The Mirror of Calabi–Yau Orbifold

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MPI/91-1

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It has recently been recognized that the relation between exactly solvable conformal field theory and Calabi–Yau (CY) spaces necessarily involves hypersurfaces in weighted projective 4–space. A surprising symmetry which pairs different CY spaces with Euler numbers differed by $\chi \longmapsto -\chi$ was found by examining a large such class of manifolds [2], [8]. This duality shows that the topologically distinct CY pair yield the isomorphic conformal theories. Such symmetry indicates that this class of CY spaces is potentially of much higher phenomenological interest for the string theorists. As the simplest example, consider the Fermat quintic in $\mathbb{P}^4$,

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \ ,$$

which is invariant under the action of the subgroup G of $SL_5(\mathbb{C})$ consisting of all the order 5 diagonal matrices. The mirror partner of this Fermat quintic is the CY resolution of its quotient variety by G. Its Euler number equals to 200, $(= -\chi$ (quintic)), by the "orbifold Euler number" formula [1], [4], [11], [12]. In this paper, we shall study the above construction on a more general setting for the hypersurfaces in weighted projective space. Consider a quasismooth hypersurface in $W\mathbb{P}^4_{(n_1,\ldots,n_5)}$ defined by a degree d polynomial $f(z) = 0$ with $d = \sum_{i=1}^{5} n_i$, g.c.d. $(n_j \mid j \neq i) = 1$ for all i . Let G be a diagonal subgroup of $SL_5(\mathbb{C})$ leaving the polynomial f(z) invariant. Let $X(\hat{f},G)$ be the CY resolution of $X(f,G) = \{[z] \in W\mathbb{P}^4_{(n_i)} \mid f(z) = 0\}/G$. The question which arises here is to what extent the mirror of $X(\hat{f},G)$, (which means a CY space with Euler number = $-\chi(X(\hat{f},G))$), be obtained through the geometrical construction. In general, the hypersurface f(z) = 0 in $W\mathbb{P}^4_{(n_i)}$ has the singularities itself. Hence a more general formula than the "orbifold Euler number" is needed for the purpose of computation while some specific example is given. As in [9], one can resolve the singularities of X(f,G) while maintaining the condition $c_1=0$ to obtain the CY space $X(\hat{f},G)$. By the same argument in [13], the general formulae for the Euler number and the Betti numbers of $X(\hat{f},G)$ are obtained, which generalize Vafa's formula for the hypersurface cases [13], [17]. The formulae are expressed in terms of the weights n_i's and the group G , and described in THEOREM 1 of this paper. These terms involve the Euler numbers of the Milnor's fibers associated to the polynomial f(z) and its intersection with a certain part of the coordinates z_i's .

For the discussion of the mirror of $X(\hat{f},G)$, we consider only the cases when d is divisible by n_i for all i , and $d_i = d/n_i \geq 3$. In this situation, each diagonal subgroup G of $SL_5(\mathbb{C})$ containing

$$\begin{bmatrix} e^{2\pi\sqrt{-1}/d_1} & & \\ & \ddots & \\ & & e^{2\pi\sqrt{-1}/d_5} \end{bmatrix},$$

is naturally corresponding to another subgroup G' of this kind such that

$$\chi(X(\hat{f},G)) = -\chi(X(\hat{f},G')) ,$$

here f, g are the G, G′ resp., invariant quasihomogeneous polynomials of degree d with the only isolated critical point at origin. The group G′ is called the dual of G . This correspondence will be described in section 3. In order to show the above symmetry $\chi \longmapsto -\chi$, by the generalized Vafa's formula we have obtained, it needs only to consider the case when both f(z), g(z) are the Fermat polynomial for the weights n_i's

$$z_1^{d_1} + z_2^{d_2} + z_3^{d_3} + z_4^{d_4} + z_5^{d_5} .$$

For the Fermat polynomial f(z) , we have the explicit correspondence between the Hodge groups of $X(\hat{f},G)$ and $X(\hat{f},G')$:

$$H^{11}(X(\hat{f},G)) \xrightarrow{\sim} H^{21}(X(\hat{f},G')) ,$$

$$H^{21}(X(\hat{f},G)) \xrightarrow{\sim} H^{11}(X(\hat{f},G')) ,$$

which implies $\chi(X(\hat{f},G)) = -\chi(X(\hat{f},G'))$ by the Hodge structures of CY spaces. The derivation of the above isomorphisms will be given in section 4. These isomorphisms are also interesting in its own, as it is known that the Fermat polynomial is corresponding to the superpotential of the Landau–Ginzburg N=2 conformal theories composed of the A–type singularities [5], [7]. Here we identify the cohomology elements of the dual CY spaces corresponding to the same massless state of the conformal theory.

From the physics point of view, the CY spaces we focus here are the lowest order approximations to string vacua constructed from the N=2 minimal models [6]. The symmetry, $\chi \longmapsto -\chi$, is simply the effect on the minimal model vacuum of reversing the sign of the left moving U(1) charge of all states. On the underlying geometry, the effect is far more pronounced. At this time of writing, it is not clear to what extent a mirror exists for a general CY space. Recent developments have indicated that the examples constructed in this paper may also be realised by polynomials in weighted $\mathbb{P}^4$ [10]. For CY hypersurfaces which are not of Fermat type, the formulation of mirrors given here cannot be applied. It would be interesting to have a more general method to obtain further examples to show the symmetry $\chi \longmapsto -\chi$ of CY manifolds. The mirrors of CY spaces in the complete–intersection case and the higher dimensional generalization have now been considered; work along these lines is in progress.

The author is most pleased to acknowledge the many fruitful discussions with Prof. B.R. Greene for explaining his and his collaborators' work on the duality $\chi \longmapsto -\chi$. Here we have given a mathematical proof of their result [8], in which the conclusions were derived from the conformal field theory based on the physicist's reasoning. I also wish to thank Prof. F. Hirzebruch for the opportunity of visiting Max–Planck–Institut für Mathematik where this work was done.

1. Preliminary

Let n_i, $1 \leq i \leq 5$, be the positive integers with g.c.d. $(n_j \mid j \neq i) = 1$ for all i. Let $d = \sum_{i=1}^{5} n_i$, $q_i = n_i/d$, $d_i = d/\text{g.c.d.}(d, n_i)$ for each i. Let $\mathbb{R}^5$ $(\mathbb{R}^{5*})$ be the vector space consisting of all 5×1 columm (1×5 row resp.) vectors, and $\{e^i \mid 1 \leq i \leq 5\}$, $\{e_i \mid 1 \leq i \leq 5\}$ be the standard base of $\mathbb{R}^5$ and $\mathbb{R}^{5*}$ respectively. Denote

$T =$ the algebraic torus $(\mathbb{C}^*)^5$,

$q = \sum_{i=1}^{5} q_i e^i$,

$D_q = \{ \begin{bmatrix} t_1 \\ \vdots \\ t_5 \end{bmatrix} \in T \mid t_i^{d_i} = 1 \text{ for all } i \}$,

$SD_q =$ the subgroup of D_q consisting all the elements $\begin{bmatrix} t_1 \\ \vdots \\ t_5 \end{bmatrix}$ with $\prod_{i=1}^{5} t_i = 1$.

$$\exp_q : \mathbb{R}^5 \longrightarrow T, \quad \exp_q(x) = \begin{bmatrix} e^{2\pi\sqrt{-1}\, q_1 x_1} \\ \vdots \\ e^{2\pi\sqrt{-1}\, q_5 x_5} \end{bmatrix},$$

$$\mathrm{tr}_q : \mathbb{R}^5 \longrightarrow \mathbb{R}, \quad \mathrm{tr}_q(x) = \sum_i q_i x_i \quad \text{for } x = \sum_i x_i e^i,$$

$Q =$ the group generated by $\exp_q(\sum_{i=1}^{5} e^i)$,

$N_1 = \exp_q^{-1}(1)$, $1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

The subgroups of T are in one–to–one correspondence with the (additive) subgroups of $\mathbb{R}^5$ containing N_1 by assigning G to $N_G \doteqdot \exp_q^{-1}(G)$, and

$$G_1 \subseteq G_2 \text{ iff } N_{G_1} \subseteq N_{G_2} \ ,$$
$$N_1 = \{ \sum_i m_i q_i^{-1} e^i \mid m_i \in \mathbb{Z} \} \ , \qquad (1)$$
$$N_{D_q} = \{ \sum_i m_i (d_i q_i)^{-1} e^i \mid m_i \in \mathbb{Z} \} \ .$$

In fact, N_G is the group of 1–parameter subgroups of the algebraic group T/G ,

$$N_G = \mathrm{Hom}_{\mathrm{alg.group}}(\mathbb{C}^*, T/G) \ .$$

Now we consider the group of characters of T/G ,

$$M_G = \mathrm{Hom}_{\mathrm{alg.group}}(T/G, \mathbb{C}^*) \ .$$

Then the subgroups of D_q are in one–one correspondence with all the sublattices of M_1 containing M_{D_q} , and

$$G_1 \subseteq G_2 \text{ iff } M_{G_1} \supseteq M_{G_2} \ ,$$
M_1 = the standard lattice in $\mathbb{R}^{5*}$ consisting of all elements $\sum_i m_i e_i$, $m_i \in \mathbb{Z}$ (2)

$$M_{D_q} = \{ \sum_i m_i d_i e_i \mid m_i \in \mathbb{Z} \} \ .$$

For our purpose, we shall consider only the subgroups lying between Q and SD_q . Throughout the discussion of this paper, G shall always denote a group with

$$Q \subsetneq G \subsetneq SD_q .$$

Let

$\mathbb{C}[z,z^{-1}]$ = the algebra consisting of all the Laurent polynomials $f(z,z^{-1})$
$(= f(z_1,z_1^{-1},\dots,z_5,z_5^{-1}))$.
$\mathbb{C}[z]$ = the algebra of all the polynomials $f(z)$ $(= f(z_1,\dots,z_5))$,
$z^k = z_1^{k_1}\dots z_5^{k_5}$ for $k = \sum_i k_i e_i \in M_1$,
$(M_G)_+ = \{ \sum_i k_i e_i \in M_G \,|\, k_i \geq 0\}$,
P_G = the $\mathbb{C}$–subspace of $\mathbb{C}[z]$ generated by z^k , $k \in (M_G)_+$ with $tr_q(k) = 1$.

Consider the natural action of G on T . It induces a G–action on the algebraic functions of T , $\mathbb{C}[z,z^{-1}]$. Then the following lemma is obvious.

<u>LEMMA 1</u>. For $k \in M_1$,

$k \in M_G \longleftrightarrow z^k \in \mathbb{C}[z,z^{-1}]^G$,
$k \in (M_G)_+ \longleftrightarrow z^k \in \mathbb{C}[z]^G$,
$f(z) \in P_G \longleftrightarrow f(z)$ is a G–invariant quasihomogeneous polynomial of degree d with weight n_i's .

2. Calabi–Yau orbifolds

Let d, n_i, G be the same as in the previous section. The linear action of G on $\mathbb{C}^5$ induces a G–action on the weighted projective space $W\,\mathbb{P}^4_{(n_i)}$. Let f(z) be a G–invariant quasihomogeneous polynomial of degree d with weights n_i's. Assume f(z) has the only isolated critical point at the origin. The polynomial f defines a quasismooth hypersurface in $W\,\mathbb{P}^4_{(n_i)}$:

$$\{\,[z] \in W\,\mathbb{P}^5_{(n_i)} \mid f(z) = 0\} \ ,$$

which is stable under the action of G. The G–quotient of this hypersurface is denoted by

$$X = X(f,G) \ .$$

Denote

$$S = \mathbb{C}^5 \backslash 0 \ ,$$
$$M\ (= M(f)) = \{z \in S \,|\, f(z) = 0\} \ ,$$
$$\{z\} = \text{the G–orbit of } z \text{ in } S/G \text{ for } z \in S \ .$$

Since the abelian group G contains Q, the $\mathbb{C}^*$–action of S,

$$\mu\cdot z \doteq \begin{bmatrix} \mu^{n_1} z_1 \\ \vdots \\ \mu^{n_5} z_5 \end{bmatrix} \quad \text{for } \mu \in \mathbb{C}^*, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_5 \end{bmatrix} \in S ,$$

induces a well–defined $\mathbb{C}^*$–action of S/G

$$\begin{aligned} &\mathbb{C}^* \times S/G \longrightarrow S/G \\ &(\lambda,\{z\}) \longmapsto \lambda*\{z\} \doteq \{\mu\cdot z\} \ \text{ with } \ \lambda = \mu^d . \end{aligned}$$

The polynomial $f(z)$ also induces the function f_G of S/G satisfying

$$\begin{array}{ccc} S & \longrightarrow & S/G \\ {\scriptstyle f}\downarrow & \circlearrowleft & \swarrow {\scriptstyle f_G} \\ \mathbb{C} & & \end{array}$$

with $f_G(\lambda*\{z\}) = \lambda f_G(\{z\})$ for $\{z\} \in S/G$. Then M/G is the set of all elements $\{z\}$ in S/G satisfying $f_G(\{z\}) = 0$. Denote $G\mathbb{C}^*$ = the abelian group of linear transformations generated by G and

$$\begin{bmatrix} \mu^{n_1} & & 0 \\ & \ddots & \\ 0 & & \mu^{n_5} \end{bmatrix}, \ \mu \in \mathbb{C}^* . \tag{3}$$

$G\mathbb{C}^*$ acts on S and M.

<u>DEFINITION</u>. For a subset I of $\{1,\ldots,5\}$,

$C_I = \{h \in G\mathbb{C}^* \,|\, h(z) = z$ for all z with $z_i = 0$, $i \in I\}$.
$c_I = |C_I|$.

For an element z of S, the isotropic subgroup $(G\mathbb{C}^*)_z$ at z finite, and equals to $C_{I(z)}$ with $I(z) = \{i \,|\, z_i = 0\}$. The index $I(z)$ depends only on the $G\mathbb{C}^*$–orbit x of z, hence we can also write $I(x) = I(z)$. Now the space X can be identified with $\mathbb{C}^*$–quotient of M/G,

$$X = (M/G)/\mathbb{C}^* = M/G\mathbb{C}^* \ .$$

LEMMA 2. X Is a V–manifold having at most abelian quotient singularity, and with the trivial canonical sheaf. Furthermore, for $x \in X$,

$$(X,x) \simeq (\mathbb{C}^3/C_{I(z)},0) \text{ for some imbedding } C_{I(x)} \subsetneq SL_3(\mathbb{C}) \ .$$

PROOF: Let x be an element of X, z be an element of M whose $G\mathbb{C}^*$–orbit equals to x. There is a small $C_{I(x)}$–invariant 4–disc in S which intersects both M and the $\mathbb{C}^*$–orbit of z normally at the point z. Let Δ be the intersection of this 4–disc and M. Then Δ is $C_{I(x)}$–invariant, and $(M,z) \simeq (\Delta,z) \times (\mathbb{C}^*,1)$. It is easy to see that $(X,x) \simeq (\Delta,z)/C_{I(x)}$. By $d = \sum_i n_i$, there is a never–vanishing holomorphic 4–form on M which is invariant under the action of $G\mathbb{C}^*$. Its interior product with the vector field generated by $\mathbb{C}^*$ gives a never–vanishing section of the canonical sheaf of X. Hence the result follows immediately. q.e.d.

By [12], the "minimal" toroidal resolution of X,

$$\sigma : \hat{X} \longrightarrow X$$

is a Calabi–Yau space. We are going to express the topological invariances of $\hat{X}$ in terms of q_i's and G.

In general, for a subset I of $\{1,\dots,5\}$, $C_I \cap G \not\subseteq C_I$.

<u>LEMMA 3</u>. When I satisfies the condition $S \cap \{z_i = 0 \mid i \in I\} \not\subseteq M$ (i.e. $f(z \mid z_i = 0$ for $i \in I)$ is a non–zero function), one has

$$C_I \subseteq G \ ,$$

and

$$g \in C_I \cap G \text{ iff } \{i \mid g(e^i) \neq e^i\} = I \ ,$$

here e^i's are the standard base of $\mathbb{C}^5$.

PROOF: Let s be an element in $S \setminus M$ with $\{i \mid s_i = 0\} = I$. For an element gg_μ in $G\mathbb{C}^*$ with $g \in G$, g_μ : the element of (3) for $\mu \in \mathbb{C}^*$. Then

$$gg_\mu \in C_I \longleftrightarrow gg_\mu(s) = s \longleftrightarrow f(s) = f(gg_\mu(s)) = f(g_\mu(s)) = \mu^d f(s) \ .$$

Hence $\mu^d = 1$, g_μ and $gg_\mu \in G$. Therefore $C_I \subseteq G$. The other conclusion follows the definition of C_I. q.e.d.

<u>LEMMA 4</u>. $\sum_{|I|=4} c_I = |G|$.

PROOF. Let $\mathbb{C}^*$ be the subgroup of $G\mathbb{C}^*$ consisting all the elements in (3). Then

$$G\mathbb{C}^*/\mathbb{C}^* \simeq G/Q$$

whose order is equal to $|G|/d$. For $1 \leq i \leq 5$, the isotropy group $(G\mathbb{C}^*)_{e^i}$ at e^i is C_I for $I = \{j \mid 1 \leq j \neq i \leq 5\}$. It is easy to see that the homomorphism

$$(G\mathbb{C}^*)_{e^i} \longrightarrow G\mathbb{C}^*/\mathbb{C}^*$$

is surjective with the kernel $(\mathbb{C}^*)_{e^i}$. Hence

$$|(G\mathbb{C}^*)_{e^i}| = |(\mathbb{C}^*)_{e^i}| \cdot |G|/d = n_i \cdot |G|/d \ .$$

By $d = \sum_i n_i$, the result follows immediately. q.e.d.

Now we are going to derive the generalized Vafa's formula of the Euler number of $\hat{X}$ just as the case of hypersurface in $W\mathbb{P}^4_{(n_i)}$ [13] [17].

<u>THEOREM 1</u>. Let q_i, G, $\hat{X}$ be the same as before. Then

(i) $$\chi(\hat{X}) = \frac{1}{|G|} \sum_{g,h\in G} \prod \{(1-\frac{1}{q_i}) \mid g(e^i) = h(e^i) = e^i\}$$

here $\{e^1,\ldots,e^5\}$ is the standard base of $\mathbb{C}^5$ acted linearly by G, and $\prod\{*\} \doteq 1$ if $\{*\} = \phi$.

(ii) For $g \in G$, define

$$\beta_g = \frac{1}{|G|} \sum_{h \in G} \prod \{(1-\frac{1}{q_i}) \,|\, g(e^i) = h(e^i) = e^i\} \;,$$

$$s_g = |\{i \,|\, 1 \leq i \leq 5,\, g(e^i) = e^i\}| \;.$$

Then

$$b^3(\hat{X}) = -\sum \{\beta_g \,|\, s_g \geq 3\}$$

$$b^2(X) = -1 + \tfrac{1}{2} \sum \{\beta_g \,|\, s_g < 3\} \;.$$

PROOF. Denote

$$Y = \{ [z_1,...,z_6] \in W\,\mathbb{P}^5_{(n_1,...,n_5,1)} \,|\, z_6^d = f(z_1,...,z_5)\} \;.$$

$$F = \{(z_1,...,z_5) \in \mathbb{C}^5 \,|\, 1 = f(z_1,...,z_5)\} \;.$$

Consider the linear transformation group of $\mathbb{C}^5$ as a subgroup of projective transformation group $W\,\mathbb{P}^5_{(n_1,...,n_5,1)}$. G acts on $W\,\mathbb{P}^5_{(n_1,...,n_5,1)}$, and leaves Y invariant. So we may consider G as an automorphism group of Y. The G–equivariant projection $[z_1,...,z_6] \longmapsto [z_1,...,z_5]$ induces

$$\pi : Y \longrightarrow W\,\mathbb{P}^4_{(n_i)}/G \doteqdot P \;.$$

For $I \subsetneq \{1,\ldots,5\}$, let

$$\begin{aligned} P_I &= S \cap \{z_i = 0 \text{ for all } i \in I\}/G\mathbb{C}^* . \\ X_I &= M \cap \{z_i = 0 \text{ for all } i \in I\}/G\mathbb{C}^* . \\ U_I &= P_I - X_I . \\ Y_I &= Y \cap \{z_i = 0 \text{ for all } i \in I\} . \\ \pi_I &: Y_I \longrightarrow P_I \text{ the restriction of } \pi . \end{aligned} \tag{4}$$

With the same argument in Theorem 1 of [13], by the description of C_I and LEMMA 4, one can obtain the formula in (i) after replacing the data in [13] by the corresponding ones we have just define in (4). Also the same procedure as Theorem 3 of [13], we obtain (ii). q.e.d.

The singularities of X

$$\mathrm{Sing}(X) = \cup \{X_I \mid c_I > 1\} ,$$

here X_I is defined in (4), or $= X \cap \{z_i = 0 , \text{ for all } i \in I\}$. An irreducible component of $\sigma^{-1}(\mathrm{Sing}(X))$ is called an exceptional divisor in $\hat{X}$. For $X_I \subsetneq \mathrm{Sing}(X)$, it is necessary that $|I| = 2,3$, and X_I is an irreducible non–singular curve when $|I| = 2$. The relation between the cohomologies (with $\mathbb{C}$–coefficient) of $\hat{X}$, the exceptional divisors, and X are described as follows.

<u>THEOREM 2</u>. For $r = 2,3$,

$$H^r(\hat{X}) \simeq H^r(X) \oplus \bigoplus_D H^{r-2}(D) \simeq H^r(X) \oplus \bigoplus_D H^{r-2}(\sigma(D)) ,$$

here the index D runs all the exceptional divisors in $\hat{X}$.

PROOF: By the construction of the toroidal resolution for $\hat{X}$, for an exceptional divisor D ,

$$\sigma : D \longrightarrow \sigma(D)$$

is a $\mathbb{P}^1$–bundle when $\sigma(D)$ = a curve, and D is a rational surface when $\sigma(D)$ is a point. Hence $H^i(D) \simeq H^i(\sigma(D))$ for $i = 0,1$. So we need only to show either one of the above isomorphisms for each r . For r = 3 , the conclusion follows from Theorem 2 of [13]. For r = 2 , by the CY property of $\hat{X}$, the second cohomology of $\hat{X}$ with $\mathbb{Z}$–coefficient is isomorphic to the group of all line bundles of $\hat{X}$, $H^1(\hat{X},\mathcal{O}^*)$. Consider the natural induced homomorphism,

$$\sigma^* : H^1(X,\mathcal{O}^*) \longrightarrow H^1(\hat{X},\mathcal{O}^*) \ ,$$

which is injective by the connectedness of the fibers of $\sigma : \hat{X} \longrightarrow X$. Applying the operation σ^* on the line bundles of X , by the normal property for X , one can show that every line bundle is uniquely expressed by $\sigma^*(\mathbb{L}) \otimes \mathcal{O}(\sum_D m_D D)$ for $\mathbb{L} \in H^1(X,\mathcal{O}^*)$, $m_D \in \mathbb{Z}$, D : exceptional divisor. Hence we obtain the above first isomorphism for r = 2 . q.e.d.

3. Dual Group

We now identify the vector spaces $\mathbb{R}^5$ with $\mathbb{R}^{5*}$ in section 1 via $e^i \longleftrightarrow e_i$. Then

$$N_1 = M_{D_q} \longleftrightarrow N_{D_q} = M_1 \longleftrightarrow$$
$$d \text{ is divisible by } n_i \text{ for all } i . \tag{5}$$

Under this condition, by (1) and (2), every subgroup G of D_q corresponds to a subgroup G' of D with

$$N_G = M_{G'} ,$$

equivalently

$$M_G = N_{G'} .$$

G' is called the dual group of G.

For the rest of this paper, we shall always assume the n_i's satisfy the condition (5), and $d/n_i \geq 3$ for all i.

The following properties for dual groups are obvious:

$$Q' = SD_q ,$$
$$(G')' = G ,$$
$$G_1 \subsetneq G_2 \text{ iff } (G_1)' \supsetneq (G_2)' .$$

Since

$$D_q/G = N_{D_q}/N_G = M_1/M_{G'} = \text{the character group of } G' ,$$

we have

$$|G| \cdot |G'| = d_1 \dots d_5 .$$

We can also obtain $N_{G'}$ from N_G by the following procedure. Denote

$$< , >_q : \mathbb{R}^5 \times \mathbb{R}^5 \longrightarrow \mathbb{R} , \quad < \sum_i x_i e^i, \sum_i y_i e^i > = \sum_i q_i x_i y_i .$$

Then $N_{G'}$ is the dual lattice of N_G with respect to $< , >_q$, i.e.

$$N_{G'} = \{x \in \mathbb{R}^5 \mid < x, N_G >_q \subseteq \mathbb{Z}\} .$$

The above properties of G and G' can also be obtained through this lattice's consideration.

Consider the quasihomogeneous polynomial functions f(z), h(z) of degree d w.r.t. n_i's , which are invariant under the action of G, G' respectively. Assume f(z), h(z) have the only isolated critical point at origin. Denote

$$\hat{X} = \text{the CY resolution of } X(= X(f,G)) ,$$
$$\hat{X}' = \text{the CY resolution of } X'(= X(h,G')) .$$

<u>THEOREM 3</u>. We have the following duality between the Hodge numbers of $\hat{X}$ and $\hat{X}'$:

$$h^{11}(\hat{X}) = h^{21}(\hat{X}'),\ h^{21}(\hat{X}) = h^{11}(\hat{X}') .$$

As a consequence, $\chi(\hat{X}) = -\chi(\hat{X}')$.

By THEOREM 1, the Hodge numbers of $\hat{X}, \hat{X}'$ depend only on the group G, G' . We may assume both polynomials $f(z)$ and $g(z)$ equal to the Fermat polynomial. Then THEOREM 3 is a consequence of the following one.

THEOREM 4. When $f(z) = g(z) =$ the Fermat polynomial for n_i's ,

$$f(z) = z_1^{d_1} + z_2^{d_2} + z_3^{d_3} + z_4^{d_4} + z_5^{d_5},\quad d_i = d/n_i \geq 3 ,$$

there exist the $\mathbb{C}$–isomorphisms between the Hodge groups of $\hat{X}$ and $\hat{X}'$:

$$H^{11}(\hat{X}) \simeq H^{21}(\hat{X}') ,$$
$$H^{21}(\hat{X}) \simeq H^{11}(\hat{X}') .$$

The next section is mainly devoted to the proof of the above theorem.

4. Fermat hypersurface

In this section, $f(z)$ always denotes the Fermat polynomial for n_i's ,

$$f(z) = z_1^{d_1} + z_2^{d_2} + z_3^{d_3} + z_4^{d_4} + z_5^{d_5},\quad d_i = d/n_i \geq 3 .$$

Let $\mathbb{R}^5$, $\exp_q$, tr_q , G , N_G , M_G be the same as in section 1,

$\sigma : \hat{X} \longrightarrow X\ (= X(f,G))$ the CY resolution of X ,

$\mathcal{J}$ = the Jacobian ring $\mathbb{C}[z_1,...,z_5]/(\frac{\partial f}{\partial z_i})$ of f(z) ,

$\mathcal{J}_I$ = the Jacobian ring of $f(z \mid z_i = 0$ for all $i \in I\}$, $I \subsetneq \{1,...,5\}$.

The action of G on $\mathbb{C}^5$ induces a G–action on $\mathbb{C}[z]$, hence on $\mathcal{J}$ and $\mathcal{J}_I$ for all I . For a non–negative integer m , let

$(\mathcal{J}_I)^{G,m}$ = the subspace of $(\mathcal{J}_I)^G$ generated by all the monomial z^k with

$$k = \sum_i k_i e^i \ , \ k_i = 0 \ \text{for } i \in I \text{ , and } \sum_{j \notin I} q_j(k_j+1) = m+1 \ .$$

$$\mathcal{J}^{G,m} = (\mathcal{J}_\phi)^{G,m} \ .$$

By $\sum_i q_i = 1$,

$\mathcal{J}^{G,m}$ = the subspace of $\mathcal{J}^G$ generated by those monomial polynomial z^k with

$$tr_q(k) = m \ .$$

By [3], [16], the Hodge structure of X and X_I's (defined in (4)) can be described by G–spaces $\mathcal{J}$, $\mathcal{J}_I$. We have

$$H^{21}(X) \simeq \mathcal{J}^{G,1} \ ,$$
$$H^{12}(X) \simeq \mathcal{J}^{G,2} \ ,$$
$$H^{10}(X_I) \simeq (\mathcal{J}_I)^{G,0} \quad \text{for} \quad |I| = 2 \ , \tag{6}$$
$$H^0(X_K) \simeq \mathbb{C} \oplus (\mathcal{J}_K)^{G,0} \quad \text{for} \quad |K| = 3 \ .$$

We are going to associate the above spaces to the certain data in M_G. For this purpose, we introduce the following notations.

DEFINITION. Let L be a lattice in $\mathbb{R}^5$ with $N_Q \subseteq L \subseteq N_{Q'}$, I be a subset of $\{1,\ldots,5\}$, k_i be the i–th coordinate of k for $k \in L$. Define

the I–th sublattice of $L = \{k \in L \,|\, k_j = 0 \ \text{for all} \ j \notin I\}$.

$F(L) = \{k \in L \,|\, tr_q(k) = 1, 0 \leq k_i \leq d_i-2 \ \text{for all} \ i \}$.

$E(L;I) = \{k \in \text{I–th sublattice of } L \,|\, tr_q(k) = 1, 0 \leq k_i \leq d_i-1, i \in I\}$, for $|I| = 2,3$.

$$E(L;K)^* = E(L;K) - \bigcup_{\substack{J \subseteq K \\ |J|=2}} E(L;J) \quad \text{for} \quad |K| = 3 \ .$$

The following technique lemmas are essential for the latter discussion.

LEMMA 5. There exists an one–one correspondence between the following subsets of L,

$$\varphi : \bigcup \{E(L;I) \,|\, |I| = 2,3\} \longrightarrow F(L) \setminus \{ \sum_i e^i \} \ .$$

PROOF. By $\sum_i q_i = 1$, there are at most two d_i with value 3. When there are two $d_i = 3$, by remunerating the indices, we shall always assume $d_1 = d_2 = 3$ in this proof. For an element k, let k_i be the i–th coordinate of k. Denote $e = \sum_i e^i$. It is easy to see that, for $k \in F(L)$,

$$k_i > 0 \text{ for all } i \longleftrightarrow k = e \ .$$

Define φ to be the identity map on $F(L) \cap (\cup \{E(L;I) \mid |I| = 2,3\})$. We need only to define φ on $E(L;I)\backslash F(L)$ for $|I| = 2,3$.

(1) For $|I| = 3$, say $I = \{1,2,3\}$.

For $k \in E(L;I)\backslash F(L)$, by the definition of $E(L;I)$ and $F(L)$,

$$k = k_1e^1 + k_2e^2 + k_3e^3 \text{ for some } k_i = d_i-1 \ , \ k_j > 0 \text{ for } 1 \leq j \leq 3 \ .$$

CLAIM: There is only one i with $k_i = d_i-1$, furthermore for $j \neq i$, $k_j < d_j-2$. Otherwise, say $k_1 = d_1-1$ and $k_2 \geq d_2-2$. We have

$$1 = tr_q(k) = k_1/d_1 + k_2/d_2 + k_3/d_3 \geq 1-1/d_1 + 1-2/d_2 + k_3/d_3 \ ,$$

$$1/d_2 > 1/d_3 + 1/d_4 + 1/d_5 \ ,$$

$$1 = 1/3 + 2/3 \geq 1/d_1 + 2/d_2 > \sum_{i=1}^{5} 1/d_i = 1 \ ,$$

which is a contradiction. we define

$$\varphi(k) = k + e - d_ie^i \text{ for } k \in E(L;I)\backslash F(L) \text{ with } i \text{ as above.}$$

Then $\varphi(k) \in F(L)\backslash\{e\}$, and

$$\varphi(E(L;I)\backslash F(L)) = \left\{ k\in F(L) \,\middle|\, \begin{array}{l} k\neq e,\ k_4=k_5=1,\ k_i=0 \text{ for some unique } i, \\ k_j\geq 2 \text{ for the other } j\text{'s} \end{array} \right\} .$$

(2) For $|I| = 2$, say $I = \{1,2\}$.

For $k \in E(L,I)\backslash F(L)$,

$$k = k_1e^1 + k_2e^2 ,\ k_i = d_i-1 \text{ for some } i .$$

CLAIM: If $k_i = d_i-1$ and the other $k_j \geq d_j-2$, then $d_1 = d_2 = 3$, and $(k_1,k_2) = (2,1)$ or $(1,2)$. Say, $k_1 = d_1-1$, $k_2 \geq d_2-2$. Then

$$1 = tr_q(k) = k_1/d_1 + k_2/d_2 \geq 1-1/d_1 + 1 - 2/d_2 ,$$

$$1/d_1 + 2/d_2 \geq 1 ,$$

hence $d_1 = d_2 = 3$, $(k_1,k_2) = (2,1)$.

Now we define

$$\varphi(k) = \begin{cases} e^1 + 2e^3 + 2e^4 + 2e^5 & \text{if } d_1 = d_2 = 3,\ k_1 = 2,\ k_2 = 1 , \\ e^2 + 2e^3 + 2e^4 + 2e^5 & \text{if } d_1 = d_2 = 3,\ k_1 = 1, k_2 = 2 , \\ k + e - d_ie^i & \text{otherwise, here } i \text{ with } k_i = d_i\text{-}1 . \end{cases}$$

Then $\varphi(k) \in F(L)\backslash\{e\}$, and

$$\varphi(E(L;I)\backslash F(L)) = \begin{cases} \{e^1+2e^3+2e^4+2e^5,\ e^2+2e^3+2e^4+2e^5\} & \text{if } d_1 = d_2 = 3 , \\ \{k\in F(L) \ \big|\ k\neq e,\ k_3=k_4=k_5=1,\ k_i=0 \text{ for some unique } i , \\ \qquad k_j\geq 2 \text{ for the other } j \} , & \text{otherwise.} \end{cases}$$

Let k be an element of $F(L)\backslash(\{e\} \cup \{E(L;I) \mid |I| = 2,3\})$. It is easy to see that $|\{i \mid k_i = 0\}| = 1$, and $|\{i \mid k_i = 1\}| \geq 2$ except $d_1 = d_2 = 3$ and $k \notin \{e^1+2e^3+2e^4+2e^5, e^2+2e^3+2e^4+2e^5\}$. Then k belongs to one and only one $\phi(E(L;I)\backslash F(L))$, for $|I| = 2,3$, which we have just described. Hence the map φ is bijective. q.e.d.

<u>LEMMA 6</u>.

(i) There exists a base of $H^{21}(X)$ which is bijective to $F(M_G)$.

(ii) The map

$$k \longmapsto k' \ \text{ with } \ k = \sum_i k_i e^i\ ,\ \ k' = \sum_i k_i' e^i,\ \ k_i+k_i' = d_i-2\ ,$$

defines an one–one correspondence between $F(M_G)$ and

$$F(M_G)' = \{k' = \sum_i k_i' e^i \in M_G \mid tr_q(k') = 2,\ 0 \leq k_i' \leq d_i-2 \ \text{ for all } i \} .$$

And $F(M_G)'$ is bijective to a base of $H^{12}(X)$.

<u>PROOF</u>. Since each element of $\mathcal{J}$ is uniquely represented by z^k with

$k = \sum_i k_i e^i$, $0 \leq k_i \leq d_i - 2$, by (6) and LEMMA 1, there is a base of $H^{21}(X)$ parametrized by the elements in $F(M_G)$. So we obtain (i). (ii) follows from $tr_q(k) + tr_q(k') = 3$ when $k_i + k_i' = d_i - 2$ for all i . q.e.d.

For the convenience of the notations, we shall consider

$$F(M_G) = \text{the base of } H^{21}(X) \ .$$

<u>LEMMA 7</u>. For I with $|I| = 2,3$, let $I' = \{1,\ldots,5\} \backslash I$. Then

$$X_I \subset \mathrm{Sing}(X) \ \text{ iff } \ E(N_G;I) \neq \phi \ ,$$

and there exists an one–one correspondence between the following sets.

(i) For $|I| = 2$,

$\{D$: the exceptional divisor in X with $\sigma(D) = X_I\} \longleftrightarrow E(N_G;I)$,
a base of $H^{10}(X_I) \longleftrightarrow E(M_G;I')^*$.

(ii) For $|K| = 3$,

$\{D$: the exceptional divisor in X with $\sigma(D) = a\} \longleftrightarrow E(N_G;K)^*$,
a base of $H^0(X_K)/\mathbb{C}\delta \longleftrightarrow E(M_G;K')$,

here a : an element of X_K ,

δ : the 0–cocycle assigning the value 1 for each element of X_K .

PROOF. By LEMMA 2, $C_I \subsetneq G$ for all I . Hence C_I is isomorphic to the quotient of the I–th sublattice of N_G by its intersection with N_1 via the map $\exp_q$. For $|I| = 2$ or 3 ,

$$X_I \subseteq \mathrm{Sing}(X) \longleftrightarrow C_I \neq \{1\} \longleftrightarrow E(N_G;I) \neq \phi \ .$$

For $|I| = 2$, X_I is an irreducible non–singular curve. By (6) and LEMMA 1, the elements of a base of $H^{10}(X_I)$ are in one–one correspondence with

$$k' = \sum_i k_i' \, e^i, \, k_i' = 0 \ \text{ for } i \in I \, , \ \sum_{j \notin I} q_j k_j' = 1, \, 1 \leq k_j' \leq d_j - 1 \ \text{ for } j \notin I \ ,$$

which is the same for $k' \in E(M_G;I')^*$.

For $X_K \subseteq \mathrm{Sing}(X)$, $|K| = 3$, X_K consists of finite elements. By the similar argument as the base of $|I| = 2$, the base of $H^0(X_K)/\mathbb{C}\delta$ is in one–one correspondence with $E(M_G;K')$. The conclusion for the exceptional divisors in the lemma follows from the construction of the toroidal resolution $\hat{X}$, which can be found in [12], [14]. q.e.d.

Again, we shall regard

$$E(M_G;I')^* = \text{the base of } H^{10}(X_I) \ \text{ for } |I| = 2 \ ,$$
$$E(M_G;K') \coprod \{\delta\} = \text{the base of } H^0(X_K) \ \text{ for } |K| = 3 \ .$$

Proof of THEOREM 4. By THEOREM 2,

$$H^{21}(\hat{X}) \simeq H^{21}(X) \oplus \oplus \left[H^{10}(\sigma(D)) \,\middle|\, \begin{matrix} D : \text{exceptional divisor with} \\ \sigma(D) = X_I \text{ for some } |I| = 2 \end{matrix} \right],$$

$$H^{11}(\hat{X}) \simeq H^{11}(X) \oplus \oplus (H^0(\sigma(D)) \mid D : \text{exceptional divisor}).$$

By LEMMA 6 and 7, we can regard

$$F(M_G) \coprod \coprod_{|I|=2} (E(N_G;I) \times E(M_G;I')) = \text{the base of } H^{21}(\hat{X}) ,$$

$$\coprod_{|K|=3} E(N_G;K)^* \coprod \coprod_{|I|=2} (E(N_G;I) \times \{\delta\}) \coprod \coprod_{|I|=2} (E(N_G;I) \times E(M_G;I')) = \text{the base of } H^{11}(\hat{X}) .$$

The similar expression for $H^{21}(\hat{X}')$ and $H^{11}(\hat{X}')$ by replacing G by G'. Since $N_G = M_{G'}$, $M_G = N_{G'}$, in order to obtain an one–one correspondence of the bases of $H^{21}(\hat{X})$ and $H^{11}(\hat{X}')$, it suffices to have a bijective map between

$$F(N_{G'}) \text{ and } \coprod_{|K|=3} E(N_{G'};K)^* \coprod \coprod_{|I|=2} (E(N_{G'};I) \times \{\delta\}) ,$$

and the similar statement for $H^{11}(\hat{X})$ and $H^{21}(\hat{X}')$ by replacing G' by G. By LEMMA 5, there exists a such bijective map of the lattice $L = N_{G'}, N_G$. Hence we obtain our result. q.e.d.

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