t\left\{\varphi(f)\right\}) \\
& \square(t\alpha_Z\mu^{\boxminus}_{\{\varphi(g)\},\{\varphi(f)\}})\square(t\alpha_Z\nu_{\{gf\},\varphi})\square(t\alpha_Z\varphi(\mu_{\{g\},\{f\}}))\square(t\alpha_Z\varphi(\xi_g\xi_f)) \\
\overset{(e)}{=} \quad & (\psi((\xi_g^{\boxminus}\xi_f^{\boxminus})\square\mu^{\boxminus}_{\{g\},\{f\}}) t\alpha_X)\square(\nu^{\boxminus}_{\{gf\},\psi} t\alpha_X)\square\mu^{\boxminus}_{\{\psi(gf)\},\alpha_X}\square\mu_{\{\psi(g)\},\{\psi(f)\}\alpha_Y} \\
& \square\sigma_{t\{\psi(g)\}t\{\alpha_Y\varphi(f)\}}(e(\{gf\}))\square\mu^{\boxminus}_{\{\psi(g)\},\alpha_Y\{\varphi(f)\}}\square\mu_{\{\psi(g)\}\alpha_Y,\{\varphi(f)\}} \\
& \square\mu^{\boxminus}_{\alpha_Z\{\varphi(g)\},\{\varphi(f)\}}\square\mu_{\alpha_Z,\{\varphi(gf)\}}\square(t\alpha_Z\nu_{\{gf\},\varphi})\square(t\alpha_Z\varphi(\mu_{\{g\}\{f\}}\square(\xi_g\xi_f))) \\
\overset{(f)}{=} \quad & \tau_{gf}
\end{aligned}$$

In (a) we apply (3.4); for (b) we use (A.4) and (3.3); in (c) we use (11.1) and (3.3); for (d) we apply (4.5), (13.6), (3.3) and (3.4); for (e) we use (13.5) and (3.3); and finally we use (3.3) for (f) as well as the naturality of α.

12 Proof of Theorem 7.12

The key of the proof of Theorem 7.12 is the following result.

Proposition 12.1. *Given a pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and a 2-cocycle*

$$s \in \bar{F}^2(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0\varphi))$$

there is a new pseudofunctor $\varphi + s\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ defined as φ on objects, maps and tracks such that for any object X in $\mathbf{A}$ we have $(\varphi + s)_X = \varphi_X$ and for maps $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ in $\mathbf{A}$

$$(\varphi + s)_{f,g} = \sigma_{\varphi(fg)}(s(\left\{f\right\},\left\{g\right\}))\square\varphi_{f,g}.$$

Moreover, for any choice of a global section as in Section 13 we have the following equality

$$b_{\varphi+s} = b_\varphi + s.$$

The proof is tedious but straightforward.

Proof of Theorem 7.12. If s is a cocycle representing ω then by Proposition 12.1 and (7.6) the homotopy class $[\varphi + s]$ satisfies the properties of $[\varphi] + \omega$ in the statement. The uniqueness as well as the fact that this defines an effective and transitive action follow easily from Propositions 7.8 and 7.9 applied to the vertical composite $0^{\square}_{\pi_0\varphi} = 0^{\square}_{\pi_0\varphi}\square 0^{\square}_{\pi_0\varphi}$. $\square$

13 Global sections and cochains

A *global section* (t, μ, ν, η) on the abelian track categories is a choice of maps and tracks

- $t\{f\} \in \{f\}$,
- $\mu_{\{f\},\{g\}} \colon t\{f\}\, t\{g\} \Rightarrow t\{fg\}$,
- $\nu_{\{f\},\varphi} \colon \varphi(t\{f\}) \Rightarrow t\{\varphi(f)\}$,
- $\eta_{X,\alpha} \colon \alpha_X \Rightarrow t\{\alpha_X\}$,

for all abelian track categories $\mathbf{A}$, objects X and (composable) morphisms $\{f\}, \{g\}$ in $\pi_0\mathbf{A}$, pseudofunctors between abelian track categories $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$, and pseudonatural transformations $\alpha\colon \varphi \Rightarrow \psi$. These choices must satisfy the following properties for all objects X and morphisms $\{f\} : X \to Y$ in $\pi_0\mathbf{A}$

$$t\{1_X\} = 1_X, \tag{13.1}$$

$$\mu_{\{f\},\{1_X\}} = 0^{\square}_{t\{f\}}, \tag{13.2}$$

$$\mu_{\{1_Y\},\{f\}} = 0^{\square}_{t\{f\}}, \tag{13.3}$$

$$\nu_{\{1_X\},\varphi} = \varphi_X. \tag{13.4}$$

Fixed a global section we define normalized cochains

$$c_{\mathbf{A}} \in \bar{F}^3(\pi_0\mathbf{A}, \pi_1\mathbf{A}),$$

$$b_\varphi \in \bar{F}^2(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0\varphi)),$$

$$o_{\phi,\varphi} \in \bar{F}^1(\pi_0\mathbf{A}, (\pi_1\mathbf{C})\mathcal{F}(\pi_0(\phi\varphi))),$$

$$e_\alpha \in \bar{F}^1(\pi_0\mathbf{A}, (\pi_1\mathbf{B})\mathcal{F}(\pi_0\alpha)),$$

for all abelian track categories $\mathbf{A}$, pseudofunctors $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and $\phi\colon \mathbf{B} \rightsquigarrow \mathbf{C}$, and pseudonatural transformations $\alpha\colon \varphi \Rightarrow \psi$. The cochain $c_{\mathbf{A}}$ is the cocycle defined in [BD89], it is given by the formula

$$\begin{aligned} c_{\mathbf{A}}(\{f\},\{g\},\{h\}) &= \sigma^{-1}_{t\{fgh\}}(\mu_{\{fg\},\{h\}}\square(\mu_{\{f\},\{g\}}t\{h\}) \\ &\quad \square(t\{f\}\,\mu^{\boxminus}_{\{g\},\{h\}})\square\mu^{\boxminus}_{\{f\},\{gh\}}). \end{aligned} \tag{13.5}$$

The formula for the cochain b_φ is

$$(13.6) \qquad \begin{aligned} b_\varphi(\{f\},\{g\}) &= \sigma^{-1}_{t\{\varphi(fg)\}}(\mu_{\{\varphi(f)\},\{\varphi(g)\}}\square(\nu_{\{f\},\varphi}\nu_{\{g\},\varphi}) \\ &\quad \square\varphi^{\boxminus}_{t\{f\},t\{g\}}\square\varphi(\mu^{\boxminus}_{\{f\},\{g\}})\square\nu^{\boxminus}_{\{fg\},\varphi}), \end{aligned}$$

for $o_{\phi,\varphi}$ we have

$$o_{\phi,\varphi}(\{f\}) = \sigma_{t\{\phi\varphi(f)\}}(\nu_{\{f\},\phi\varphi}\square\phi(\nu^{\boxminus}_{\{f\},\varphi})\square\nu^{\boxminus}_{\{\varphi(f)\},\phi}),$$

and finally if $\{f\} : Y \to X$ is a morphism in $\pi_0\mathbf{A}$

$$\begin{aligned} e_\alpha(\{f\}) &= \sigma^{-1}_{t\{\alpha_X\varphi(f)\}}(\mu_{\{\psi(f)\},\{\alpha_Y\}}\square(\nu_{\{f\},\psi}\eta_{Y,\alpha})\square\alpha_{t\{f\}} \\ &\quad \square(\eta^{\boxminus}_{X,\alpha}\nu^{\boxminus}_{\{f\},\varphi})\square\mu^{\boxminus}_{\{\alpha_X\},\{\varphi(f)\}}). \end{aligned}$$

The reader can check that these cochains are indeed normalized by using (13.1), . . . , (13.4), (A.2), (A.3) and (A.6).

Proposition 13.7. *The following equalities hold*

1. $\delta(b_\varphi) = (\pi_1\varphi)_* c_{\mathbf{A}} - (\pi_0\varphi)^* c_{\mathbf{B}}$,
2. $\delta(e_\alpha) = -((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi}, \alpha))_* b_\varphi - \alpha^{\#} c_{\mathbf{B}} + ((\pi_1\mathbf{B})\mathcal{F}(\alpha, 0^{\square}_{\pi_0\psi}))_* b_\psi$,
3. $\delta(o_{\phi,\varphi}) = b_{\phi\varphi} - ((\pi_1\phi)\mathcal{F}(\pi_0\varphi))_* b_\varphi - (\pi_0\varphi)^* b_\phi$.

Proof. The proof of this proposition is a tedious but straightforward computation which uses intensively the properties of linear track extensions of categories, pseudofunctors and pseudonatural transformations, see Section 3 and the Appendix. In fact the proof of (2) consists essentially of reversing the long computation carried out in the proof of Theorem 7.7, see Section 11. So that the reader can get an idea of how to carry out these calculations we include here a detailed proof of (3).

Given composable morphisms $\bullet \overset{\{f\}}{\to} \bullet \overset{\{g\}}{\to} \bullet$ in $\pi_0\mathbf{A}$

$$\begin{aligned} \delta(o)(\{f\},\{g\}) &= \{f\}_*\, o(\{g\}) - o(\{fg\}) + \{g\}^*\, o(\{f\}) \\ &= \{f\}_*\, \sigma_{t\{\phi\varphi(g)\}}(\nu_{\{g\},\phi\varphi}\square\phi(\nu^{\boxminus}_{\{g\},\varphi})\square\nu^{\boxminus}_{\{\varphi(g)\},\phi}) \\ &\quad -\sigma_{t\{\phi\varphi(fg)\}}(\nu_{\{fg\},\phi\varphi}\square\phi(\nu^{\boxminus}_{\{fg\},\varphi})\square\nu^{\boxminus}_{\{\varphi(fg)\},\phi}) \\ &\quad + \{g\}^*\, \sigma_{t\{\phi\varphi(f)\}}(\nu_{\{f\},\phi\varphi}\square\phi(\nu^{\boxminus}_{\{f\},\varphi})\square\nu^{\boxminus}_{\{\varphi(f)\},\phi}) \\ &\overset{\text{(a)}}{=} \sigma_{t\{\phi\varphi(fg)\}}(\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\square(t\,\{\phi\varphi(f)\}\,\nu_{\{g\},\phi\varphi}) \\ &\quad \square(t\,\{\phi\varphi(f)\}\,\phi(\nu^{\boxminus}_{\{g\},\varphi}))\square(t\,\{\phi\varphi(f)\}\,\nu^{\boxminus}_{\{\varphi(g)\},\phi}) \\ &\quad \square\mu^{\boxminus}_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\square\nu_{\{\varphi(fg)\},\phi}\square\phi(\nu_{\{fg\},\varphi})\square\nu^{\boxminus}_{\{fg\},\phi\varphi} \\ &\quad \square\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}(\nu_{\{f\},\phi\varphi}t\,\{\phi\varphi(g)\})\square(\phi(\nu^{\boxminus}_{\{f\},\varphi})t\,\{\phi\varphi(g)\}) \\ &\quad \square(\nu^{\boxminus}_{\{\varphi(f)\},\phi}t\,\{\phi\varphi(g)\})\square\mu^{\boxminus}_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}) \\ &\overset{\text{(b)}}{=} \sigma_{t\{\phi\varphi(fg)\}}(\nu_{\{\varphi(fg)\},\phi}\square\phi(\nu_{\{fg\},\varphi})\square\nu^{\boxminus}_{\{fg\},\phi\varphi} \\ &\quad \square\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\square(\nu_{\{f\},\phi\varphi}\nu_{\{g\},\phi\varphi}) \\ &\quad \square(\phi(\nu^{\boxminus}_{\{f\},\varphi})\phi(\nu^{\boxminus}_{\{g\},\varphi}))\square(\nu^{\boxminus}_{\{\varphi(f)\},\phi}\nu^{\boxminus}_{\{\varphi(g)\},\phi}) \\ &\quad \square\mu^{\boxminus}_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}) \end{aligned}$$

In (a) we use (3.3) and (3.4) and in (b) we apply (3.3) again.

On the other hand

$$\begin{aligned}
& b_{\phi\varphi}(\{f\},\{g\}) \\
& -((\pi_1\phi)\mathcal{F}(\pi_0\varphi))_*(b_\varphi)(\{f\},\{g\}) \\
& -(\pi_0\varphi)^*(b_\phi)(\{f\},\{g\}) \overset{\text{(c)}}{=} \sigma^{-1}_{t\{\phi\varphi(fg)\}}(\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\Box(\nu_{\{f\},\phi\varphi}\nu_{\{g\},\phi\varphi}) \\
& \qquad \Box(\phi\varphi)^{\boxminus}_{t\{f\},t\{g\}}\Box\phi\varphi(\mu^{\boxminus}_{\{f\},\{g\}})\Box\nu^{\boxminus}_{\{fg\},\phi\varphi}) \\
& \qquad -\sigma^{-1}_{\phi(t\{\varphi(fg)\})}(\phi(\mu_{\{\varphi(f)\},\{\varphi(g)\}})\Box\phi(\nu_{\{f\},\varphi}\nu_{\{g\},\varphi}) \\
& \qquad \Box(\phi\varphi)^{\boxminus}_{t\{f\},t\{g\}}\Box\phi\varphi(\mu^{\boxminus}_{\{f\},\{g\}})\Box\phi(\nu^{\boxminus}_{\{fg\},\varphi})) \\
& \qquad -\sigma^{-1}_{t\{\phi\varphi(fg)\}}(\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\Box(\nu_{\{\varphi(f)\},\phi}\nu_{\{\varphi(g)\},\phi}) \\
& \qquad \Box\phi^{\boxminus}_{t\{\varphi(f)\},t\{\varphi(g)\}}\Box\phi(\mu^{\boxminus}_{\{\varphi(f)\},\{\varphi(g)\}})\Box\nu^{\boxminus}_{\{\varphi(fg)\},\phi}) \\
& \overset{\text{(d)}}{=} \sigma^{-1}_{t\{\phi\varphi(fg)\}}(\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\Box(\nu_{\{f\},\phi\varphi}\nu_{\{g\},\phi\varphi}) \\
& \qquad \Box\phi^{\boxminus}_{\varphi(t\{f\}),\varphi(t\{g\})}\Box\phi(\varphi^{\boxminus}_{t\{f\},t\{g\}})\Box\phi\varphi(\mu^{\boxminus}_{\{f\},\{g\}}) \\
& \qquad \Box\nu^{\boxminus}_{\{fg\},\phi\varphi}\Box\nu_{\{\varphi(fg)\},\phi}\Box\phi(\nu_{\{fg\},\varphi})\Box\phi\varphi(\mu_{\{f\},\{g\}}) \\
& \qquad \Box\phi(\varphi_{t\{f\},t\{g\}})\Box\phi(\nu^{\boxminus}_{\{f\},\varphi}\nu^{\boxminus}_{\{g\},\varphi}) \\
& \qquad \Box\phi_{t\{\varphi(f)\},t\{\varphi(g)\}}\Box(\nu^{\boxminus}_{\{\varphi(f)\},\phi}\nu^{\boxminus}_{\{\varphi(g)\},\phi}) \\
& \qquad \Box\mu^{\boxminus}_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}) \\
& \overset{\text{(e)}}{=} \sigma^{-1}_{\phi\varphi(t\{fg\})}(\nu^{\boxminus}_{\{fg\},\phi\varphi}\Box\nu_{\{\varphi(fg)\},\phi}\Box\phi(\nu_{\{fg\},\varphi})) \\
& \qquad +\sigma^{-1}_{\phi(\varphi(t\{f\})\varphi(t\{g\}))}((\phi(\nu^{\boxminus}_{\{f\},\varphi})\phi(\nu^{\boxminus}_{\{g\},\varphi})) \\
& \qquad \Box(\nu^{\boxminus}_{\{\varphi(f)\},\phi}\nu^{\boxminus}_{\{\varphi(g)\},\phi})\Box(\nu_{\{f\},\phi\varphi}\nu_{\{g\},\phi\varphi})) \\
& \overset{\text{(f)}}{=} \sigma_{t\{\phi\varphi(fg)\}}(\nu_{\{\varphi(fg)\},\phi}\Box\phi(\nu_{\{fg\},\varphi})\Box\nu^{\boxminus}_{\{fg\},\phi\varphi} \\
& \qquad \Box\mu_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}}\Box(\nu_{\{f\},\phi\varphi}\nu_{\{g\},\phi\varphi}) \\
& \qquad \Box(\phi(\nu^{\boxminus}_{\{f\},\varphi})\phi(\nu^{\boxminus}_{\{g\},\varphi}))\Box(\nu^{\boxminus}_{\{\varphi(f)\},\phi}\nu^{\boxminus}_{\{\varphi(g)\},\phi}) \\
& \qquad \Box\mu^{\boxminus}_{\{\phi\varphi(f)\},\{\phi\varphi(g)\}})
\end{aligned}$$

In (c) we use (3.7) and (A.5); for (d) we apply (3.3); in (e) we use (A.4) as well as (3.3); and finally for (f) we use (3.3) again. These two chains of equalities establish (3). ☐

The following proposition follows from Proposition 13.7 (1), (4.2), (4.9) and the fact that $c_{\mathbf{A}}$ and $c_{\mathbf{B}}$ are cocycles.

Proposition 13.8. *If $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ are pseudofunctors between abelian track categories and $\alpha\colon \pi\varphi \Rightarrow \pi\psi$ is a 2-cell in* **nat** *then*

$$\delta(((\pi_1\mathbf{B})\mathcal{F}(0^{\Box}_{\pi_0\varphi},\alpha))_* b_\varphi - ((\pi_1\mathbf{B})\mathcal{F}(\alpha, 0^{\Box}_{\pi_0\psi}))_* b_\psi) = -\delta\alpha^{\#}(c_{\mathbf{B}}).$$

Let $(\bar{t},\bar{\mu},\bar{\nu},\bar{\eta})$ be another global section. A *transition* from $(\bar{t},\bar{\mu},\bar{\nu},\bar{\eta})$ to (t,μ,ν,η) is given by tracks

$$\rho_{\{f\}}\colon t\{f\} \Rightarrow \bar{t}\{f\}$$

indexed by the morphisms $\{f\}$ in $\pi_0\mathbf{A}$ for all abelian track categories $\mathbf{A}$. Such a transition determines cochains

$$v_{\mathbf{A}} \in \bar{F}^2(\pi_0\mathbf{A},\pi_1\mathbf{A}),$$

$$u_\varphi \in \bar{F}^2(\pi_0\mathbf{A},(\pi_1\mathbf{B})\mathcal{F}(\pi_0\varphi)),$$

where $\mathbf{A}$ is any abelian track category and $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is an arbitrary pseudofunctor between abelian track categories. These cochains are defined by the following formulas

$$v_{\mathbf{A}}(\{f\},\{g\}) = \sigma^{-1}_{\bar{t}\{fg\}}(\bar{\mu}_{\{f\},\{g\}}\square(\rho_{\{f\}}\rho_{\{g\}})\square\mu^{\boxminus}_{\{f\},\{g\}}\square\rho^{\boxminus}_{\{fg\}}),$$

$$u_{\varphi}(\{f\}) = \sigma^{-1}_{\bar{t}\{f\}}(\bar{\nu}_{\{f\},\varphi}\square\varphi(\rho_{\{f\}})\square\nu^{\boxminus}_{\{f\},\varphi}\square\rho^{\boxminus}_{\{\varphi(f)\}}).$$

Let us call $\bar{c}_{\mathbf{A}}$ and $\bar{b}_{\varphi}$ the cochains defined as above by the global section $(\bar{t},\bar{\mu},\bar{\nu},\bar{\eta})$.

Proposition 13.9. *The following equalities hold*

1. $\delta(v_{\mathbf{A}}) = c_{\mathbf{A}} - \bar{c}_{\mathbf{A}}$,
2. $\delta(u_{\varphi}) = -b_{\varphi} + \bar{b}_{\varphi} - (\pi_0\varphi)^* v_{\mathbf{B}} + (\pi_1\varphi)_* v_{\mathbf{A}}$.

The proof of this proposition is tedious but straightforward, as the proof of Proposition 13.7. We leave it to the reader.

Proposition 13.10. *Given pseudofunctors $\varphi,\psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ between abelian track categories and a 2-cell $\alpha\colon \pi\varphi \Rightarrow \pi\psi$ in* **nat** *the following equality holds*

$$\begin{aligned} \delta(((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_* u_{\varphi} - \alpha^{\#} v_{\mathbf{B}} & \\ -((\pi_1\mathbf{B})\mathcal{F}(\alpha,0^{\square}_{\pi_0\psi}))_* u_{\psi}) &= -((\pi_1\mathbf{B})\mathcal{F}(0^{\square}_{\pi_0\varphi},\alpha))_*(b_{\varphi} - \bar{b}_{\varphi}) \\ &\quad -\alpha^{\#}(c_{\mathbf{B}} - \bar{c}_{\mathbf{B}}) \\ &\quad +((\pi_1\mathbf{B})\mathcal{F}(\alpha,0^{\square}_{\pi_0\psi}))_*(b_{\psi} - \bar{b}_{\psi}). \end{aligned}$$

This proposition can be checked by using Proposition 13.9, (4.2) and (4.9).

A Pseudofunctors and pseudonatural transformations

A *pseudofunctor* $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ between track categories is an assignment of objects, maps and tracks together with additional tracks

$$\varphi_{f,g}\colon \varphi(f)\varphi(g) \Rightarrow \varphi(fg) \quad \text{and} \quad \varphi_X\colon \varphi(1_X) \Rightarrow 1_{\varphi(X)}$$

for all objects X and composable maps $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ in $\mathbf{A}$. These tracks must satisfy the following conditions: the following three composite tracks are identity tracks for all maps $f\colon X \to Y$ and composable maps $\bullet \xrightarrow{h} \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ in $\mathbf{A}$,

$$\begin{tikzcd}[column sep=large]
\bullet \arrow[r, "\varphi(h)"'] \arrow[rr, bend left=40, "\varphi(gh)"] \arrow[rrr, bend left=60, "\varphi(fgh)"] \arrow[rrr, bend right=60, "\varphi(fgh)"'] \arrow[rr, phantom, bend left=20, "\Downarrow\varphi^{\boxminus}_{g,h}"] \arrow[rrr, phantom, bend left=45, "\Downarrow\varphi^{\boxminus}_{f,gh}"] \arrow[rrr, phantom, bend right=45, "\varphi_{fg,h}\Downarrow"] & \bullet \arrow[r, "\varphi(g)"] \arrow[rr, bend right=40, "\varphi(fg)"'] \arrow[rr, phantom, bend right=20, "\Downarrow\varphi_{f,g}"] & \bullet \arrow[r, "\varphi(f)"] & \bullet
\end{tikzcd},$$

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\varphi(1_X)"'] \arrow[r, bend left=60, "1_{\varphi(X)}"] \arrow[r, phantom, shift left=3, "\varphi_X^{\boxminus}\Downarrow"] \arrow[rr, bend right=40, "\varphi(f)"'] \arrow[rr, phantom, shift right=4, "\Downarrow \varphi_{f,1_X}"] & \varphi(X) \arrow[r, "\varphi(f)"] & \varphi(Y)
\end{tikzcd} ,$$

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\varphi(f)"] \arrow[rr, bend right=40, "\varphi(f)"'] \arrow[rr, phantom, shift right=4, "\varphi_{1_Y,f}\Downarrow"] & \varphi(Y) \arrow[r, "\varphi(1_Y)"'] \arrow[r, bend left=60, "1_{\varphi(Y)}"] \arrow[r, phantom, shift left=3, "\varphi_Y^{\boxminus}\Downarrow"] & \varphi(Y)
\end{tikzcd} ,$$

For any horizontally composable tracks α, β in $\mathbf{A}$ the following composite track is $\varphi(\alpha\beta)$

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "\varphi(g)"] \arrow[r, bend right, "\varphi(g')"'] \arrow[r, phantom, "\Downarrow \varphi(\beta)"] \arrow[rr, bend left=70, "\varphi(fg)"] \arrow[rr, phantom, shift left=6, "\Downarrow \varphi^{\boxminus}_{f,g}"] \arrow[rr, bend right=70, "\varphi(f'g')"'] \arrow[rr, phantom, shift right=6, "\varphi_{f',g'}\Downarrow"] & \bullet \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(f')"'] \arrow[r, phantom, "\Downarrow \varphi(\alpha)"] & \bullet
\end{tikzcd} ,$$

and φ preserves vertical composition of tracks as well as identity tracks. These conditions can be summarized in the following six equations:

(A.1) $$\varphi_{fg,h}\Box(\varphi_{f,g}\varphi(h)) = \varphi_{f,gh}\Box(\varphi(f)\varphi_{g,h}),$$

(A.2) $$\varphi_{f,1_X} = \varphi(f)\varphi_X,$$

(A.3) $$\varphi_{1_Y,f} = \varphi_Y\varphi(f),$$

(A.4) $$\varphi_{f',g'}\Box(\varphi(\alpha)\varphi(\beta)) = \varphi(\alpha\beta)\Box\varphi_{f,g},$$

(A.5) $$\varphi(\gamma\Box\varepsilon) = \varphi(\gamma)\Box\varphi(\varepsilon),$$

(A.6) $$\varphi(0^{\Box}_f) = 0^{\Box}_{\varphi(f)}.$$

Notice that a *track functor* $\varphi\colon \mathbf{A} \to \mathbf{B}$ is exactly a pseudofunctor such that $\varphi_{f,g}$ and φ_X are always identity tracks.

Consider now two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$. A *pseudonatural transformation* $\alpha\colon \varphi \Rightarrow \psi$ is given by maps $\alpha_X\colon \varphi(X) \to \psi(X)$ for all objects X in $\mathbf{A}$ together with tracks

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\alpha_X"] \arrow[d, "\varphi(f)"'] & \psi(X) \arrow[d, "\psi(f)"] \\
\varphi(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] & \psi(Y)
\end{tikzcd}$$

for all maps $f\colon X \to Y$ in $\mathbf{A}$. These tracks satisfy the following properties: for all tracks $\tau\colon f \Rightarrow g$ in $\mathbf{A}$ the following composite track coincides with α_g

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\alpha_X"] \arrow[d, "\varphi(f)"] \arrow[d, bend right=70, "\varphi(g)"'] \arrow[d, phantom, shift right=6ex, "\overset{\varphi(\tau)^{\boxminus}}{\Rrightarrow}"] & \psi(X) \arrow[d, "\psi(f)"'] \arrow[d, bend left=70, "\psi(g)"] \arrow[d, phantom, shift left=6ex, "\underset{\psi(\tau)}{\Rrightarrow}"] \\
\varphi(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, shorten=4mm, "\alpha_f"'] & \psi(Y)
\end{tikzcd}$$

Given composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{A}$ the following composite track coincides with α_{fg}

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\alpha_X"] \arrow[d, "\varphi(f)"'] \arrow[dd, bend right=70, "\varphi(fg)"'] & \psi(X) \arrow[d, "\psi(f)"] \arrow[dd, bend left=70, "\psi(fg)"] \\
\arrow[l, phantom, "\underset{\varphi_{g,f}^{\boxminus}}{\Rrightarrow}"] \varphi(Y) \arrow[r, "\alpha_Y"] \arrow[d, "\varphi(g)"'] \arrow[ur, Rightarrow, shorten=4mm, "\alpha_f"'] & \psi(Y) \arrow[d, "\psi(g)"] \arrow[r, phantom, "\underset{\psi_{g,f}}{\Rrightarrow}"] & {} \\
\varphi(Z) \arrow[r, "\alpha_Z"] \arrow[ur, Rightarrow, shorten=4mm, "\alpha_g"'] & \psi(Z)
\end{tikzcd}$$

For any object X in $\mathbf{A}$ the following composite track is the identity track $0^{\square}_{\alpha_X}$.

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\alpha_X"] \arrow[d, "\varphi(1_X)"] \arrow[d, bend right=70, "1_{\varphi(X)}"'] \arrow[d, phantom, shift right=6ex, "\overset{\varphi_X^{\boxminus}}{\Rrightarrow}"] & \psi(X) \arrow[d, "\psi(1_X)"'] \arrow[d, bend left=70, "1_{\psi(X)}"] \arrow[d, phantom, shift left=6ex, "\underset{\psi_X}{\Rrightarrow}"] \\
\varphi(X) \arrow[r, "\alpha_X"'] \arrow[ur, Rightarrow, shorten=4mm, "\alpha_{1_X}"'] & \psi(X)
\end{tikzcd}$$

These three conditions can be restated as the following equations

$$(\psi(\tau)\alpha_X)\square\alpha_f = \alpha_g\square(\alpha_Y\varphi(\tau)), \tag{A.7}$$

$$(\psi_{g,f}\alpha_X)\square(\psi(g)\alpha_f)\square(\alpha_g\varphi(f)) = \alpha_{gf}\square(\alpha_Z\varphi_{g,f}), \tag{A.8}$$

$$\varphi_X = (\psi_X\alpha_X)\square\alpha_{1_X}. \tag{A.9}$$

If φ and ψ were track functors, α would be a *track natural transformation* provided α_f were always an identity track.

Recall that a *homotopy* $\xi\colon \varphi \Rightarrow \psi$ between pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ has been defined as a pseudonatural transformation with $\xi_X = 1_{\varphi(X)} = 1_{\psi(X)}$ for all objects X in $\mathbf{A}$. In order to regard homotopies as a special kind of pseudofunctors we define the *reduced cylinder track category* $\mathbb{D}$. It has a unique 0-cell $*$, a unique non-trivial morphism $\imath\colon * \to *$ with $\iota^2 = \iota$, and only one non-identity track $\ell\colon 1_* \Rightarrow \imath$ which also satisfies $\ell^2 = \ell$. The reader can check that this category can actually be drawn as a reduced cylinder, that is, a cylinder over the circle with a shrunk segment. There are exactly two pseudofunctors from the final track category $j^0, j^1\colon * \rightsquigarrow \mathbb{D}$. One of them, let us say j^0, is a track functor, and the other one is not. The unique pseudofunctor in the opposite direction is denoted by $q\colon \mathbb{D} \rightsquigarrow *$. Notice that all these pseudofunctors induce the identity $\pi\mathbb{D} = \pi*$ when applying the functor π, that is, they are weak equivalences.

Proposition A.10. *Any homotopy* $\xi\colon \varphi \Rightarrow \psi$ *induces a pseudofunctor* $\bar{\xi}\colon \mathbf{A}\times\mathbb{D} \rightsquigarrow \mathbf{B}$ *with* $\bar{\xi}j^0 = \varphi$ *and* $\bar{\xi}j^1 = \psi$. *Conversely a pseudofunctor* $\zeta\colon \mathbf{A}\times\mathbb{D} \rightsquigarrow \mathbf{B}$ *induces a homotopy* $\bar{\zeta}\colon \zeta j^0 \Rightarrow \zeta j^1$ *with* $\bar{\zeta}_f = \zeta(0^{\square}_f, \ell)$. *Moreover* $\bar{\bar{\xi}} = \xi$ *and* $\bar{\bar{\zeta}} = \zeta$.

The homotopy $\bar{\zeta}$ is completely defined in the statement, so all one has to do is to check that the axioms of a pseudonatural transformation hold. We leave this task to the reader. We also leave to the reader to check that there is a unique choice for $\bar{\xi}$ satisfying the properties of the statement.

The next proposition shows that pseudofunctors are flabby, in the sense that one can change de definition on maps within the same homotopy class in a quite general way.

Proposition A.11. *Let* $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *be a pseudofunctor and* $\xi_f\colon \varphi(f) \Rightarrow \psi^f$ *a collection of arbitrary tracks indexed by the maps of* $\mathbf{A}$. *There is a well-defined pseudofunctor* $\psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ *which coincides with* φ *on objects given on maps by* $\psi(f) = \psi^f$ *on tracks* $\alpha\colon f \Rightarrow g$ *by* $\psi(\alpha) = \xi_g\varphi(\alpha)\xi_f^{\boxminus}$, *and by the formulas* $\psi_{f,g} = \xi_{fg}\square\varphi_{f,g}\square(\xi_f^{\boxminus}\xi_g^{\boxminus})$, *and* $\psi_X = \varphi_X\xi^{\boxminus}_{1_X}$. *Moreover, the tracks* ξ_f *define a homotopy* $\xi\colon \varphi \Rightarrow \psi$.

A *reduced pseudofunctor* φ is a pseudofunctor that preserves identity maps $\varphi(1_X) = 1_{\varphi(X)}$ and $\varphi_X = 0^{\square}_{1_X}$.

Proposition A.12. *Any pseudofunctor is homotopic to a reduced one.*

Proof. It is enough to apply Proposition A.11 to the tracks $\xi_f\colon \varphi(f) \Rightarrow \psi^f$ such that $\psi^f = \varphi(f)$ and $\xi_f = 0^{\square}_{\varphi(f)}$ if f is not an identity map, $\psi^{1_X} = 1_{\varphi(X)}$ and $\xi_{1_X} = \varphi_X$. □

A track category $\mathbf{A}$ has a *strict zero object* $*$ if the morphism groupoids $[\![X,*]\!]_{\mathbf{A}}$ and $[\![*,X]\!]_{\mathbf{A}}$ are trivial for all objects X. The strict zero object is unique up to isomorphism in $\mathbf{A}$. In this case all morphism groupoids $[\![X,Y]\!]_{\mathbf{A}}$ have a distinguished map $0_{X,Y}\colon X \to Y$ which is the unique one that factors as $X \to * \to Y$.

Proposition A.13. *Let* $\mathbf{A}$ *be an abelian track category such that* $\pi_0\mathbf{A}$ *has a zero object* $*$ *and* $\pi_1\mathbf{A}$ *vanishes over all maps* $X \to *$ *and* $* \to X$, *then there is an abelian track category* $\mathbf{B}$ *with a strict zero object* $*$ *and a track functor* $\varphi\colon \mathbf{A} \to \mathbf{B}$ *with* $\pi_0\varphi = 1_{\pi_0\mathbf{A}}$ *and* $\pi_1\varphi = 0^{\square}_{\pi_1\mathbf{A}}$, *in particular* φ *is a weak equivalence and* $\mathbf{A}$ *is homotopy equivalent to* $\mathbf{B}$.

Proof. The objects of $\mathbf{B}$ are the same as in $\mathbf{A}$. In order to define maps in $\mathbf{B}$ we say that two maps $f, g\colon X \to Y$ in $\mathbf{A}$ are equivalent $f \sim g$ if $f = g$ or if there is a diagram

(a)
$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

in $\mathbf{A}$ where the composition of the upper arrows is f and the composition of the lower ones is g. Notice that the tracks in (a) are unique by the vanishing condition imposed to $\pi_1\mathbf{A}$. The reader can check that this is a natural equivalence relation on the ordinary category $\mathbf{A}_0$, therefore the quotient category $\mathbf{B}_0 = \mathbf{A}_0/\sim$ is defined. Let us write $[f]$ for the equivalence class of a map f in $\mathbf{A}$. Tracks $[f] \Rightarrow [g]$ in $\mathbf{B}$ are defined as tracks $f \Rightarrow g$ in $\mathbf{A}$ between two arbitrarily chosen representatives. This definition is consistent because, as we noticed before, diagram (a) is unique when it exists, hence vertical composition of tracks in $\mathbf{B}$ can be defined in a unique reasonable way. □

Suppose that $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is a pseudofunctor between track categories with strict zero object $*$ such that φ preserves the zero object $\varphi(*) = *$. The pseudofunctor φ is *normalized at zero maps* if it preserves zero maps $\varphi(0_{X,Y}) = 0_{\varphi(X),\varphi(Y)}$, and given maps $f\colon Y \to Z$ and $g\colon W \to X$ the equalities $\varphi_{f,0_{X,Y}} = 0^{\square}_{0_{X,Z}}$ and $\varphi_{0_{X,Y},g} = 0^{\square}_{0_{W,X}}$ hold.

Proposition A.14. *Any pseudofunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ preserving strict zero objects is homotopic to a pseudofunctor ψ normalized at zero maps.*

Proof. The pseudofunctor ψ is obtained by applying Proposition A.11 to the tracks $\xi_f\colon \varphi(f) \Rightarrow \psi^f$ such that $\psi^f = \varphi(f)$ and $\xi_f = 0^{\square}_{\varphi(f)}$ if f is not a zero map, $\psi^{0_{X,Y}} = 0_{\varphi(X),\varphi(Y)}$ and $\xi_{0_{X,Y}} = \varphi^{\boxminus}_{0_{*,Y},0_{X,*}}$. The reader can check that ψ is indeed normalized by using (A.4) and the fact that $*$ is a strict zero object. $\square$

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[IV]

Cohomologically triangulated categories

H.-J. Baues and F. Muro

Abstract

We define cohomologically triangulated categories as triples $(\mathbf{A}, t, \nabla)$ given by an additive category $\mathbf{A}$, an additive equivalence $t\colon \mathbf{A} \to \mathbf{A}$ and a cohomology class ∇ in the translation cohomology $H^3(\mathbf{A}, t)$. Here ∇ is triangulated if it is represented by a good translation track category $(\mathbf{B}, s)$ satisfying two natural axioms (TTr1) and (TTr2). It is proved that in this case $(\mathbf{A}, t)$ is a triangulated category in which exact triangles depend only on the class ∇.

Introduction

A stable homotopy theory (like the homotopy category of spectra or chain complexes) yields a groupoid-enriched category $\mathbf{B}$ in which 2-cells are tracks, i. e. homotopy classes of homotopies. The suspension determines a pseudofunctor $s\colon \mathbf{B} \rightsquigarrow \mathbf{B}$ such that by the classification result in [III] one obtains a characteristic cohomology class

$$\nabla = \langle \mathbf{B}, [s] \rangle \in H^3(\mathbf{A}, t).$$

Here $\mathbf{A}$ is the homotopy category of $\mathbf{B}$ and t is induced by s. Extending the main result in [I] we show that if the pair $(\mathbf{B}, s)$ satisfies two natural axioms (TTr1) and (TTr2) on the so-called track triangles then $(\mathbf{A}, t)$ is a triangulated category in which exact triangles are induced by the track triangles. Moreover, we show that the exact triangles are determined by the cohomology class ∇ and do not depend on the choice of $(\mathbf{B}, s)$ representing ∇. Therefore we call the triple $(\mathbf{A}, t, \nabla)$ a cohomologically triangulated category and we call ∇ a triangulated cohomology class. In [V] we characterize triangulated cohomology classes in a purely cohomological way.

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Notation

In this paper the arrow $\to$ is used for morphisms in ordinary categories, 1-cells in 2-categories and functors. We use $\Rightarrow$ for 2-cells in 2-categories and natural transformations, and $\rightsquigarrow$ for pseudofunctors. Identity morphisms or 1-cells are denoted by 1, with the object as a subscript when necessary. For the identity 2-cells and natural

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transformations we use the symbol $0^{\Box}$. Horizontal composition in 2-categories is denoted by juxtaposition, as the composition law in ordinary categories, while we use the symbol $\Box$ for vertical composition. The vertical inverse of an invertible 2-cell or natural transformation α is $\alpha^{\boxminus}$.

1 Triangulated categories

Let $\mathbf{A}$ be an additive category with zero object $*$ and let $t\colon \mathbf{A} \to \mathbf{A}$ be an additive equivalence, hence

$$t\colon \operatorname{Hom}_{\mathbf{A}}(X, Y) \cong \operatorname{Hom}_{\mathbf{A}}(tX, tY)$$

is an isomorphism of abelian groups and $t(X \oplus Y) = (tX) \oplus (tY)$. A *small candidate triangle* in $\mathbf{A}$ is a diagram in $\mathbf{A}$ of the form

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

such that $if = 0$, $qi = 0$ and $(tf)q = 0$. A morphism k between small candidate triangles is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] \arrow[d, "k_2"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"] & \bar{B} \arrow[r, "\bar{i}"] & \bar{C} \arrow[r, "\bar{q}"] & t\bar{A}
\end{tikzcd}$$

It is an isomorphism if the vertical arrows are isomorphisms in $\mathbf{A}$. The category of small candidate triangles will be denoted by $\mathbf{cand}(\mathbf{A}, t)$.

Definition 1.1. We say that $\mathbf{A}$ is a *pretriangulated category* if it is equipped with a distinguished family $\mathcal{E}$ of candidate triangles (called *exact* triangles) which is subject to the following axioms:

(Tr0) Any candidate triangle isomorphic to an exact triangle is exact. For any object A in $\mathbf{A}$ the following candidate triangle is exact

$$A \xrightarrow{1_A} A \longrightarrow * \longrightarrow tA.$$

(Tr1) Any morphism $f\colon A \to B$ in $\mathbf{A}$ is part of an exact triangle

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA.$$

(Tr2) A candidate triangle $A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$ is exact if and only if $B \xrightarrow{i} C \xrightarrow{q} tA \xrightarrow{-tf} tB$ is exact.

(Tr3) Any commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & t\bar{A}
\end{tikzcd}$$

whose rows are exact triangles can be extended to a morphism k between candidate triangles.

We say that $\mathbf{A}$ is a *triangulated category* if in addition it satisfies:

(Tr4) The *octahedral axiom*. Given two morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{A}$, if the rows of the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "i_f"] \arrow[d, "g"] & C_f \arrow[r, "q_f"] & tA \arrow[d, equal] \\
A \arrow[r, "gf"] \arrow[d, "f"] & C \arrow[r, "i_{gf}"] \arrow[d, equal] & C_{gf} \arrow[r, "q_{gf}"] & tA \arrow[d, "tf"] \\
B \arrow[r, "g"] & C \arrow[r, "i_g"] & C_g \arrow[r, "q_g"] & tB
\end{tikzcd}$$

are exact triangles then this diagram admits a commutative extension

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "i_f"] \arrow[d, "g"] & C_f \arrow[r, "q_f"] \arrow[d, "\bar{g}"] & tA \arrow[d, equal] \\
A \arrow[r, "gf"] \arrow[d, "f"] & C \arrow[r, "i_{gf}"] \arrow[d, equal] & C_{gf} \arrow[r, "q_{gf}"] \arrow[d, "\bar{f}"] & tA \arrow[d, "tf"] \\
B \arrow[r, "g"] & C \arrow[r, "i_g"] & C_g \arrow[r, "q_g"] & tB
\end{tikzcd}$$

such that

$$C_f \xrightarrow{\bar{g}} C_{gf} \xrightarrow{\bar{f}} C_g \xrightarrow{t(i_f)q_g} tC_f$$

is an exact triangle.

2 Triangulated track categories and cohomologically triangulated categories

A *track category* $\mathbf{A}$ is a category enriched in groupoids. Hence for objects X, Y in $\mathbf{A}$ one has the hom-groupoid $[\![X, Y]\!]_{\mathbf{A}}$ in $\mathbf{A}$. The objects $f\colon X \to Y$ of this groupoid are called *maps*, and the morphisms $\alpha\colon f \Rightarrow g$ are called *tracks*. Let $[\![X, Y]\!]_{\mathbf{A}}(f, g)$ be the set of all tracks $f \Rightarrow g$. The symbol for the identity track in f is $0^{\square}_{f}$. Equivalently, a track category is a 2-category all of whose 2-cells are invertible. Ordinary categories can be regarded as track categories with only the identity tracks.

A *pseudofunctor* between track categories $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is an assignment of objects, maps and tracks which preserves composition and identities only up to certain given tracks

$$\varphi_{f,g}\colon \varphi(f)\varphi(g) \Rightarrow \varphi(fg) \text{ and } \varphi_X\colon \varphi(1_X) \Rightarrow 1_{\varphi(X)}.$$

These tracks must satisfy well-known coherence and naturality properties, see for example the Appendix of [III] for details. A *pseudonatural transformation* $\alpha\colon \varphi \Rightarrow \psi$ between two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is a collection of maps $\alpha_X\colon \varphi(X) \to \psi(X)$ in $\mathbf{B}$ indexed by the objects of $\mathbf{A}$ such that the usual squares only commute up to given tracks $\alpha_f\colon \alpha_Y\varphi(f) \Rightarrow \psi(f)\alpha_X$ indexed by the maps $f\colon X \to Y$ in $\mathbf{A}$. These tracks must satisfy certain axioms, see also the Appendix of [III].

A *homotopy* $\xi\colon \varphi \Rightarrow \psi$ between two pseudofunctors $\varphi, \psi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is a pseudonatural transformation ξ such that the maps $\xi_X\colon \varphi(X) \to \psi(X)$ are identities, i.

e. $\varphi(X) = \psi(X)$ and $\xi_X = 1_{\varphi(X)} = 1_{\psi(X)}$. Two pseudofunctors φ and ψ are *homotopic* $\varphi \simeq \psi$ if there exists a homotopy between them. Since homotopies are invertible pseudonatural transformations the homotopy relation is a natural equivalence relation in the category of track categories and pseudofunctors. See [III] 3 for details.

In the rest of this section $(\mathbf{A}, t)$ will be a translation category with $\mathbf{A}$ and t additive and t an equivalence. We will consider the following $\mathbf{A}$-bimodule (i. e. functor $\mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}$ where $\mathbf{Ab}$ is the category of abelian groups)

$$\mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t, -) \tag{2.1}$$

and the natural transformation

$$\bar{t}\colon \mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t, -) \Rightarrow \mathrm{Hom}_{\mathbf{A}}(t^2, t) = \mathrm{Hom}^t(t, t) \tag{2.2}$$

given by $(-1)t$. Recall that in this case the translation cohomology of $(t, \bar{t})$ defined in [III] 5 is denoted by

$$H^*(\mathbf{A}, t) = H^*(t, \bar{t}).$$

Definition 2.3. A *translation track category* $(\mathbf{B}, [s])$ over $(\mathbf{A}, t)$ is a linear track extension in the sense of [BD89]

$$\mathrm{Hom}^t \xrightarrow{\sigma} \mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \xrightarrow{p} \mathbf{A} \tag{2.4}$$

consisting of a track category $\mathbf{B}$, a functor $p\colon \mathbf{B}_0 \to \mathbf{A}$ from the underlying ordinary category $\mathbf{B}_0$ of $\mathbf{B}$ such that p is the identity on objects and $p(f) = p(g)$ if and only if there exists a track $f \Rightarrow g$, and isomorphisms

$$\sigma_f\colon \mathrm{Hom}^t(X, Y) \cong \mathrm{Aut}_{[\![X,Y]\!]_{\mathbf{B}}}(f) = [\![X, Y]\!]_{\mathbf{B}}(f, f)$$

for all maps $f\colon X \to Y$ in $\mathbf{B}$. These isomorphisms satisfy the following properties, given a track $\alpha\colon f \Rightarrow g$ in $\mathbf{B}$,

$$\alpha \square \sigma_f(x) = \sigma_g(x) \square \alpha; \tag{2.5}$$

and given composable maps $\bullet \xrightarrow{h} \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ in $\mathbf{B}$

$$f\sigma_g(x) = \sigma_{fg}(p(f)_* x) \text{ and } \sigma_g(x)h = \sigma_{gh}(p(h)^* x). \tag{2.6}$$

Also as part of the structure of a translation track category there is a homotopy class of pseudofunctors $[s]\colon \mathbf{B} \rightsquigarrow \mathbf{B}$ such that for some representative s (and hence for anyone) one has that for all maps $f\colon X \to Y$ in $\mathbf{B}$ and $a \in \mathrm{Hom}^t(X, Y)$ the equalities $p(s(f)) = t(p(f))$ and $s(\sigma_f(a)) = \sigma_{s(f)}(\bar{t}(a))$ hold.

Two translation track categories $(\mathbf{B}, [s])$ and $(\mathbf{C}, [r])$ over $(\mathbf{A}, t)$ are equivalent if there exists a pseudofunctor $\varphi\colon \mathbf{B} \rightsquigarrow \mathbf{C}$ such that for all maps $f\colon X \to Y$ in $\mathbf{B}$ and $a \in \mathrm{Hom}^t(X, Y)$ the equalities $p(\varphi(f)) = p(f)$ and $\sigma_{\varphi(f)}(a) = \varphi(\sigma_f(a))$ are satisfied and φs is homotopic to $r\varphi$.

The next result is the main theorem in [III].

Theorem 2.7. *Equivalence classes of translation track categories over* $(\mathbf{A}, t)$ *are in bijective correspondence with the translation cohomology classes in* $H^3(\mathbf{A}, t)$*. This bijection is determined by the characteristic translation cohomology class of a translation track category* $(\mathbf{B}, [s])$ *over* $(\mathbf{A}, t)$

$$\langle \mathbf{B}, [s] \rangle \in H^3(\mathbf{A}, t)$$

defined in the proof of [III] 6.3.

Hence for each class $\nabla \in H^3(\mathbf{A}, t)$ we can choose an associated translation track category $(\mathbf{B}, [s])$ with

$$\langle \mathbf{B}, [s] \rangle = \nabla.$$

By [III] A.13 and A.14 we can always assume that the zero object $*$ of $\mathbf{A}$ is a strict zero object in $\mathbf{B}$, i. e. $[\![*, X]\!]_{\mathbf{B}}$ and $[\![X, *]\!]_{\mathbf{B}}$ are the trivial groupoids for any object X in $\mathbf{B}$, in particular the zero map $0\colon X \to Y$ is defined for any two objects X, Y in $\mathbf{B}$ as the composite $X \to * \to Y$. We can also assume that the pseudofunctor s representing the homotopy class $[s]$ (which preserves $*$, since s coincides on objects with t) is normalized at zero maps, i. e. $s(0) = 0$, $s_{0,f} = 0_0^{\square}$ and $s_{f,0} = 0_0^{\square}$ for any map f in $\mathbf{B}$. In this situation we say tat $(\mathbf{B}, s)$ is a *good translation track category*.

A *track triangle* in a good translation track category $(\mathbf{B}, s)$, compare [I], is a diagram in $\mathbf{B}$

(2.8)
$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow H_0"] & B \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow H_1"] & C \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow H_2"] & sA \arrow[r, "s(f)"'] & sB
\end{tikzcd}$$

such that the following equalities hold

$$\begin{aligned}
(qH_0)\square(H_1^{\boxminus}f) &= \sigma_0(1_{tA}),\\
(s(f)H_1)\square(H_2^{\boxminus}i) &= \sigma_0(-1_{tB}),\\
(s(i)H_2)\square((s(H_0)\square s_{i,f})^{\boxminus}q) &= \sigma_0(1_{tC}).
\end{aligned}$$

A morphism of track triangles is a diagram in $\mathbf{B}$

(2.9)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1" description] \arrow[dl, Rightarrow, shorten=3mm, "K_0"] & C \arrow[r, "q"] \arrow[d, "k_2" description] \arrow[dl, Rightarrow, shorten=3mm, "K_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, shorten=3mm, "K_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

such that the next three equalities are satisfied

$$\begin{aligned}
k_2H_0 &= (\bar{H}_0k_0)\square(\bar{i}K_0)\square(K_1f),\\
s(k_0)H_1 &= (\bar{H}_1k_1)\square(\bar{q}K_1)\square(K_2i),\\
s(k_1)H_2 &= (\bar{H}_2k_2)\square(s(\bar{f})K_2)\square((s_{\bar{f},k_0}^{\boxminus}\square s(K_0)\square s_{k_1,f})q).
\end{aligned}$$

Here $\bar{H}_i$ are the structure tracks of the lower track triangle in (2.9). Another morphism

(2.10)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bar{k}_0"'] & B \arrow[r, "i"] \arrow[d, "\bar{k}_1" description] \arrow[dl, Rightarrow, shorten=3mm, "\bar{K}_0"] & C \arrow[r, "q"] \arrow[d, "\bar{k}_2" description] \arrow[dl, Rightarrow, shorten=3mm, "\bar{K}_1"] & sA \arrow[d, "s\bar{k}_0"] \arrow[dl, Rightarrow, shorten=3mm, "\bar{K}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

is *homotopic* to the morphism in (2.9) if there exist tracks $L_i\colon k_i \Rightarrow \bar{k}_i$ $(i = 0, 1, 2)$ such that the following equations are satisfied

$$\begin{aligned}
(\bar{f}L_0)\Box K_0 &= \bar{K}_0(L_1 f),\\
(\bar{i}L_1)\Box K_1 &= \bar{K}_1(L_2 i),\\
(\bar{q}L_2)\Box K_2 &= \bar{K}_2(s(L_0)q).
\end{aligned}$$

Notice that if the tracks L_i are given first then the diagram (2.10) defined by the previous three equations automatically defines a morphism of track triangles which, in addition, is homotopic to (2.9).

The category of track triangles in $(\mathbf{B}, s)$ and homotopy classes of morphisms will be denoted by $\mathbf{candt}(\mathbf{B}, s)$. There is an obvious forgetful functor

$$\varrho = \varrho_{(\mathbf{B},s)}\colon \mathbf{candt}(\mathbf{B}, s) \longrightarrow \mathbf{cand}(\mathbf{A}, t) \tag{2.11}$$

sending a morphism in the source category represented by (2.9) to the following morphism of small candidate triangles in the sense of Section 1

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, "p(k_0)"'] & B \arrow[r, "p(i)"] \arrow[d, "p(k_1)"] & C \arrow[r, "p(q)"] \arrow[d, "p(k_2)"] & tA \arrow[d, "tp(k_0)"] \\
\bar{A} \arrow[r, "p(\bar{f})"'] & \bar{B} \arrow[r, "p(\bar{i})"'] & \bar{C} \arrow[r, "p(\bar{q})"'] & t\bar{A}
\end{tikzcd}$$

Proposition 2.12. *Let $(\mathbf{B}, s)$ and $(\mathbf{C}, r)$ be good translation track categories representing the same cohomology class in $H^3(\mathbf{A}, t)$. Then a small candidate triangle in $(\mathbf{A}, t)$ is in the image of $\varrho_{(\mathbf{B},s)}$ if and only if it is in the image of $\varrho_{(\mathbf{C},r)}$.*

Proof. Let $\varphi\colon \mathbf{B} \rightsquigarrow \mathbf{C}$ be a pseudofunctor realizing the equivalence between $(\mathbf{B}, s)$ and $(\mathbf{C}, r)$ as in Definition 2.3 and let $\xi\colon r\varphi \Rightarrow \varphi s$ be a homotopy. Recall that the pseudofunctors s and r coincide on objects (they both actually coincide on objects with t) and that φ is the identity on objects. In particular by [III] A.14 we can suppose without loss of generality that φ is normalized with respect to zero maps. The pseudofunctor φ and the homotopy ξ induce a functor

$$(\varphi, \xi)_*\colon \mathbf{candt}(\mathbf{B}, s) \longrightarrow \mathbf{candt}(\mathbf{C}, r)$$

sending a track triangle

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0", ""{name=U0, below}] & B \arrow[r, "i"] \arrow[rr, bend right=60, "0"', ""{name=D1, above}] & C \arrow[r, "q"] \arrow[rr, bend left=60, "0", ""{name=U2, below}] & sA \arrow[r, "s(f)"'] & sB
\arrow[from=1-2, to=U0, phantom, "\Uparrow H_0"]
\arrow[from=1-3, to=D1, phantom, "\Downarrow H_1"]
\arrow[from=1-4, to=U2, phantom, "\Uparrow H_2"]
\end{tikzcd}$$

to

$$\begin{tikzcd}[column sep=large]
A \arrow[r, "\varphi(f)"'] \arrow[rr, bend left=80, "0" {name=Z0}] \arrow[rr, bend left=40, "\varphi(if)" {description, name=IF}] & B \arrow[r, "\varphi(i)" description] \arrow[rr, bend right=40, "\varphi(qi)" {description, name=QI}] \arrow[rr, bend right=80, "0"' {name=Z1}] & C \arrow[r, "\varphi(q)" description] \arrow[rr, bend left=80, "0" {name=Z2}] \arrow[rr, bend left=40, "\varphi(s(f)q)" {description, name=SFQ}] & rA \arrow[r, "r\varphi(f)" {swap, name=RPF}] \arrow[r, bend left=50, "\varphi s(f)" {description, name=PSF}] & rB
\arrow[from=1-2, to=IF, Rightarrow, "\varphi_{i,f}"']
\arrow[from=IF, to=Z0, Rightarrow, "\varphi(H_0)"']
\arrow[from=1-3, to=QI, Rightarrow, "\varphi_{q,i}"]
\arrow[from=QI, to=Z1, Rightarrow, "\varphi(H_1)"]
\arrow[from=1-4, to=SFQ, Rightarrow, "\varphi_{s(f),q}"]
\arrow[from=SFQ, to=Z2, Rightarrow, "\varphi(H_2)"']
\arrow[from=RPF, to=PSF, Rightarrow, "\xi_f"]
\end{tikzcd}$$

The functor $(\varphi,\xi)_*$ sends the homotopy class of a track triangle morphism in $(\mathbf{B},s)$ given by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1" description] \arrow[dl, Rightarrow, shorten=3mm, "K_0"] & C \arrow[r, "q"] \arrow[d, "k_2" description] \arrow[dl, Rightarrow, shorten=3mm, "K_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, shorten=3mm, "K_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

to the homotopy class of track triangle morphisms in $(\mathbf{C},r)$ given by

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "\varphi(f)"] \arrow[d, "\varphi(k_0)" description] \arrow[dr, bend left=30, "\varphi(k_1f)" {description, name=K1F}] \arrow[dr, bend right=30, "\varphi(\bar{f}k_0)" {description, name=FK0}] & B \arrow[r, "\varphi(i)"] \arrow[d, "\varphi(k_1)" description] \arrow[dr, bend left=30, "\varphi(k_2i)" {description, name=K2I}] \arrow[dr, bend right=30, "\varphi(\bar{i}k_1)" {description, name=IK1}] & C \arrow[r, "\varphi(q)"] \arrow[d, "\varphi(k_2)" description] \arrow[dr, bend left=30, "\varphi(s(k_0)q)" {description, name=SKQ}] \arrow[dr, bend right=30, "\varphi(\bar{q}k_2)" {description, name=QK2}] & rA \arrow[d, "\varphi s(k_0)" {description, name=PSK}] \arrow[d, bend left=80, "r\varphi(k_0)" {description, name=RPK}] \\
\bar{A} \arrow[r, "\varphi(\bar{f})"'] & \bar{B} \arrow[r, "\varphi(\bar{i})"'] & \bar{C} \arrow[r, "\varphi(\bar{q})"'] & r\bar{A}
\arrow[from=1-2, to=K1F, Rightarrow, "\varphi_{k_1,f}"']
\arrow[from=K1F, to=FK0, Rightarrow, "\varphi(K_0)"']
\arrow[from=FK0, to=2-1, Rightarrow, "\varphi^{\boxminus}_{\bar{f},k_0}"]
\arrow[from=1-3, to=K2I, Rightarrow, "\varphi_{k_2,i}"']
\arrow[from=K2I, to=IK1, Rightarrow, "\varphi(K_1)"']
\arrow[from=IK1, to=2-2, Rightarrow, "\varphi^{\boxminus}_{\bar{i},k_1}"]
\arrow[from=1-4, to=SKQ, Rightarrow, "\varphi_{s(k_0),q}"']
\arrow[from=SKQ, to=QK2, Rightarrow, "\varphi(K_2)"']
\arrow[from=QK2, to=2-3, Rightarrow, "\varphi^{\boxminus}_{\bar{q},k_2}"]
\arrow[from=RPK, to=PSK, Rightarrow, "\xi_{k_0}"]
\end{tikzcd}$$

The following diagram of functors is commutative

$$\begin{tikzcd}
\mathbf{candt}(\mathbf{B},s) \arrow[dr, "\varrho_{(\mathbf{B},s)}"] \arrow[dd, "(\varphi,\xi)_*"'] & \\
& \mathbf{cand}(\mathbf{A},t) \\
\mathbf{candt}(\mathbf{C},r) \arrow[ur, "\varrho_{(\mathbf{C},r)}"'] &
\end{tikzcd}$$

therefore the small candidate triangles in $(\mathbf{A},t)$ in the image of $\varrho_{(\mathbf{B},s)}$ are also in the image of $\varrho_{(\mathbf{C},r)}$. The other inclusion follows by symmetry. $\square$

Definition 2.13. Given a cohomology class $\nabla \in H^3(\mathbf{A}, t)$ we say that a small candidate triangle

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

in $(\mathbf{A}, t)$ is a *small* ∇*-triangle* if it is in the image of $\varrho_{(\mathbf{B},s)}$ for some (and hence for any) good translation track category $(\mathbf{B}, s)$ representing ∇. This definition is consistent by Proposition 2.12.

The following definition generalizes the concept of a triangulated track category as introduced in [I] to the case of a "non-strict" translation functor.

Definition 2.14. A *pretriangulated track category* is a good translation track category $(\mathbf{B}, s)$ together with a distinguished family of track triangles which is subject to the following axiom:

(TTr1) For each map $f\colon A \to B$ in $\mathbf{B}$ there exists a distinguished track triangle

$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(f)"'] & sB \\
 & & \Downarrow H_1 & &
\end{tikzcd} .$$

It is a *triangulated track category* if in addition the following axiom is satisfied:

(TTr2) Given distinguished track triangles

$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(f)"'] & sB \\
 & & \Downarrow H_1 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \Uparrow \bar{H}_0 & & \Uparrow \bar{H}_2 & \\
\bar{A} \arrow[r, "\bar{f}"'] \arrow[rr, bend left=60, "0"] & \bar{B} \arrow[r, "\bar{i}" description] \arrow[rr, bend right=60, "0"'] & \bar{C} \arrow[r, "\bar{q}" description] \arrow[rr, bend left=60, "0"] & s\bar{A} \arrow[r, "s(\bar{f})"'] & s\bar{B} \\
 & & \Downarrow \bar{H}_1 & &
\end{tikzcd} ,$$

any diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[d, "k_1"] \arrow[dl, Rightarrow, shorten=10pt, "K_0"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B}
\end{tikzcd}$$

in **B** extends to a track triangle morphism

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"] \arrow[dl, Rightarrow, shorten=4mm, "K_0"] & C \arrow[r, "q"] \arrow[d, "k_2"] \arrow[dl, Rightarrow, shorten=4mm, "K_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, shorten=4mm, "K_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

Theorem 2.15. *Let $(\mathbf{B}, s)$ and $(\mathbf{C}, r)$ be good translation track categories representing the same cohomology class in $H^3(\mathbf{A}, t)$. Then $(\mathbf{B}, s)$ admits the structure of a (pre)triangulated track category if and only if $(\mathbf{C}, r)$ does.*

The proof of this theorem will be given in Section 4.

Definition 2.16. Let $\mathbf{A}$ be an additive category, $t\colon \mathbf{A} \to \mathbf{A}$ an additive equivalence and $\nabla \in H^3(\mathbf{A}, t)$ a translation cohomology class. We say that $(\mathbf{A}, t, \nabla)$ is a *cohomologically (pre)triangulated category* if for one (and hence for any) good translation track category $(\mathbf{B}, s)$ representing ∇ the pair $(\mathbf{B}, s)$ admits a (pre)triangulated track category structure. In this case ∇ is called a *(pre)triangulated cohomology class.* This definition is consistent by Theorem 2.15.

Generalizing the main result in [I] we show:

Theorem 2.17. *If $(\mathbf{A}, t, \nabla)$ is a cohomologically (pre)triangulated category and $\mathcal{E}_\nabla$ is the class of small ∇-triangles then $(\mathbf{A}, t, \mathcal{E}_\nabla)$ is a (pre)triangulated category.*

Proof. Axiom (Tr1) follows from (TTr1) and (Tr3) is a consequence of Proposition 4.18. Moreover, (Tr0) and (Tr2) will be proved in [V] 5.8 by using a purely cohomological characterization of small ∇-triangles. We prove now that the octahedral axiom holds provided $(\mathbf{A}, t, \nabla)$ is cohomologically triangulated. Let $(\mathbf{B}, s)$ be a good translation track category representing ∇. The pair $(\mathbf{B}, s)$ admits a triangulated track category structure by Definition 2.16. Let us fix such a structure.

Consider morphisms $A \overset{p(f)}{\to} B \overset{p(g)}{\to} C$ in $\mathbf{A}$. Indeed one can always suppose that this diagram comes from a diagram $A \overset{f}{\to} B \overset{g}{\to} C$ in $\mathbf{B}$. By (TTr1) and (TTr2) there is a diagram of track triangle morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "i_f"] \arrow[d, "g"] & C_f \arrow[r, "q_f"] \arrow[d, "\bar{g}"] \arrow[dl, Rightarrow, shorten=4mm, "F"] & sA \arrow[d, equal] \arrow[dl, Rightarrow, shorten=4mm, "G"] \\
A \arrow[r, "gf"] \arrow[d, "f"'] & C \arrow[r, "i_{gf}"] \arrow[d, equal] & C_{gf} \arrow[r, "q_{gf}"] \arrow[d, "\bar{f}"] \arrow[dl, Rightarrow, shorten=4mm, "F'"] & sA \arrow[d, "sf"] \arrow[dl, Rightarrow, shorten=4mm, "G'"] \\
B \arrow[r, "g"'] & C \arrow[r, "i_g"'] & C_g \arrow[r, "q_g"'] & sB
\end{tikzcd}$$

where the rows are distinguished track triangles, and now by Proposition 5.1

$$C_f \xrightarrow{p(\bar{g})} C_{gf} \xrightarrow{p(\bar{f})} C_g \xrightarrow{(tp(i_f))p(q_g)} tC_f$$

is a small ∇-triangle. This means that axiom (Tr4) holds for the diagram

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, equal] & B \arrow[r, "p(i_f)"] \arrow[d, "g"] & C_f \arrow[r, "p(q_f)"] & tA \arrow[d, equal] \\
A \arrow[r, "p(gf)"] \arrow[d, "p(f)"] & C \arrow[r, "p(i_{gf})"] \arrow[d, equal] & C_{gf} \arrow[r, "p(q_{gf})"] & tA \arrow[d, "tp(f)"] \\
B \arrow[r, "p(g)"] & C \arrow[r, "p(i_g)"] & C_g \arrow[r, "p(q_g)"] & tB
\end{tikzcd}$$

but it also must hold for any other diagram

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & \bar{C}_f \arrow[r] & tA \arrow[d, equal] \\
A \arrow[r, "p(gf)"] \arrow[d, "p(f)"] & C \arrow[r] \arrow[d, equal] & \bar{C}_{gf} \arrow[r] & tA \arrow[d, "tp(f)"] \\
B \arrow[r, "p(g)"] & C \arrow[r] & \bar{C}_g \arrow[r] & tB
\end{tikzcd}$$

whose rows are small ∇-triangles, however this is a consequence of the fact that the extension of a morphism $h\colon X \to Y$ in $\mathbf{A}$ to a small ∇-triangle given by (Tr1) is unique up to an isomorphism of small ∇-triangles given by the identity on X and Y since the functor ς in (4.15) restricted to small ∇-triangles fits into a linear extension of categories defined in Proposition 4.22, and therefore it reflects isomorphisms, see [Bau89] IV.4.11. ☐

Notice that by Proposition 2.12 the (pre)triangulated structure determined on $\mathbf{A}$ by this theorem only depends on the translation cohomology class $\nabla \in H^3(\mathbf{A}, t)$.

3 Obstruction theory for track triangles

Track triangles, introduced in Section 2, are the basic elements for the definition of a (pre)triangulated track category. In this section develop the obstruction theory for the realization of small candidate triangle morphisms by track triangle morphisms through the functor ϱ in (2.11). This obstruction theory will play an important role in the purely cohomological characterization of cohomologically triangulated categories carried out in [V].

In the statement of the following proposition we use the $\mathbf{cand}(\mathbf{A}, t)$-bimodules

$$H^*(sh^3, [-,-]^*\bar{t})$$

defined by the translation cohomology groups of the triangle category, see [V] 4. Actually we only need them in dimensions 1 and 2. The cochain complex $\tilde{G}^*([T, \bar{T}]^*\bar{t})$ computing the cohomology groups $H^*(sh^3, [T, \bar{T}]^*\bar{t})$ for two small candidate triangles

$$T = \left\{A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA\right\},$$

$$\bar{T} = \left\{\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{i}} \bar{C} \xrightarrow{\bar{q}} t\bar{A}\right\},$$

is given in low dimensions by

$$\begin{tikzcd}
\tilde{G}^0([T,\bar{T}]^*\bar{t}) \arrow[d, "\delta"] & = & \mathrm{Hom}_{\mathbf{A}}(tA,\bar{A}) \oplus \mathrm{Hom}_{\mathbf{A}}(tB,\bar{B}) \oplus \mathrm{Hom}_{\mathbf{A}}(tC,\bar{C}) \arrow[d, "\begin{pmatrix} \bar{f}_* & -f^* & 0 \\ 0 & \bar{i}_* & -i^* \\ -q^*\bar{t}_{(A,\bar{A})} & 0 & \bar{q}_* \end{pmatrix}"] \\
\tilde{G}^1([T,\bar{T}]^*\bar{t}) \arrow[d, "\delta"] & = & \mathrm{Hom}_{\mathbf{A}}(tA,\bar{B}) \oplus \mathrm{Hom}_{\mathbf{A}}(tB,\bar{C}) \oplus \mathrm{Hom}_{\mathbf{A}}(tC,t\bar{A}) \arrow[d, "\begin{pmatrix} \bar{i}_* & f^* & 0 \\ 0 & \bar{q}_* & i^* \\ q^*\bar{t}_{(A,\bar{B})} & 0 & (t\bar{f})_* \end{pmatrix}"] \\
\tilde{G}^2([T,\bar{T}]^*\bar{t}) \arrow[d, "\delta"] & = & \mathrm{Hom}_{\mathbf{A}}(tA,\bar{C}) \oplus \mathrm{Hom}_{\mathbf{A}}(tB,t\bar{A}) \oplus \mathrm{Hom}_{\mathbf{A}}(tC,t\bar{B}) \arrow[d, "\begin{pmatrix} \bar{q}_* & -f^* & 0 \\ 0 & (t\bar{f})_* & -i^* \\ -q^*\bar{t}_{(A,\bar{C})} & 0 & (t\bar{i})_* \end{pmatrix}"] \\
\tilde{G}^3([T,\bar{T}]^*\bar{t}) & = & \mathrm{Hom}_{\mathbf{A}}(tA,t\bar{A}) \oplus \mathrm{Hom}_{\mathbf{A}}(tB,t\bar{B}) \oplus \mathrm{Hom}_{\mathbf{A}}(tC,t\bar{C})
\end{tikzcd}$$

See [V] 4.14. Here $(-)^*$ and $(-)_*$ denote morphisms induced by the $\mathbf{A}$-bimodule $\mathrm{Hom}_{\mathbf{A}}(t,-)$. The functoriality on T and $\bar{T}$ is clear.

Proposition 3.1. *The functor (2.11) fits into an exact sequence*

$$H^1(sh^3,[-,-]^*\bar{t}) \xrightarrow{+} \mathbf{candt}(\mathbf{B},s) \xrightarrow{\varrho} \mathbf{cand}(\mathbf{A},t) \xrightarrow{\vartheta} H^2(sh^3,[-,-]^*\bar{t})$$

in the sense of [Bau89] IV.4.10. Moreover, the action + is effective.

An exact sequence for a functor encodes the usual properties of obstruction theory in homotopy theory. The abstract definition will not be recalled here, we refer the reader to [Bau89] IV.4.10. However the verification of the axioms for the case of Proposition 3.1 constitutes the statement of the following five lemmas, which prove Proposition 3.1.

In the proof of the following results we will use a fixed global section (ℓ,μ,ν,η) in the sense of [III] 14. We are actually interested in ℓ, μ and part of ν only. This is given by a choice of a map $\ell f\colon A \to B$ in $\mathbf{B}$ for any $f\colon A \to B$ in $\mathbf{A}$ with $p(\ell f) = f$ and $\ell(1_X) = 1_X$, tracks $\mu_{f,g}\colon (\ell f)(\ell g) \Rightarrow \ell(fg)$ with $\mu_{f,1} = 0^{\square}_{\ell f} = \mu_{1,f}$, and tracks $\nu_{f,s}\colon s(\ell f) \Rightarrow \ell(tf)$ with $\nu_{1_X,s} = s_X$. Since $\mathbf{B}$ has a strict zero object $*$ which is preserved by s and s is normalized at zero maps we can also impose here $\ell(0) = 0$, $\mu_{f,0} = 0^{\square}_0 = \mu_{0,f}$ and $\nu_{0,s} = 0^{\square}_0$. We will also need a collection γ of tracks $\gamma_f\colon \ell p(f) \Rightarrow f$ indexed by all maps f in $\mathbf{B}$ such that $\gamma_{\ell g} = 0^{\square}_{\ell g}$ for any morphism g in $\mathbf{A}$.

Lemma 3.2. *Let T and $\bar{T}$ be track triangles given by*

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_0"] & B \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow H_1"] & C \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_2"] & sA \arrow[r, "s(f)"'] & sB
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{f}"'] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow \bar{H}_0"] & \bar{B} \arrow[r, "\bar{i}" description] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow \bar{H}_1"] & \bar{C} \arrow[r, "\bar{q}" description] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow \bar{H}_2"] & s\bar{A} \arrow[r, "s(\bar{f})"'] & s\bar{B}
\end{tikzcd}$$

For any morphism $k\colon \varrho T \to \varrho\bar{T}$ of small candidate triangles as in the following diagram

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, "k_0"] & B \arrow[r, "p(i)"] \arrow[d, "k_1"] & C \arrow[r, "p(q)"] \arrow[d, "k_2"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "p(\bar{f})"] & \bar{B} \arrow[r, "p(\bar{i})"] & \bar{C} \arrow[r, "p(\bar{q})"] & t\bar{A}
\end{tikzcd}$$

there is a well-defined cohomology class

$$\vartheta_{T,\bar{T}}(k) \in H^2(sh^3, [\varrho T, \varrho\bar{T}]^*\bar{t})$$

which vanishes if and only if k is the image by ϱ of a track triangle morphism $T \to \bar{T}$.

Proof. The element $\vartheta_{T,\bar{T}}(k)$ measures the failure in

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "f"] \arrow[r, bend right=40, "\ell p(f)" description] \arrow[r, phantom, bend right=20, "\Downarrow\gamma^{\boxminus}_f"] \arrow[d, "\ell k_0"'] \arrow[dr, "\ell(k_1 p(f))" description] \arrow[dr, phantom, "\mu_{k_1,p(f)}" near start, shift left=4] \arrow[dr, phantom, "\mu^{\boxminus}_{p(\bar{f}),k_0}" near end, shift right=4] & B \arrow[r, "i"] \arrow[r, bend right=40, "\ell p(i)" description] \arrow[r, phantom, bend right=20, "\Downarrow\gamma^{\boxminus}_i"] \arrow[d, "\ell k_1"'] \arrow[dr, "\ell(k_2 p(i))" description] \arrow[dr, phantom, "\mu_{k_2,p(i)}" near start, shift left=4] \arrow[dr, phantom, "\mu^{\boxminus}_{p(\bar{i}),k_1}" near end, shift right=4] & C \arrow[r, "q"] \arrow[r, bend right=40, "\ell q" description] \arrow[r, phantom, bend right=20, "\Downarrow\gamma^{\boxminus}_q"] \arrow[d, "\ell k_2"'] \arrow[dr, "\ell(p(\bar{q})k_2)" description] \arrow[dr, phantom, "\mu_{tk_0,p(q)}" near start, shift left=4] \arrow[dr, phantom, "\mu^{\boxminus}_{p(\bar{q}),k_2}" near end, shift right=4] & sA \arrow[d, "s\ell k_0"] \arrow[d, bend right=40, "\ell tk_0"'] \arrow[d, phantom, bend right=20, "\overset{\nu_{k_0,s}}{\Leftarrow}"] \\
\bar{A} \arrow[r, "\bar{f}"'] \arrow[r, bend left=40, "\ell p(\bar{f})" description] \arrow[r, phantom, bend left=20, "\Downarrow\gamma_{\bar{f}}"] & \bar{B} \arrow[r, "\bar{i}"'] \arrow[r, bend left=40, "\ell p(\bar{i})" description] \arrow[r, phantom, bend left=20, "\Downarrow\gamma_{\bar{i}}"] & \bar{C} \arrow[r, "\bar{q}"'] \arrow[r, bend left=40, "\ell p(\bar{q})" description] \arrow[r, phantom, bend left=20, "\Downarrow\gamma_{\bar{q}}"] & s\bar{A}
\end{tikzcd}$$

to define a morphism of track triangles, i. e. it is represented by the cocycle $(b_{-2}, b_{-1}, b_0) \in \tilde{G}^2([\varrho T, \varrho\bar{T}]^*\bar{t})$ with

$$\begin{aligned}
b_{-2} &= \sigma_0^{-1}((\bar{H}_0(\ell k_0))\square(\bar{i}M_0)\square(M_1 f)\square((\ell k_2)H_0^{\boxminus})),\\
b_{-1} &= \sigma_0^{-1}((\bar{H}_1(\ell k_1))\square(\bar{q}M_1)\square(M_2 i)\square(s(\ell k_0)H_1^{\boxminus})),\\
b_0 &= \sigma_0^{-1}((\bar{H}_2(\ell k_2))\square(s(\bar{f})M_2)\square((s^{\boxminus}_{\bar{f},\ell k_0}\square s(M_0)\square s_{\ell k_1,f})q)\square(s(\ell k_1)H_2^{\boxminus})),
\end{aligned}$$

where

$$\begin{aligned}
M_0 &= (\gamma_{\bar{f}}(\ell k_0))\square\mu^{\boxminus}_{p(\bar{f}),k_0}\square\mu_{k_1,p(f)}\square((\ell k_1)\gamma_f^{\boxminus}),\\
M_1 &= (\gamma_{\bar{i}}(\ell k_1))\square\mu^{\boxminus}_{p(\bar{i}),k_1}\square\mu_{k_2,p(i)}\square((\ell k_2)\gamma_i^{\boxminus}),\\
M_2 &= (\gamma_{\bar{q}}(\ell k_2))\square\mu^{\boxminus}_{p(\bar{q}),k_2}\square\mu_{tk_0,p(q)}\square(\nu_{k_0,s}\gamma_q^{\boxminus}).
\end{aligned}$$

If the cocycle (b_{-2}, b_{-1}, b_0) is the coboundary of $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^1([\varrho T, \varrho\bar{T}]^*\bar{t})$ then k is the image under ϱ of the homotopy class of morphisms of track triangles $T \to \bar{T}$ represented by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\ell k_0"'] \arrow[dr, phantom, "\swarrow K_0"] & B \arrow[r, "i"] \arrow[d, "\ell k_1"'] \arrow[dr, phantom, "\swarrow K_1"] & C \arrow[r, "q"] \arrow[d, "\ell k_2"'] \arrow[dr, phantom, "\swarrow K_2"] & sA \arrow[d, "s\ell k_0"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

with

$$\begin{aligned} K_0 &= \sigma_{\bar{f}(\ell k_0)}(-c_{-2})\Box M_0, \\ K_1 &= \sigma_{\bar{i}(\ell k_1)}(-c_{-1})\Box M_1, \\ K_2 &= \sigma_{\bar{q}(\ell k_2)}(-c_0)\Box M_2. \end{aligned}$$

On the other hand if k is the image by ϱ of the track triangle morphism

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bar{k}_0"'] & B \arrow[r, "i"] \arrow[d, "\bar{k}_1"] \arrow[dl, Rightarrow, shorten=4mm, "\bar{K}_0"] & C \arrow[r, "q"] \arrow[d, "\bar{k}_2"] \arrow[dl, Rightarrow, shorten=4mm, "\bar{K}_1"] & sA \arrow[d, "s\bar{k}_0"] \arrow[dl, Rightarrow, shorten=4mm, "\bar{K}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

then the cochain $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^1([\varrho T, \varrho\bar{T}]^*\bar{t})$ defined by

$$\begin{aligned} -c_{-2} &= \sigma^{-1}_{\bar{f}\bar{k}_0}(\bar{K}_0\Box(\gamma_{\bar{k}_1}f)\Box M_0^{\boxminus}\Box(\bar{f}\gamma^{\boxminus}_{\bar{k}_0})), \\ -c_{-1} &= \sigma^{-1}_{\bar{i}\bar{k}_1}(\bar{K}_1\Box(\gamma_{\bar{k}_2}i)\Box M_1^{\boxminus}\Box(\bar{i}\gamma^{\boxminus}_{\bar{k}_1})), \\ -c_0 &= \sigma^{-1}_{\bar{q}\bar{k}_2}(\bar{K}_2\Box(s(\gamma_{\bar{k}_0})q)\Box M_2^{\boxminus}\Box(\bar{q}\gamma^{\boxminus}_{\bar{k}_2})), \end{aligned}$$

satisfies $\delta(c_{-2}, c_{-1}, c_0) = (b_{-2}, b_{-1}, b_0)$ so $\vartheta_{T,\bar{T}}(k) = 0$.

The reader can check that the cohomology class $\vartheta_{T,\bar{T}}(k)$ does not depend on the choices made for its definition. □

Lemma 3.3. *In the conditions of the statement of Lemma 3.2 if another track triangle*

$$\begin{tikzcd}[column sep=large]
\bar{\bar{A}} \arrow[r, "\bar{\bar{f}}"'] \arrow[rr, bend left=70, "0", ""{name=U0, below}] & \bar{\bar{B}} \arrow[r, "\bar{\bar{i}}"] \arrow[rr, bend right=70, "0"', ""{name=D1, above}] \arrow[to=U0, Rightarrow, "\bar{\bar{H}}_0"] & \bar{\bar{C}} \arrow[r, "\bar{\bar{q}}"] \arrow[rr, bend left=70, "0", ""{name=U2, below}] \arrow[to=D1, Rightarrow, "\bar{\bar{H}}_1"] & s\bar{\bar{A}} \arrow[r, "s(\bar{\bar{f}})"'] \arrow[to=U2, Rightarrow, "\bar{\bar{H}}_2"] & s\bar{\bar{B}}
\end{tikzcd}$$

and another morphism $\bar{k}\colon \varrho\bar{T} \to \varrho\bar{\bar{T}}$ of small candidate triangles

$$\begin{tikzcd}
\bar{A} \arrow[r, "p(\bar{f})"] \arrow[d, "\bar{k}_0"] & \bar{B} \arrow[r, "p(\bar{i})"] \arrow[d, "\bar{k}_1"] & \bar{C} \arrow[r, "p(\bar{q})"] \arrow[d, "\bar{k}_2"] & t\bar{A} \arrow[d, "t\bar{k}_0"] \\
\bar{\bar{A}} \arrow[r, "p(\bar{\bar{f}})"] & \bar{\bar{B}} \arrow[r, "p(\bar{\bar{i}})"] & \bar{\bar{C}} \arrow[r, "p(\bar{\bar{q}})"] & t\bar{\bar{A}}
\end{tikzcd}$$

are given, then the following equality holds

$$\vartheta_{T,\bar{\bar{T}}}(\bar{k}k) = \bar{k}_*\vartheta_{T,\bar{T}}(k) + k^*\vartheta_{\bar{T},\bar{\bar{T}}}(\bar{k}).$$

Proof. Following the definition of ϑ in the proof of Lemma 3.2 the reader can check that

(a) $$\bar{k}_*\vartheta_{T,\bar{T}}(k) + k^*\vartheta_{\bar{T},\bar{\bar{T}}}(\bar{k})$$

measures the failure in

$$\begin{tikzcd}[column sep=10em, row sep=10em]
A \arrow[r, "f"] \arrow[r, bend right, "\ell p(f)" description] \arrow[r, phantom, "\Downarrow\gamma_f^{\boxminus}"'] \arrow[d, "\ell k_0"'] \arrow[dr, "\ell(k_1p(f))" description] \arrow[dr, phantom, "\mu_{k_1,p(f)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar f),k_0}", near end, shift right=4ex] & B \arrow[r, "i"] \arrow[r, bend right, "\ell p(i)" description] \arrow[r, phantom, "\Downarrow\gamma_i^{\boxminus}"'] \arrow[d, "\ell k_1"'] \arrow[dr, "\ell(k_2p(i))" description] \arrow[dr, phantom, "\mu_{k_2,p(i)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar i),k_1}", near end, shift right=4ex] & C \arrow[r, "q"] \arrow[r, bend right, "\ell q" description] \arrow[r, phantom, "\Downarrow\gamma_q^{\boxminus}"'] \arrow[d, "\ell k_2"'] \arrow[dr, "\ell(p(\bar q)k_2)" description] \arrow[dr, phantom, "\mu_{tk_0,p(q)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar q),k_2}", near end, shift right=4ex] & sA \arrow[d, "s\ell k_0"] \arrow[d, bend right, "\ell tk_0"'] \arrow[d, phantom, "\overset{\nu_{k_0,s}}{\Leftarrow}"] \\
\bar A \arrow[r, "\ell p(\bar f)" description] \arrow[d, "\ell\bar k_0"'] \arrow[dr, "\ell(\bar k_1p(\bar f))" description] \arrow[dr, phantom, "\mu_{\bar k_1,p(\bar f)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar{\bar f}),\bar k_0}", near end, shift right=4ex] & \bar B \arrow[r, "\ell p(\bar i)" description] \arrow[d, "\ell\bar k_1"'] \arrow[dr, "\ell(\bar k_2p(\bar i))" description] \arrow[dr, phantom, "\mu_{\bar k_2,p(\bar i)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar{\bar i}),\bar k_1}", near end, shift right=4ex] & \bar C \arrow[r, "\ell p(\bar q)" description] \arrow[d, "\ell\bar k_2"'] \arrow[dr, "\ell(\bar{\bar q}\bar k_2)" description] \arrow[dr, phantom, "\mu_{t\bar k_0,p(\bar q)}\swarrow", near start, shift left=4ex] \arrow[dr, phantom, "\swarrow\mu^{\boxminus}_{p(\bar{\bar q}),\bar k_2}", near end, shift right=4ex] & s\bar A \arrow[d, "s\ell\bar k_0"] \arrow[d, bend right, "\ell t\bar k_0"'] \arrow[d, phantom, "\overset{\nu_{\bar k_0,s}}{\Leftarrow}"] \\
\bar{\bar A} \arrow[r, "\bar{\bar f}"'] \arrow[r, bend left, "\ell p(\bar{\bar f})" description] \arrow[r, phantom, "\Downarrow\gamma_{\bar{\bar f}}"] & \bar{\bar B} \arrow[r, "\bar{\bar i}"'] \arrow[r, bend left, "\ell p(\bar{\bar i})" description] \arrow[r, phantom, "\Downarrow\gamma_{\bar{\bar i}}"] & \bar{\bar C} \arrow[r, "\bar{\bar q}"'] \arrow[r, bend left, "\ell p(\bar{\bar q})" description] \arrow[r, phantom, "\Downarrow\gamma_{\bar{\bar q}}"] & s\bar{\bar A}
\end{tikzcd}$$

to define a track triangle morphism $T \to \bar{\bar T}$, i. e. (a) is represented by $(\bar b_{-2}, \bar b_{-1}, \bar b_0) \in \tilde G^2([\varrho T, \varrho\bar{\bar T}]^*\bar t)$ defined by

$$\begin{aligned}
\bar b_{-2} &= \sigma_0^{-1}((\bar{\bar H}_0(\ell\bar k_0)(\ell k_0))\square(\bar{\bar i}\bar M_0)\square(\bar M_1 f)\square((\ell\bar k_2)(\ell k_2)H_0^{\boxminus})),\\
\bar b_{-1} &= \sigma_0^{-1}((\bar{\bar H}_1(\ell\bar k_1)(\ell k_1))\square(\bar{\bar q}\bar M_1)\square(\bar M_2 i)\square(s((\ell\bar k_0)(\ell k_0))H_1^{\boxminus})),\\
\bar b_0 &= \sigma_0^{-1}((\bar{\bar H}_2(\ell\bar k_2)(\ell k_2))\square(s(\bar{\bar f})\bar M_2)\square((s^{\boxminus}_{\bar{\bar f},(\ell\bar k_0)(\ell\bar k_0)}\square s(\bar M_0)\square s_{(\ell\bar k_1)(\ell k_1),f})q)\\
&\quad\square(s((\ell\bar k_1)(\ell k_1))H_2^{\boxminus})),
\end{aligned}$$

where

$$\begin{aligned}
\bar M_0 &= (((\gamma_{\bar{\bar f}}(\ell\bar k_0))\square\mu^{\boxminus}_{p(\bar{\bar f}),\bar k_0}\square\mu_{\bar k_1,p(\bar f)})(\ell k_0))\square((\ell\bar k_1)(\mu^{\boxminus}_{p(\bar f),k_0}\square\mu_{k_1,p(f)}\square((\ell k_1)\gamma_f^{\boxminus}))),\\
\bar M_1 &= (((\gamma_{\bar{\bar i}}(\ell\bar k_1))\square\mu^{\boxminus}_{p(\bar{\bar i}),\bar k_1}\square\mu_{\bar k_2,p(\bar i)})(\ell k_1))\square((\ell\bar k_2)(\mu^{\boxminus}_{p(\bar i),k_1}\square\mu_{k_2,p(i)}\square((\ell k_2)\gamma_i^{\boxminus}))),\\
\bar M_2 &= (((\gamma_{\bar{\bar q}}(\ell\bar k_2))\square\mu^{\boxminus}_{p(\bar{\bar q}),\bar k_2}\square\mu_{t\bar k_0,p(\bar q)}\square(\nu_{\bar k_0,s}(\ell p(\bar q))))(\ell k_2))\\
&\quad\square(s(\ell\bar k_0)(\mu^{\boxminus}_{p(\bar q),k_2}\square\mu_{tk_0,p(q)}\square(\nu_{k_0,s}\gamma_q^{\boxminus}))).
\end{aligned}$$

On the other hand $\vartheta_{T,\bar{\bar{T}}}(\bar{k}k)$ measures the failure in

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[rr, "f"] \arrow[rr, bend right, "\ell p(f)" description] \arrow[rr, phantom, bend right=10, "\Downarrow\gamma_f^{\boxminus}"] \arrow[dd, "\ell(\bar{k}_0k_0)"'] \arrow[ddrr, "\ell(\bar{k}_1k_1p(f))" description] & & B \arrow[rr, "i"] \arrow[rr, bend right, "\ell p(i)" description] \arrow[rr, phantom, bend right=10, "\Downarrow\gamma_i^{\boxminus}"] \arrow[dd, "\ell(\bar{k}_1k_1)" description] \arrow[ddrr, "\ell(\bar{k}_2k_2p(i))" description] & & C \arrow[rr, "q"] \arrow[rr, bend right, "\ell q" description] \arrow[rr, phantom, bend right=10, "\Downarrow\gamma_q^{\boxminus}"] \arrow[dd, "\ell(\bar{k}_2k_2)" description] \arrow[ddrr, "\ell(p(\bar{\bar{q}})\bar{k}_2k_2)" description] & & sA \arrow[dd, "s\ell(\bar{k}_0k_0)"] \arrow[dd, bend right=50, "\ell t(\bar{k}_0k_0)" description] \\
& \mu_{\bar{k}_1k_1,p(f)} \Swarrow \quad \Swarrow \mu^{\boxminus}_{p(\bar{\bar{f}}),\bar{k}_0k_0} & & \mu_{\bar{k}_2k_2,p(i)} \Swarrow \quad \Swarrow \mu^{\boxminus}_{p(\bar{\bar{i}}),\bar{k}_1k_1} & & \mu_{t(\bar{k}_0k_0),p(q)} \Swarrow \quad \Swarrow \mu^{\boxminus}_{p(\bar{q}),\bar{k}_2k_2} & \overset{\nu_{\bar{k}_0k_0,s}}{\Leftarrow} \\
\bar{\bar{A}} \arrow[rr, "\bar{\bar{f}}"'] \arrow[rr, bend left, "\ell p(\bar{\bar{f}})" description] \arrow[rr, phantom, bend left=10, "\Downarrow\gamma_{\bar{\bar{f}}}"] & & \bar{\bar{B}} \arrow[rr, "\bar{\bar{i}}"'] \arrow[rr, bend left, "\ell p(\bar{\bar{i}})" description] \arrow[rr, phantom, bend left=10, "\Downarrow\gamma_{\bar{\bar{i}}}"] & & \bar{\bar{C}} \arrow[rr, "\bar{\bar{q}}"'] \arrow[rr, bend left, "\ell p(\bar{q})" description] \arrow[rr, phantom, bend left=10, "\Downarrow\gamma_{\bar{\bar{q}}}"] & & s\bar{\bar{A}}
\end{tikzcd}$$

to define a track triangle morphism $T \to \bar{\bar{T}}$, i. e. $\vartheta_{T,\bar{\bar{T}}}(\bar{k}k)$ is represented by $(b_{-2}, b_{-1}, b_0) \in \tilde{G}^2([\varrho T, \varrho\bar{\bar{T}}]^*\bar{t})$ with

$$\begin{aligned}
b_{-2} &= \sigma_0^{-1}((\bar{\bar{H}}_0(\ell(\bar{k}_0k_0)))\square(\bar{\bar{i}}M_0)\square(M_1f)\square((\ell(\bar{k}_2k_2))H_0^{\boxminus})),\\
b_{-1} &= \sigma_0^{-1}((\bar{\bar{H}}_1(\ell(\bar{k}_1k_1)))\square(\bar{\bar{q}}M_1)\square(M_2i)\square(s(\ell(\bar{k}_0k_0))H_1^{\boxminus})),\\
b_0 &= \sigma_0^{-1}((\bar{\bar{H}}_2(\ell(\bar{k}_2k_2)))\square(s(\bar{\bar{f}})M_2)\square((s^{\boxminus}_{\bar{\bar{f}},\ell(\bar{k}_0k_0)}\square s(M_0)\square s_{\ell(\bar{k}_1k_1),f})q)\\
&\quad \square(s(\ell(\bar{k}_1k_1))H_0^{\boxminus}))),
\end{aligned}$$

where

$$\begin{aligned}
M_0 &= (\gamma_{\bar{\bar{f}}}(\ell(\bar{k}_0k_0)))\square\mu^{\boxminus}_{p(\bar{\bar{f}}),\bar{k}_0k_0}\square\mu_{\bar{k}_1k_1,p(f)}\square((\ell(\bar{k}_1k_1))\gamma_f^{\boxminus}),\\
M_1 &= (\gamma_{\bar{\bar{i}}}(\ell(\bar{k}_1k_1)))\square\mu^{\boxminus}_{p(\bar{\bar{i}}),\bar{k}_1k_1}\square\mu_{\bar{k}_2k_2,p(i)}\square((\ell(\bar{k}_2k_2))\gamma_i^{\boxminus}),\\
M_2 &= (\gamma_{\bar{\bar{q}}}(\ell(\bar{k}_2k_2)))\square\mu^{\boxminus}_{p(\bar{\bar{q}}),\bar{k}_2k_2}\square\mu_{t(\bar{k}_0k_0),p(q)}\square(\nu_{\bar{k}_0k_0,s}\gamma_q^{\boxminus}).
\end{aligned}$$

Now a laborious but straightforward computation shows that the cochain $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^1([\varrho T, \varrho\bar{\bar{T}}]^*\bar{t})$ given by

$$\begin{aligned}
c_{-2} &= \sigma^{-1}_{\bar{\bar{f}}\ell(\bar{k}_0k_0)}(M_0\square(\mu_{\bar{k}_1,k_1}f)\square\bar{M}_0^{\boxminus}\square(\bar{\bar{f}}\mu^{\boxminus}_{\bar{k}_0,k_0}))\\
c_{-1} &= \sigma^{-1}_{\bar{\bar{i}}\ell(\bar{k}_1k_1)}(M_1\square(\mu_{\bar{k}_2,k_2}i)\square\bar{M}_1^{\boxminus}\square(\bar{\bar{i}}\mu^{\boxminus}_{\bar{k}_1,k_1}))\\
c_0 &= \sigma^{-1}_{\bar{\bar{q}}\ell(\bar{k}_2k_2)}(M_2\square((s(\mu_{\bar{k}_0,k_0})\square s_{\ell\bar{k}_0,\ell k_0})q)\square\bar{M}_2^{\boxminus}\square(\bar{\bar{f}}\mu^{\boxminus}_{\bar{k}_2,k_2}))
\end{aligned}$$

satisfies

$$\delta(c_{-2}, c_{-1}, c_0) = (b_{-2}, b_{-1}, b_0) - (\bar{b}_{-2}, \bar{b}_{-1}, \bar{b}_0),$$

therefore the lemma follows. $\square$

Lemma 3.4. *For any candidate triangle T given by*

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow H_0"] & B \arrow[r, "i"] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow H_1"] & C \arrow[r, "q"] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow H_2"] & sA \arrow[r, "s(f)"'] & sB
\end{tikzcd}$$

and any $c \in H^2(sh^3, [\varrho T, \varrho T]^*\bar{t})$ *there is a candidate triangle* $\bar{T}$ *as follows*

$$\begin{tikzcd}
 & \Uparrow \bar{H}_0 & & \Uparrow \bar{H}_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i"] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "q"] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(f)"'] & sB \\
 & & \Downarrow \bar{H}_1 & &
\end{tikzcd}$$

such that $\theta_{T,\bar{T}}(1_{\varrho T}) = c$.

Proof. If c represented by the cocycle $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^2([\varrho T, \varrho T]^*\bar{t})$ then it is enough to define

$$\begin{aligned}
\bar{H}_0 &= \sigma_0(c_{-2})\Box H_0,\\
\bar{H}_1 &= \sigma_0(c_{-1})\Box H_1,\\
\bar{H}_2 &= \sigma_0(c_0)\Box H_2.
\end{aligned}$$

□

Lemma 3.5. *Given track triangles* T *and* $\bar{T}$ *defined by*

$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i"] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "q"] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(f)"'] & sB \\
 & & \Downarrow H_1 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \Uparrow \bar{H}_0 & & \Uparrow \bar{H}_2 & \\
\bar{A} \arrow[r, "\bar{f}"'] \arrow[rr, bend left=60, "0"] & \bar{B} \arrow[r, "\bar{i}"] \arrow[rr, bend right=60, "0"'] & \bar{C} \arrow[r, "\bar{q}"] \arrow[rr, bend left=60, "0"] & s\bar{A} \arrow[r, "s(\bar{f})"'] & s\bar{B} \\
 & & \Downarrow \bar{H}_1 & &
\end{tikzcd}$$

the abelian group $H^1(sh^3, [\varrho T, \varrho\bar{T}]^*\bar{t})$ *acts effectively on the set of homotopy classes of track triangle morphisms* $T \to \bar{T}$ *in such a way that given* $c \in H^1(sh^3, [\varrho T, \varrho\bar{T}]^*\bar{t})$ *and a homotopy class* $k\colon T \to \bar{T}$ *then* $\varrho(k+c) = \varrho(k)$ *and if* $\bar{k}\colon T \to \bar{T}$ *is another homotopy class with* $\varrho(k) = \varrho(\bar{k})$ *then there exists (a unique)* $c \in H^1(sh^3, [\varrho T, \varrho\bar{T}]^*\bar{t})$ *such that* $\bar{k} = k + c$.

Proof. The action $+$ is defined as follows: let $k\colon T \to \bar{T}$ be a homotopy class of morphisms of track triangles represented by

(a)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"] \arrow[dl, Rightarrow, "K_0"] & C \arrow[r, "q"] \arrow[d, "k_2"] \arrow[dl, Rightarrow, "K_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, "K_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

and let $(c_{-2},c_{-1},c_0)\in\tilde{G}^1([\varrho T,\varrho\bar{T}]^*\bar{t})$ be a cocycle representing the cohomology class $c\in H^1(sh^3,[\varrho T,\varrho\bar{T}]^*\bar{t})$, then $k+c\colon T\to\bar{T}$ is represented by the diagram

(b)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"'] \arrow[dl, Rightarrow, "\bar{K}_0"] & C \arrow[r, "q"] \arrow[d, "k_2"'] \arrow[dl, Rightarrow, "\bar{K}_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, "\bar{K}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

with

$$\begin{aligned}
\bar{K}_0 &= \sigma_{\bar{f}k_0}(c_{-2})\Box K_0,\\
\bar{K}_1 &= \sigma_{\bar{i}k_1}(c_{-1})\Box K_1,\\
\bar{K}_2 &= \sigma_{\bar{q}k_2}(c_0)\Box K_2.
\end{aligned}$$

Actually, the homotopy class of (b) only depends on c, because if $(\bar{c}_{-2},\bar{c}_{-1},\bar{c}_0)-(c_{-2},c_{-1},c_0)\in\tilde{G}^1([\varrho T,\varrho\bar{T}]^*\bar{t})$ is the coboundary of $(e_{-2},e_{-1},e_0)\in\tilde{G}^0([\varrho T,\varrho\bar{T}]^*\bar{t})$ and

(c)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"'] \arrow[dl, Rightarrow, "\bar{\bar{K}}_0"] & C \arrow[r, "q"] \arrow[d, "k_2"'] \arrow[dl, Rightarrow, "\bar{\bar{K}}_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, "\bar{\bar{K}}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

is given by

$$\begin{aligned}
\bar{\bar{K}}_0 &= \sigma_{\bar{f}k_0}(\bar{c}_{-2})\Box K_0,\\
\bar{\bar{K}}_1 &= \sigma_{\bar{i}k_1}(\bar{c}_{-1})\Box K_1,\\
\bar{\bar{K}}_2 &= \sigma_{\bar{q}k_2}(\bar{c}_0)\Box K_2,
\end{aligned}$$

then the tracks $\sigma_{k_0}(e_{-2})\colon k_0\Rightarrow k_0$, $\sigma_{k_1}(e_{-1})\colon k_1\Rightarrow k_1$ and $\sigma_{k_2}(e_0)\colon k_2\Rightarrow k_2$ prove that (b) and (c) are homotopic.

Moreover, if (a) is homotopic to

(d)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bar{k}_0"'] & B \arrow[r, "i"] \arrow[d, "\bar{k}_1"'] \arrow[dl, Rightarrow, "\tilde{K}_0"] & C \arrow[r, "q"] \arrow[d, "\bar{k}_2"'] \arrow[dl, Rightarrow, "\tilde{K}_1"] & sA \arrow[d, "s\bar{k}_0"] \arrow[dl, Rightarrow, "\tilde{K}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

by the tracks $L_i\colon k_i\Rightarrow\bar{k}_i$ $(i=0,1,2)$ then (b) is homotopic to

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bar{k}_0"'] & B \arrow[r, "i"] \arrow[d, "\bar{k}_1"'] \arrow[dl, Rightarrow, "\tilde{\tilde{\bar{K}}}_0"] & C \arrow[r, "q"] \arrow[d, "\bar{k}_2"'] \arrow[dl, Rightarrow, "\tilde{\tilde{\bar{K}}}_1"] & sA \arrow[d, "s\bar{k}_0"] \arrow[dl, Rightarrow, "\tilde{\tilde{\bar{K}}}_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

with

$$\begin{aligned}
\tilde{\tilde{\bar{K}}}_0 &= \sigma_{\bar{f}k_0}(c_{-2})\Box\tilde{K}_0,\\
\tilde{\tilde{\bar{K}}}_1 &= \sigma_{\bar{i}k_1}(c_{-1})\Box\tilde{K}_1,\\
\tilde{\tilde{\bar{K}}}_2 &= \sigma_{\bar{q}k_2}(c_0)\Box\tilde{K}_2,
\end{aligned}$$

also by the tracks L_i. Therefore $k+c$ does not depend on the choice of a representative for the homotopy class $k\colon T \to \bar{T}$.

If (a) and (b) where homotopic the homotopy would be necessarily given by tracks $\sigma_{k_0}(b_{-2})\colon k_0 \Rightarrow k_0$, $\sigma_{k_1}(b_{-1})\colon k_1 \Rightarrow k_1$ and $\sigma_{k_2}(b_0)\colon k_2 \Rightarrow k_2$. In this case it is easy to check that $\delta(b_{-2}, b_{-1}, b_0) = (c_{-2}, c_{-1}, c_0)$, so $c = 0$. This proves the effectiveness.

Finally suppose that we have two homotopy classes of morphisms $k, \bar{k}\colon T \to \bar{T}$ defined by (a) and (d), respectively, such that $p(k_i) = p(\bar{k}_i)$ $(i = 0, 1, 2)$. Then if we define

$$\begin{aligned}
c_{-2} &= \sigma_{\bar{f}k_0}((\bar{f}(\gamma_{k_0}\square\gamma^{\boxminus}_{\bar{k}_0}))\square\tilde{K}_0\square((\gamma_{\bar{k}_1}\square\gamma^{\boxminus}_{k_1})f)\square K^{\boxminus}_0),\\
c_{-1} &= \sigma_{\bar{i}k_1}((\bar{i}(\gamma_{k_1}\square\gamma^{\boxminus}_{\bar{k}_1}))\square\tilde{K}_1\square((\gamma_{\bar{k}_2}\square\gamma^{\boxminus}_{k_2})i)\square K^{\boxminus}_1),\\
c_0 &= \sigma_{\bar{q}k_2}((\bar{q}(\gamma_{k_2}\square\gamma^{\boxminus}_{\bar{k}_2}))\square\tilde{K}_2\square(s(\gamma_{\bar{k}_0}\square\gamma^{\boxminus}_{k_0})q)\square K^{\boxminus}_2),
\end{aligned}$$

it is not had to check that $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^2([\varrho T, \varrho\bar{T}]^*\bar{t})$ is a cocycle and its cohomology class c satisfies $\bar{k} = k + c$, since the track triangle morphism (b) defined by (a) and (c_{-2}, c_{-1}, c_0) is homotopic to (d) by $\gamma_{\bar{k}_i}\square\gamma^{\boxminus}_{k_i}\colon k_i \Rightarrow \bar{k}_i$ $(i = 0, 1, 2)$. □

Lemma 3.6. *Given homotopy classes of track triangle morphisms*

$$T \xrightarrow{k} \bar{T} \xrightarrow{\bar{k}} \bar{\bar{T}}$$

and cohomology classes $c \in H^1(sh^3, [\varrho T, \varrho\bar{T}]^*\bar{t})$ *and* $\bar{c} \in H^1(sh^3, [\varrho\bar{T}, \varrho\bar{\bar{T}}]^*\bar{t})$ *then the action defined in Lemma 3.5 satisfies the following linear distributivity law*

$$(\bar{k} + \bar{c})(k + c) = \bar{k}k + (\varrho(k)^*\bar{c} + \varrho(\bar{k})_*c).$$

This lemma follows easily from the definition of the action + in the proof of Lemma 3.5.

4 Small ∇-triangles and homotopy pairs

Consider an additive category $\mathbf{A}$, an additive equivalence $t\colon \mathbf{A} \to \mathbf{A}$, a translation cohomology class $\nabla \in H^3(\mathbf{A}, t)$, and a good translation track category $(\mathbf{B}, s)$ representing ∇. These data will remain fixed throughout this section. Here we define the category of homotopy pairs in $\mathbf{B}$. We establish relations between this category, the category of track triangles on $(\mathbf{B}, s)$, and the category of small ∇-triangles. These relations will play an important role in the proof of the main theorems of [V]. We also characterize (pre)triangulated categories in terms of homotopy pairs. By using this characterization we prove Theorem 2.15 at the end of this section.

We are going to use the concept of (weak) linear extension of categories in [Bau89] IV.3.2 that we now recall. At the same time we are going to define the weaker concept of (weak) linear action on a functor, which is however different from the linear actions considered in [Bau89] IV.2.8. A *linear extension* (resp. *action*)

(4.1) $$D \xrightarrow{+} \mathbf{D} \xrightarrow{p} \mathbf{C}$$

is given by a full functor p which is the identity on objects, a $\mathbf{C}$-bimodule D (i. e. a functor $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$), and (effective) right actions of $D(pX, pY)$ on the morphism sets $\mathbf{D}(X, Y)$ for all objects X and Y in $\mathbf{D}$ such that the orbits coincide with (resp.

are contained in) the fibers of the map $p\colon \mathbf{D}(X,Y) \twoheadrightarrow \mathbf{C}(pX,pY)$. These actions must satisfy the following *linear distributivity law*: given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathbf{D}$ and elements $a \in D(pX,pY)$ and $b \in D(pY,pZ)$ we have

$$(g+b)(f+a) = gf + (p(f)^*b + p(g)_*a).$$

In a *weak linear extension* (resp. *action*) we do not require p to be the identity on objects but only that any object in $\mathbf{C}$ is isomorphic to an object in the image of p. A weak linear extension can always be replaced by a honest linear extension up to equivalence of categories. This definition of weak linear extension is equivalent to the original definition in [Bau89] IV.3.3.

Let $\mathbf{T}$ be any category. The category $\mathbf{Pair}(\mathbf{T})$ of pairs on $\mathbf{T}$ is defined as follows. Objects are morphisms in $\mathbf{T}$ and a morphism $(h,k)\colon f \to g$ is given by a commutative square in $\mathbf{T}$

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "k"] & \bullet
\end{tikzcd}$$

If is $\mathbf{T}$ additive and D is a biadditive $\mathbf{T}$-bimodule we define $D^\#$ as the $\mathbf{Pair}(\mathbf{T})$-bimodule given by

$$D^\#(f,g) = D(A,D)/(g_*D(A,C) + f^*D(B,D))$$

for $f\colon A \to B$ and $g\colon C \to D$ in $\mathbf{T}$.

The category $\mathbf{Hopair}(\mathbf{B})$ of *homotopy pairs* in the track category $\mathbf{B}$ is defined as follows. Objects are maps $f\colon A \to B$ in $\mathbf{B}$. A morphism $[k_0,k_1,K]\colon f \to \bar{f}$ is represented by a diagram

(4.2)

$$\begin{tikzcd}
A \arrow[r, "k_0"] \arrow[d, "f"'] & \bar{A} \arrow[d, "\bar{f}"] \\
B \arrow[r, "k_1"'] \arrow[ur, Rightarrow, "K"'] & \bar{B}
\end{tikzcd}$$

in $\mathbf{B}$. Moreover, $[k_0,k_1,K] = [\bar{k}_0,\bar{k}_1,\bar{K}]$ if and only if there are tracks $L_0\colon k_0 \Rightarrow \bar{k}_0$ and $L_1\colon k_1 \Rightarrow \bar{k}_1$ such that $(\bar{f}L_0)\Box K = \bar{K}\Box(L_1 f)$. The obvious forgetful functor $\tilde{p}_{\mathbf{B}}\colon \mathbf{Hopair}(\mathbf{B}) \to \mathbf{Pair}(\mathbf{A})$ fits into a weak linear extension

$$(\mathrm{Hom}^t)^\# \xrightarrow{+} \mathbf{Hopair}(\mathbf{B}) \xrightarrow{\tilde{p}_{\mathbf{B}}} \mathbf{Pair}(\mathbf{A}), \tag{4.3}$$

compare [Bau97]. The action is defined as follows, given a morphism $[k_0,k_1,K]$ represented by a diagram as above and $\tilde{g} \in (\mathrm{Hom}^t)^\#(p(f),p(\bar{f}))$ represented by $g\colon tA \to \bar{B}$ then $[k_0,k_1,K] + \tilde{g} = [k_0,k_1,K\Box\sigma_{k_1 f}(g)]$.

The category $\mathbf{Pair}(\mathbf{A})$ is obviously additive and $(\mathrm{Hom}^t)^\#$ is a biadditive $\mathbf{Pair}(\mathbf{A})$-bimodule. By [BHP97] 6.2 the category $\mathbf{Hopair}(\mathbf{B})$ is also additive in a unique compatible way with the structure of weak linear extension in (4.3). One can easily check that the full subcategory of $\mathbf{Hopair}(\mathbf{B})$ given by the maps $\iota X\colon * \to X$ is isomorphic to $\mathbf{A}$. We denote

$$\iota\colon \mathbf{A} \longrightarrow \mathbf{Hopair}(\mathbf{B})$$

to the full additive inclusion. The *cone functor* associated to a map $f\colon A \to B$ in $\mathbf{B}$ is the additive functor

$$Cone_f = \mathrm{Hom}_{\mathbf{Hopair(B)}}(f, \iota)\colon \mathbf{A} \longrightarrow \mathbf{Ab}. \tag{4.4}$$

Cone functors are introduced in [I] in a different but equivalent way. There $Cone_f(X)$ is defined as the quotient set

$$Cone_f(X) = \{(\alpha, \hat{\alpha})\,;\, \alpha\colon B \to X \text{ in } \mathbf{B} \text{ and } \hat{\alpha}\colon \alpha f \Rightarrow 0\} / \simeq$$

where $(\alpha, \hat{\alpha}) \simeq (\beta, \hat{\beta})$ if there exists $H\colon \alpha \Rightarrow \beta$ such that $\hat{\alpha} = \hat{\beta}\square(Hf)$. The equivalence class represented by $(\alpha, \hat{\alpha})$ is denoted by $\{\alpha, \hat{\alpha}\}$. the correspondence with the definition given here is determined by the identification $\{\alpha, \hat{\alpha}\} = [0, \alpha, \hat{\alpha}]\colon f \to \iota X$.

Recall that $\mathbf{A}$-modules are additive functors $\mathbf{A} \to \mathbf{Ab}$. The category of $\mathbf{A}$-modules and natural transformations between them is denoted by $\mathbf{mod}(\mathbf{A})$. The category $\mathbf{A}^{op}$ is fully included in $\mathbf{mod}(\mathbf{A})$ through the functor $X \mapsto \mathrm{Hom}_{\mathbf{A}}(X, -)$. The advantage of the new approach to cone functors is that we immediately obtain the following result.

Proposition 4.5. *Cone functors define an additive functor*

$$Cone\colon \mathbf{Hopair(B)}^{op} \longrightarrow \mathbf{mod}(\mathbf{A}).$$

Proof. The functor *Cone* corresponds to

$$\mathrm{Hom}_{\mathbf{Hopair(B)}}(-, \iota)\colon \mathbf{Hopair(B)}^{op} \times \mathbf{A} \longrightarrow \mathbf{Ab}$$

by the exponential law. With the notation in [I] a morphism of homotopy pairs $[k_0, k_1, K]\colon f \to \bar{f}$ induces the $\mathbf{A}$-module morphism

$$[k_0, k_1, K]^*\colon Cone_{\bar{f}} \longrightarrow Cone_f$$

defined by $\{\alpha, \hat{\alpha}\} \mapsto \{\alpha k_1, (\hat{\alpha}k_0)\square(\alpha K)\}$. □

Cone functors are embedded in exact sequences of $\mathbf{A}$-modules

$$\mathrm{Hom}_{\mathbf{A}}(tB, -) \overset{(tp(f))^*}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}(tA, -) \overset{\tilde{q}_f}{\longrightarrow} Cone_f \overset{\tilde{i}_f}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}(B, -) \overset{p(f)^*}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}(A, -). \tag{4.6}$$

Here $\tilde{q}_f(g) = \{0, \sigma_0(g)\}$ and $\tilde{i}_f\,\{\alpha, \hat{\alpha}\} = p(\alpha)$, see [I] 3.4. In fact the exactness of this sequence is an immediate consequence of the axioms of a weak linear extension for (4.3) and the compatibility with the additive structure on $\mathbf{Hopair(B)}$, see [BHP97] 6.2.

Remark 4.7. This exact sequence is natural in f in the following sense, given a morphism of homotopy pairs $[k_0, k_1, K]\colon f \to \bar{f}$ represented by a diagram as in (4.2) the following diagram of $\mathbf{A}$-modules commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{A}}(tB, -) \arrow[r, "(tp(f))^*"] & \mathrm{Hom}_{\mathbf{A}}(tA, -) \arrow[r, "\tilde{q}_f"] & Cone_f \arrow[r, "\tilde{i}_f"] & \mathrm{Hom}_{\mathbf{A}}(B, -) \arrow[r, "p(f)^*"] & \mathrm{Hom}_{\mathbf{A}}(A, -) \\
\mathrm{Hom}_{\mathbf{A}}(t\bar{B}, -) \arrow[r, "(tp(\bar{f}))^*"'] \arrow[u, "(tp(k_1))^*"'] & \mathrm{Hom}_{\mathbf{A}}(t\bar{A}, -) \arrow[r, "\tilde{q}_{\bar{f}}"'] \arrow[u, "(tp(k_0))^*"'] & Cone_{\bar{f}} \arrow[r, "\tilde{i}_{\bar{f}}"'] \arrow[u, "{[k_0,k_1,K]^*}"'] & \mathrm{Hom}_{\mathbf{A}}(\bar{B}, -) \arrow[r, "p(\bar{f})^*"'] \arrow[u, "p(k_1)^*"'] & \mathrm{Hom}_{\mathbf{A}}(\bar{A}, -) \arrow[u, "p(k_0)^*"']
\end{tikzcd}$$

Remark 4.8. Let $\mathbf{Hopair}^{rep}(\mathbf{B}) \subset \mathbf{Hopair}(\mathbf{B})$ be the full subcategory given by those maps in $\mathbf{B}$ such that the associated cone functor is representable, for any such a map f in $\mathbf{B}$ one can choose an $\mathbf{A}$-module isomorphism

$$\chi_f \colon \operatorname{Hom}_{\mathbf{A}}(C_f, -) \cong Cone_f.$$

A particular choice determines a functor

$$C \colon \mathbf{Hopair}^{rep}(\mathbf{B}) \longrightarrow \mathbf{A} \tag{4.9}$$

such that the restriction of $Cone$ in Proposition 4.5 to $\mathbf{Hopair}^{rep}(\mathbf{B})^{op}$ coincides up to natural equivalence with

$$\mathbf{Hopair}^{rep}(\mathbf{B})^{op} \xrightarrow{C^{op}} \mathbf{A}^{op} \subset \mathbf{mod}(\mathbf{A}).$$

The natural equivalence χ is given by the representations above. By using C we can define a functor

$$\zeta \colon \mathbf{Hopair}^{rep}(\mathbf{B}) \longrightarrow \mathbf{cand}(\mathbf{A}, t) \tag{4.10}$$

as follows. Given an object $f \colon A \to B$ in $\mathbf{Hopair}^{rep}(\mathbf{B})$ the small candidate triangle $\zeta(f)$ is

$$A \xrightarrow{p(f)} B \xrightarrow{i_f} C_f \xrightarrow{q_f} tA, \tag{4.11}$$

where i_f and q_f are determined by the $\mathbf{A}$-module morphisms $\tilde{i}_f$ and $\tilde{q}_f$ in (4.6), respectively, the isomorphism χ_f and Yoneda's lemma. The functor ζ sends a morphism $[k_0, k_1, K]$ in $\mathbf{Hopair}^{rep}(\mathbf{B})$ represented by diagram (4.2) to the following morphism of small candidate triangles

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, "p(k_0)"] & B \arrow[r, "i_f"] \arrow[d, "p(k_1)"] & C_f \arrow[r, "q_f"] \arrow[d, "{C[k_0,k_1,K]}"] & tA \arrow[d, "tp(k_0)"] \\
\bar{A} \arrow[r, "p(\bar{f})"'] & \bar{B} \arrow[r, "i_{\bar{f}}"'] & C_{\bar{f}} \arrow[r, "q_{\bar{f}}"'] & t\bar{A}
\end{tikzcd}$$

This diagram commutes by Remark 4.7. The functor ζ does not depend on the choice of the isomorphisms χ_f, up to natural equivalence.

The following proposition shows that track triangles determine representations of cone functors.

Proposition 4.12. *Given a track triangle in* $(\mathbf{B}, s)$

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, shift left=4ex, "\Uparrow H_0"] & B \arrow[r, "i"] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow H_1"] & C \arrow[r, "q"] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, shift left=4ex, "\Uparrow H_2"] & sA \arrow[r, "s(f)"'] & sB
\end{tikzcd}$$

the $\mathbf{A}$*-module morphism*

$$\chi_{\{i, H_0\}} \colon \operatorname{Hom}_{\mathbf{A}}(C, -) \longrightarrow Cone_f$$

represented by $\{i, H_0\} \in Cone_f(C)$ *is an isomorphism.*

Proof. The inverse sends $\{\alpha, \hat{\alpha}\} \in Cone_f(X)$ to the unique morphism $l\colon C \to X$ in $\mathbf{A}$ such that $\sigma_0(\bar{t}l) = ((s(\hat{\alpha})\square s_{\alpha,f})q)\square(\alpha H_2^{\boxminus})$. Compare [I] 4. $\square$

Remark 4.13. In the conditions of the statement of Proposition 4.12 the following diagram of $\mathbf{A}$-modules commutes

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathbf{A}}(C,-) \arrow[dr, "p(i)^*"] \arrow[dd, "\chi_{\{i,H_0\}}", "\cong"'] & \\
\mathrm{Hom}_{\mathbf{A}}(tA,-) \arrow[ur, "p(q)^*"] \arrow[dr, "\tilde{q}_f"'] & & \mathrm{Hom}_{\mathbf{A}}(B,-) \\
 & Cone_f \arrow[ur, "\tilde{i}_f"'] &
\end{tikzcd}$$

By Proposition 4.12 there is an obvious forgetful functor

$$\bar{\varrho} = \bar{\varrho}_{(\mathbf{B},s)}\colon \mathbf{candt}(\mathbf{B}, s) \longrightarrow \mathbf{Hopair}^{rep}(\mathbf{B}) \tag{4.14}$$

sending the homotopy class of a track triangle morphism as in (2.9) to $[k_0, k_1, K_0]$.

There is also an obvious functor

$$\varsigma\colon \mathbf{cand}(\mathbf{A}, t) \longrightarrow \mathbf{Pair}(\mathbf{A}) \tag{4.15}$$

sending a small candidate triangle

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

to $f\colon A \to B$ regarded as an object of $\mathbf{Pair}(\mathbf{A})$.

Proposition 4.16. *In the following diagram of functors*

$$\begin{tikzcd}
 & \mathbf{Hopair}^{rep}(\mathbf{B}) \arrow[dr, "\tilde{p}"] \arrow[dd, "\zeta"] & \\
\mathbf{candt}(\mathbf{B}, s) \arrow[ur, "\bar{\varrho}"] \arrow[dr, "\varrho"'] & & \mathbf{Pair}(\mathbf{A}) \\
 & \mathbf{cand}(\mathbf{A}, t) \arrow[ur, "\varsigma"'] &
\end{tikzcd}$$

the triangle in the left commutes up to natural equivalence and the triangle in the right is strictly commutative.

Proof. The statement about the triangle in the right-hand side is obvious. Let

$$\begin{tikzcd}
 & {} & & {} & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[u, Rightarrow, "H_0"] \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] & D \arrow[d, Rightarrow, "H_1"] \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] & sA \arrow[u, Rightarrow, "H_2"] \arrow[r, "s(f)"'] & sB \\
 & & {} & &
\end{tikzcd} \tag{a}$$

be any track triangle. Possibly the representation

$$\chi_f\colon \mathrm{Hom}_{\mathbf{A}}(C_f, -) \cong Cone_f$$

chosen for the definition of C in (4.9) and ζ in (4.10) is different from

$$\chi_{\{i,H_0\}}\colon \operatorname{Hom}_{\mathbf{A}}(D,-)\cong \mathit{Cone}_f$$

in Proposition 4.12. However if the morphism $l\colon D\to C_f$ in $\mathbf{A}$ represents

$$\operatorname{Hom}_{\mathbf{A}}(C_f,-)\overset{\chi_f}{\cong}\mathit{Cone}_f\overset{\chi^{-1}_{\{i,H_0\}}}{\cong}\operatorname{Hom}_{\mathbf{A}}(D,-)$$

then the following isomorphism of small candidate triangles

$$\begin{tikzcd}
A \arrow[r, "p(f)"] \arrow[d, equal] & B \arrow[r, "p(i)"] \arrow[d, equal] & D \arrow[r, "p(q)"] \arrow[d, "l", "\cong"'] & tA \arrow[d, equal] \\
A \arrow[r, "p(f)"'] & B \arrow[r, "i_f"'] & C_f \arrow[r, "q_f"'] & tA
\end{tikzcd}$$

is natural in the track triangle (a). This can be checked by using the definition of ζ and the next lemma. ☐

Lemma 4.17. *Given a morphism between track triangles*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"'] & B \arrow[r, "i"] \arrow[d, "k_1"] \arrow[dl, Rightarrow, "K_0"] & C \arrow[r, "q"] \arrow[d, "k_2"] \arrow[dl, Rightarrow, "K_1"] & sA \arrow[d, "sk_0"] \arrow[dl, Rightarrow, "K_2"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & s\bar{A}
\end{tikzcd}$$

the composite $\mathbf{A}$*-module morphism*

$$\operatorname{Hom}_{\mathbf{A}}(\bar{C},-)\overset{\chi_{\{\bar{i},\bar{H}_0\}}}{\cong}\mathit{Cone}_{\bar{f}}\xrightarrow{[k_0,k_1,K_0]^*}\mathit{Cone}_f\overset{\chi^{-1}_{\{i,H_0\}}}{\cong}\operatorname{Hom}_{\mathbf{A}}(C,-)$$

is represented by $p(k_2)\colon C\to\bar{C}$.

Proof. It is enough to check that $[k_0,k_1,K_0]^*\left\{\bar{i},\bar{H}_0\right\}=\left\{\bar{i}k_1,\bar{H}_0\square(\bar{i}K_0)\right\}$ coincides with $\chi_{\{i,H_0\}}(p(k_2))=\{k_2i,k_2H_0\}$ in $\mathit{Cone}_f(\bar{C})$. For this one uses $K_1\colon k_2i\Rightarrow\bar{i}k_1$ and the definition of track triangle morphism. ☐

The following result allows us to prove (Tr3) for cohomologically pretriangulated categories. Let $\mathbf{cand}(\nabla)$ be the category of small ∇-triangles.

Proposition 4.18. *The functor* ς *is full over the full subcategory* $\mathbf{cand}(\nabla)$ *of small* ∇*-triangles.*

Proof. By Proposition 4.12, Remark 4.13 and Yoneda's lemma any small ∇-triangle is isomorphic to an object in the image of the functor ζ in (4.10). By Proposition 4.16 the composite functor $\varsigma\zeta$ coincides with $\tilde{p}$, which is known to be full, hence this proposition follows. ☐

Suppose now that $(\mathbf{A},t,\nabla)$ is cohomologically pretriangulated. By (TTr1) and Proposition 4.12 all cone functors are representable. By Remark 4.8 a choice of representations determines a functor

$$\zeta\colon \mathbf{Hopair}(\mathbf{B})\to\mathbf{cand}(\mathbf{A},t)$$

which is well defined up to natural equivalence.

Proposition 4.19. *If $(\mathbf{A}, t, \nabla)$ is cohomologically pretriangulated then the functor ζ takes values in small ∇-triangles.*

Proof. By Proposition 4.12, Remark 4.13 and Yoneda's lemma for any map f in $\mathbf{B}$ the small candidate triangle $\zeta(f)$ is isomorphic to a small ∇-triangle, now the proposition follows from Theorem 2.17 (pretriangulated case) and (Tr0). ☐

In the next proposition we show that the functor $\bar{\varrho}$ in (4.14) is compatible with the actions of $H^1(sh^3, [-,-]^*\bar{t})$ and $(\mathrm{Hom}^t)^\#$ in Lemma 3.5 and (4.3), respectively. For this we consider the natural transformation of $\mathbf{cand}(\mathbf{A}, t)$-bimodules

$$(4.20) \qquad \varpi\colon H^1(sh^3, [-,-]^*\bar{t}) \Rightarrow (\mathrm{Hom}^t)^\#(\varsigma, \varsigma)$$

sending a cohomology class represented by a 1-cocycle $(c_{-2}, c_{-1}, c_0) \in \tilde{G}^1([T, \bar{T}]^*\bar{t})$ to the element represented by c_{-2} in $(\mathrm{Hom}^t)^\#(\varsigma T, \varsigma \bar{T})$.

Proposition 4.21. *For any morphism $f\colon T \to \bar{T}$ in $\mathbf{candt}(\mathbf{B}, s)$ and any $\alpha \in H^1(sh^3, [\varrho T, \varrho \bar{T}]^*\bar{t})$ we have that*

$$\bar{\varrho}(f + \alpha) = \bar{\varrho}(f) + \varpi_{(\varrho T, \varrho \bar{T})}(\alpha).$$

This result follows immediately from the definition of the linear actions.

There is a similar result for the functor ζ in (4.10). In order to state it we construct first a weak linear action on the functor ς in (4.15). For this we consider the $\mathbf{Pair}(\mathbf{A})$-bimodule

$$\mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker}).$$

Here $\mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}$ denotes morphism abelian groups in the category of $\mathbf{A}^{op}$-modules, and

$$\mathrm{Ker}\, t, \mathrm{Coker}\colon \mathbf{Pair}(\mathbf{A}) \longrightarrow \mathbf{mod}(\mathbf{A}^{op})$$

send a morphism $f\colon A \to B$ in $\mathbf{A}$ to the kernel of tf and the cokernel of f in the abelian category of $\mathbf{A}^{op}$-modules through the Yoneda full inclusion $\mathbf{A} \subset \mathbf{mod}(\mathbf{A}^{op})$, respectively.

Proposition 4.22. *There is a weak linear action*

$$\mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker}) \xrightarrow{+} \mathbf{cand}(\mathbf{A}, t) \xrightarrow{\varsigma} \mathbf{Pair}(\mathbf{A}).$$

Moreover, if $(\mathbf{A}, t, \nabla)$ is a cohomologically pretriangulated category then this weak linear action restricts to a weak linear extension on the full subcategory of small ∇-triangles

$$\mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker}) \xrightarrow{+} \mathbf{cand}(\nabla) \xrightarrow{\varsigma} \mathbf{Pair}(\mathbf{A}).$$

Proof. The second part of this proposition actually holds for any pretriangulated category if we change $\mathbf{cand}(\nabla)$ by the category of exact triangles. For any small candidate triangle

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

we denote by

$$\hat{\imath}\colon \mathrm{Coker}\, f \longrightarrow \mathrm{Hom}_{\mathbf{A}}(-, C)$$

and

$$\hat{q}\colon \mathrm{Hom}_{\mathbf{A}}(-, C) \longrightarrow \mathrm{Ker}\, tf$$

to the $\mathbf{A}^{op}$-module morphisms induced by the equalities $if = 0$ and $(tf)q = 0$, respectively. Given a morphism k between small candidate triangles

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] \arrow[d, "k_2"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"] & \bar{B} \arrow[r, "\bar{i}"] & \bar{C} \arrow[r, "\bar{q}"] & t\bar{A}
\end{tikzcd}$$

and $\alpha \in \mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, tf, \mathrm{Coker}\, \bar{f})$ the morphism $k + \alpha$ is given by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] \arrow[d, "k_2+\hat{\bar{i}}\alpha\hat{q}"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"] & \bar{B} \arrow[r, "\bar{i}"] & \bar{C} \arrow[r, "\bar{q}"] & t\bar{A}
\end{tikzcd}$$

where we identify the $\mathbf{A}^{op}$-module morphism

$$\hat{\bar{i}}\alpha\hat{q} \colon \mathrm{Hom}_{\mathbf{A}}(-, C) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(-, \bar{C})$$

with the corresponding morphism $C \to \bar{C}$ in $\mathbf{A}$ by Yoneda's lemma.

The proof of the second part is an easy exercise of pretriangulated category theory. $\square$

Let

$$\Xi \colon (\mathrm{Hom}^t)^\# \Rightarrow \mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker}) \tag{4.23}$$

be the natural transformation of $\mathbf{Pair}(\mathbf{A})$-bimodules defined as follows: given $f \colon A \to B$ and $\bar{f} \colon \bar{A} \to \bar{B}$ in $\mathbf{A}$ and an element $\alpha \in (\mathrm{Hom}^t)^\#(f, \bar{f})$ represented by $\tilde{\alpha} \colon tA \to \bar{B}$ in $\mathbf{A}$ then $\Xi(\alpha) \in \mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, tf, \mathrm{Coker}\, \bar{f})$ is represented by the composite

$$\mathrm{Ker}\, tf \hookrightarrow \mathrm{Hom}_{\mathbf{A}}(-, tA) \xrightarrow{\tilde{\alpha}} \mathrm{Hom}_{\mathbf{A}}(-, \bar{B}) \twoheadrightarrow \mathrm{Coker}\, \bar{f},$$

where the first and the last arrows are the inclusion and the projection of the kernel and the cokernel, respectively.

Proposition 4.24. *Given a morphism* $[k_0, k_1, K] \colon f \to \bar{f}$ *in* $\mathbf{Hopair}^{rep}(\mathbf{B})$ *and* $\alpha \in (\mathrm{Hom}^t)^\#(\tilde{p}(f), \tilde{p}(\bar{f}))$ *then*

$$\zeta([k_0, k_1, K] + \alpha) = \zeta([k_0, k_1, K]) + \Xi_{(\tilde{p}(f), \tilde{p}(\bar{f}))}(\alpha).$$

Proof. By using the compatibility of the additive structure on $\mathbf{Hopair}(\mathbf{B})$ given by [BHP97] 6.2 with the linear extension of categories (4.3) and our new definition of cone functors in (4.4) one readily checks that if $\tilde{\alpha} \colon tA \to \bar{B}$ represents α then

$$([k_0, k_1, K] + \alpha)^* = [k_0, k_1, K]^* + \tilde{q}_f \tilde{\alpha}^* \tilde{i}_{\bar{f}} \colon \mathit{Cone}_{\bar{f}} \longrightarrow \mathit{Cone}_f.$$

Here we use the $\mathbf{A}^{op}$-module morphisms in the exact sequence (4.6). By definition of C in (4.9) this implies that

$$C([k_0, k_1, K] + \alpha) = C[k_0, k_1, K] + i_{\bar{f}} \tilde{\alpha} q_f$$

where $i_{\bar{f}}$ and q_f are given by the definition of ζ on objects in (4.11). This last equation yields the formula in the statement. $\square$

We record the following further property of the natural transformations ϖ and Ξ which is going to be relevant in [V].

Lemma 4.25. *The composition of the two following* $\mathbf{cand}(\mathbf{A}, t)$*-bimodule morphisms is trivial*

$$H^1(sh^3, [-,-]^*\bar{t}) \xrightarrow{\varpi} (\mathrm{Hom}^t)^\#(\varsigma, \varsigma) \overset{\Xi(\varsigma^{op}\times\varsigma)}{\longrightarrow} \mathrm{Hom}_{\mathbf{mod}(\mathbf{A})}(\mathrm{Ker}\, t\varsigma, \mathrm{Coker}\, \varsigma).$$

Moreover, if $(\mathbf{A}, t, \nabla)$ *is cohomologically pretriangulated and we restrict to the full subcategory* $\mathbf{cand}(\nabla)$ *of small* ∇*-triangles then the corresponding sequence of* $\mathbf{cand}(\nabla)$*-bimodule morphisms is exact.*

Proof. The first part of the statement is an easy exercise based on the definition of the cochain complex $\tilde{G}^*$ in Section 3, and we leave it to the reader.

Suppose now that $(\mathbf{A}, t, \nabla)$ is cohomologically pretriangulated and that we have two small ∇-triangles

$$T = \left\{A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA\right\},$$

$$\bar{T} = \left\{\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{i}} \bar{C} \xrightarrow{\bar{q}} t\bar{A}\right\},$$

and an element $\alpha \in (\mathrm{Hom}^t)^\#(\varsigma T, \varsigma\bar{T})$ represented by $\tilde{\alpha}\colon tA \to \bar{B}$ such that $\Xi(\alpha) = 0$, i. e. the composite

$$\text{(a)} \qquad \mathrm{Ker}\, tf \hookrightarrow \mathrm{Hom}_{\mathbf{A}}(-, tA) \xrightarrow{\tilde{\alpha}} \mathrm{Hom}_{\mathbf{A}}(-, \bar{B}) \twoheadrightarrow \mathrm{Coker}\, \bar{f}$$

vanishes. Since T and $\bar{T}$ are small ∇-triangles by Theorem 2.17 (pretriangulated case) and well-known properties of pretriangulated categories the sequences of $\mathbf{A}^{op}$-modules

$$\mathrm{Hom}_{\mathbf{A}}(-, C) \xrightarrow{q} \mathrm{Hom}_{\mathbf{A}}(-, tA) \xrightarrow{tf} \mathrm{Hom}_{\mathbf{A}}(-, tB),$$

$$\text{(b)} \qquad \mathrm{Hom}_{\mathbf{A}}(-, \bar{A}) \xrightarrow{\bar{f}} \mathrm{Hom}_{\mathbf{A}}(-, \bar{B}) \xrightarrow{\bar{i}} \mathrm{Hom}_{\mathbf{A}}(-, \bar{C}),$$

are exact, therefore the vanishing of (a) is equivalent to the identity $\bar{i}\tilde{\alpha}q = 0$. Since (b) is exact and $\bar{i}\tilde{\alpha}q = 0$ then by Yoneda's lemma there exists a commutative diagram in $\mathbf{A}$

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "k_0"'] & tA \arrow[r, "tf"] \arrow[d, "\tilde{\alpha}"] & tB \arrow[r, "ti"] & tC \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & t\bar{A}
\end{tikzcd}$$

Now by (Tr0) and (Tr2) the upper row is an exact triangle in the pretriangulated structure for $\mathbf{A}$ given by Theorem 2.17, hence by (Tr3) there is a commutative extension

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "k_0"'] & tA \arrow[r, "tf"] \arrow[d, "\tilde{\alpha}"] & tB \arrow[r, "ti"] \arrow[d, "k_2"] & tC \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{i}"'] & \bar{C} \arrow[r, "\bar{q}"'] & t\bar{A}
\end{tikzcd}$$

By using the commutativity of this diagram the reader can easily check that $(\tilde{\alpha}, -k_2, tk_0) \in \tilde{G}^1([T, \bar{T}]^*\bar{t})$ is a 1-cocycle whose cohomology class c satisfies $\varpi(c) = \alpha$.

□

The following lemmas state translations of axioms (TTr1) and (TTr2).

Lemma 4.26. *The pair* $(\mathbf{B}, s)$ *admits a pretriangulated track category structure if and only if all cone functors are representable and the functor*

$$\zeta\colon \mathbf{Hopair}(\mathbf{B}) \longrightarrow \mathbf{cand}(\mathbf{A}, t)$$

defined in (4.10) takes values in small ∇*-triangles.*

Proof. In order to prove the "if" part we only need to construct a class of distinguished track triangles satisfying (TTr1). For any map f in $\mathbf{B}$ the small candidate triangle $\zeta(f)$ is a small ∇-triangle so it is the image by ϱ of a track triangle

$$\begin{tikzcd}
A \arrow[r, "\tilde f"'] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=35, "\Uparrow H_0"] & B \arrow[r, "i" description] \arrow[rr, bend right=70, "0"'] \arrow[rr, phantom, bend right=35, "\Downarrow H_1"] & C \arrow[r, "q" description] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=35, "\Uparrow H_2"] & sA \arrow[r, "s(\tilde f)"'] & sB
\end{tikzcd}$$

In particular $p(\tilde f) = f$. We choose a track $\gamma\colon f \Rightarrow \tilde f$ and set the track triangle

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[r, bend left=40, "\tilde f" description] \arrow[r, phantom, bend left=20, "\Uparrow\gamma"] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=40, "\Uparrow H_0"] & B \arrow[r, "i" description] \arrow[rr, bend right=70, "0"'] \arrow[rr, phantom, bend right=35, "\Downarrow H_1"] & C \arrow[r, "q" description] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=40, "H_2 \Uparrow"] & sA \arrow[r, "s(f)"'] \arrow[r, bend left=40, "s(\tilde f)" description] \arrow[r, phantom, bend left=20, "\Uparrow s(\gamma)"] & sB
\end{tikzcd}$$

to be distinguished. Now (TTr1) is clearly satisfied.

The "only if" part follows from Propositions 4.12 and 4.19. ☐

Lemma 4.27. *The pair* $(\mathbf{B}, s)$ *admits a triangulated track category structure if and only if there is a full subcategory* $\mathbf{dist}(\mathbf{B}, s)$ *of* $\mathbf{candt}(\mathbf{B}, s)$ *such that the composite functor*

$$\mathbf{dist}(\mathbf{B}, s) \subset \mathbf{candt}(\mathbf{B}, s) \xrightarrow{\bar\varrho_{(\mathbf{B},s)}} \mathbf{Hopair}(\mathbf{B}) \xrightarrow{\tilde p_{\mathbf{B}}} \mathbf{Pair}(\mathbf{A}) \tag{a}$$

is surjective on objects and the restriction of $\bar\varrho_{(\mathbf{B},s)}$ *to* $\mathbf{dist}(\mathbf{B}, s)$ *is full.*

Proof. If $(\mathbf{B}, s)$ is a triangulated track category and $\mathbf{dist}(\mathbf{B}, s)$ is given by distinguished track triangles then (a) is surjective on objects by (TTr1), since $\tilde p$ is already known to be surjective on objects, and $\bar\varrho_{(\mathbf{B},s)}$ restricted to $\mathbf{dist}(\mathbf{B}, s)$ is full by (TTr2).

Conversely if (a) is surjective on objects we can define distinguished track triangles as follows. We choose for any morphism $f\colon A \to B$ in $\mathbf{A}$ a track triangle in $\mathbf{dist}(\mathbf{B}, s)$

$$\begin{tikzcd}
A \arrow[r, "\ell f"'] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=35, "\Uparrow H_0^f"] & B \arrow[r, "i_f" description] \arrow[rr, bend right=70, "0"'] \arrow[rr, phantom, bend right=35, "\Downarrow H_1^f"] & C \arrow[r, "q_f" description] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, bend left=35, "\Uparrow H_2^f"] & sA \arrow[r, "s(\ell f)"'] & sB
\end{tikzcd} \tag{b}$$

with $p(\ell f) = f$. Moreover, for any map $g\colon A \to B$ in $\mathbf{B}$ we choose a track $\gamma_g\colon \ell p(g) \Rightarrow g$ with $\gamma_{\ell f} = 0^{\square}_{\ell f}$. Now we define the distinguished track triangles as the track triangles (c)

$$\begin{tikzcd}
A \arrow[r, "g"'] \arrow[r, bend left=50, "\ell p(g)"{name=U}] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i_{p(g)}"] \arrow[rr, bend right=40, "0"'] & C \arrow[r, "q_{p(g)}"] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "sg"'] \arrow[r, bend left=50, "s\ell p(g)"] & sB
\arrow[from=1-1, to=1-2, phantom, "\Uparrow \gamma_g^{\boxminus}"]
\arrow[from=1-2, to=1-2, phantom, "\Uparrow H_0^{p(g)}"]
\arrow[from=1-3, to=1-3, phantom, "\Downarrow H_1^{p(g)}"']
\arrow[from=1-4, to=1-4, phantom, "\Uparrow H_2^{p(g)}"]
\arrow[from=1-4, to=1-5, phantom, "\Uparrow s\gamma_g^{\boxminus}"]
\end{tikzcd}$$

for all maps g in $\mathbf{B}$. The axiom (TTr1) is now clearly satisfied. Moreover, (TTr2) also holds since (c) is isomorphic to (b) for $f = p(g)$. The isomorphism is given by diagram (c) itself. ☐

Now we are ready to prove Theorem 2.15.

Proof of Theorem 2.15. As in the proof of Proposition 2.12 we choose a pseudofunctor $\varphi\colon \mathbf{B} \rightsquigarrow \mathbf{C}$ in the conditions of Definition 2.3 realizing the equivalence between $(\mathbf{B}, s)$ and $(\mathbf{C}, r)$ and a homotopy $\xi\colon r\varphi \Rightarrow \varphi s$. We recall that the pseudofunctors s and r coincide on objects (they both actually coincide with t) and that φ is the identity on objects, in particular by [III] A.14 we can suppose that φ is normalized with respect to zero maps. In the proof of Proposition 2.12 we defined a functor

$$(\varphi, \xi)_*\colon \mathbf{candt}(\mathbf{B}, s) \longrightarrow \mathbf{candt}(\mathbf{C}, r)$$

induced by φ and ξ. The pseudofunctor φ also induces a functor between the categories of homotopy pairs

$$\varphi_*\colon \mathbf{Hopair}(\mathbf{B}) \longrightarrow \mathbf{Hopair}(\mathbf{C}).$$

This functor sends a map $f\colon A \to B$ in $\mathbf{B}$ regarded as an object in $\mathbf{Hopair}(\mathbf{B})$ to $\varphi(f)\colon A \to B$. The image of the morphism represented by the diagram

$$\begin{tikzcd}
A \arrow[r, "k_0"] \arrow[d, "f"'] & \bar{A} \arrow[d, "\bar{f}"] \\
B \arrow[r, "k_1"'] \arrow[ur, Rightarrow, shorten=4mm, "K"'] & \bar{B}
\end{tikzcd}$$

in $\mathbf{B}$ is given by

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "\varphi(k_0)"] \arrow[d, "\varphi(f)"'] \arrow[dr, bend left=25, "\varphi(\bar{f}k_0)"] \arrow[dr, bend right=25, "\varphi(k_1 f)"'] & \bar{A} \arrow[d, "\varphi(\bar{f})"] \\
B \arrow[r, "\varphi(k_1)"'] & \bar{B}
\arrow[from=2-1, to=1-2, phantom, "\overset{\varphi^{\boxminus}_{\bar{f},k_0}}{\Nearrow}\ \overset{\varphi(K)}{\Nearrow}\ \overset{\varphi_{k_1,f}}{\Nearrow}"]
\end{tikzcd}$$

The following diagram of functors is commutative

(a)
$$\begin{tikzcd}
\mathbf{candt}(\mathbf{B},s) \arrow[r, "\bar{\varrho}_{(\mathbf{B},s)}"] \arrow[d, "(\varphi,\xi)_*"'] & \mathbf{Hopair}(\mathbf{B}) \arrow[d, "\varphi_*"] \\
\mathbf{candt}(\mathbf{C},r) \arrow[r, "\bar{\varrho}_{(\mathbf{C},r)}"'] & \mathbf{Hopair}(\mathbf{C})
\end{tikzcd}$$

Moreover, φ_* induces a map of weak linear extensions

(b)
$$\begin{tikzcd}
(\mathrm{Hom}^t)^\# \arrow[r, "+"] \arrow[d, equal] & \mathbf{Hopair}(\mathbf{B}) \arrow[r, "\tilde{p}_{\mathbf{B}}"] \arrow[d, "\varphi_*"] & \mathbf{Pair}(\mathbf{A}) \arrow[d, equal] \\
(\mathrm{Hom}^t)^\# \arrow[r, "+"] & \mathbf{Hopair}(\mathbf{C}) \arrow[r, "\tilde{p}_{\mathbf{C}}"] & \mathbf{Pair}(\mathbf{A})
\end{tikzcd}$$

In particular φ_* is an equivalence of categories.

The equivalence φ_* in diagram (b) determines isomorphisms between the cone functors defined by $\mathbf{B}$ and by $\mathbf{C}$ since cone functors are restriction of Hom-functors in the category of homotopy pairs, see (4.4). In particular φ_* restricts to the full subcategories given by maps with representable associated cone functor and the functors ζ defined in Remark 4.8 fit into a diagram commuting up to equivalence

$$\begin{tikzcd}
\mathbf{Hopair}^{rep}(\mathbf{B}) \arrow[dd, "\varphi_*", "\sim"'] \arrow[dr, "\zeta"] & \\
& \mathbf{cand}(\mathbf{A},t) \\
\mathbf{Hopair}^{rep}(\mathbf{C}) \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

This and Lemma 4.26 prove that $(\mathbf{B},s)$ admits a pretriangulated track category structure provided $(\mathbf{C},r)$ does. The converse follows by symmetry.

Suppose now that $(\mathbf{B},s)$ admits a triangulated track category structure. Then by Lemma 4.27 there is a full subcategory $\mathbf{dist}(\mathbf{B},s)$ of $\mathbf{candt}(\mathbf{B},s)$ such that, over $\mathbf{dist}(\mathbf{B},s)$, the functor $\tilde{p}_{\mathbf{B}}\bar{\varrho}_{(\mathbf{B},s)}$ is surjective on objects and $\bar{\varrho}_{(\mathbf{B},s)}$ is full. If we define $\mathbf{dist}(\mathbf{C},r)$ as the full subcategory of $\mathbf{candt}(\mathbf{C},r)$ given by the images by $(\varphi,\xi)_*$ of the objects in $\mathbf{dist}(\mathbf{B},s)$ then the commutativity of (a) and (b) and the fact that φ_* is an equivalence imply that, over $\mathbf{dist}(\mathbf{C},r)$, the functor $\tilde{p}_{\mathbf{C}}\bar{\varrho}_{(\mathbf{C},r)}$ is surjective on objects and $\bar{\varrho}_{(\mathbf{C},r)}$ is full, therefore again by Lemma 4.26 we have that $(\mathbf{C},r)$ admits a triangulated track category structure. The converse follows also by symmetry. ☐

5 The octahedral axiom

This section contains the results which allow us to prove the octahedral axiom for cohomologically triangulated categories. In this section $\mathbf{A}$ is always an additive category and $t\colon \mathbf{A} \to \mathbf{A}$ an additive equivalence.

Proposition 5.1. *Let $(\mathbf{B},s)$ be a good translation track category representing $\nabla \in H^*(\mathbf{A},t)$. Suppose in addition that $(\mathbf{A},t,\nabla)$ is cohomologically pretriangulated. Con-*

sider two composable morphisms of track triangles as follows

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "i_f"] \arrow[d, "g"] & C_f \arrow[r, "q_f"] \arrow[d, "\bar{g}"] \arrow[dl, Rightarrow, "F"] & sA \arrow[d, equal] \arrow[dl, Rightarrow, "G"] \\
A \arrow[r, "gf" description] \arrow[d, "f"'] & C \arrow[r, "i_{gf}" description] \arrow[d, equal] & C_{gf} \arrow[r, "q_{gf}" description] \arrow[d, "\bar{f}"] \arrow[dl, Rightarrow, "F'"] & sA \arrow[d, "sf"] \arrow[dl, Rightarrow, "G'"] \\
B \arrow[r, "g"'] & C \arrow[r, "i_g"'] & C_g \arrow[r, "q_g"'] & sB
\end{tikzcd}$$

then

$$C_f \xrightarrow{p(\bar{g})} C_{gf} \xrightarrow{p(\bar{f})} C_g \xrightarrow{(tp(i_f))p(q_g)} tC_f$$

is a small ∇-triangle.

For the proof of this proposition we need the exact sequence in the following lemma.

Lemma 5.2. *Let $(\mathbf{B}, s)$ be a good translation track category over $(\mathbf{A}, t)$ and let t^{-1} be an inverse equivalence for t with $\mu\colon tt^{-1} \cong 1_{\mathbf{A}}$ a natural isomorphism. Given morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{B}$ there is an exact sequence of $\mathbf{A}$-modules*

$$Cone_{gf}t^{-1} \xrightarrow{[1_A,g,0^{\square}_{gf}]^*t^{-1}} Cone_f t^{-1} \xrightarrow{\partial} Cone_g \xrightarrow{[f,1_C,0^{\square}_{gf}]^*} Cone_{gf} \xrightarrow{[1_A,g,0^{\square}_{gf}]^*} Cone_f$$

where ∂ is the composite

$$\partial\colon Cone_f t^{-1} \overset{\tilde{i}_f t^{-1}}{\to} \mathrm{Hom}_{\mathbf{A}}(B, t^{-1}) \overset{t}{\cong} \mathrm{Hom}_{\mathbf{A}}(tB, tt^{-1}) \overset{\mu_*}{\cong} \mathrm{Hom}_{\mathbf{A}}(tB, -) \overset{\tilde{q}_g}{\to} Cone_g.$$

Compare [I] 3.6.

Proof of Proposition 5.1. This proof is essentially and adaptation of [I] 9 to the more general case considered in this paper. Consider the following diagram

$$\begin{tikzcd}
Cone_{gf}t^{-1} \arrow[r, "{[1_A,g,0^{\square}_{gf}]^*t^{-1}}"] \arrow[dd, "\chi^{-1}_{\{i_{gf},H_0^{gf}\}}t^{-1}", "\cong"'] \arrow[ddr, phantom, "\spadesuit"] & Cone_f t^{-1} \arrow[rr, "\partial"] \arrow[dr, "\tilde{i}_f t^{-1}"] \arrow[dd, "\chi^{-1}_{\{i_f,H_0^f\}}t^{-1}", "\cong"'] \arrow[ddr, phantom, "\clubsuit"] & & Cone_g \arrow[r, "{[f,1_C,0^{\square}_{gf}]^*}"] \arrow[dddddd, "\chi^{-1}_{\{i_g,H_0^g\}}"', "\cong"] \arrow[dddddd, phantom, bend left=10, "\spadesuit"] & Cone_{gf} \arrow[r, "{[1_A,g,0^{\square}_{gf}]}"] \arrow[dddddd, "\chi^{-1}_{\{i_{gf},H_0^{gf}\}}"', "\cong"] \arrow[dddddd, phantom, bend left=10, "\spadesuit"] & Cone_f \arrow[dddddd, "\chi^{-1}_{\{i_f,H_0^f\}}"', "\cong"] \\
& & \mathrm{Hom}_{\mathbf{A}}(B,t^{-1}) \arrow[dd, "t", "\cong"'] & & & \\
\mathrm{Hom}_{\mathbf{A}}(C_{gf},t^{-1}) \arrow[r, "p(\bar{g})^*"] \arrow[dd, "t", "\cong"'] & \mathrm{Hom}_{\mathbf{A}}(C_f,t^{-1}) \arrow[ur, "p(i_f)^*"'] \arrow[dd, "t", "\cong"'] & & & & \\
& & \mathrm{Hom}_{\mathbf{A}}(tB,tt^{-1}) \arrow[dd, "\mu_*", "\cong"'] \arrow[ddr, phantom, "\clubsuit"] & & & \\
\mathrm{Hom}_{\mathbf{A}}(tC_{gf},tt^{-1}) \arrow[r, "(tp(\bar{g}))^*"] \arrow[dd, "\mu_*", "\cong"'] & \mathrm{Hom}_{\mathbf{A}}(tC_f,tt^{-1}) \arrow[ur, "(tp(i_f))^*"'] \arrow[dd, "\mu_*", "\cong"'] & & & & \\
& & \mathrm{Hom}_{\mathbf{A}}(tB,-) \arrow[dr, "p(q_g)^*"] \arrow[uuuuur, bend right=10, "\tilde{q}_g"] & & & \\
\mathrm{Hom}_{\mathbf{A}}(tC_{gf},-) \arrow[r, "(tp(\bar{g}))^*"] & \mathrm{Hom}_{\mathbf{A}}(tC_f,-) \arrow[ur, "(tp(i_f))^*"] \arrow[rr, "((tp(i_f))p(q_g))^*"'] & & \mathrm{Hom}_{\mathbf{A}}(C_g,-) \arrow[r, "p(\bar{f})^*"] & \mathrm{Hom}_{\mathbf{A}}(C_{gf},-) \arrow[r, "p(\bar{g})^*"] & \mathrm{Hom}_{\mathbf{A}}(C_f,-)
\end{tikzcd}$$

(a)

This diagram is commutative. The subdiagrams labeled with a ♠ commute by Lemma 4.17, and the subdiagrams labeled with a ♣ by Remark 4.13. The commutativity of the other subdiagrams is trivial.

Consider now the diagram

(b)

$$\begin{tikzcd}
Cone_{gf}t^{-1} \arrow[r, "{[1_A,g,0^{\square}_{gf}]^*t^{-1}}"] & Cone_f t^{-1} \arrow[rr, "\partial"] \arrow[dr, "\tilde{i}_f t^{-1}"] & & Cone_g \arrow[r, "{[f,1_C,0^{\square}_{gf}]^*}"] & Cone_{gf} \arrow[r, "{[1_A,g,0^{\square}_{gf}]}"] & Cone_f \\
 & & \mathrm{Hom}_{\mathbf{A}}(B,t^{-1}) \arrow[dd, "t", "\cong"'] & & & \\
\mathrm{Hom}_{\mathbf{A}}(C_{gf},t^{-1}) \arrow[uu, "\chi_{\left\{i_{gf},H_0^{gf}\right\}}^{t^{-1}}", "\cong"'] \arrow[r, "p(\bar{g})^*"] & \mathrm{Hom}_{\mathbf{A}}(C_f,t^{-1}) \arrow[uu, "\chi_{\left\{i_f,H_0^{f}\right\}}^{t^{-1}}", "\cong"'] \arrow[ur, "p(i_f)^*"'] & & & & \\
 & & \mathrm{Hom}_{\mathbf{A}}(tB,tt^{-1}) \arrow[dd, "\mu_*", "\cong"'] & & & \\
\mathrm{Hom}_{\mathbf{A}}(tC_{gf},tt^{-1}) \arrow[uu, "t^{-1}", "\cong"'] \arrow[r, "(tp(\bar{g}))^*"] & \mathrm{Hom}_{\mathbf{A}}(tC_f,tt^{-1}) \arrow[uu, "t^{-1}", "\cong"'] \arrow[ur, "(tp(i_f))^*"'] & \arrow[r, phantom, "\clubsuit"] & \arrow[r, phantom, "\spadesuit"] & & \\
 & & \mathrm{Hom}_{\mathbf{A}}(tB,-) \arrow[uuuuur, bend right, "\tilde{q}_g"] & & & \\
\mathrm{Hom}_{\mathbf{A}}(tC_{gf},-) \arrow[uu, "\mu_*^{-1}", "\cong"'] \arrow[r, "(tp(\bar{g}))^*"] & \mathrm{Hom}_{\mathbf{A}}(tC_f,-) \arrow[uu, "\mu_*^{-1}", "\cong"'] \arrow[ur, "(tp(i_f))^*"] \arrow[rr, "\tilde{q}_{\bar{g}}"'] & & Cone_{\bar{g}} \arrow[uuuuuu, "{[i_f,i_{gf},F^{\boxminus}]^*}"'] \arrow[r, "\tilde{i}_{\bar{g}}"] & \mathrm{Hom}_{\mathbf{A}}(C_{gf},-) \arrow[uuuuuu, "\chi_{\left\{i_{gf},H_0^{gf}\right\}}", "\cong"'] \arrow[r, "p(\bar{g})^*"] & \mathrm{Hom}_{\mathbf{A}}(C_f,-) \arrow[uuuuuu, "\chi_{\left\{i_f,H_0^{f}\right\}}", "\cong"']
\end{tikzcd}$$

We claim that this diagram also commutes. Actually we only have to check the commutativity of ♣ and ♠. The commutativity of ♣ follows from the next equalities:

$$\begin{aligned}
[f,1_C,0^{\square}_{gf}]^*\tilde{q}_{\bar{g}}(h) &= [f,1_C,0^{\square}_{gf}]^*\{0,\sigma_0(h)\} \\
&= \left\{0i_{gf},\sigma_0(h)\square(0F^{\boxminus})\right\} \\
&= \{0,\sigma_0(h)i_f\} \\
&= \{0,\sigma_0(h(tp(i_f)))\} \\
&= \tilde{q}_g(h(tp(i_f))) \\
&= \tilde{q}_g(tp(i_f))^*(h).
\end{aligned}$$

Given now $\{\alpha,\hat{\alpha}\}\in Cone_{\bar{g}}(X)$ we have that $[f,1_C,0^{\square}_{gf}]^*[i_f,i_{gf},F^{\boxminus}]^*\{\alpha,\hat{\alpha}\}$ is represented by

$$\begin{tikzcd}
 & C_f \arrow[r, "\bar{g}"] \arrow[rr, bend left=60, "0"] & C_{gf} \arrow[r, "\alpha"'] \arrow[u, phantom, "\Uparrow \tilde{\alpha}"] & X \\
A \arrow[r, "f"'] & B \arrow[u, "i_f"] \arrow[r, "g"'] & C \arrow[u, "i_{gf}"'] \arrow[ul, Rightarrow, "F^{\boxminus}"'] &
\end{tikzcd}$$

This composite track coincides clearly with

$$\begin{tikzcd}
 & C_f \arrow[r, "\bar{g}"] \arrow[rr, bend left=60, "0"] & C_{gf} \arrow[r, "\alpha"'] \arrow[u, phantom, "\Uparrow \tilde{\alpha}"] & X \\
A \arrow[ur, bend left, "0"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\Uparrow H_0^f"] & B \arrow[u, "i_f"] \arrow[r, "g"'] & C \arrow[u, "i_{gf}"'] \arrow[ul, Rightarrow, "F^{\boxminus}"'] &
\end{tikzcd}$$

This other one is obviously the same as

$$\begin{tikzcd}
 & C_f \arrow[r, "\bar{g}"] & C_{gf} \arrow[r, "\alpha"] & X \\
A \arrow[ur, bend left, "0"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\Uparrow H_0^f"] & B \arrow[u, "i_f"] \arrow[r, "g"'] & C \arrow[u, "i_{gf}"'] \arrow[ul, Rightarrow, "F^{\boxminus}"'] &
\end{tikzcd}$$

The conditions defining track triangle morphisms show that this composite track coincides with

$$\begin{tikzcd}
A \arrow[r, "gf"'] \arrow[rr, bend left=60, "0"] & C \arrow[r, "i_{gf}"'] \arrow[u, phantom, "\Uparrow H_0^{gf}"] & C_{gf} \arrow[r, "\alpha"] & X
\end{tikzcd}$$

and this composite track represents $\chi_{\{i_{gf},H_0^{gf}\}}\tilde{i}_{\bar{g}}\{\alpha,\hat{\alpha}\}\in Cone_{gf}(X)$, so ♠ commutes.

The commutativity of (b), Lemma 5.2, (4.6) and the five lemma show that $[i_f,i_{gf},F^{\boxminus}]^*$ is an isomorphism. Moreover, since $(\mathbf{A},t,\nabla)$ is cohomologically pretriangulated there is a track triangle

$$\begin{tikzcd} C_f \arrow[r, "\bar{g}"'] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_0^{\bar{g}}"] & C_{gf} \arrow[r, "i_{\bar{g}}"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow H_1^{\bar{g}}"] & C_{\bar{g}} \arrow[r, "q_{\bar{g}}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow H_2^{\bar{g}}"] & sC_f \arrow[r, "s(\bar{g})"'] & sC_{gf} \end{tikzcd} .$$

and a commutative diagram

$$\begin{tikzcd} & & \mathrm{Hom}_{\mathbf{A}}(C_{\bar{g}},-) \arrow[dr, "p(i_{\bar{g}})^*"] \arrow[dd, "\chi\{i_{\bar{g}},H_0^{\bar{g}}\}", "\cong"'] & & \\ \mathrm{Hom}_{\mathbf{A}}(tC_{gf},-) \arrow[r, "(tp(\bar{g}))^*"] & \mathrm{Hom}_{\mathbf{A}}(tC_f,-) \arrow[ur, "p(q_{\bar{g}})^*"] \arrow[dr, "\tilde{q}_{\bar{g}}"'] & & \mathrm{Hom}_{\mathbf{A}}(C_{gf},-) \arrow[r, "p(\bar{g})^*"] & \mathrm{Hom}_{\mathbf{A}}(C_f,-), \\ & & Cone_{\bar{g}} \arrow[ur, "\tilde{i}_{\bar{g}}"'] & & \end{tikzcd}$$

(c)

see Remark 4.13. Therefore composing (a), (b) and (c) and applying Yoneda's lemma we obtain an isomorphism of small candidate triangles

$$\begin{tikzcd}
C_f \arrow[r, "p(\bar{g})"] \arrow[d, equal] & C_{gf} \arrow[r, "p(\bar{f})"] \arrow[d, equal] & C_g \arrow[r, "(tp(i_f))p(q_g)"] \arrow[d, "l", "\cong"'] & tC_f \arrow[d, equal] \\
C_f \arrow[r, "p(\bar{g})"'] & C_{gf} \arrow[r, "p(i_{\bar{g}})"'] & C_{\bar{g}} \arrow[r, "p(q_{\bar{g}})"'] & tC_f
\end{tikzcd}$$

where $l\colon C_g \to C_{\bar{g}}$ represents

$$\chi^{-1}_{\left\{i_g, H_0^g\right\}}[i_f, i_{gf}, F^{\boxminus}]^* \chi_{\left\{i_{\bar{g}}, H_0^{\bar{g}}\right\}}\colon \operatorname{Hom}_{\mathbf{A}}(C_{\bar{g}}, -) \cong \operatorname{Hom}_{\mathbf{A}}(C_g, -).$$

Since the lower candidate triangle is a small ∇-triangle, the upper one is also a small ∇-triangle by Theorem 2.17 (pretriangulated case) and (Tr0), hence the proof is finished. ☐

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Triangulated cohomology classes

H.-J. Baues and F. Muro

Abstract

A cohomologically triangulated category is an additive category **A** together with a translation functor t and a cohomology class $\nabla \in H^3(\mathbf{A}, t)$. We show that ∇ satisfies certain purely cohomological conditions if and only if ∇ is represented by a triangulated track category. This yields a purely cohomologically characterization of triangulated cohomology classes.

Introduction

In [IV] we define a cohomologically triangulated category as a triple $(\mathbf{A}, t, \nabla)$ given by an additive category **A**, an additive equivalence $t\colon \mathbf{A} \to \mathbf{A}$ and a cohomology class

$$\nabla \in H^3(\mathbf{A}, t)$$

in the translation cohomology of the pair $(\mathbf{A}, t)$ in the sense of [III] such that ∇ is represented by a good translation track category satisfying two axioms (TTr1) and (TTr2). For this we use the main result of [III], which says that 3-dimensional translation cohomology classes classify translation track categories.

The aim of this paper is to characterize cohomologically triangulated categories in a purely cohomological way, i. e. without invoking the fact that translation cohomology classes are represented by translation track categories. This is achieved in Theorem 7.1 as a consequence of Theorems 6.11 and 6.14. There we obtain vanishing conditions on some new cohomology classes associated to ∇ which imply that $(\mathbf{A}, t, \nabla)$ is a cohomologically triangulated category. These new cohomology classes are obtained from ∇ by using spectral sequences constructed in Section 8, see Section 7.

With this new characterization one can forget about the techniques of track categories in the previous papers since the axioms (TTr1) and (TTr2) are completely encoded in the translation cohomology class ∇. The original definition in terms of track categories is the most convenient one to check that stable homotopy categories give rise to cohomologically triangulated categories. However the cohomological approach will be more convenient for a further study of cohomologically triangulated categories, like their localizations, categories of diagrams over them, t-structures, monoidal structures, etc.

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Notation

In this paper the arrow → is used for morphisms in ordinary categories, 1-cells in 2-categories and functors. We use ⇒ for 2-cells in 2-categories and natural transformations, and ⇝ for pseudofunctors. Identity morphisms or 1-cells are denoted by 1, with the object as a subscript when necessary. For the identity 2-cells and natural transformations we use the symbol $0^\square$. Horizontal composition in 2-categories is denoted by juxtaposition, as the composition law in ordinary categories, while we use the symbol $\square$ for vertical composition. The vertical inverse of an invertible 2-cell or natural transformation α is $\alpha^\boxminus$.

1 Translation categories

A *translation category* $(\mathbf{A}, t)$ is a category $\mathbf{A}$ with a zero object $*$ together with an endofunctor $t\colon \mathbf{A} \to \mathbf{A}$ preserving the zero object $*$. Notice that (pre)triangulated categories yield examples of translation categories.

It is well known that the category $\mathbf{Cat}$ of all small categories with zero object is a 2-category with zero-preserving functors as 1-cells and natural transformations as 2-cells. Small translation categories also form a 2-category $\mathbf{TCat}$. A morphism (or 1-cell) of translation categories $(\varphi, \alpha)\colon (\mathbf{A}, t) \to (\mathbf{B}, s)$ is a functor $\varphi\colon \mathbf{A} \to \mathbf{B}$ preserving zero objects together with a natural isomorphism $\alpha\colon s\varphi \Rightarrow \varphi t$, i. e. the diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "t"] \arrow[d, "\varphi"'] & \mathbf{A} \arrow[d, "\varphi"] \\
\mathbf{B} \arrow[r, "s"'] & \mathbf{B}
\end{tikzcd}$$

commutes up to the natural isomorphism α. Composition of morphisms is defined by

$$(\varphi, \alpha)(\psi, \beta) = (\varphi\psi, (\varphi\beta)\square(\alpha\psi)).$$

A 2-cell $\gamma\colon (\varphi, \alpha) \Rightarrow (\psi, \beta)$ between two morphisms $(\varphi, \alpha), (\psi, \beta)\colon (\mathbf{A}, t) \to (\mathbf{B}, s)$ in $\mathbf{TCat}$ is a natural transformation $\gamma\colon \varphi \Rightarrow \psi$ such that

$$\beta\square(s\gamma) = (\gamma t)\square\alpha.$$

Vertical and horizontal composition of 2-cells in $\mathbf{TCat}$ are given by the corresponding composition laws between natural transformations in $\mathbf{Cat}$.

Ordinary categories $\mathbf{C}$ with zero object can be regarded as translation categories by using identity functors $\mathbf{C} = (\mathbf{C}, 1_\mathbf{C})$. This defines an inclusion of the 2-category $\mathbf{Cat}$ into $\mathbf{TCat}$. The 2-category $\mathbf{TCat}$ has products $(\mathbf{A}, t) \times (\mathbf{B}, s) = (\mathbf{A} \times \mathbf{B}, t \times s)$.

Morphisms between translation categories and 2-cells between them are also defined for non-small translation categories exactly in the same way.

We shall need the cohomology of translation categories, in particular, the cohomology of a triangulated category. For this we recall first the concept of cohomology of categories, see [BW85].

2 Cohomology of categories

Given a category $\mathbf{C}$ a $\mathbf{C}$-*bimodule* D is a functor $D\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$ where $\mathbf{Ab}$ is the category of abelian groups. We write $f^* = D(f, 1)$ and $g_* = D(1, g)$ for the induced homomorphisms. The cohomology $H^*(\mathbf{C}, D)$ of $\mathbf{C}$ with coefficients in D is defined by the cochain complex $F^*(\mathbf{C}, D)$ concentrated in dimensions ≥ 0. This complex is given in dimension n by the following product indexed by all sequences of morphisms of length n in $\mathbf{C}$,

$$F^n(\mathbf{C}, D) = \prod_{X_0 \overset{\sigma_1}{\leftarrow} \cdots \overset{\sigma_n}{\leftarrow} X_n} D(X_n, X_0). \tag{2.1}$$

Here we understand that a sequence of length 0 is the identity morphism of an object X in $\mathbf{C}$. The coordinate of $c \in F^n(\mathbf{C}, D)$ in $X_0 \overset{\sigma_1}{\leftarrow} \cdots \overset{\sigma_n}{\leftarrow} X_n$ will be denoted by $c(\sigma_1, \dots, \sigma_n) \in D(X_n, X_0)$. The differential δ is defined as

$$\begin{aligned} \delta(c)(\sigma_1, \dots, \sigma_{n+1}) \quad = \quad & \sigma_{1*} c(\sigma_2, \dots, \sigma_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i c(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ & + (-1)^{n+1} \sigma_{n+1}^* c(\sigma_1, \dots, \sigma_n) \end{aligned}$$

over an n-cochain c for $n \geq 1$, and $\delta(c)(\sigma) = \sigma_* c(X) - \sigma^* c(Y)$ for $n = 0$ and $\sigma \colon X \to Y$.

In order $H^*(\mathbf{C}, D)$ to be a set we assume that $\mathbf{C}$ is small, one can use this notation accordingly if $\mathbf{C}$ is small with respect to some fixed universe.

This is a particular case of the Baues-Wirsching cohomology of $\mathbf{C}$, see [BW85]. It also coincides with the Hochschild-Mitchell cohomology of the free ringoid $\mathbb{Z}\mathbf{C}$ generated by the category $\mathbf{C}$, see [Mit72].

A functor $\varphi \colon \mathbf{D} \to \mathbf{C}$ and a natural transformation $\lambda \colon D \Rightarrow E$ induce the following cochain homomorphisms where X_n is the source of σ_n and X_0 is the target of σ_1,

$$\varphi^* \colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D(\varphi, \varphi)), \;\; \varphi^*(c)(\sigma_1, \dots, \sigma_n) = c(\varphi(\sigma_1), \dots, \varphi(\sigma_n));$$

$$\lambda_* \colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{C}, E), \;\; \lambda_*(c)(\sigma_1, \dots, \sigma_n) = \lambda_{X_n, X_0}(c(\sigma_1, \dots, \sigma_n)); \tag{2.2}$$

and the corresponding homomorphisms in cohomology,

$$\begin{aligned} \varphi^* \colon H^*(\mathbf{C}, D) &\longrightarrow H^*(\mathbf{D}, D(\varphi, \varphi)), \\ \lambda_* \colon H^*(\mathbf{C}, D) &\longrightarrow H^*(\mathbf{C}, E). \end{aligned}$$

A cochain $c \in F^n(\mathbf{C}, D)$ is said to be *normalized* (at identities) if $c(\sigma_1, \dots, \sigma_n) = 0$ provided σ_i is an identity morphism for some $1 \leq i \leq n$. The inclusion of the subcomplex $\bar{F}^*(\mathbf{C}, D) \subset F^*(\mathbf{C}, D)$ of normalized cochains induces an isomorphism in cohomology, see [BD89]. Moreover, the induced cochain homomorphisms above restrict to normalized cochains.

In this paper we are mainly interested in categories $\mathbf{C}$ with a zero object $*$. In this case morphism sets $\mathbf{C}(X, Y)$ in $\mathbf{C}$ are pointed by the zero morphisms $0_{X,Y} = 0 \colon X \to * \to Y$. Suppose that the $\mathbf{C}$ bimodule D vanishes whenever the zero object is in one side $D(X, *) = 0 = D(*, X)$. In this case we say that D is $*$-*normalized*. A cochain $c \in F^n(\mathbf{C}, D)$ is said to be $*$-*normalized* (i. e. normalized at identities and zero morphisms) if $c(\sigma_1, \dots, \sigma_n) = 0$ provided σ_i is an identity or a zero morphism for some $1 \leq i \leq n$. If D is $*$-normalized the inclusion of the subcomplex $\tilde{F}^*(\mathbf{C}, D) \subset F^*(\mathbf{C}, D)$ of $*$-normalized cochains also induces an isomorphism in cohomology, see [BD89]. Moreover, the induced cochain homomorphisms φ^* and λ_* above restrict to $*$-normalized cochains when φ preserves the zero object and E is $*$-normalized.

A natural transformation $\alpha\colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi\colon \mathbf{D} \to \mathbf{C}$ induces a cochain homomorphism

$$\alpha^*\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D(\varphi, \psi))$$

defined as

$$\alpha^*(c)(\sigma_1, \cdots, \sigma_n) = \alpha_{X_0 *}(c(\varphi(\sigma_1), \dots, \varphi(\sigma_n))).$$

This is $F^*(\alpha, 0^{\square}_{D(\varphi,\psi)})$ in [Mur04] 4.1. Here X_0 is the target of σ_1. The induced homomorphism in cohomology is denoted in the same way

$$\alpha^*\colon H^*(\mathbf{C}, D) \longrightarrow H^*(\mathbf{D}, D(\varphi, \psi)).$$

Notice that the cochain homomorphism defined by an identity natural transformation $0^{\square}_{\varphi}\colon \varphi \Rightarrow \varphi$ coincides with the cochain homomorphism defined by the corresponding functor, i. e. $(0^{\square}_{\varphi})^* = \varphi^*$. The natural transformation $\alpha\colon \varphi \Rightarrow \psi$ also induces cochain homotopies given by degree -1 homomorphisms

$$\alpha_{\#}, \alpha^{\#}\colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, D(\varphi, \psi)),$$

with

$$\delta\alpha_{\#} + \alpha_{\#}\delta = -(D(0^{\square}_{\varphi}, \alpha))_* \varphi^* + \alpha^*, \tag{2.3}$$

$$\delta\alpha^{\#} + \alpha^{\#}\delta = -(D(\alpha, 0^{\square}_{\psi}))_* \psi^* + \alpha^*. \tag{2.4}$$

They are defined by the following formulas

$$\alpha_{\#}(c)(\sigma_1, \dots, \sigma_n) = \sum_{i=0}^{n} (-1)^i \alpha_{X_0 *}(c(\varphi(\sigma_1), \dots, \varphi(\sigma_i), 1_{\varphi(X_i)}, \varphi(\sigma_{i+1}), \dots, \varphi(\sigma_n))),$$

$$\alpha^{\#}(c)(\sigma_1, \dots, \sigma_n) = \sum_{i=0}^{n} (-1)^i c(\psi(\sigma_1), \dots, \psi(\sigma_i), \alpha_{X_i}, \varphi(\sigma_{i+1}), \dots, \varphi(\sigma_n)).$$

Here X_i is the source of σ_i and/or the target of σ_{i+1}. These homotopies are part of the 2-functorial properties of the Baues-Wirsching complex $F^*(\mathbf{C}, D)$ studied in [Mur04]. More precisely, $\alpha_{\#}$ and $\alpha^{\#}$ are $h_{(0^{\square}_{\varphi}, \alpha)}$ and $h_{(\alpha, 0^{\square}_{\psi})}$ in [Mur04] 4.1, respectively. Notice that these induced cochain homomorphisms and homotopies restrict always to the subcomplex of normalized cochains, and also to the subcomplex of $*$-normalized cochains if the functors φ and ψ preserve the zero object. Moreover, as one can readily notice on the normalized cochain complex the formulas $\alpha_{\#} = 0$, $(0^{\square}_{\varphi})^{\#} = 0$ and $\alpha^* = (D(0^{\square}_{\varphi}, \alpha))_* \varphi^*$ hold, in particular by (2.3) and (2.4) we have that

$$\delta\alpha^{\#} + \alpha^{\#}\delta = -(D(\alpha, 0^{\square}_{\psi}))_* \psi^* + (D(0^{\square}_{\varphi}, \alpha))_* \varphi^*. \tag{2.5}$$

3 Translation cohomology

Let $(\mathbf{A}, t)$ be a translation category in the sense of Section 1, D a $*$-normalized $\mathbf{A}$-bimodule and $\bar{t}\colon D \Rightarrow D(t, t)$ a natural transformation. Then the pair $(t, \bar{t})$ is termed a *translation* on $(\mathbf{A}, D)$, see [III] 6. The translation cohomology $H^*(t, \bar{t})$ of $(t, \bar{t})$, introduced in [III], is the cohomology of the cochain complex $\tilde{F}(t, \bar{t})$ which is the homotopy fiber of the cochain homomorphism

$$\bar{t}_* - t^* \colon \tilde{F}^*(\mathbf{A}, D) \longrightarrow \tilde{F}^*(\mathbf{A}, D(t, t)).$$

We will sometimes refer to the translation cohomology $H^*(t, \bar{t})$ as the translation cohomology of the translation category $(\mathbf{A}, t)$ with coefficients in the natural transformation $\bar{t}$. In dimension n this homotopy fiber is

$$\tilde{F}^n(t, \bar{t}) = \tilde{F}^n(\mathbf{A}, D) \oplus \tilde{F}^{n-1}(\mathbf{A}, D(t, t))$$

and the differential is the matrix

$$\begin{pmatrix} \delta & 0 \\ \bar{t}_* - t^* & -\delta \end{pmatrix}.$$

In particular there is a long exact sequence ($n \in \mathbb{Z}$)

$$\text{(3.1)} \qquad \cdots \to H^n(t, \bar{t}) \xrightarrow{j} H^n(\mathbf{A}, D) \xrightarrow{\bar{t}_* - t^*} H^n(\mathbf{A}, D(t, t)) \xrightarrow{\partial} H^{n+1}(t, \bar{t}) \to \cdots .$$

Given a morphism $(\varphi, \alpha) \colon (\mathbf{B}, s) \to (\mathbf{A}, t)$ in **TCat** there is an induced translation $(s, (\varphi, \alpha)^* \bar{t})$ on $(\mathbf{B}, D(\varphi, \varphi))$ with

$$\text{(3.2)} \qquad (\varphi, \alpha)^* \bar{t} = (D(\alpha^{\boxminus}, \alpha)) \square (\bar{t}(\varphi^{op} \times \varphi)),$$

and an induced cochain homomorphism

$$(\varphi, \alpha)^* \colon \tilde{F}^*(t, \bar{t}) \longrightarrow \tilde{F}^*(s, (\varphi, \alpha)^* \bar{t})$$

given by the matrix

$$(\varphi, \alpha)^* = \begin{pmatrix} \varphi^* & 0 \\ (D(\alpha^{\boxminus}, 0^{\square}_{\varphi s}))_* \alpha^{\#} & (D(\alpha^{\boxminus}, \alpha))_* \varphi^* \end{pmatrix}.$$

Here we use (2.5). The induced homomorphisms in translation cohomology will be denoted in the same way

$$(\varphi, \alpha)^* \colon H^*(t, \bar{t}) \longrightarrow H^*(s, (\varphi, \alpha)^* \bar{t}).$$

Similarly two natural transformations $\lambda \colon D \Rightarrow E$ and $\bar{s} \colon E \Rightarrow E(t, t)$ between $*$-normalized $\mathbf{A}$-bimodules on $\mathbf{A}$ such that $(\lambda(t^{op} \times t)) \square \bar{t} = \bar{s} \square \lambda$ give rise to a translation $(t, \bar{s})$ on $(\mathbf{A}, E)$ together with a cochain homomorphism

$$\lambda_* \colon \tilde{F}^*(t, \bar{t}) \longrightarrow \tilde{F}^*(t, \bar{s})$$

given by the matrix

$$\text{(3.3)} \qquad \lambda_* = \begin{pmatrix} \lambda_* & 0 \\ 0 & (\lambda(t^{op} \times t))_* \end{pmatrix}.$$

The corresponding homomorphism in translation cohomology will also be denoted by

$$\lambda_* \colon H^*(t, \bar{t}) \longrightarrow H^*(t, \bar{s}).$$

These homomorphisms can be used to describe the simple version of the functorial properties of translation cohomology, see [III] 6.2.

Two morphisms $(\varphi,\alpha),(\psi,\beta)\colon(\mathbf{B},s)\to(\mathbf{A},t)$ in **TCat** also induce a translation $(s,[(\varphi,\alpha),(\psi,\beta)]^*\bar{t})$ on $(\mathbf{B},D(\varphi,\psi))$ given by

$$[(\varphi,\alpha),(\psi,\beta)]^*\bar{t}=(D(\alpha^{\boxminus},\beta))\Box(\bar{t}(\varphi^{op}\times\psi)). \tag{3.4}$$

Notice that

$$[(\varphi,\alpha),(\varphi,\alpha)]^*\bar{t}=(\varphi,\alpha)^*\bar{t}. \tag{3.5}$$

In [III] we consider cochains which are only normalized with respect to identities since we are not considering categories with zero objects there. More precisely, in the cited paper we define the translation cochain complex $\bar{F}(t,\bar{t})$ as the homotopy fiber of

$$\bar{t}_*-t^*\colon\bar{F}^*(\mathbf{A},D)\longrightarrow\bar{F}^*(\mathbf{A},D(t,t)).$$

Clearly $\tilde{F}^*(t,\bar{t})\subset\bar{F}^*(t,\bar{t})$ and this inclusion induces isomorphisms in all cohomology groups. This follows from the exact sequence (3.1), the five lemma, and the fact that the inclusion of the subcomplexes of normalized and $*$-normalized cochains in ordinary cohomology of categories induce isomorphisms in cohomology, see Section 2.

Remark 3.6. As we pointed out in [III] 6.6 the translation cochain complex is not a functor itself since, in general, composition of induced cochain homomorphisms is only associative up to cochain homotopy, compare with the proof of [III] 6.4. However it is immediate to notice by using (2.2) and (3.3) that once we fix a translation category $(\mathbf{A},t)$ then $\bar{F}(t,-)$ is a functor on the category whose objects are natural transformations of $\mathbf{A}$-bimodules $\bar{t}\colon D\Rightarrow D(t,t)$, and morphisms $\lambda\colon\bar{t}\to\bar{s}$ with $\bar{s}\colon E\Rightarrow E(t,t)$ are natural transformations $\lambda\colon D\Rightarrow E$ with $(\lambda(t^{op}\times t))\Box\bar{t}=\bar{s}\Box\lambda$. The $*$-normalized translation cochain complex $\tilde{F}(t,-)$ is also a functor if we restrict to $*$-normalized $\mathbf{A}$-bimodules.

4 The cohomology of the triangle category

The *triangle translation category* $(\triangle,sh^3)$ is a universal model of candidate triangles in triangulated categories. The set of objects of $\triangle$ is $\mathbb{Z}\cup\{*\}$, $*$ is the zero object of $\triangle$ and the unique morphisms apart from identities and zeroes are $(n\in\mathbb{Z})$

$$\imath=\imath_n\colon n-1\longrightarrow n,$$

in particular $\imath^2=0$. The zero-object-preserving functor (*shift functor*)

$$sh\colon\triangle\longrightarrow\triangle$$

is defined by $sh(n)=n+1$ and $sh(\imath_n)=\imath_{n+1}$. This is actually an isomorphism of categories and sh^3 denotes its third power.

Let $\mathbf{A}$ be a category with zero object. A functor $X\colon\triangle\to\mathbf{A}$ preserving the zero object coincides with a *cochain complex* in $\mathbf{A}$, i. e. a diagram $(n\in\mathbb{Z})$

$$\cdots\to X_{n-1}\xrightarrow{d}X_n\xrightarrow{d}X_{n+1}\to\cdots$$

with $d^2=0$. We write $X_n=X(n)$ for the values of X on objects in $\triangle$ and $d=d_n=d_n^X=X(\imath_n)$ for the morphisms induced by the non-trivial morphisms in the category $\triangle$.

Let $(\mathbf{A}, t)$ be a translation category. A *candidate triangle* (X, α) in $(\mathbf{A}, t)$ is a cochain complex X in $\mathbf{A}$

$$\cdots \to X_{n-1} \xrightarrow{d} X_n \xrightarrow{d} X_{n+1} \to \cdots$$

together with a degree 3 cochain isomorphism $\alpha \colon tX \cong X$, i. e. a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & tX_n \arrow[r, "td"] \arrow[d, "\alpha_n", "\cong"'] & tX_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}", "\cong"'] & \cdots \\
\cdots \arrow[r] & X_{n+3} \arrow[r, "d"] & X_{n+4} \arrow[r] & \cdots
\end{tikzcd}$$

Morphisms of candidate triangles $f \colon (X, \alpha) \to (Y, \beta)$ are cochain homomorphisms $f \colon X \to Y$ such that $\beta(tf) = f\alpha$.

Let $\mathbf{Cand}(\mathbf{A}, t)$ be the category of candidate triangles. If $\mathbf{A}$ is additive and t is an additive equivalence then $\mathbf{Cand}(\mathbf{A}, t)$ is equivalent to the category of small candidate triangles $\mathbf{cand}(\mathbf{A}, t)$ as defined in [IV] 1, compare [I]. Recall that an object in this category is a diagram in $\mathbf{A}$ of the form

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

such that $if = 0$, $qi = 0$ and $(tf)q = 0$. A morphism k between small candidate triangles is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] \arrow[d, "k_2"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"] & \bar{B} \arrow[r, "\bar{i}"] & \bar{C} \arrow[r, "\bar{q}"] & t\bar{A}
\end{tikzcd}$$

There is an equivalence of categories,

$$\mathbf{Cand}(\mathbf{A}, t) \longrightarrow \mathbf{cand}(\mathbf{A}, t) \tag{4.1}$$

sending a candidate triangle (X, α) to the small candidate triangle

$$X_{-2} \xrightarrow{d_{-1}} X_{-1} \xrightarrow{d_0} X_0 \xrightarrow{\alpha_{-2}^{-1} d_1} tX_{-2}.$$

In order to construct the inverse equivalence

$$\mathbf{cand}(\mathbf{A}, t) \longrightarrow \mathbf{Cand}(\mathbf{A}, t) \tag{4.2}$$

we need to choose an inverse equivalence $t^{-1} \colon \mathbf{A} \to \mathbf{A}$ for t and a natural isomorphism $\mu \colon tt^{-1} \cong 1_{\mathbf{A}}$. With these choices the equivalence (4.2) sends a small candidate triangle $A \overset{f}{\to} B \overset{i}{\to} C \overset{q}{\to} tA$ to the candidate triangle (X, α) with $X_{3n} = t^n C$, $X_{3n+1} = t^{n+1} A$, $X_{3n+2} = t^{n+1} B$, $d_{3n} = t^n i$, $d_{3n+1} = t^n q$ and $d_{3n+2} = t^{n+1} f$ for all $n \in \mathbb{Z}$, $\alpha_n = 1_{tX_n}$ if $n \geq -2$, and $\alpha_n = \mu_{X_{n+3}}$ for $n < -2$.

Notice that a candidate triangle (X, α) as above is the same as a morphism of translation categories $(X, \alpha) \colon (\triangle, sh^3) \to (\mathbf{A}, t)$ in the sense of Section 1. Moreover, the category $\mathbf{Cand}(\mathbf{A}, t)$ of candidate triangles on $(\mathbf{A}, t)$ coincides with the category given by morphisms from $(\triangle, sh^3)$ to $(\mathbf{A}, t)$ and 2-cells between them in the 2-category $\mathbf{TCat}$ defined in Section 1,

$$\mathbf{Cand}(\mathbf{A}, t) = \mathbf{TCat}((\triangle, sh^3), (\mathbf{A}, t)).$$

The translation cohomology of the triangle translation category $(\triangle, sh^3)$ is relevant in this paper. Below we identify its translation cochain complex. For this we begin by considering the cohomology of the category $\triangle$.

Let D be a $*$-normalized $\triangle$-bimodule. The reader can readily notice that there is an isomorphism

$$\tilde{F}^n(\triangle, D) \cong \prod_{m\in\mathbb{Z}} D(m, m+n), \tag{4.3}$$

which we use as an identification, where the m^{th} factor of the product corresponds to the sequence

$$m+n \overset{\imath_{m+n}}{\longleftarrow} \cdots \overset{\imath_{m+1}}{\longleftarrow} m.$$

If we write $c(m)$ for the m^{th} coordinate of an n-cochain c in the product (4.3) then

$$(\delta(c))(m) = \imath_{m+n+1*}c(m) + (-1)^{n+1}\imath_{m+1}^{*}c(m+1).$$

The identification in (4.3) is clearly natural in D, in particular, given a natural transformation $\bar{t}\colon D \Rightarrow D(sh^3, sh^3)$ there is an identification

$$\tilde{F}^n(sh^3, \bar{t}) \cong \prod_{m\in\mathbb{Z}} D(m, m+n) \oplus \prod_{m\in\mathbb{Z}} D(m+3, m+n+2) \tag{4.4}$$

for $n > 0$ and

$$\tilde{F}^0(sh^3, \bar{t}) \cong \prod_{m\in\mathbb{Z}} D(m, m). \tag{4.5}$$

The translation differential is given by

$$\begin{aligned}(\delta(c,b))(m) \quad = \quad & (\imath_{m+n+1*}c(m) + (-1)^{n+1}\imath_{m+1}^{*}c(m+1), \\ & \bar{t}_{(m,m+n)}c(m) - c(m+3) - \imath_{m+n*}b(m) - (-1)^{n}\imath_{m+1}^{*}b(m+1))\end{aligned}$$

for an n-cochain (c, b) with $n > 0$, and

$$(\delta c)(m) = (\imath_{m+1*}c(m) - \imath_{m+1}^{*}c(m+1), \bar{t}_{(m,m)}c(m) - c(m+3))$$

for a 0-cochain c. These identifications in the translation cochain complex are natural in $\bar{t}$, see Remark 3.6.

Lemma 4.6. *If $\bar{t}$ is a natural isomorphism then the cochain homomorphism $\bar{t}_* - sh^{3^*}\colon \tilde{F}^*(\triangle, D) \to \tilde{F}^*(\triangle, D(sh^3, sh^3))$ is surjective.*

Proof. Actually there are levelwise sections $\omega\colon \tilde{F}^n(\triangle, D(sh^3, sh^3)) \to \tilde{F}^n(\triangle, D)$ $(n \in \mathbb{Z})$ defined as follows $(m \in \mathbb{Z}, 0 \leq i \leq 2)$

$$\omega(a)(3m-i) = \begin{cases} -\sum_{j=1}^{m} \bar{t}^{m-j}(a(3(j-1)-i)), & \text{if } m > 0; \\ 0, & \text{if } m = 0; \\ \sum_{j=m+1}^{0} \bar{t}^{m-j}(a(3(j-1)-i)), & \text{if } m < 0. \end{cases}$$

The reader can easily check that $(\bar{t}_* - sh^{3^*})\omega = 1$. Notice however that ω does not define a cochain homomorphism. □

Consider now the cochain complex $\tilde{G}^*(\bar{t})$ defined by

$$\tilde{G}^n(\bar{t}) = D(-2, n-2) \oplus D(-1, n-1) \oplus D(0, n),$$

with differential

$$\delta = \left(\begin{array}{ccc} \imath_{n-1*} & (-1)^{n+1}\imath_{-1}^* & 0 \\ 0 & \imath_{n*} & (-1)^{n+1}\imath_0^* \\ (-1)^{n+1}\imath_1^*\bar{t}_{(0,n)} & 0 & \imath_{n+1*} \end{array} \right)$$

on dimension n.

Proposition 4.7. *If $\bar{t}$ is a natural isomorphism then the cohomology of $\tilde{G}^*(\bar{t})$ is isomorphic to the translation cohomology $H^*(sh^3, \bar{t})$.*

Proof. By Lemma 4.6 we can replace the homotopy fiber of the cochain homomorphism $\bar{t}_* - sh^{3^*}$ by its kernel $\mathrm{Ker}[\bar{t}_* - sh^{3^*}]$. More precisely, the obvious inclusion

$$\mathrm{Ker}[\bar{t}_* - sh^{3^*}] \subset \tilde{F}^*(sh^3, \bar{t})$$

induces isomorphisms in all cohomology groups. The reader can easily check that the cochain homomorphism

$$\kappa \colon \tilde{G}^*(\bar{t}) \hookrightarrow \tilde{F}^*(\triangle, D)$$

defined by $\kappa(c_{-2}, c_{-1}, c_0)(3m - i) = \bar{t}^m(c_i)$ $(m \in \mathbb{Z}, 0 \leq i \leq 2)$ is actually a kernel for $\bar{t}_* - sh^{3^*}$ hence the proposition follows. □

Remark 4.8. Notice that in fact we have established an explicit quasi-isomorphism in the proof of Proposition 4.7

$$\upsilon = (\kappa, 0) \colon \tilde{G}^*(\bar{t}) \longrightarrow \tilde{F}^*(sh^3, \bar{t}).$$

Moreover, the cochain complexes $\tilde{G}^*$ define a functor on the category of natural transformations $\bar{t} \colon D \Rightarrow D(sh^3, sh^3)$ between $*$-normalized $\triangle$-bimodules, the same as $\tilde{F}^*(sh^3, -)$, see Remark 3.6. Recall that morphisms $\lambda \colon \bar{t} \to \bar{s}$ between natural transformations $\bar{t} \colon D \Rightarrow D(sh^3, sh^3)$ and $\bar{s} \colon E \Rightarrow E(sh^3, sh^3)$ in this category are natural transformations $\lambda \colon D \Rightarrow E$ such that

$$(\lambda(sh^{3^{op}} \times sh^3)) \Box \bar{t} = \bar{s} \Box \lambda.$$

The induced homomorphism

$$\lambda_* \colon \tilde{G}^*(\bar{t}) \longrightarrow \tilde{G}^*(\bar{s})$$

in dimension n is given by the matrix

$$\lambda_* = \left(\begin{array}{ccc} \lambda_{(-2,n-2)} & 0 & 0 \\ 0 & \lambda_{(-1,n-1)} & 0 \\ 0 & 0 & \lambda_{(0,n)} \end{array} \right).$$

Therefore the quasi-isomorphism υ and hence the isomorphism in Proposition 4.7 are natural in $\bar{t}$.

Unfortunately υ in Remark 4.8 is not a chain homotopy equivalence, so we have not a direct way of representing a cocycle in $\tilde{F}^*(sh^3, \bar{t})$ by a cocycle coming from $\tilde{G}^*(\bar{t})$. The following lemma solves this problem.

Lemma 4.9. *If* $\bar{t}\colon D \Rightarrow D(sh^3, sh^3)$ *is a natural isomorphism between* $*$*-normalized* $\triangle$*-bimodules and* $(c,b) \in \tilde{F}^n(sh^3, \bar{t})$ *is a cocycle then*

$$(c(-2), c(-1), c(0) - D(\imath_1, 1_n)b(-2)) \in \tilde{G}^n(\bar{t})$$

is a cocycle whose image by υ *in Remark 4.8 represents the same cohomology class as* (c,b)*.*

Proof. With the notation of the proof of Lemma 4.6 it is easy to check that

$$\upsilon(c(-2), c(-1), c(0) - D(\imath_1, 1_n)b(-2)) = (c,b) - \delta(\omega(b), 0),$$

hence this lemma follows. □

We now consider a special example of a natural transformation as in Proposition 4.7. Suppose that $(\mathbf{A}, t)$ is a translation category given by an additive category $\mathbf{A}$ and an additive equivalence of categories $t\colon \mathbf{A} \to \mathbf{A}$. We then have the abelian group $\mathrm{Hom}_{\mathbf{A}}(X,Y)$ of morphisms in $\mathbf{A}$. Let Hom^t be the $\mathbf{A}$-bimodule defined by t in $\mathbf{A}$ as follows

$$\mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t, -), \tag{4.10}$$

and let

$$\bar{t}\colon \mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t, -) \Rightarrow \mathrm{Hom}_{\mathbf{A}}(t^2, t) = \mathrm{Hom}^t(t, t) \tag{4.11}$$

be given by $(-1)t$. By (3.4) a candidate triangle (X, α) in $(\mathbf{A}, t)$ yields a natural transformation $(X,\alpha)^*\bar{t}$ which can be used as a coefficient object for the computation of translation cohomology groups of $(\triangle, sh^3)$. The 3-dimensional abelian group of the cochain complex $\tilde{G}^*((X,\alpha)^*\bar{t})$ is

$$\tilde{G}^3((X,\alpha)^*\bar{t}) = \mathrm{Hom}_{\mathbf{A}}(tX_{-2}, X_1) \oplus \mathrm{Hom}_{\mathbf{A}}(tX_{-1}, X_2) \oplus \mathrm{Hom}_{\mathbf{A}}(tX_0, X_3).$$

There is a canonical element in this abelian group determined by the cochain isomorphism α, namely

$$(\alpha_{-2}, -\alpha_{-1}, \alpha_0) \in \tilde{G}^3((X,\alpha)^*\bar{t}).$$

Lemma 4.12. *The* 3*-cochain* $(\alpha_{-2}, -\alpha_{-1}, \alpha_0)$ *is indeed a cocycle.*

The cohomology class represented by this cochain will be denoted by

$$\mathrm{id}(X,\alpha) \in H^3(sh^3, (X,\alpha)^*\bar{t}) \tag{4.13}$$

and will be called the *class of identities*.

Remark 4.14. If (X,α) is the candidate triangle corresponding to the small candidate triangle

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

under the equivalence (4.2) then the cochain complex $\tilde{G}^*((X,\alpha)^*\bar{t})$ is given as follows $(n \geq 0)$

$$\tilde{G}^{3n+i}((X,\alpha)^*\bar{t}) = \begin{cases} \mathrm{Hom}_{\mathbf{A}}(tA, t^nA) \oplus \mathrm{Hom}_{\mathbf{A}}(tB, t^nB) \oplus \mathrm{Hom}_{\mathbf{A}}(tC, t^nC), & \text{if } i = 0; \\ \mathrm{Hom}_{\mathbf{A}}(tA, t^nB) \oplus \mathrm{Hom}_{\mathbf{A}}(tB, t^nC) \oplus \mathrm{Hom}_{\mathbf{A}}(tC, t^{n+1}A), & \text{if } i = 1; \\ \mathrm{Hom}_{\mathbf{A}}(tA, t^nC) \oplus \mathrm{Hom}_{\mathbf{A}}(tB, t^{n+1}A) \oplus \mathrm{Hom}_{\mathbf{A}}(tC, t^{n+1}B), & \text{if } i = 2. \end{cases}$$

Moreover, the class of identities $\mathrm{id}(X,\alpha)$ is represented by the 3-cocycle

$$(1_{tA}, -1_{tB}, 1_{tC}) \in \tilde{G}^3((X,\alpha)^*\bar{t}).$$

For this reason it is called class of identities.

We will use later another cocycle representing the class of identities given in the following lemma.

Lemma 4.15. *The cochain* $b = (b_1, 0) \in \tilde{F}(sh^3, (X,\alpha)^*\bar{t})$ *defined by* $b_1(m) = (-1)^m\alpha_m$ *is a cocycle representing the class of identities.*

Proof. We defined $\mathrm{id}(X,\alpha)$ as the cohomology class represented by $(\alpha_{-2}, -\alpha_{-1}, \alpha_0) \in \tilde{G}^3((X,\alpha)\bar{t})$. The quasi-isomorphism

$$v\colon \tilde{G}^*((X,\alpha)^*\bar{t}) \longrightarrow \tilde{F}^*(sh^3, (X,\alpha)^*\bar{t})$$

in Remark 4.8 sends $(\alpha_{-2}, -\alpha_{-1}, \alpha_0)$ to the cochain

$$\begin{aligned} v(\alpha_0, -\alpha_1, \alpha_2) = (c, 0) \in \tilde{F}^3(sh^3, (X,\alpha)^*\bar{t}) \quad &= \quad \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{A}}(tX_m, X_{m+3}) \\ &\quad \oplus \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{A}}(tX_{m+3}, X_{m+5}) \end{aligned}$$

defined as $c(3m-i) = ((X,\alpha)^*\bar{t})^m((-1)^i\alpha_i)$ for all $m \in \mathbb{Z}$ and $0 \leq i \leq 2$. We check by induction on m that $c(3m-i) = (-1)^{3m-i}\alpha_{3m-i}$. This is obvious for $m = 0$. If it is true for m then

$$\begin{aligned} c(3(m+1)-i) &= ((X,\alpha)^*\bar{t})^{m+1}((-1)^i\alpha_i) \\ &= ((X,\alpha)^*\bar{t})_{(3m-i,3m-i+3)}(c(3m-i)) \\ &= ((X,\alpha)^*\bar{t})_{(3m-i,3(m+1)-i)}((-1)^{3m-i}\alpha_{3m-i}) \\ &= \alpha_{3(m+1)-i}(-(-1)^{3m-i}t\alpha_{3m-i})(t\alpha_{3m-i}^{-1}) \\ &= (-1)^{3(m+1)-i}\alpha_{3(m+1)-i}. \end{aligned}$$

□

A small candidate triangle is said to be $*$-*trivial* if it has one of the following forms:

$$A \xrightarrow{1_A} A \longrightarrow * \longrightarrow tA,$$

$$A \longrightarrow * \longrightarrow tA \xrightarrow{1_{tA}} tA,$$

$$* \longrightarrow A \xrightarrow{1_A} A \longrightarrow *.$$

Notice that these are exactly those triangles which are required to be exact in any (pre)triangulated category. Analogously a candidate triangle (X,α) is $*$-trivial if there exists $i \in \mathbb{Z}/3$ such that $d_{n+1}\colon X_n \to X_{n+1}$ is an identity morphism and $X_{n+2} = *$ provided $n = i \mod 3$. The functors (4.1) and (4.2) preserve $*$-trivial objects.

Lemma 4.16. *If* (X,α) *is a* $*$*-trivial candidate triangle then* $H^n(sh^3, (X,\alpha)^*\bar{t}) = 0$ *for all* $n \geq 0$.

This lemma is a consequence of the property of the class of identities stated in the following lemma. Recall that a cochain complex X in $\mathbf{A}$ is *contractible* if there are morphisms $h_i\colon X_i \to X_{i-1}$ such that $1_{X_i} = h_{i+1}d_{i+1} + d_ih_i$ for all $i \in \mathbb{Z}$. A candidate triangle (X,α) is said to be contractible if X is contractible and the morphisms h_i can be chosen in such a way that $\alpha_{i-1}(th_i) = h_{i+3}\alpha_i$ for all $i \in \mathbb{Z}$. The $*$-trivial candidate triangles are the most basic examples of contractible candidate triangles. The reader can easily check that under the equivalences of categories in (4.1) and (4.2) contractible candidate triangles correspond to small contractible candidate triangles in the sense of [Nee90] 1.5.

Lemma 4.17. *Let (X,α) be a candidate triangle. The following statements are equivalent:*

1. $\mathrm{id}(X,\alpha)=0$,
2. (X,α) *is contractible,*
3. X *is contractible,*
4. $X\,sh^n$ *is contractible for all* $n\in\mathbb{Z}$,
5. $H^*(\triangle, D(X,-))=0$ *for any functor* $D\colon \mathbf{A}^{op}\times\triangle\rightarrow\mathbf{Ab}$ *additive in the first variable and such that* $D(-,*)=0$,
6. $H^*(\triangle, D(-,X))=0$ *for any functor* $D\colon\ \triangle\times\mathbf{A}\rightarrow\mathbf{Ab}$ *additive in the second variable and such that* $D(*,-)=0$,
7. $H^*(sh^3,[(X,\alpha),(Y,\beta)]^*\bar{s})=0$ *for any natural transformation* $\bar{s}\colon D\Rightarrow D(t,t)$ *with* D *a biadditive* $\mathbf{A}$*-bimodule and any candidate triangle* (Y,β),
8. $H^*(sh^3,[(Y,\beta),(X,\alpha)]^*\bar{s})=0$ *for any natural transformation* $\bar{s}\colon D\Rightarrow D(t,t)$ *with* D *a biadditive* $\mathbf{A}$*-bimodule and any candidate triangle* (Y,β),
9. $H^*(sh^3,(X,\alpha)^*\bar{s})=0$ *for any natural transformation* $\bar{s}\colon D\Rightarrow D(t,t)$ *with* D *a biadditive* $\mathbf{A}$*-bimodule.*

Proof. Obviously (2) ⇒ (3), (4) ⇒ (3), (7) ⇒ (9), (8) ⇒ (9) and (9) ⇒ (1) are trivial. It is elementary to check (3) ⇒ (4). If X is contractible and D is as in (5) then the cochain complex of abelian groups $\tilde{F}^*(\triangle,D(X,-))$ is also contractile. The contraction is given by the homomorphisms $\bar{h}_i\colon\tilde{F}^i(\triangle,D(X,-))\rightarrow\tilde{F}^{i-1}(\triangle,D(X,-))$ defined by $(\bar{h}_i(c))(m)=(-1)^ih_m^*(c(m-1))$. This proves (3) ⇒ (5). Similarly if X is contractible and D is as in (6) the cochain complex of abelian groups $\tilde{F}^*(\triangle,D(-,X))$ is also contractile and contraction is given by the homomorphisms $\hat{h}_i\colon\tilde{F}^i(\triangle,D(-,X))\rightarrow\tilde{F}^{i-1}(\triangle,D(-,X))$ defined by $(\hat{h}_i(c))(m)=h_{m+i*}(c(m))$. This proves (3) ⇒ (6). Since we have already showed (3) ⇔ (4) and (3) ⇒ (5) (resp. (3) ⇒ (6)) the exact sequence (3.1) proves that (5) ⇒ (7) (resp. (6) ⇒ (8)).

Suppose now that $\mathrm{id}(X,\alpha)=0$. By Lemma 4.15 we see that there are morphisms $k_i\colon tX_i\rightarrow X_{i+2}$ such that

$$d_{i+3}k_i-k_{i+1}(td_{i+1})=(-1)^i\alpha_i,\tag{a}$$

$$-\alpha_{i+2}(tk_i)(t\alpha_i^{-1})-k_{i+3}=0\tag{b}$$

for all $i\in\mathbb{Z}$. Let us check that the morphisms

$$h_i=(-1)^ik_{i-3}\alpha_{i-3}^{-1}\colon X_i\rightarrow X_{i-1}$$

determine a contraction of (X,α). On one hand

$$\begin{aligned}\alpha_{i-1}(th_i) &= \alpha_{i-1}(-1)^i(tk_{i-3})(t\alpha_{i-3}^{-1})\\ &\overset{\text{(b)}}{=} -(-1)^ik_i\\ &= h_{i+3}\alpha_i,\end{aligned}$$

and on the other hand

$$\begin{aligned} h_{i+1}d_{i+1}+d_ih_i &= (-1)^{i+1}k_{i-2}\alpha_{i-2}^{-1}d_{i+1}+d_i(-1)^ik_{i-3}\alpha_{i-3}^{-1} \\ &= (-1)^i(-k_{i-2}(td_{i-2})+d_ik_{i-3})\alpha_{i-3}^{-1} \\ &\overset{\text{(a)}}{=} (-1)^i(-1)^i\alpha_{i-3}\alpha_{i-3}^{-1} \\ &= 1_{X_i}. \end{aligned}$$

Here in the second equality we use that α is a cochain homomorphism. This proves (1) ⇒ (2). Hence the proof of this lemma is finished. □

The implication (2) ⇒ (9) should be compared, after Section 5, to the fact that contractible small candidate triangles are exact in all triangulated categories, see [Nee90] 1.6.

5 ∇-Triangles

Here we obtain a purely cohomological characterization of small ∇-triangles defined in [IV] 2.13 (Proposition 5.7).

Remark 5.1. Small ∇-triangles determine the "underlying" ordinary (pre)triangulated structure of a cohomologically (pre)triangulated category in the sense of [IV] 2.16, see [IV] 2.17. They are defined in [IV] 2.13 by using the main result in [III], which says that 3-dimensional translation cohomology classes are represented by translation track categories. We already know by [IV] 2.12 that nevertheless the class of small ∇-triangles is independent of the choice of a translation track category representing ∇. However the proof of [IV] 2.12 is still track-theoretical.

In this section $(\mathbf{A}, t)$ will be a translation category given by an additive category $\mathbf{A}$ and an additive equivalence t. Let $\mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t, -)$ be the $\mathbf{A}$-bimodule defined in (4.10) and $\bar{t}\colon \mathrm{Hom}^t = \mathrm{Hom}_{\mathbf{A}}(t,-) \Rightarrow \mathrm{Hom}_{\mathbf{A}}(t^2, t) = \mathrm{Hom}^t(t,t)$ the natural transformation in (4.11) given by $(-1)t$. The translation cohomology of $(t, \bar{t})$ will be denoted in this paper by

$$H^*(\mathbf{A}, t) = H^*(t, \bar{t}). \tag{5.2}$$

Given two candidate triangles (X, α) and (Y, β) in the sense of Section 4 the translation cohomology groups $(n \in \mathbb{Z})$

$$H^n(sh^3, [(X,\alpha),(Y,\beta)]^*\bar{t})$$

are defined by (3.4). This actually defines a $\mathbf{Cand}(\mathbf{A}, t)$-bimodule

$$H^n(sh^3, [-,-]^*\bar{t}), \tag{5.3}$$

more precisely, given a diagram in $\mathbf{Cand}(\mathbf{A}, t)$

$$(X,\alpha) \xrightarrow{f} (Y,\beta) \xrightarrow{g} (Z,\gamma)$$

the induced morphisms g_* and f^* are defined by

$$g_* = \mathrm{Hom}^t(X,g)_*\colon H^n(sh^3,[(X,\alpha),(Y,\beta)]^*\bar{t}) \longrightarrow H^n(sh^3,[(X,\alpha),(Z,\gamma)]^*\bar{t}),$$

$$f^* = \mathrm{Hom}^t(f,Z)_*\colon H^n(sh^3,[(X,\alpha),(Z,\gamma)]^*\bar{t}) \longrightarrow H^n(sh^3,[(Y,\beta),(Z,\gamma)]^*\bar{t}).$$

Notice that by (3.5)

$$H^n(sh^3, [(X,\alpha),(X,\alpha)]^*\bar{t}) = H^n(sh^3, (X,\alpha)^*\bar{t}),$$

therefore the classes of identities in (4.13) define a 0-cochain of the category $\mathbf{Cand}(\mathbf{A},t)$ with coefficients in $H^3(sh^3,[-,-]^*\bar{t})$.

Lemma 5.4. *The classes of identities defined in (4.13) yield a cohomology class*

$$\mathrm{id} \in H^0(\mathbf{Cand}(\mathbf{A},t), H^3(sh^3,[-,-]^*\bar{t})).$$

This follows from Lemma 8.16.

Let us fix a translation cohomology class

$$\nabla \in H^3(\mathbf{A},t).$$

Lemma 5.5. *The translation cohomology class* ∇ *yields a cohomology class*

$$\bar{\nabla} \in H^0(\mathbf{Cand}(\mathbf{A},t), H^3(sh^3,[-,-]^*\bar{t}))$$

defined over any candidate triangle (X,α) *by the formula*

$$\bar{\nabla}(X,\alpha) = (X,\alpha)^*\nabla \in H^3(sh^3,(X,\alpha)^*\bar{t}).$$

This follows from Lemma 8.12.

A ∇-*triangle* is a candidate triangle (X,α) in $(\mathbf{A},t)$ which is annihilated by the cohomology class

$$\mathrm{id} + \bar{\nabla} \in H^0(\mathbf{Cand}(\mathbf{A},t), H^3(sh^3,[-,-]^*\bar{t})),$$

or equivalently, (X,α) satisfies the equation

$$0 = \mathrm{id}(X,\alpha) + (X,\alpha)^*\nabla \in H^3(sh^3,(X,\alpha)^*\bar{t}). \tag{5.6}$$

Proposition 5.7. *Small* ∇*-triangles are small candidate triangles corresponding to* ∇*-triangles by the equivalence of categories (4.2).*

The proof of this proposition is in Section 10.

This result allows to prove the following proposition, which completes the proof of the main theorem in [IV].

Proposition 5.8. *Let* $(\mathbf{A},t)$ *be a translation category where* $\mathbf{A}$ *is additive and* t *is an additive equivalence and let* ∇ *be a translation cohomology class in* $H^3(\mathbf{A},t)$*. Then the class* $\mathcal{E}_\nabla$ *of small* ∇*-triangles is defined and* $(\mathbf{A},t,\mathcal{E}_\nabla)$ *satisfies the axioms (Tr0), (Tr2) and (Tr3) in the definition of a pretriangulated category, see [IV] 1.1.*

Proof. Let us prove the first part of (Tr0). For this we consider an isomorphism of candidate triangles $f\colon (X,\alpha)\to(Y,\beta)$. Since $\mathrm{id}+\bar{\nabla}$ is a 0-dimensional cohomology class we have that

$$f_*(\mathrm{id}(X,\alpha)+(X,\alpha)^*\nabla) = f^*(\mathrm{id}(Y,\beta)+(Y,\beta)^*\nabla) \in H^3(sh^3,[(X,\alpha),(Y,\beta)]^*\bar{t}),$$

but f_* and f^* are isomorphisms since f is already an isomorphism, therefore $\mathrm{id}(X,\alpha)+(X,\alpha)^*\nabla$ vanishes if and only if $\mathrm{id}(Y,\beta)+(Y,\beta)^*\nabla$ vanishes, i. e. (X,α) is a ∇-triangle if and only if (Y,β) is a ∇-triangle. The second part follows from Lemma 4.16. Axiom (Tr2) follows from Proposition 9.1 in Section 9. And finally (Tr3) is a consequence of Proposition [IV] 4.18, as we already pointed out in the proof of [IV] 2.17. □

6 Algebraic homotopy pairs and cohomologically triangulated categories

Suppose that **A** is an additive category, $t\colon \mathbf{A} \to \mathbf{A}$ is an additive equivalence and $\nabla \in H^3(\mathbf{A}, t)$. The homomorphism j in the exact sequence (3.1) determines a cohomology class

$$j\nabla \in H^3(\mathbf{A}, \mathrm{Hom}^t).$$

Recall that the objects of the additive category $\mathbf{Pair}(\mathbf{A})$ of pairs in **A** are the morphisms in **A**, and morphisms $(h,k)\colon f \to g$ are commutative squares

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"] & D \end{tikzcd}$$

The $\mathbf{Pair}(\mathbf{A})$-bimodule $(\mathrm{Hom}^t)^\#$ is defined by

$$(\mathrm{Hom}^t)^\#(f,g) = \mathrm{Hom}_{\mathbf{A}}(tA, D)/(g_* \mathrm{Hom}_{\mathbf{A}}(tA, C) + (tf)^* \mathrm{Hom}_{\mathbf{A}}(tB, D)).$$

There are natural homomorphisms

$$\lambda\colon H^{n+1}(\mathbf{A}, \mathrm{Hom}^t) \longrightarrow H^n(\mathbf{Pair}(\mathbf{A}), (\mathrm{Hom}^t)^\#), \quad n \geq 0,$$

defined in [Bau97]. These homomorphisms can also be obtained from the Künneth spectral sequence for cohomology of categories in Corollary 8.2, see Remark 8.3. In particular we can consider the cohomology class

$$\lambda j\nabla \in H^2(\mathbf{Pair}(\mathbf{A}), (\mathrm{Hom}^t)^\#).$$

This class will be called the *characteristic class of algebraic homotopy pairs*. It corresponds to a well-defined isomorphism class of linear extensions of the category $\mathbf{Pair}(\mathbf{A})$ by $(\mathrm{Hom}^t)^\#$, see [Bau89] IV.6. Any 2-cocycle $\aleph$ representing $\lambda j\nabla$ determines such a linear extension

$$(\mathrm{Hom}^t)^\# \xrightarrow{+} \mathbf{AHp}(\aleph) \xrightarrow{\bar{p}} \mathbf{Pair}(\mathbf{A}). \tag{6.1}$$

Here $\mathbf{AHp}(\aleph)$ is the *category of algebraic homotopy pairs* associated to the cocycle $\aleph$. It has the same objects as $\mathbf{Pair}(\mathbf{A})$ and morphisms $(h,k,a)\colon f \to g$ are given by a morphism $(h,k)\colon f \to g$ in $\mathbf{Pair}(\mathbf{A})$ together with an element $a \in (\mathrm{Hom}^t)^\#(f,g)$. Composition is defined by the following formula

$$(\bar{h}, \bar{k}, \bar{a})(h,k,a) = (h\bar{h}, \bar{k}k, (\bar{h},\bar{k})_* a + (h,k)^*\bar{a} - \aleph((\bar{h},\bar{k}),(h,k))).$$

The functor $\bar{p}$ is the identity on objects and $\bar{p}(h,k,a) = (h,k)$. The category $\mathbf{AHp}(\aleph)$ is well defined by $\lambda j\nabla$ up to isomorphism over $\mathbf{Pair}(\mathbf{A})$ equivariant with respect to the action of $(\mathrm{Hom}^t)^\#$ in the third coordinate of morphisms, see [Bau89] IV.3.4.

By [BHP97] 6.2 we have that $\mathbf{AHp}(\aleph)$ has a unique additive category structure such that $\bar{p}$ is an additive functor and

$$(\mathrm{Hom}^t)^\#(f,g) \hookrightarrow \mathrm{Hom}_{\mathbf{AHp}(\aleph)}(f,g) \overset{\bar{p}}{\twoheadrightarrow} \mathrm{Hom}_{\mathbf{Pair}(\mathbf{A})}(f,g) \tag{6.2}$$

is a short exact sequence of abelian groups where the first homomorphism is defined by $b \mapsto (0,0,b)$.

There is a full additive inclusion

$$\iota\colon \mathbf{A} \longrightarrow \mathbf{AHp}(\aleph)$$

given by $\iota(A)\colon * \to A$. In particular for any morphism $f\colon A \to B$ in $\mathbf{A}$ we can consider the *algebraic cone functor*

$$Cone_f^{\mathrm{Alg}} = \mathrm{Hom}_{\mathbf{AHp}(\aleph)}(f,\iota)\colon \mathbf{A} \longrightarrow \mathbf{Ab}. \tag{6.3}$$

Recall that an $\mathbf{A}$-module is just an additive functor $\mathbf{A} \to \mathbf{Ab}$, therefore algebraic cone functors are $\mathbf{A}$-modules. Morphisms between $\mathbf{A}$-modules are natural transformations and the category of $\mathbf{A}$-modules is designated by $\mathbf{mod}(\mathbf{A})$. There is a Yoneda full inclusion $\mathbf{A}^{op} \subset \mathbf{mod}(\mathbf{A})$ sending an object A to the $\mathbf{A}$-module $\mathrm{Hom}_{\mathbf{A}}(A,-)$. Algebraic cone functors determine an additive functor

$$Cone^{\mathrm{Alg}}\colon \mathbf{AHp}(\aleph)^{op} \longrightarrow \mathbf{mod}(\mathbf{A}) \tag{6.4}$$

in the obvious way.

The exactness of (6.2) implies that for any $f\colon A \to B$ in $\mathbf{A}$ there is an exact sequence of $\mathbf{A}$-modules

(6.5)

$$\mathrm{Hom}_{\mathbf{A}}(tB,-) \xrightarrow{(tf)^*} \mathrm{Hom}_{\mathbf{A}}(tA,-) \xrightarrow{\tilde{q}_f} Cone_f^{\mathrm{Alg}} \xrightarrow{\tilde{i}_f} \mathrm{Hom}_{\mathbf{A}}(B,-) \xrightarrow{f^*} \mathrm{Hom}_{\mathbf{A}}(A,-).$$

This exact sequence is natural in the category of algebraic homotopy pairs, i. e. for any morphism $(h,k,a)\colon f \to \bar{f}$ in $\mathbf{AHp}(\aleph)$ the following diagram commutes

(6.6)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{A}}(tB,-) \arrow[r, "(tf)^*"] & \mathrm{Hom}_{\mathbf{A}}(tA,-) \arrow[r, "\tilde{q}_f"] & Cone_f^{\mathrm{Alg}} \arrow[r, "\tilde{i}_f"] & \mathrm{Hom}_{\mathbf{A}}(B,-) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathbf{A}}(A,-) \\
\mathrm{Hom}_{\mathbf{A}}(t\bar{B},-) \arrow[r, "(t\bar{f})^*"'] \arrow[u, "(tk)^*"'] & \mathrm{Hom}_{\mathbf{A}}(t\bar{A},-) \arrow[r, "\tilde{q}_{\bar{f}}"'] \arrow[u, "(th)^*"'] & Cone_{\bar{f}}^{\mathrm{Alg}} \arrow[r, "\tilde{i}_{\bar{f}}"'] \arrow[u, "(h,k,a)^*"'] & \mathrm{Hom}_{\mathbf{A}}(\bar{B},-) \arrow[r, "\bar{f}^*"'] \arrow[u, "k^*"'] & \mathrm{Hom}_{\mathbf{A}}(\bar{A},-) \arrow[u, "h^*"']
\end{tikzcd}$$

Let $\mathbf{AHp}^{rep}(\aleph) \subset \mathbf{AHp}(\aleph)$ be the full subcategory whose objects are morphisms f in $\mathbf{A}$ such that the associated algebraic cone functor $Cone_f^{\mathrm{Alg}}$ is representable. For each object f in $\mathbf{AHp}^{rep}(\aleph)$ we choose an $\mathbf{A}$-module isomorphism

$$\chi_f\colon \mathrm{Hom}_{\mathbf{A}}(C_f,-) \cong Cone_f^{\mathrm{Alg}}. \tag{6.7}$$

A particular choice determines a functor

$$\bar{C}\colon \mathbf{AHp}^{rep}(\aleph) \longrightarrow \mathbf{A},\ \ f \mapsto C_f,$$

such that the restriction to $\mathbf{AHp}^{rep}(\aleph)$ of $Cone^{\mathrm{Alg}}$ in (6.4) factors up the natural equivalence χ as

$$\mathbf{AHp}^{rep}(\aleph)^{op} \xrightarrow{\bar{C}^{op}} \mathbf{A}^{op} \subset \mathbf{mod}(\mathbf{A}).$$

By using $\bar{C}$ we can define a functor

$$\bar{\zeta}\colon \mathbf{AHp}^{rep}(\aleph) \longrightarrow \mathbf{cand}(\mathbf{A},t) \tag{6.8}$$

as follows: given an object $f\colon A \to B$ in $\mathbf{AHp}^{rep}(\aleph)$ the small candidate triangle $\bar{\zeta}(f)$ is given by

$$A \xrightarrow{f} B \xrightarrow{i_f} C_f \xrightarrow{q_f} tA$$

where i_f and q_f are determined by the **A**-module morphisms $\tilde{i}_f$ and $\tilde{q}_f$ in (6.5), respectively, the isomorphism χ_f in (6.7) and Yoneda's lemma; and given a morphism $(h,k,a)\colon f \to \bar{f}$ in $\mathbf{AHp}^{rep}(\aleph)$ then $\bar{\zeta}(h,k,a)$ is the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[r, "i_f"] \arrow[d, "k"] & C_f \arrow[r, "q_f"] \arrow[d, "\bar{C}(h,k,a)"] & tA \arrow[d, "th"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "i_{\bar{f}}"'] & C_{\bar{f}} \arrow[r, "q_{\bar{f}}"'] & t\bar{A}
\end{tikzcd}$$

This diagram commutes by (6.6). The functor $\bar{\zeta}$ does not depend on the different choices made for its definition, up to natural equivalence.

In [IV] 4.22 we defined a linear action of the $\mathbf{Pair}(\mathbf{A})$-bimodule

$$\mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker})$$

on the obvious forgetful functor

$$\varsigma\colon \mathbf{cand}(\mathbf{A},t) \longrightarrow \mathbf{Pair}(\mathbf{A})$$

defined in [IV] 4.15. Moreover, in [IV] 4.23 we define a natural transformation

$$\Xi\colon (\mathrm{Hom}^t)^{\#} \Rightarrow \mathrm{Hom}_{\mathbf{mod}(\mathbf{A}^{op})}(\mathrm{Ker}\, t, \mathrm{Coker}).$$

This natural transformation is compatible with the functor $\bar{\zeta}$ in the following sense.

Proposition 6.9. *Given a morphism* $(h,k,a)\colon f \to \bar{f}$ *in* $\mathbf{AHp}^{rep}(\aleph)$ *and* $\alpha \in (\mathrm{Hom}^t)^{\#}(\bar{p}(f), \bar{p}(\bar{f}))$ *then*

$$\bar{\zeta}((h,k,a)+\alpha) = \bar{\zeta}(h,k,a) + \Xi_{(\bar{p}(f),\bar{p}(\bar{f}))}(\alpha).$$

This proposition follows by the same kind of arguments as in the proof of [IV] 4.24.

We also defined in [IV] 4.20 a natural transformation

$$\varpi\colon H^1(sh^3, [-,-]^*\bar{t}) \Rightarrow (\mathrm{Hom}^t)^{\#}(\varsigma,\varsigma).$$

We can use this natural transformation to define the following natural equivalence relation in $\mathbf{AHp}(\aleph)$: two morphisms $(h,k,a), (\bar{h},\bar{k},\bar{a})\colon f \to \bar{f}$ in $\mathbf{AHp}(\aleph)$ are equivalent if $(h,k,a)+\varpi(b) = (\bar{h},\bar{k},\bar{a})$ for some $b \in H^1(sh^3, [\bar{\zeta}(f), \bar{\zeta}(\bar{f})]^*\bar{t})$. The quotient category will be denoted by

$$\mathbf{AHp}(\aleph)/(\varpi,\bar{\zeta}).$$

Moreover, we denote

$$\mathbf{AHp}^{rep}(\aleph)/(\varpi,\bar{\zeta})$$

to the full subcategory of the quotient whose objects are the morphisms f in **A** with representable associated algebraic cone functor.

By Proposition 6.9 and the first part of [IV] 4.25 the functor $\bar{\zeta}$ factors through this quotient category

$$\bar{\zeta}\colon \mathbf{AHp}^{rep}(\aleph) \twoheadrightarrow \mathbf{AHp}^{rep}(\aleph)/(\varpi,\bar{\zeta}) \xrightarrow{\hat{\zeta}} \mathbf{cand}(\mathbf{A},t). \tag{6.10}$$

Here the first functor is the projection onto the quotient category and $\hat{\zeta}$ is uniquely determined by this factorization.

The following theorem is one of the main results of this paper.

Theorem 6.11. *Let* $\mathbf{A}$ *be an additive category,* $t\colon \mathbf{A} \to \mathbf{A}$ *an additive equivalence and* $\nabla \in H^3(\mathbf{A}, t)$*. Then* $(\mathbf{A}, t, \nabla)$ *is cohomologically pretriangulated if and only if all algebraic cone functors defined in (6.3) are representable and, using the equivalence of categories in (4.2) as an identification, the pull-back of the cohomology class* $\mathrm{id} + \bar{\nabla}$ *in Section 5 by the functor* $\hat{\zeta}$ *in (6.10) vanishes*

$$0 = \hat{\zeta}^*(\mathrm{id} + \bar{\nabla}) \in H^0(\mathbf{AHp}(\aleph)/(\varpi, \bar{\zeta}), H^3(sh^3, [\bar{\zeta}, \bar{\zeta}]^*\bar{t})).$$

The proof of this result will be given in Section 10.

Remark 6.12. At a first glance the vanishing condition in Theorem 6.11 seems to depend on the choice of the cocycle $\aleph$ representing $\lambda j \nabla$ and the representations χ_f in (6.7) used in the definition of the functor $\bar{\zeta}\colon \mathbf{AHp}(\aleph) \to \mathbf{cand}(\mathbf{A}, t)$. However it is easy to check that if $\bar{\aleph}$ is another cocycle representing $\lambda j \nabla$ and $\bar{\xi}\colon \mathbf{AHp}(\bar{\aleph}) \to \mathbf{cand}(\mathbf{A}, t)$ is defined as $\bar{\zeta}$ then for any isomorphism of linear extensions

$$\begin{tikzcd}
(\mathrm{Hom}^t)^{\#} \arrow[r, "+"] \arrow[d, equal] & \mathbf{AHp}(\aleph) \arrow[r] \arrow[d, "\cong"'] & \mathbf{Pair}(\mathbf{A}) \arrow[d, equal] \\
(\mathrm{Hom}^t)^{\#} \arrow[r, "+"] & \mathbf{AHp}(\bar{\aleph}) \arrow[r] & \mathbf{Pair}(\mathbf{A})
\end{tikzcd}$$

the isomorphism of categories in the middle induces isomorphisms between the algebraic cone functors given by $\aleph$ and $\bar{\aleph}$, therefore the first ones are representable if and only if the second ones are. Moreover, the diagram

$$\begin{tikzcd}
\mathbf{AHp}^{rep}(\aleph) \arrow[dd, "\cong"'] \arrow[dr, "\bar{\zeta}"] & \\
& \mathbf{cand}(\mathbf{A}, t) \\
\mathbf{AHp}^{rep}(\bar{\aleph}) \arrow[ur, "\bar{\xi}"'] &
\end{tikzcd}$$

commutes up to natural equivalence. This second diagram induces another one

$$\begin{tikzcd}
\mathbf{AHp}^{rep}(\aleph)/(\varpi, \bar{\zeta}) \arrow[dd, "\cong"'] \arrow[dr, "\hat{\zeta}"] & \\
& \mathbf{cand}(\nabla) \\
\mathbf{AHp}^{rep}(\bar{\aleph})/(\varpi, \bar{\xi}) \arrow[ur, "\hat{\xi}"'] &
\end{tikzcd}$$

which is also commutative up to equivalence. If algebraic cone functors are representable the behavior of cohomology of categories with respect to natural transformations, see [Mur04] 4.3, implies that the cohomology groups where the classes $\hat{\zeta}^*(\mathrm{id}+\bar{\nabla})$ and $\hat{\xi}^*(\mathrm{id}+\bar{\nabla})$ live are isomorphic under an isomorphism carrying one of these classes to the other, therefore

$$\hat{\zeta}^*(\mathrm{id} + \bar{\nabla}) = 0 \Leftrightarrow \hat{\xi}^*(\mathrm{id} + \bar{\nabla}) = 0,$$

so the vanishing condition in Theorem 6.11 is independent of all choices.

Suppose now that $(\mathbf{A}, t, \nabla)$ is cohomologically pretriangulated. By Theorem 6.11 and Proposition 5.7 the functor $\hat{\zeta}$ in (6.10) takes values in small ∇-triangles

$$\hat{\zeta}\colon \mathbf{AHp}(\aleph)/(\varpi, \bar{\zeta}) \longrightarrow \mathbf{cand}(\nabla).$$

By [Bau89] IV.4.10 (e) the exact sequence for a functor constructed in [IV] 3.1 gives rise to a well-defined cohomology class

$$\theta_\nabla \in H^1(\mathbf{cand}(\nabla), H^2(sh^3, [i_\nabla, i_\nabla]^*\bar{t})). \tag{6.13}$$

Here $i_\nabla\colon \mathbf{cand}(\nabla) \subset \mathbf{cand}(\mathbf{A}, t)$ is the inclusion of the full subcategory of small ∇-triangles.

The following theorem is also a main result in this paper.

Theorem 6.14. *Let $(\mathbf{A}, t, \nabla)$ be a cohomologically pretriangulated category. Then it is cohomologically triangulated if and only if*

$$0 = \hat{\zeta}^*\theta_\nabla \in H^1(\mathbf{AHp}(\aleph)/(\varpi, \bar{\zeta}), H^2(sh^3, [i_\nabla\hat{\zeta}, i_\nabla\hat{\zeta}]^*\bar{t})).$$

The proof of this theorem will be given in Section 10.

As in the case of small ∇-triangles, the original definition of θ_∇ is based on the fact that 3-dimensional translation cohomology classes are represented by translation track categories. Nevertheless, as we did with small ∇-triangles in Section 5, in the next section we give a purely cohomological procedure to obtain θ_∇ from ∇ by using a spectral sequence.

Remark 6.12 also applies to Theorem 6.14 with the appropriate modifications.

7 The main spectral sequence

Cohomologically triangulated categories $(\mathbf{A}, t, \nabla)$ were defined in [IV] by using the representability of 3-dimensional translation cohomology classes by translation track categories. By means of the classes of identities in Section 4 we were able to show in Proposition 5.7 that small ∇-triangles (the exact triangles of cohomologically triangulated categories) can be obtained from ∇ by purely cohomological methods. Theorem 6.14 characterizes cohomologically triangulated categories in strictly cohomological terms by using a 1-dimensional cohomology class θ_∇. This class is obtained from an exact sequence for functors in [IV] 3.1 encoding the obstruction theory for the category of track triangles associated to a good translation track category representing ∇. However the reader may be missing a more intrinsical approach to cohomologically triangulated categories, let us say a construction of θ_∇ from ∇ by using only techniques from cohomology of categories. We address this issue in this section, which is mainly expository. The most of the proofs and the technical constructions appear in the next one. As a consequence of these results we give a new characterization of cohomologically (pre)triangulated categories in terms of the filtration associated to the spectral sequences constructed in the next section. This will be the main result of this paper.

In the next section we construct some spectral sequences for the computation of the (translation) cohomology of products. There we also define for any translation category $(\mathbf{A}, t)$ with $\mathbf{A}$ additive and t an additive equivalence a canonical translation cohomology class

$$\mathrm{Id} \in H^3(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}),$$

which we call the *global class of identities*, see (8.15).

Fixed a translation cohomology class

$$\nabla \in H^3(\mathbf{A}, t)$$

in this section we show how the spectral sequences in Section 8 together with ∇ and the global class of identities determine the cohomology classes $\mathrm{id} + \bar{\nabla}$ and θ_∇ which define ∇-triangles and are used to detect when $(\mathbf{A}, t, \nabla)$ is a cohomologically (pre)triangulated category, see Theorems 6.11 and 6.14.

In Remark 8.9 we define an "evaluation" translation category morphism

$$(ev, \epsilon)\colon \mathbf{Cand}(\mathbf{A}, t) \times (\triangle, sh^3) \longrightarrow (\mathbf{A}, t).$$

As we explain in that remark the pull-back of $\bar{t}$ in (4.11) along (ev, ϵ) is denoted by

$$(ev, \epsilon)^*\bar{t} = [-,-]^*\bar{t},$$

since it is given by the natural transformations between $\triangle$-bimodules defined by (3.4). This "evaluation" morphism can be used to pull back the translation cohomology class ∇ to the same cohomology group where the global class of identities lives

$$(ev, \epsilon)^*\nabla \in H^3(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}).$$

The following theorem is the main result of this paper.

Theorem 7.1. *Let $\mathbf{A}$ be an additive category, $t\colon \mathbf{A} \to \mathbf{A}$ an additive equivalence and $\nabla \in H^3(\mathbf{A}, t)$. Then $(\mathbf{A}, t, \nabla)$ is cohomologically triangulated (resp. pretriangulated) if and only if all algebraic cone functors defined in (6.3) are representable and the cohomology class*

$$(\hat{\zeta} \times 1_\triangle)^*(\mathrm{Id} + (ev, \epsilon)^*\nabla) \in H^3(1_{\mathbf{AHp}(\aleph)/(\varpi,\bar{\zeta})} \times sh^3, [\hat{\zeta}, \hat{\zeta}]^*\bar{t}),$$

where $\hat{\zeta}$ is defined in (6.10), lies in the third (resp. second) term $D^{2,1}$ (resp. $D^{1,2}$) of the filtration

$$0 = D^{4,-1} \subset D^{3,0} \subset D^{2,1} \subset D^{1,2} \subset D^{0,3} = H^3(1_{\mathbf{AHp}(\aleph)/(\varpi,\bar{\zeta})} \times sh^3, [\hat{\zeta}, \hat{\zeta}]^*\bar{t})$$

given by the spectral sequence in Proposition 8.8.

This theorem is a consequence of Theorems 6.11 and 6.14 together with Lemma 7.2, (7.4), Proposition 7.5 and the naturality of the spectral sequence in Proposition 8.8 below.

Remark 8.9 and Proposition 8.8 yield a spectral sequence

$$E_2^{p,q} = H^p(\mathbf{Cand}(\mathbf{A}, t), H^q(sh^3, [-,-]^*\bar{t})) \Longrightarrow H^{p+q}(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}).$$

Let

$$0 = D^{n+1,-1} \subset D^{n,0} \subset \cdots \subset D^{p,q} \subset \cdots \subset D^{0,n} = H^n(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t})$$

be the filtration such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$. There are homomorphisms ($n \geq 0$)

$$\xi_0\colon H^n(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}) \longrightarrow H^0(\mathbf{Cand}(\mathbf{A}, t), H^n(sh^3, [-,-]^*\bar{t}))$$

given by

$$D^{0,n} \twoheadrightarrow D^{0,n}/D^{1,n-1} = E_\infty^{0,n} \subset E_2^{0,n}.$$

The homomorphisms ξ_0 satisfy the property stated in the following lemma.

Lemma 7.2. *In the conditions above, given $b \in H^n(\mathbf{A}, t)$ and a candidate triangle (X, α) in $(\mathbf{A}, t)$ we have that*

$$\xi_0((ev, \epsilon)^* b)(X, \alpha) = (X, \alpha)^* b \in H^n(sh^3, (X, \alpha)^* \bar{t}).$$

This is Lemma 8.12, proved in the next section. Lemma 5.5 follows from this one. Moreover, the global class of identities determines the classes of identities through ξ_0.

Lemma 7.3. *For any candidate triangle (X, α) we have $(\xi_0(\mathrm{Id}))(X, \alpha) = \mathrm{id}(X, \alpha)$.*

This is Lemma 8.16, also proved in the next section. This implies that the 0-cochain id determined by the classes of identities in (4.13) is actually a cohomology class as stated in Lemma 5.4, and moreover

$$\xi_0(\mathrm{Id}) = \mathrm{id}. \tag{7.4}$$

In particular by using these lemmas we notice that ∇-triangles are the candidate triangles annihilated by the cohomology class

$$\xi_0(\mathrm{Id} + (ev, \epsilon)^* \nabla) \in H^0(\mathbf{Cand}(\mathbf{A}, t), H^3(sh^3, [-, -]^* \bar{t})).$$

Let $i_\nabla \colon \mathbf{Cand}(\nabla) \to \mathbf{Cand}(\mathbf{A}, t)$ be the inclusion of the full subcategory of ∇-triangles. By Proposition 8.8 there is also a spectral sequence

$$E_\infty^{p,q} = H^p(\mathbf{Cand}(\nabla), H^q(sh^3, [i_\nabla, i_\nabla]^* \bar{t})) \Longrightarrow H^{p+q}(1_{\mathbf{Cand}(\nabla)} \times sh^3, [i_\nabla, i_\nabla]^* \bar{t}).$$

We also denote by

$$0 = D^{n+1,-1} \subset D^{n,0} \subset \cdots \subset D^{p,q} \subset \cdots \subset D^{0,n} = H^n(1_{\mathbf{Cand}(\nabla)} \times sh^3, [i_\nabla, i_\nabla]^* \bar{t})$$

to the filtration such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$ and by ξ_0 to the homomorphisms $(n \geq 0)$

$$\xi_0 \colon H^n(1_{\mathbf{Cand}(\nabla)} \times sh^3, [i_\nabla, i_\nabla]^* \bar{t}) \longrightarrow H^0(\mathbf{Cand}(\nabla), H^n(sh^3, [i_\nabla, i_\nabla]^* \bar{t}))$$

given by

$$D^{0,n} \twoheadrightarrow D^{0,n}/D^{1,n-1} = E_\infty^{0,n} \subset E_2^{0,n}.$$

The category $\mathbf{Cand}(\nabla)$ is the biggest full subcategory such that its inclusion functor i_∇ satisfies

$$0 = i_\nabla^* \xi_0(\mathrm{Id} + (ev, \epsilon)^* \nabla) = \xi_0(i_\nabla \times 1_\triangle, 0^\square_{i_\nabla \times sh^3})^* (\mathrm{Id} + (ev, \epsilon)^* \nabla).$$

Here for the second equality we use the naturality of the spectral sequence in Proposition 8.8. This shows that

$$(i_\nabla \times 1_\triangle, 0^\square_{i_\nabla \times sh^3})^* (\mathrm{Id} + (ev, \epsilon)^* \nabla) \in D^{1,2} \subset H^3(1_{\mathbf{Cand}(\nabla)} \times sh^3, [i_\nabla, i_\nabla]^* \bar{t}).$$

Let ξ_1 be the homomorphism given by

$$\xi_1 \colon D^{1,2} \twoheadrightarrow D^{1,2}/D^{2,1} = E_\infty^{1,2} \subset E_2^{1,2} = H^1(\mathbf{Cand}(\nabla), H^2(sh^3, [i_\nabla, i_\nabla]^* \bar{t})).$$

Proposition 7.5. *Let*

$$\theta_\nabla \in H^1(\mathbf{Cand}(\nabla), H^2(sh^3, [i_\nabla, i_\nabla]^* \bar{t}))$$

be the cohomology class determined by θ_∇ in (6.13), the equivalence of categories in (4.1) and Proposition 5.7. Then the following equality holds

$$\theta_\nabla = \xi_1(i_\nabla \times 1_\triangle, 0^\square_{i_\nabla \times sh^3})^* (\mathrm{Id} + (ev, \epsilon)^* \nabla).$$

This theorem is proved in Section 10

In particular we obtain that the cohomology class θ_∇, which was defined by using a translation track category representing ∇, is indeed well defined by ∇.

8 The construction of the spectral sequences and the global class of identities

The normalized cochain complex $\bar{F}^*(\mathbf{C}, D)$ defined in Section 2 for the computation of cohomology of categories is the normalization of a cosimplicial abelian group $F^\bullet(\mathbf{C}, D)$, see [BD89] Appendix B,

$$\bar{F}^*(\mathbf{C}, D) = NF^\bullet(\mathbf{C}, D).$$

We refer the reader to the Appendix of this paper for basic facts and notation concerning (bi)cosimplicial abelian groups, normalized (bi)complexes, diagonals, and total complexes. Here $F^\bullet(\mathbf{C}, D)$ is given by the abelian groups in (2.1). The coboundaries d_i and codegeneracies s_i are given by

$$\begin{array}{rcl} (d_i c)(\sigma_1, \dots, \sigma_n) & = & \left\{ \begin{array}{ll} \sigma_{1*} c(\sigma_2, \dots, \sigma_n), & i = 0; \\ c(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_n), & 0 < i < n; \\ \sigma_n^* c(\sigma_1, \dots, \sigma_{n-1}), & i = n; \end{array} \right. \\ (s_i c)(\sigma_1, \dots, \sigma_{n-1}) & = & c(\sigma_1, \dots, \sigma_i, 1_{X_i}, \sigma_{i+1}, \dots, \sigma_{n-1}). \end{array}$$

where $c \in F^n(\mathbf{C}, D)$ and X_i is the source of σ_i and/or the target of σ_{i+1}. The ordinary functorial properties of this bicomplex are the same as for the cochain complexes $\bar{F}^*(\mathbf{C}, D)$ and $F^*(\mathbf{C}, D)$, see [BW85] or [Mur04].

Let $\mathbf{C}$ and $\mathbf{B}$ be categories and D a $(\mathbf{C} \times \mathbf{B})$-bimodule. The functorial properties of $F^\bullet$ yield a cosimplicial $\mathbf{C}$-bimodule $F^\bullet(\mathbf{B}, D)$. Moreover, by applying $F^\bullet(\mathbf{C}, -)$ to this simplicial $\mathbf{C}$-bimodule we obtain a bicosimplicial abelian group

$$F^\bullet(\mathbf{C}, F^\bullet(\mathbf{B}, D)).$$

Lemma 8.1. *There is a natural isomorphism of cosimplicial abelian groups*

$$\mathrm{Diag} F^\bullet(\mathbf{C}, F^\bullet(\mathbf{B}, D)) \cong F^\bullet(\mathbf{C} \times \mathbf{B}, D).$$

Proof. A cochain $c \in F^n(\mathbf{C}, F^n(\mathbf{B}, D))$ corresponds to the cochain $\bar{c} \in F^n(\mathbf{C} \times \mathbf{B}, D)$ defined by

$$\bar{c}((\sigma_1, \tau_1), \dots, (\sigma_n, \tau_n)) = (c(\sigma_1, \dots, \sigma_n))(\tau_1, \dots, \tau_n).$$

The reader can easily check that this identification is compatible with the cosimplicial operators. □

The isomorphism in this lemma will be used as an identification. Moreover, this lemma and the cosimplicial Eilenberg-Zilber-Cartier theorem described in the Appendix yield the following result.

Corollary 8.2 (Künneth spectral sequence). *Given categories* $\mathbf{C}$ *and* $\mathbf{D}$ *and a* $(\mathbf{C} \times \mathbf{B})$*-bimodule* D*, there is a natural spectral sequence*

$$E_2^{p,q} = H^p(\mathbf{C}, H^q(\mathbf{B}, D)) \Longrightarrow H^{p+q}(\mathbf{C} \times \mathbf{B}, D).$$

Proof. This is the spectral sequence of the bicomplex $\bar{F}^n(\mathbf{C}, \bar{F}^n(\mathbf{B}, D))$. The abutment is indeed isomorphic to the cohomology of $\bar{F}^*(\mathbf{C} \times \mathbf{B}, D)$ by Lemma 8.1 and Theorem A.1. □

This spectral sequence can be regarded as a particular case of the spectral sequence constructed in [PR05].

Remark 8.3. For any category $\mathbf{C}$ the category of pairs $\mathbf{Pair}(\mathbf{C})$ introduced in Section 6 is nothing but the category of functors $\mathbb{I} \to \mathbf{C}$ where $\mathbb{I}$ is the unit interval category, with only two objects $0, 1$ and a unique non-trivial morphism $0 \to 1$. In particular there is an evaluation functor

$$\overline{ev}\colon \mathbf{Pair}(\mathbf{C}) \times \mathbb{I} \longrightarrow \mathbf{C}.$$

For any $\mathbf{C}$-bimodule D there is an associated $\mathbf{Pair}(\mathbf{C})$-bimodule $D^{\#}$ constructed in [Bau97] together with natural homomorphisms

$$\lambda\colon H^{n+1}(\mathbf{C}, D) \longrightarrow H^n(\mathbf{Pair}(\mathbf{C}), D^{\#}), \quad n \geq 0.$$

For $n = 2$ the homomorphism λ plays a role in Section 6. The reader can check by using the normalized cochain complex $\bar{F}^*(\mathbb{I}, -)$ that there is a natural identification

$$D^{\#} = H^1(\mathbb{I}, \overline{ev}^* D).$$

Moreover, $H^n(\mathbb{I}, -)$ vanishes for all $n > 1$ and the homomorphisms

$$H^{n+1}(\mathbf{C}, D) \twoheadrightarrow E_\infty^{n,1} \hookrightarrow E_2^{n,1} = H^n(\mathbf{Pair}(\mathbf{C}), H^1(\mathbb{I}, \overline{ev}^* D)) = H^n(\mathbf{Pair}(\mathbf{C}), D^{\#})$$

induced by the Künneth spectral sequence in Corollary 8.2 coincide with λ for all $n \geq 0$.

Suppose now that $\mathbf{C}$ is a category with zero object, $(\mathbf{B}, s)$ is a translation category, D is a $(\mathbf{C} \times \mathbf{B})$-bimodule, and $\bar{s}\colon D \Rightarrow D(1 \times s, 1 \times s)$ is a natural transformation. Then there is a morphism of cosimplicial $\mathbf{C}$-bimodules

$$\bar{s}_* - s^*\colon F^\bullet(\mathbf{B}, D) \longrightarrow F^\bullet(\mathbf{B}, D(1 \times s, 1 \times s)).$$

By applying $F^\bullet(\mathbf{C}, -)$ we obtain now a bicosimplicial abelian group homomorphism

$$F^\bullet(\mathbf{C}, \bar{s}_* - s^*)\colon F^\bullet(\mathbf{C}, F^\bullet(\mathbf{B}, D)) \longrightarrow F^\bullet(\mathbf{C}, F^\bullet(\mathbf{B}, D(1 \times s, 1 \times s))).$$

On the other hand the functorial properties of the translation cochain complex $\bar{F}(s, -)$ described in Remark 3.6 imply that $\bar{F}^*(s, \bar{s})$ is a cochain complex of $\mathbf{C}$-bimodules, therefore if we apply $\bar{F}^*(\mathbf{C}, -)$ we obtain a bicomplex

$$\bar{F}^*(\mathbf{C}, \bar{F}^*(s, \bar{s})). \tag{8.4}$$

The translation cochain complex

$$\bar{F}^*(1_{\mathbf{C}} \times s, \bar{s}) \tag{8.5}$$

is also defined.

In the next two lemmas we identify (8.5) and the total complex of (8.4) as the homotopy fibers of the two possible ways of converting $F^\bullet(\mathbf{C}, \bar{s}_* - s^*)$ into a cochain homomorphism. For this we recall that given a cochain morphism

$$\xi\colon A^* \longrightarrow B^*$$

its *homotopy fiber* $\mathrm{Fib}(\xi)$ is the cochain complex defined by

$$\mathrm{Fib}^n(\xi) = A^n \oplus B^{n-1}$$

and differential

$$\begin{pmatrix} d & 0 \\ \xi & -d \end{pmatrix}.$$

Lemma 8.6. *There is a natural identification*

$$\mathrm{FibTot}NF^\bullet(\mathbf{C},\bar{s}_*-s^*)=\mathrm{Tot}\bar{F}^*(\mathbf{C},\bar{F}^*(s,\bar{s})).$$

Proof. We have the following equalities

$$\begin{aligned}
\mathrm{Fib}^n\mathrm{Tot}NF^\bullet(\mathbf{C},\bar{s}_*-s^*) &= \mathrm{Tot}^nNF^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D))\\
&\quad\oplus\mathrm{Tot}^{n-1}NF^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D(1\times s,1\times s)))\\
&= \bigoplus_{p+q=n}N^{p,q}F^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D))\\
&\quad\oplus\bigoplus_{p+q=n-1}N^{p,q}F^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D(1\times s,1\times s)))\\
&= \bigoplus_{p+q=n}\bar{F}^p(\mathbf{C},\bar{F}^q(\mathbf{B},D))\\
&\quad\oplus\bigoplus_{p+q=n-1}\bar{F}^p(\mathbf{C},\bar{F}^q(\mathbf{B},D(1\times s,1\times s)))\\
&= \bigoplus_{p+q=n}\left(\bar{F}^p(\mathbf{C},\bar{F}^q(\mathbf{B},D))\oplus\bar{F}^p(\mathbf{C},\bar{F}^{q-1}(\mathbf{B},D(1\times s,1\times s)))\right)\\
&= \bigoplus_{p+q=n}\bar{F}^p(\mathbf{C},\bar{F}^q(s,\bar{s}))\\
&= \mathrm{Tot}^n\bar{F}^*(\mathbf{C},\bar{F}^*(s,\bar{s})).
\end{aligned}$$

We leave to the reader to check the compatibility with the differentials. □

Lemma 8.7. *There is a natural identification*

$$\mathrm{Fib}N\mathrm{Diag}F^\bullet(\mathbf{C},\bar{s}_*-s^*)=\bar{F}^*(1_{\mathbf{C}}\times s,\bar{s}).$$

Proof. We have the following identifications

$$\begin{aligned}
\mathrm{Fib}^nN\mathrm{Diag}F^\bullet(\mathbf{C},\bar{s}_*-s^*) &= N^n\mathrm{Diag}F^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D))\\
&\quad\oplus N^{n-1}\mathrm{Diag}F^\bullet(\mathbf{C},F^\bullet(\mathbf{B},D(1\times s,1\times s)))\\
&\overset{\text{(a)}}{=} N^nF^\bullet(\mathbf{C}\times\mathbf{B},D)\\
&\quad\oplus N^{n-1}F^\bullet(\mathbf{C}\times\mathbf{B},D(1\times s,1\times s))\\
&= \bar{F}^n(\mathbf{C}\times\mathbf{B},D)\oplus\bar{F}^{n-1}(\mathbf{C}\times\mathbf{B},D(1\times s,1\times s))\\
&= \bar{F}^n(1_{\mathbf{C}}\times s,\bar{s}).
\end{aligned}$$

Here we use Lemma 8.1 in (a). The reader can check that this is compatible with the differentials. □

By using these two lemmas and the Eilenberg-Zilber-Cartier theorem in the Appendix we obtain the following result.

Proposition 8.8. *Consider a category* $\mathbf{C}$ *with zero object, a translation category* $(\mathbf{B},s)$*, a* $(\mathbf{C}\times\mathbf{B})$*-bimodule* D*, and a natural transformation* $\bar{s}\colon D\Rightarrow D(1\times s,1\times s)$*. Then there is a natural spectral sequence*

$$E_2^{p,q}=H^p(\mathbf{C},H^q(s,\bar{s}))\Longrightarrow H^{p+q}(1_{\mathbf{C}}\times s,\bar{s}).$$

Proof. This is the spectral sequence of the bicomplex (8.4). Lemmas 8.6 and 8.7 together with the naturality of the cochain homotopy equivalences in Theorem A.1, the long exact sequence for the cohomology of a fibration of cochain complexes, and the five lemma imply that the cochain homomorphism given by the matrices

$$\begin{pmatrix}\mathfrak{F}_n & 0\\ 0 & \mathfrak{F}_{n-1}\end{pmatrix},\quad n\in\mathbb{Z},$$

induces an isomorphism between the cohomology of (8.5) and the cohomology of the total complex of (8.4), therefore the abutment of the spectral sequence is as in the statement of this proposition. □

We will write

$$0 = D^{n+1,-1} \subset D^{n,0} \subset \cdots \subset D^{p,q} \subset \cdots \subset D^{0,n} = H^n(1_{\mathbf{C}} \times s, \bar{s})$$

for the filtration such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$.

Remark 8.9. We are specially interested in the case where $(\mathbf{B}, s) = (\triangle, sh^3)$ and $\mathbf{C} = \mathbf{Cand}(\mathbf{A}, t)$ for some translation category $(\mathbf{A}, t)$ as in Section 5. There is a translation category morphism

$$(ev, \epsilon)\colon \mathbf{Cand}(\mathbf{A}, t) \times (\triangle, sh^3) \longrightarrow (\mathbf{A}, t)$$

defined by the zero-object-preserving "evaluation" functor

$$ev\colon \mathbf{Cand}(\mathbf{A}, t) \times \triangle \longrightarrow \mathbf{A}$$

given by $ev((X,\alpha),n) = X_n$ on objects and given a morphism $f\colon (X,\alpha) \to (Y,\beta)$ of candidate triangles $ev(f, 1_n) = f_n$, $ev(1_{(X,\alpha)}, \imath_n) = d_n$, and the natural transformation

$$\epsilon\colon t\; ev \Rightarrow ev(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3)$$

with $\epsilon_{((X,\alpha),n)} = \alpha_n\colon tX_n \to X_{n+3}$. We therefore can pull back the natural transformation $\bar{t}\colon \operatorname{Hom}^t \Rightarrow \operatorname{Hom}^t(t,t)$, see (4.10) and (4.11), along (ev, ϵ)

$$(ev,\epsilon)^*\bar{t}\colon \operatorname{Hom}_{\mathbf{A}}(t\; ev, ev) \Rightarrow \operatorname{Hom}_{\mathbf{A}}(t\; ev(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3), ev(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3)).$$

If we evaluate this natural transformation of $(\mathbf{Cand}(\mathbf{A}, t) \times \triangle)$-bimodules on two candidate triangles (X,α) and (Y,β), leaving free the second variable, we obtain the following natural transformation of $\triangle$-bimodules

$$((ev,\epsilon)^*\bar{t})_{(((X,\alpha),-),((Y,\beta),-))} = [(X,\alpha),(Y,\beta)]^*\bar{t},$$

see (3.4), hence we can write

$$(ev,\epsilon)^*\bar{t} = [-,-]^*\bar{t}.$$

This remark and Proposition 8.8 yield a spectral sequence

(8.10)
$$E_\infty^{p,q} = H^p(\mathbf{Cand}(\mathbf{A},t), H^q(sh^3, [-,-]^*\bar{t})) \Longrightarrow H^{p+q}(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}).$$

In particular there are homomorphisms ($n \geq 0$)

(8.11) $$\xi_0\colon H^n(1_{\mathbf{Cand}(\mathbf{A},t)} \times sh^3, [-,-]^*\bar{t}) \longrightarrow H^0(\mathbf{Cand}(\mathbf{A},t), H^n(sh^3, [-,-]^*\bar{t}))$$

given by

$$D^{0,n} \twoheadrightarrow D^{0,n}/D^{1,n-1} = E_\infty^{0,n} \subset E_2^{0,n}.$$

The homomorphisms ξ_0 satisfy the property stated in the following lemma, see (3.5).

Lemma 8.12. *In the conditions above, given $b \in H^n(\mathbf{A}, t)$ and a candidate triangle (X,α) in $(\mathbf{A}, t)$ we have that*

$$\xi_0((ev,\epsilon)^*b)(X,\alpha) = (X,\alpha)^*b \in H^n(sh^3, (X,\alpha)^*\bar{t}).$$

Proof. The homomorphisms ξ_0 are given as follows. Let

$$\begin{array}{rcl}\bar{c}=(\bar{c}_1,\bar{c}_2) & \in & \bar{F}^n(1_{\mathbf{Cand}(\mathbf{A},t)}\times sh^3,[-,-]^*\bar{t}) \\ & = & \bar{F}^n(\mathbf{Cand}(\mathbf{A},t)\times\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev,ev)) \\ & & \oplus\bar{F}^{n-1}(\mathbf{Cand}(\mathbf{A},t)\times\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev(1\times sh^3),ev(1\times sh^3)))\end{array}$$

be a cocycle representing $c\in H^n(1_{\mathbf{Cand}(\mathbf{A},t)}\times sh^3,[-,-]^*\bar{t})$. We are going to use the identification in Lemma 8.1 as well as the homotopy equivalence $\mathfrak{F}$ in Theorem A.1. The cochains

$$\begin{array}{rcl}\mathfrak{F}(\bar{c}_1) & \in & \mathrm{Tot}^n\bar{F}^*(\mathbf{Cand}(\mathbf{A},t),\bar{F}^*(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev,ev))) \\ & = & \displaystyle\bigoplus_{p+q=n}\bar{F}^p(\mathbf{Cand}(\mathbf{A},t),\bar{F}^q(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev,ev))), \\ \mathfrak{F}(\bar{c}_2) & \in & \mathrm{Tot}^{n-1}\bar{F}^*(\mathbf{Cand}(\mathbf{A},t),\bar{F}^*(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev(1\times sh^3),ev(1\times sh^3)))) \\ & = & \displaystyle\bigoplus_{p+q=n-1}\bar{F}^p(\mathbf{Cand}(\mathbf{A},t),\bar{F}^q(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev(1\times sh^3),ev(1\times sh^3)))),\end{array}$$

have components

$$\begin{array}{rcl}\mathfrak{F}(\bar{c}_1)^{0,n} & \in & \bar{F}^0(\mathbf{Cand}(\mathbf{A},t),\bar{F}^n(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev,ev))), \\ \mathfrak{F}(\bar{c}_2)^{0,n-1} & \in & \bar{F}^0(\mathbf{Cand}(\mathbf{A},t),\bar{F}^{n-1}(\triangle,\mathrm{Hom}_{\mathbf{A}}(t\;ev(1\times sh^3),ev(1\times sh^3)))).\end{array}$$

The element $\xi_0(c)(X,\alpha)$ is represented by the cocycle

$$(\mathfrak{F}(\bar{c}_1)^{0,n}(X,\alpha),\mathfrak{F}(\bar{c}_2)^{0,n-1}(X,\alpha))\in\bar{F}^n(sh^3,[-,-]^*\bar{t}).$$

By using the definition of $\mathfrak{F}$ in the statement of Theorem A.1 we see that this cochain is given as follows

$$\begin{array}{rcl}(\mathfrak{F}(\bar{c}_1)^{0,n}(X,\alpha))(\sigma_1,\dots,\sigma_n) & = & \bar{c}_1((1_{(X,\alpha)},\sigma_1),\dots,(1_{(X,\alpha)},\sigma_n)), \\ (\mathfrak{F}(\bar{c}_2)^{0,n-1}(X,\alpha))(\sigma_1,\dots,\sigma_{n-1}) & = & \bar{c}_2((1_{(X,\alpha)},\sigma_1),\dots,(1_{(X,\alpha)},\sigma_{n-1})).\end{array}$$

Suppose now that

$$\bar{b}=(\bar{b}_1,\bar{b}_2)\in\bar{F}^n(t,\bar{t})=\bar{F}^n(\mathbf{A},\mathrm{Hom}_{\mathbf{A}}(t,-))\oplus\bar{F}^{n-1}(\mathbf{A},\mathrm{Hom}_{\mathbf{A}}(t^2,t))$$

is a cocycle representing $b\in H^n(\mathbf{A},t)$. Then $c=(ev,\epsilon)^*b$ is represented by

$$\bar{c}=(\bar{c}_1,\bar{c}_2)=(ev,\epsilon)^*\bar{b}=(ev^*\bar{b}_1,\mathrm{Hom}_{\mathbf{A}}(t\epsilon^{-1},1)_*\epsilon^{\#}(\bar{b}_1)+\mathrm{Hom}_{\mathbf{A}}(t\epsilon^{-1},\epsilon)_*ev^*\bar{b}_2),$$

and using the definition of (ev,ϵ) in Remark 8.9 we obtain

$$\begin{array}{rcl}(ev^*\bar{b}_1)((1_{(X,\alpha)},\sigma_1),\dots,(1_{(X,\alpha)},\sigma_n)) & = & \bar{b}_1(X(\sigma_1),\dots,X(\sigma_n)) \\ & = & (X^*\bar{b}_1)(\sigma_1,\dots,\sigma_n), \\ (\mathrm{Hom}_{\mathbf{A}}(t\epsilon^{-1}_{((X,\alpha),k-n+1)},1)_*\epsilon^{\#}(\bar{b}_1)) & & \\ ((1_{(X,\alpha)},\sigma_1),\dots,(1_{(X,\alpha)},\sigma_{n-1})) & = & \displaystyle\sum_{i=0}^{n-1}(t\alpha^{-1}_{k-n+1})^*(-1)^i\bar{b}_1(Xsh^3(\sigma_1),\dots, \\ & & Xsh^3(\sigma_i),\alpha_{k-i},tX(\sigma_{i+1}),\dots,tX(\sigma_{n-1})) \\ & = & \mathrm{Hom}_{\mathbf{A}}(t\alpha^{-1}_{k-n+1},1)_*\alpha^{\#}(\sigma_1,\dots,\sigma_{n-1}), \\ (\mathrm{Hom}_{\mathbf{A}}(t\epsilon^{-1}_{((X,\alpha),k-n+1)},\epsilon_{((X,\alpha),k)})_*ev^*\bar{b}_2) & & \\ ((1_{(X,\alpha)},\sigma_1),\dots,(1_{(X,\alpha)},\sigma_{n-1})) & = & (t\alpha^{-1}_{k-n+1})^*\alpha_{k*}\bar{b}_2(X(\sigma_1),\dots,X(\sigma_{n-1})) \\ & = & (\mathrm{Hom}_{\mathbf{A}}(t\alpha_{k-n+1},\alpha_k)_*X^*\bar{b}_2)(\sigma_1,\dots,\sigma_{n-1}),\end{array}$$

where $k-i$ is the source of σ_i and/or the target of σ_{i+1}. Hence we observe that

$$(\mathfrak{F}(\bar{c}_1)^{0,n}(X,\alpha), \mathfrak{F}(\bar{c}_2)^{0,n-1}(X,\alpha)) = (X,\alpha)^*\bar{b},$$

i. e. the cocycle representative of $\xi_0((ev,\epsilon)^*b)(X,\alpha)$ also represents $(X,\alpha)^*b$, hence the proof is finished. □

We now consider the bicomplex

$$\tilde{F}^*(\mathbf{Cand}(\mathbf{A},t), \tilde{F}^*(sh^3, [-,-]^*\bar{t})). \tag{8.13}$$

which is the $*$-normalized version of (8.4) for the special case considered in Remark 8.9. In bidegree (p,q) for $q>0$ this bicomplex is

$$\begin{array}{l}\tilde{F}^p(\mathbf{Cand}(\mathbf{A},t), \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{A}}(t\ ev(-,m), ev(-,m+q))) \\ \oplus\tilde{F}^p(\mathbf{Cand}(\mathbf{A},t), \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{A}}(t\ ev(-,m+3), ev(-,m+q+2))).\end{array}$$

We now define the $(0,3)$-cochain

$$\widetilde{\mathrm{Id}} = (\overline{\mathrm{Id}}, 0) \in \tilde{F}^0(\mathbf{Cand}(\mathbf{A},t), \tilde{F}^3(sh^3, [-,-]^*\bar{t}))$$

by

$$\overline{\mathrm{Id}}(X,\alpha)(m) = (-1)^m\alpha_m \in \mathrm{Hom}_{\mathbf{A}}(tX_m, X_{m+3})$$

for any candidate triangle (X,α) and $m\in\mathbb{Z}$.

Proposition 8.14. *The cochain* $\widetilde{\mathrm{Id}}$ *is a cocycle in the total complex of (8.13).*

Proof. The horizontal differential d_h of (8.13) is given by the cohomology of $\mathbf{Cand}(\mathbf{A},t)$ and the vertical differential d_v is determined by the translation cohomology of $(\triangle, sh^3)$. Therefore given a morphism of candidate triangles $f\colon (X,\alpha)\to(Y,\beta)$ we have

$$(d_h\widetilde{\mathrm{Id}})(f) = ((\delta\overline{\mathrm{Id}})(f), 0)$$

and for any $m\in\mathbb{Z}$

$$\begin{array}{rcl}((\delta\overline{\mathrm{Id}})(f))(m) & = & f_{m+3_*}(\overline{\mathrm{Id}}(X,\alpha))(m) - f_m^*(\overline{\mathrm{Id}}(Y,\beta))(m) \\ & = & f_{m+3}\alpha_m - \beta_m(tf_m) \\ & = & 0.\end{array}$$

This vanishes because all morphisms of candidate triangles satisfy $\beta(tf) = f\alpha$, see Section 4.

On the other hand for any candidate triangle (X,α) we have

$$(d_v(\widetilde{\mathrm{Id}}))(X,\alpha) = (\delta(\overline{\mathrm{Id}}(X,\alpha)), ((X,\alpha)^*\bar{t})_*(\overline{\mathrm{Id}}(X,\alpha)) - sh^{3^*}(\overline{\mathrm{Id}}(X,\alpha))).$$

The first coordinate is

$$\begin{array}{rcl}(\delta(\overline{\mathrm{Id}}(X,\alpha)))(m) & = & d_{m+4_*}\overline{\mathrm{Id}}(X,\alpha)(m) + d_{m+1}^*\overline{\mathrm{Id}}(X,\alpha)(m+1) \\ & = & (-1)^m d_{m+4}\alpha_m + (-1)^{m+1}\alpha_{m+1}(td_{m+1}) \\ & = & 0\end{array}$$

This vanishes since α is a degree 3 cochain homomorphism. For the second coordinate by (3.4) we have

$$\begin{aligned}
(((X,\alpha)^*\bar{t})_*(\overline{\mathrm{Id}}(X,\alpha)))(m) & \\
-(sh^{3^*}(\overline{\mathrm{Id}}(X,\alpha)))(m) &= (-1)^m\alpha_{m+3}\bar{t}_{(X_m,X_{m+3})}(\alpha_m)(t\alpha_m^{-1}) \\
&\quad -(-1)^{m+3}\alpha_{m+3} \\
&= -(-1)^m\alpha_{m+3}(t\alpha_m)(t\alpha_m^{-1})-(-1)^{m+3}\alpha_{m+3} \\
&= 0.
\end{aligned}$$

Now the proof is finished. ☐

The inclusion of bicomplexes

$$\tilde{F}^*(\mathbf{Cand}(\mathbf{A},t),\tilde{F}^*(sh^3,[-,-]^*\bar{t}))\subset\bar{F}^*(\mathbf{Cand}(\mathbf{A},t),\bar{F}^*(sh^3,[-,-]^*\bar{t}))$$

induces a quasi-isomorphism in total complexes since we know that the corresponding map of spectral sequences induces isomorphisms in the E_2-term. Hence, by using Lemmas 8.6 and 8.7 and the cosimplicial Eilenberg-Zilber-Cartier theorem in the Appendix, we see that the cochain $\widetilde{\mathrm{Id}}$ represents a cohomology class

$$\mathrm{Id}\in H^3(1_{\mathbf{Cand}(\mathbf{A},t)}\times sh^3,[-,-]^*\bar{t}). \tag{8.15}$$

We will refer to it as the *global class of identities*.

The following lemma proves that the global class of identities determines all classes of identities defined in (4.13) through the homomorphism ξ_0 in (8.11).

Lemma 8.16. *For any candidate triangle (X,α) we have $(\xi_0(\mathrm{Id}))(X,\alpha)=\mathrm{id}(X,\alpha)$.*

Proof. The cohomology class $(\xi_0(\mathrm{Id}))(X,\alpha)$ is represented by the cochain

$$(\overline{\mathrm{Id}}(X,\alpha),0)\in\tilde{F}^3(sh^3,(X,\alpha)^*\bar{t}).$$

This cochain is the same as the cochain in Lemma 4.15, hence the proof is finished. ☐

Let $i_\nabla\colon\mathbf{Cand}(\nabla)\hookrightarrow\mathbf{Cand}(\mathbf{A},t)$ be the inclusion of the full subcategory of ∇-triangles. By definition of ∇-triangle and Lemma 8.12 if $b=(b_1,b_2)\in\tilde{F}^3(t,\bar{t})$ is a representative of $\nabla\in H^3(\mathbf{A},t)$ then there exists a cochain

$$e=(e_1,e_2)\in\tilde{F}^0(\mathbf{Cand}(\nabla),\tilde{F}^2(sh^3,[i_\nabla,i_\nabla]^*\bar{t}))$$

such that for any ∇-triangle (X,α) the equation

$$\delta(e_1(X,\alpha),e_2(X,\alpha))=\tilde{\mathrm{Id}}(X,\alpha)+(X,\alpha)^*b\in\tilde{F}^3(sh^3,(X,\alpha)^*\bar{t}) \tag{8.17}$$

holds.

Lemma 8.18. *Let e be a cochain as above. The cohomology class $\xi_1(i_\nabla\times 1_\triangle,0^\square_{i_\nabla\times sh^3})^*(\mathrm{Id}+(ev,\epsilon)^*\nabla)$ is represented by the 1-cochain*

$$\theta\in\tilde{F}^1(\mathbf{Cand}(\nabla),H^2(sh^3,[i_\nabla,i_\nabla]^*\bar{t}))$$

such that for any morphism of candidate triangles $l\colon(X,\alpha)\to(Y,\beta)$ the cohomology class $\theta(l)\in H^2(sh^3,[(X,\alpha),(Y,\beta)]^*\bar{t})$ is represented by the 2-cocycle $\bar{\theta}(l)\in$

$\tilde{F}^2(sh^3, [(X,\alpha),(Y,\beta)]^*\bar{t})$ *given by*

$$\begin{array}{rcl}
(\bar{\theta}(l))(m) & = & (b_1(d_{m+2}, d_{m+1}, l_m) - b_1(d_{m+2}, l_{m+1}, d_{m+1}) \\
 & & +b_1(l_{m+2}, d_{m+2}, d_{m+1}) \\
 & & -l_{m+2}(e_1(X,\alpha)(m)) + (e_1(Y,\beta)(m))(tl_m), \\
 & & b_1(d_{m+4}, \beta_m, tl_m)(t\alpha_m^{-1}) - b_1(\beta_{m+1}, td_{m+1}, tl_m)(t\alpha_m^{-1}) \\
 & & +b_1(\beta_{m+1}, tl_{m+1}, td_{m+1})(t\alpha_m^{-1}) \\
 & & -b_1(d_{m+4}, l_{m+3}, \alpha_m)(t\alpha_m^{-1}) + b_1(l_{m+4}, d_{m+4}, \alpha_m)(t\alpha_m^{-1}) \\
 & & -b_1(l_{m+4}, \alpha_{m+1}, td_{m+1})(t\alpha_m^{-1}) \\
 & & -\beta_{m+1}b_2(d_{m+1}, l_m)(t\alpha_m^{-1}) + \beta_{m+1}b_2(l_{m+1}, d_{m+1})(t\alpha_m^{-1}) \\
 & & -l_{m+1}(e_2(X,\alpha)(m)) + (e_2(Y,\beta)(m))(tl_m)).
\end{array}$$

The proof of this lemma is technically just a continuation of the proof of Lemma 8.12. One only needs to use again two elementary but tedious formulas, namely the homotopy equivalence $\mathfrak{F}$ in Theorem A.1 and the construction of the spectral sequence of a bicomplex. We leave the details to the reader.

9 The axiom (Tr2)

Let $(\mathbf{A}, t)$ be a translation category with $\mathbf{A}$ additive and t an additive equivalence. There is an isomorphism

$$\ominus\colon \mathbf{Cand}(\mathbf{A}, t) \longrightarrow \mathbf{Cand}(\mathbf{A}, t)$$

defined on objects as

$$\ominus(X,\alpha) = (1_{\mathbf{A}}, -0_t^{\square})(X,\alpha)(sh, 0_{sh^4}^{\square}) = (Xsh, -\alpha\, sh)$$

and on morphisms f as $\ominus(f) = f\,sh$. This endofunctor is related to the axiom (Tr2) in the definition of pretriangulated categories, see [IV] 1. More precisely, if

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

is the small candidate triangle corresponding to (X,α) by the equivalence of categories in (4.1), then the small candidate triangle corresponding to $\ominus(X,\alpha)$ is isomorphic to

$$B \xrightarrow{i} C \xrightarrow{q} tA \xrightarrow{-tf} tB.$$

The isomorphism is given by the following commutative diagram in $\mathbf{A}$

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, equal] & C \arrow[r, "q"] \arrow[d, equal] & tA \arrow[r, "-tf"] \arrow[d, equal] & tB \arrow[d, equal] \\
X_{-1} \arrow[r, "d_0"] \arrow[d, equal] & X_0 \arrow[r, "\alpha_{-2}^{-1}d_1"] \arrow[d, equal] & tX_{-2} \arrow[r, "-td_{-1}"] \arrow[d, "\alpha_{-2}", "\cong"'] & tX_{-1} \arrow[d, equal] \\
X_{-1} \arrow[r, "d_0"] & X_0 \arrow[r, "d_1"] & X_1 \arrow[r, "-\alpha_{-1}^{-1}d_2"] & tX_{-1}
\end{tikzcd}$$

Proposition 9.1. *Let (X,α) be a candidate triangle, then (X,α) is a ∇-triangle if and only if $\ominus(X,\alpha)$ is a ∇-triangle.*

For the proof of this proposition we need the following two lemmas.

Lemma 9.2. *The homomorphism induced by* $(1_{\mathbf{A}}, -0_t^{\square})$ *on translation cohomology is the identity*

$$1 = (1_{\mathbf{A}}, -0_t^{\square})^* \colon H^*(\mathbf{A}, t) \longrightarrow H^*(\mathbf{A}, t).$$

Proof. We have to prove that the homomorphism given by the identity matrix is homotopic to

$$\begin{aligned}(1_{\mathbf{A}}, -0_t^{\square})^* &= \begin{pmatrix} 1 & 0 \\ (\mathrm{Hom}^t(-0_t^{\square}, 0_t^{\square}))_*(-0_t^{\square})^{\#} & (\mathrm{Hom}^t(-0_t^{\square}, -0_t^{\square}))_* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -(-0_t^{\square})^{\#} & 1 \end{pmatrix}.\end{aligned}$$

First of all we restrict ourselves to cochains which are normalized not only with respect to identities and zero morphisms but also with respect to products in $\mathbf{A}$ in the sense of [BT96]. By using [BT96] A.11 we see there is no lose of generality in this restriction.

We claim that a homotopy is given by

$$\begin{pmatrix} 0 & 0 \\ ((0_t^{\square}, 0_t^{\square}), \binom{0_t^{\square}}{-0_t^{\square}})^{\%} - ((0_t^{\square}, 0_t^{\square}), \binom{0_t^{\square}}{0})^{\%} - ((0_t^{\square}, 0_t^{\square}), \binom{0}{-0_t^{\square}})^{\%} & 0 \end{pmatrix}.$$

Here $(-,-)^{\%}$ are the cochain homotopies in [III] 5 and $0\colon t \Rightarrow t$ is the natural transformation given by zero morphisms.

We have the following equalities

$$\begin{aligned}
&\delta((0_t^{\square}, 0_t^{\square}), \tbinom{0_t^{\square}}{-0_t^{\square}})^{\%} - ((0_t^{\square}, 0_t^{\square}), \tbinom{0_t^{\square}}{-0_t^{\square}})^{\%}\delta \\
&-\delta((0_t^{\square}, 0_t^{\square}), \tbinom{0_t^{\square}}{0})^{\%} + ((0_t^{\square}, 0_t^{\square}), \tbinom{0_t^{\square}}{0})^{\%}\delta \\
&-\delta((0_t^{\square}, 0_t^{\square}), \tbinom{0}{-0_t^{\square}})^{\%} + ((0_t^{\square}, 0_t^{\square}), \tbinom{0}{-0_t^{\square}})^{\%}\delta \\
\overset{(a)}{=}\ & - \mathrm{Hom}^t(\tbinom{0_t^{\square}}{-0_t^{\square}}, 0_t^{\square})_*(0_t^{\square}, 0_t^{\square})^{\#} \\
&- \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0_t^{\square}}{-0_t^{\square}}^{\#} + 0^{\#} \\
&+ \mathrm{Hom}^t(\tbinom{0_t^{\square}}{0}, 0_t^{\square})_*(0_t^{\square}, 0_t^{\square})^{\#} \\
&+ \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0_t^{\square}}{0}^{\#} - (0_t^{\square})^{\#} \\
&+ \mathrm{Hom}^t(\tbinom{0}{-0_t^{\square}}, 0_t^{\square})_*(0_t^{\square}, 0_t^{\square})^{\#} \\
&+ \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0}{-0_t^{\square}}^{\#} - (0_t^{\square})\# \\
\overset{(b)}{=}\ & - \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0_t^{\square}}{-0_t^{\square}}^{\#} + 0^{\#} \\
&+ \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0_t^{\square}}{0}^{\#} - (0_t^{\square})^{\#} \\
&+ \mathrm{Hom}^t(0_t^{\square}, (0_t^{\square}, 0_t^{\square}))_* \tbinom{0}{-0_t^{\square}}^{\#} - (0_t^{\square})\# \\
\overset{(c)}{=}\ & -(-0_t^{\square})^{\#}.
\end{aligned}$$

Here we use [III] (5.11) for (a) and the biadditivity of Hom^t in (b). Normalization with respect to identities shows that $(0_t^{\square})^{\#} = 0$, see [III] (5.7), normalization with respect to zero morphisms implies that $0^{\#} = 0$, and these and normalization with respect to products show that

$$\begin{pmatrix} 0_t^{\square} \\ 0 \end{pmatrix}^{\#} = 0, \begin{pmatrix} 0 \\ -0_t^{\square} \end{pmatrix}^{\#} = \begin{pmatrix} 0 \\ (-0_t^{\square})^{\#} \end{pmatrix} \text{ and } \begin{pmatrix} 0_t^{\square} \\ -0_t^{\square} \end{pmatrix}^{\#} = \begin{pmatrix} 0 \\ (-0_t^{\square})^{\#} \end{pmatrix},$$

therefore (c) follows easily.

The previous chain of equalities proves the claim, and hence the lemma. □

Lemma 9.3. *For any candidate triangle* (X, α) *the equality*

$$(sh, 0_{sh^4}^{\square})^* \mathrm{id}(X, \alpha) = \mathrm{id}(\ominus(X, \alpha))$$

holds in $H^3(sh^3, (sh, 0_{sh^4}^{\square})^*(X, \alpha)^* \bar{t}) = H^3(sh^3, (\ominus(X, \alpha))^* \bar{t})$.

Proof. By Lemma 4.15 $\mathrm{id}(X, \alpha)$ and $\mathrm{id}(\ominus(X, \alpha))$ are represented by the translation cocycles $(b, 0)$ and $(c, 0)$, respectively, defined as

$$b(m) = (-1)^m \alpha_m \text{ and } c(m) = (-1)^m(-\alpha_{m+1}),$$

hence the equality already holds at the level of representative cocycles $(sh, 0_{sh^4}^{\square})^*(b, 0) = (c, 0)$. □

Proof of Proposition 9.1. By Lemmas 9.2 and 9.3 we have that

$$\mathrm{id}(\ominus(X, \alpha)) + (\ominus(X, \alpha))^* \nabla = (sh^3, 0_{sh^4}^{\square})^*(\mathrm{id}(X, \alpha) + (X, \alpha)^* \nabla)$$

and $(sh^3, 0_{sh^4}^{\square})^*$ is an isomorphism because $(sh^3, 0_{sh^4}^{\square})$ is already an isomorphism, therefore the proposition follows. □

10 Proofs of Propositions 5.7 and 7.5 and Theorems 6.11 and 6.14

In this section we concentrate all proofs where the use of the track-theoretical techniques developed in [III] and [IV] is strictly necessary. We will freely use the definitions and notation in [IV] 2 and 3.

In this section $(\mathbf{B}, s)$ will always be a good translation track category representing a translation cohomology class $\nabla \in H^3(\mathbf{A}, t)$.

In the proof of the following results we will use a fixed global section (ℓ, μ, ν, η) in the sense of [III] 14. We are actually interested in ℓ, μ and part of ν only. This is given by a choice of a morphism $\ell f \colon A \to B$ in $\mathbf{B}$ for any $f \colon A \to B$ in $\mathbf{A}$ with $p(\ell f) = f$ and $\ell(1_X) = 1_X$, tracks $\mu_{f,g} \colon (\ell f)(\ell g) \Rightarrow \ell(fg)$ with $\mu_{f,1} = 0_{\ell f}^{\square} = \mu_{1,f}$, and tracks $\nu_{f,s} \colon s(\ell f) \Rightarrow \ell(tf)$ with $\nu_{1_X,s} = s_X$. Since $\mathbf{B}$ has a strict zero object $*$ which is preserved by s and s is normalized at zero maps we can also impose here $\ell(0) = 0$, $\mu_{f,0} = 0_0^{\square} = \mu_{0,f}$ and $\nu_{0,s} = 0_0^{\square}$. We recall from [III] that ∇ is represented by the cocycle $(c_{\mathbf{B}}, b_s) \in \tilde{F}^3(t, \bar{t})$ with

$$\begin{aligned} c_{\mathbf{A}}(f, g, h) &= \sigma_{\ell(fgh)}^{-1}(\mu_{fg,h} \square (\mu_{f,g}(\ell h)) \square ((\ell f)\mu_{g,h}^{\boxminus}) \square \mu_{f,gh}^{\boxminus}), \\ b_s(f, g) &= \sigma_{\ell(t(fg))}^{-1}(\mu_{t(f),t(g)} \square (\nu_{f,s}\nu_{g,s}) \square s_{\ell f,\ell g}^{\boxminus} \square s(\mu_{f,g}^{\boxminus}) \square \nu_{fg,s}^{\boxminus}). \end{aligned}$$

The cocycle $(c_{\mathbf{B}}, b_s)$ is in principle only normalized with respect to identities, but the extra conditions required here to the global section clearly imply that it is also normalized with respect to zero morphisms. We will also need a collection γ of tracks $\gamma_f \colon \ell p(f) \Rightarrow f$ indexed by all maps f in $\mathbf{B}$ such that $\gamma_{\ell g} = 0_{\ell g}^{\square}$ for any morphism g in $\mathbf{A}$.

Proof of Proposition 5.7. Let

(a)
$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "i" description] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "q" description] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(f)"'] & sB \\
 & & \Downarrow H_1 & &
\end{tikzcd}$$

be a track triangle and let (X, α) be the candidate triangle associated to the small candidate triangle

(a)
$$A \xrightarrow{p(f)} B \xrightarrow{p(i)} C \xrightarrow{p(q)} tA$$

under the equivalence (4.2). The reader can check that the coboundary of the 2-cochain

$$\begin{aligned}
&(\sigma_0^{-1}(H_0\Box(\gamma_i\gamma_f)\Box\mu^{\boxminus}_{p(i),p(f)}), \sigma_0^{-1}(H_1\Box(\gamma_q\gamma_i)\Box\mu^{\boxminus}_{p(q),p(i)}),\\
&\qquad \sigma_0^{-1}(H_2\Box(s(\gamma_f)\gamma_q)\Box(\nu^{\boxminus}_{p(f),s}(\ell p(q)))\Box\mu^{\boxminus}_{tp(f),p(q)})) \quad \in \quad \tilde{G}^2((X,\alpha)^*\bar{t})
\end{aligned}$$

is

$$\begin{aligned}
&(1_{tA}, -1_{tB}, 1_{tC})\\
&+(c_{\mathbf{B}}(p(q),p(i),p(f)), c_{\mathbf{B}}(tp(f),p(q),p(i)), c_{\mathbf{B}}(tp(i),tp(f),p(q)) - b_s(p(i),p(f))(tp(q))),
\end{aligned}$$

and by Lemma 4.9 and Remark 4.14 this represents the cohomology class $\mathrm{id}(X,\alpha) + (X,\alpha)^*\nabla$, therefore (X,α) is a ∇-triangle.

On the other hand, if the "long" candidate triangle associated to the small one

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

is a small ∇-triangle we can choose a 2-cochain $(e_{-2}, e_{-1}, e_0) \in \tilde{G}^2((X,\alpha)^*\bar{t})$ such that

$$\delta(e_{-2}, e_{-1}, e_0) = (1_{tA}, -1_{tB}, 1_{tC}) + (c_{\mathbf{B}}(q,i,f), c_{\mathbf{B}}(tf,q,i), c_{\mathbf{B}}(ti,tf,q) - b_s(i,f)(tq)).$$

Now it is not difficult to check that

$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "\ell f"'] \arrow[rr, bend left=60, "0"] & B \arrow[r, "\ell i" description] \arrow[rr, bend right=60, "0"'] & C \arrow[r, "\ell q" description] \arrow[rr, bend left=60, "0"] & sA \arrow[r, "s(\ell f)"'] & sB \\
 & & \Downarrow H_1 & &
\end{tikzcd}$$

with

$$\begin{aligned}
H_0 &= \sigma_0(e_{-2})\Box\mu_{i,f},\\
H_1 &= \sigma_0(e_{-1})\Box\mu_{q,i},\\
H_2 &= \sigma_0(e_0)\Box\mu_{tf,q}\Box(\nu_{f,s}(\ell q)),
\end{aligned}$$

is a track triangle. $\square$

Proof of Proposition 7.5. As we remark in the paragraph preceding the statement of Lemma 8.18 there is a cochain

$$e = (e_1, e_2) \in \tilde{F}^0(\mathbf{Cand}(\nabla), \tilde{F}^2(sh^3, [-,-]^*\bar{t}))$$

such that for any ∇-triangle (X, α) the equation

$$\delta(e_1(X,\alpha), e_2(X,\alpha)) = \widetilde{\mathrm{Id}}(X,\alpha) + (X,\alpha)^*(c_{\mathbf{B}}, b_s)$$

is satisfied. Suppose that (X,α) is the image of the small ∇-triangle T given by

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} tA$$

by the equivalence of categories in (4.2). The previous equation implies that in the cochain complex $\tilde{G}^*((X,\alpha)^*\bar{t})$ the coboundary of

$$(e_1(X,\alpha)(-2), e_1(X,\alpha)(-1), e_1(X,\alpha)(0) - e_2(X,\alpha)(-2)(td_1))$$

is

$$(1_{tA}, -1_{tB}, 1_{tC}) + (c_{\mathbf{B}}(q,i,f), c_{\mathbf{B}}(tf,q,i), c_{\mathbf{B}}(ti,tf,q) - b_s(i,f)(tq)),$$

therefore the diagram

$$\begin{tikzcd}[column sep=large]
A \arrow[r, "\ell f"'] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, shift left=4ex, "\Uparrow H_0"] & B \arrow[r, "\ell i"] \arrow[rr, bend right=70, "0"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow H_1"] & C \arrow[r, "\ell q"] \arrow[rr, bend left=70, "0"] \arrow[rr, phantom, shift left=4ex, "\Uparrow H_2"] & sA \arrow[r, "s(\ell f)"'] & sB
\end{tikzcd}$$

with

$$\begin{aligned}
H_0 &= \sigma_0(e_1(X,\alpha)(-2))\square\mu_{i,f}, \\
H_1 &= \sigma_0(e_1(X,\alpha)(-1))\square\mu_{q,i}, \\
H_2 &= \sigma_0(e_1(X,\alpha)(0) - e_2(X,\alpha)(-2)(tq))\square\mu_{tf,q}\square(\nu_{f,s}(\ell q)),
\end{aligned}$$

is a track triangle that we call $\wp T$, see the proof of Proposition 5.7.

With this choice of a track triangle $\wp T$ for any small ∇-triangle T we can obtain a 1-cocycle θ representing the cohomology class θ_∇ in (6.13), see the proof of that corollary. More precisely, given a morphism $k\colon T \to \bar{T}$ between small ∇-triangles as in the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k_0"] & B \arrow[r, "i"] \arrow[d, "k_1"] & C \arrow[r, "q"] \arrow[d, "k_2"] & tA \arrow[d, "tk_0"] \\
\bar{A} \arrow[r, "\bar{f}"] & \bar{B} \arrow[r, "\bar{i}"] & \bar{C} \arrow[r, "\bar{q}"] & t\bar{A}
\end{tikzcd}$$

if we write (Y,β) for the ∇-triangle corresponding to $\bar{T}$ under the equivalence of categories in (4.2) then the cohomology class $\theta(k) = \vartheta_{\wp T, \wp \bar{T}}(k) \in H^2(sh^3, [(X,\alpha),(Y,\beta)]^*\bar{t})$

is represented by the cocycle $(b_{-2}, b_{-1}, b_0) \in \tilde{G}^2([(X,\alpha),(Y,\beta)]^*\bar{t})$ with

$$\begin{aligned}
b_{-2} &= \sigma_0^{-1}(((\sigma_0(e_1(Y,\beta)(-2))\square\mu_{\bar{i},\bar{f}})(\ell k_0))\square((\ell\bar{i})(\mu^{\boxminus}_{\bar{f},k_0}\square\mu_{k_1,f}))\\
&\quad \square((\mu^{\boxminus}_{\bar{i},k_1}\square\mu_{k_2,i})(\ell f))\square((\ell k_2)(\mu^{\boxminus}_{i,f}\square\sigma_0(-e_1(X,\alpha)(-2))))),\\
b_{-1} &= \sigma_0^{-1}(((\sigma_0(e_1(Y,\beta)(-1))\square\mu_{\bar{q},\bar{i}})(\ell k_1))\square((\ell\bar{q})(\mu^{\boxminus}_{\bar{i},k_1}\square\mu_{k_2,i}))\\
&\quad \square((\mu^{\boxminus}_{\bar{q},k_2}\square\mu_{tk_0,q})(\ell i))\square((\ell tk_0)(\mu^{\boxminus}_{q,i}\square\sigma_0(-e_1(X,\alpha)(-1))))),\\
b_0 &= \sigma_0^{-1}(((\sigma_0(e_1(Y,\beta)(0)-e_2(Y,\beta)(-2)(t\bar{q}))\square\mu_{t\bar{f},\bar{q}}\square(\nu_{\bar{f},s}(\ell\bar{q})))(\ell k_2))\\
&\quad \square(s(\ell\bar{f})(\mu^{\boxminus}_{\bar{q},k_2}\square\mu_{tk_0,q}\square(\nu_{k_0,s}(\ell q))))\square((s^{\boxminus}_{\ell\bar{f},\ell k_0}\square s(\mu^{\boxminus}_{\bar{f},k_0}\square\mu_{k_1,f})\square s_{\ell k_1,\ell f})(\ell q))\\
&\quad \square(s(\ell k_1)((\nu^{\boxminus}_{f,s}(\ell q))\square\mu^{\boxminus}_{tf,q}\square\sigma_0(-e_1(X,\alpha)(0)+e_2(X,\alpha)(-2)(tq))))).
\end{aligned}$$

The reader can check that

$$\begin{aligned}
b_{-2} &= c_{\mathbf{B}}(\bar{i},\bar{f},k_0) - c_{\mathbf{B}}(\bar{i},k_1,f) + c_{\mathbf{B}}(k_2,i,f)\\
&\quad -k_2(e_1(X,\alpha)(-2)) + (e_1(Y,\beta)(-2))(tk_0),\\
b_{-1} &= c_{\mathbf{B}}(\bar{q},\bar{i},k_1) - c_{\mathbf{B}}(\bar{q},k_2,i) + c_{\mathbf{B}}(tk_0,q,i)\\
&\quad -(tk_0)(e_1(X,\alpha)(-1)) + (e_1(Y,\beta)(-1))(tk_2),\\
b_0 &= c_{\mathbf{B}}(t\bar{f},\bar{q},k_2) - c_{\mathbf{B}}(t\bar{f},tk_0,q) + c_{\mathbf{B}}(tk_1,tf,q)\\
&\quad -(tk_1)(e_1(X,\alpha)(0)) + (e_1(Y,\beta)(0))(tk_2)\\
&\quad +b_s(\bar{f},k_0)(tq) - b_s(k_1,f)(tq)\\
&\quad +k_1(e_2(X,\alpha)(-2))(tq) - (e_2(Y,\beta)(-2))(t(\bar{q}k_2)),
\end{aligned}$$

therefore this proposition follows from Lemmas 8.18 and 4.9 and the fact that $(c_{\mathbf{B}}, b_s)$ is normalized. $\square$

Proof of Theorem 6.11. By [Bau97] 2.4 and 2.5 there is a map from the weak linear extension of homotopy pairs in [IV] 4.3 to the linear extension of algebraic homotopy pairs in (6.1)

(a)
$$\begin{tikzcd}
(\mathrm{Hom}^t)^{\#} \arrow[r, "+"] \arrow[d, equal] & \mathbf{Hopair}(\mathbf{B}) \arrow[r, "\tilde{p}"] \arrow[d, "m", "\sim"'] & \mathbf{Pair}(\mathbf{A}) \arrow[d, equal] \\
(\mathrm{Hom}^t)^{\#} \arrow[r, "+"] & \mathbf{AHp}(\aleph) \arrow[r, "\bar{p}"] & \mathbf{Pair}(\mathbf{A})
\end{tikzcd}$$

Here m is an equivalence of categories. We point out that the category of homotopy pairs considered in [Bau97] does not coincide strictly with our **Hopair**(**B**) defined in [IV] 4. However it is equivalent to **Hopair**(**B**) in an obvious way which is compatible with the (weak) linear extension structures above.

The equivalence of categories m in (a) induces isomorphisms between the track-theoretical cone functors in [IV] 4.4 and the algebraic ones in (6.4), i. e. for any map f in $\mathbf{B}$ we have

(c)
$$m\colon \mathit{Cone}_f \cong \mathit{Cone}^{\mathrm{Alg}}_{p(f)}.$$

In particular Cone_f is representable if and only if $\mathit{Cone}^{\mathrm{Alg}}_{p(f)}$ is. These isomorphisms show that the following diagram commutes up to equivalence

(d)
$$\begin{tikzcd}
\mathbf{Hopair}(\mathbf{B})^{op} \arrow[dd, "\sim"', "m^{op}"] \arrow[dr, "Cone"] & \\
& \mathbf{mod}(\mathbf{A}) \\
\mathbf{AHp}(\aleph)^{op} \arrow[ur, "Cone^{\mathrm{Alg}}"'] &
\end{tikzcd}$$

As we saw in [IV] 4.8 and Section 6 diagram (d) restricted to (algebraic) homotopy pairs with representable associated (algebraic) cone functor can be extended in a commutative way up to equivalence

$$\begin{tikzcd}
\mathbf{Hopair}^{rep}(\mathbf{B})^{op} \arrow[dd, "\sim"', "m^{op}"] \arrow[dr, "C^{op}"] \arrow[drr, bend left, "Cone"] & & \\
& \mathbf{A}^{op} \arrow[r] & \mathbf{mod}(\mathbf{A}) \\
\mathbf{AHp}^{rep}(\aleph)^{op} \arrow[ur, "\bar{C}^{op}"'] \arrow[urr, bend right, "Cone^{\mathrm{Alg}}"'] & &
\end{tikzcd}$$

The functors C and $\bar{C}$ were used to define functors ζ and $\bar{\zeta}$ that therefore fit in a diagram commuting up to equivalence

(e)
$$\begin{tikzcd}
\mathbf{Hopair}^{rep}(\mathbf{B}) \arrow[dd, "\sim"', "m"] \arrow[dr, "\zeta"] & \\
& \mathbf{cand}(\mathbf{A}, t) \\
\mathbf{AHp}^{rep}(\aleph) \arrow[ur, "\bar{\zeta}"'] &
\end{tikzcd}$$

Now by Proposition 5.7 the vanishing condition in the statement of Theorem 6.11 is equivalent to the fact that $\hat{\zeta}$ takes values in small ∇-triangles. This is obviously the same as saying that $\bar{\zeta}$ takes values in small ∇-triangles and by the commutativity properties of (e) this happens if and only if ζ takes values in small ∇-triangles, therefore the theorem follows from [IV] 4.26. □

Proof of Theorem 6.14. This proof is technically a continuation of the proof of Theorem 6.11 above since we need here all arguments already considered there. In this proof we will pay attention to the functors in the following extended diagram

(f)

$$\begin{tikzcd}
 & \mathbf{candt}(\mathbf{B},t) \arrow[dl, "\bar\varrho"'] \arrow[dd, "\varrho"] \\
\mathbf{Hopair}(\mathbf{B}) \arrow[dr, "\zeta"] \arrow[dd, "m", "\sim"'] & \text{(g)} \\
 & \mathbf{cand}(\nabla) \\
\mathbf{AHp}(\aleph) \arrow[ur, "\bar\zeta"'] \arrow[dr, two heads] & \\
 & \mathbf{AHp}(\aleph)/(\varpi,\bar\zeta) \arrow[uu, hook, "\hat\zeta"']
\end{tikzcd}$$

The lower triangle is strictly commutative, see Section 6, and (g) commutes up to equivalence by [IV] 4.16. Moreover, $\hat\zeta$ is faithful here by the second part of [IV] 4.25.

By [IV] 4.27 what we have to prove is the following: $\hat\zeta^*\theta_\nabla = 0$ if and only if there exists a full subcategory $\mathbf{dist}(\mathbf{B},s) \subset \mathbf{candt}(\mathbf{B},s)$ such that

$$\mathbf{dist}(\mathbf{B},s) \subset \mathbf{candt}(\mathbf{B},s) \xrightarrow{\bar\varrho} \mathbf{Hopair}(\mathbf{B}) \xrightarrow{\tilde p} \mathbf{Pair}(\mathbf{A}) \tag{h}$$

is surjective on objects and

$$\mathbf{dist}(\mathbf{B},s) \subset \mathbf{candt}(\mathbf{B},s) \xrightarrow{\bar\varrho} \mathbf{Hopair}(\mathbf{B}) \tag{i}$$

is full.

For this we give the following description of a normalized 1-cocycle θ representing θ_∇ in terms of the obstruction operator ϑ in the exact sequence for the functor ϱ in [IV] 3.1. For each small ∇-triangle T we choose a track triangle $\wp T$ such that $\varrho\wp T = T$. Then given a morphism of small ∇-triangles $k\colon T \to \bar T$ we have

$$\theta(f) = \vartheta_{\wp T,\wp\bar T}(k).$$

Actually all cocycles representing θ_∇ can be obtained in this way, compare [Bau89] IV.4.10 (d).

Suppose first that there exists a full subcategory $\mathbf{dist}(\mathbf{B},s)$ such that the functors (h) and (i) satisfy the required properties. For any morphism f in $\mathbf{A}$ we can choose the representation

$$\chi_f\colon \operatorname{Hom}_{\mathbf{A}}(C_f,-) \cong \mathit{Cone}_f^{\mathrm{Alg}}$$

used in the definition of $\bar\zeta$ to be induced by the isomorphism in (c) and a track triangle $\wp\bar\zeta(f)$ in $\mathbf{dist}(\mathbf{B},s)$ whose image by $\tilde p\bar\varrho$ is f, see [IV] 4.12. Now it is immediate to notice by using [IV] 3.2 and the commutativity properties of (f) that the pull-back of the 1-cocycle θ by the functor $\hat\zeta$ is already trivial $\hat\zeta^*\theta = 0$, in particular $\hat\zeta^*\theta_\nabla = 0$.

Suppose now that $\hat\zeta^*\theta_\nabla = 0$. We choose a 0-cocycle c such that $\delta(c) = \hat\zeta^*\theta$ and a natural isomorphism $\mathfrak{q}\colon \varrho \cong \zeta\bar\varrho$. We take $\mathbf{dist}(\mathbf{B},s)$ to be the full subcategory given by the track triangles

$$\tilde T_f = \wp\hat\zeta(f) + (-c(f))$$

in the sense of [Bau89] IV.4.10 (d), where f runs along all morphisms in $\mathbf{A}$. With this definition (h) is obviously surjective on objects, since $\tilde{p}\bar{\varrho}\tilde{T}_f = \tilde{p}\bar{\varrho}\wp\hat{\zeta}(f) = f$. Let us check that (i) is full. We have to show that any morphism of homotopy pairs $[k_0, k_1, K]\colon \bar{\varrho}\tilde{T}_f \to \bar{\varrho}\tilde{T}_g$ is the image by $\bar{\varrho}$ of a track triangle morphism $\tilde{k}\colon \tilde{T}_f \to \tilde{T}_g$. Consider the next chain of equalities

$$\begin{array}{rcl}
\vartheta_{\tilde{T}_f,\tilde{T}_g}(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f}) & = & \vartheta_{\wp\hat{\zeta}(f),\wp\hat{\zeta}(g)}(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f}) \\
 & & +(-(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f})_*c(f) \\
 & & +(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f})^*c(g)) \\
 & = & \theta(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f}) \\
 & & -(\delta(c))(\mathfrak{q}_{\tilde{T}_g}^{-1}\zeta[k_0,k_1,K]\mathfrak{q}_{\tilde{T}_f}) \\
 & = & 0.
\end{array}$$

Here we use [IV] 3.3 for the first equality. By [IV] 3.2, and since the triangle (g) in diagram (f) commutes up to the natural equivalence $\mathfrak{q}$, we see that $\zeta[k_0, k_1, K]\colon \zeta\bar{\varrho}\tilde{T}_f \to \zeta\bar{\varrho}\tilde{T}_g$ in $\mathbf{cand}(\nabla)$ is the image by $\zeta\bar{\varrho}$ of a morphism $\check{k}\colon \tilde{T}_f \to \tilde{T}_g$ in $\mathbf{candt}(\mathbf{B}, s)$, however $\bar{\varrho}\check{k}$ need not be the same as $[k_0, k_1, K]$. Nevertheless by the commutativity properties of (f), since $\hat{\zeta}$ is faithful and m is an equivalence fitting into the map of linear extensions (a) then there exists $b \in H^1(sh^3, [\hat{\zeta}(f), \hat{\zeta}(g)]^*\bar{t})$ such that

$$[k_0, k_1, K] = \bar{\varrho}(\check{k}) + \varpi_{(\hat{\zeta}(f),\hat{\zeta}(g))}(b),$$

therefore by [IV] 4.21 the morphism

$$\tilde{k} = \check{k} + b\colon \tilde{T}_f \longrightarrow \tilde{T}_g$$

satisfies $\bar{\varrho}(\tilde{k}) = [k_0, k_1, K]$ as desired. Now the proof is finished. □

A The cosimplicial Eilenberg-Zilber-Cartier theorem

Let Δ be the usual simplicial category. A *bicosimplicial abelian group* is a functor $A^{\bullet,\bullet}\colon \Delta\times\Delta \to \mathbf{Ab}$. This functor is completely determined by the abelian groups $A^{p,q}$ ($p, q \geq 0$), the horizontal cosimplicial coboundaries d_i^h and codegeneracies s_i^h defining the horizontal cosimplicial abelian groups $A^{\bullet,q}$ ($q \geq 0$), and the vertical coboundaries d_i^v and codegeneracies s_i^v which give rise to the vertical cosimplicial abelian groups $A^{p,\bullet}$ ($p \geq 0$). Horizontal and vertical operators commute.

The *diagonal* $\mathrm{Diag}(A^{\bullet,\bullet})$ of a bicosimplicial abelian group is the cosimplicial abelian group given by $\mathrm{Diag}^n(A^{\bullet,\bullet}) = A^{n,n}$ with coboundaries $d_i = d_i^h d_i^v = d_i^v d_i^h$ and codegeneracies $s_i = s_i^h s_i^v = s_i^v s_i^h$.

The *normalized cochain complex* $N(B^\bullet)$ of a cosimplicial abelian group $B^\bullet$ is given by

$$N^n(B^\bullet) = \bigcap_{i=0}^{n-1} \mathrm{Ker}[s_i\colon B^n \to B^{n-1}]$$

with differential $d = \sum_{i=0}^{n+1}(-1)^i d_i\colon N^n(B^\bullet) \to N^{n+1}(B^\bullet)$.

One can also define the *normalized bicomplex* $N(A^{\bullet,\bullet})$ of a bicosimplicial abelian group as

$$N^{p,q}(A^{\bullet,\bullet}) = \left(\bigcap_{i=0}^{p-1} \mathrm{Ker}[s_i^h \colon A^{p,q} \to A^{p-1,q}]\right) \cap \left(\bigcap_{i=0}^{q-1} \mathrm{Ker}[s_i^v \colon A^{p,q} \to A^{p,q-1}]\right).$$

The horizontal differential $d_h \colon N^{p,q}(A^{\bullet,\bullet}) \to N^{p+1,q}(A^{\bullet,\bullet})$ is given by $d_h = \sum_{i=0}^{p+1}(-1)^i d_i^h$ and the vertical differential $d_v \colon N^{p,q}(A^{\bullet,\bullet}) \to N^{p,q+1}(A^{\bullet,\bullet})$ is $d_v = \sum_{i=0}^{q+1}(-1)^i d_i^v$. Notice that horizontal and vertical differentials commute.

Recall that for any bicomplex $C^{*,*}$ with commuting horizontal d_h and vertical d_v differentials $d_h d_v = d_v d_h$ the *total complex* $\mathrm{Tot}C^{*,*}$ is the cochain complex defined by

$$\mathrm{Tot}^n C^{*,*} = \bigoplus_{p+q=n} C^{p,q}$$

and such that the differential of an element $a \in C^{p,q}$ is $d(a) = d_h(a) + (-1)^p d_v(a)$.

The following well known theorem is the dual of the Eilenberg-Zilber-Cartier theorem, see [DP61] 2.9. For the statement we recall that a (p,q)*-shuffle* σ is a permutation of the set $\{0, \dots, p+q-1\}$ such that

$$\sigma(0) < \cdots < \sigma(p-1) \text{ and } \sigma(p) < \cdots < \sigma(p+q-1).$$

The parity of the (p,q)-shuffle σ as a permutation is denoted by $\epsilon(\sigma) \in \{0,1\}$. It is known that $\epsilon(\sigma) = \sum_{i=0}^{p-1}(\sigma(i) - i) \mod 2$.

Theorem A.1. *Let $A^{\bullet,\bullet}$ be a bicosimplicial abelian group. There are mutually inverse natural cochain homotopy equivalences*

$$\mathrm{Tot}N(A^{\bullet,\bullet}) \underset{\mathfrak{G}}{\overset{\mathfrak{F}}{\leftrightarrows}} N\mathrm{Diag}(A^{\bullet,\bullet})$$

defined by

$$\mathfrak{F}(a) = \sum_{p+q=n} \sum_{\sigma \in \{(p,\,q)\text{-}shuffles\}} (-1)^{\epsilon(\sigma)} s^v_{\sigma(0)} \cdots s^v_{\sigma(p-1)} s^h_{\sigma(p)} \cdots s^h_{\sigma(p+q-1)}(a), \quad a \in A^{n,n};$$

$$\mathfrak{G}(b) = d^h_{p+q} \cdots d^h_{p+1} d^v_{p-1} \cdots d^v_0(b), \quad b \in A^{p,q}.$$

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