TIGHT SMOOTHING OF SOME POLYHEDRAL SURFACES

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1. INTRODUCTION AND RESULTS

The main goal of this paper is to approximate certain tight polyhedral surfaces in euclidean 3-space E^3 by tight smooth surfaces of the same topological type: we call this *tight smoothing*. This smoothing procedure gives a convenient tool for constructing examples of tight smooth surfaces. In particular we will construct in this way tight smooth surfaces of odd Euler characteristic χ. So far the existence of tight surfaces with odd Euler characteristic depended on a very complicated construction due to N.H.Kuiper ([9]). Although we have no doubt that the idea behind this construction is essentially correct, it seems to be very difficult to prove rigorously that the described surface is tight. Our examples are very explicit and have moreover a three-fold symmetry.

A smooth compact surface M immersed into E^3 is called *tight* if its total absolute curvature $\int |K|\, do$ equals the minimal possible value $2\pi(4 - \chi(M))$. Equivalent conditions (also applicable to nonsmooth surfaces) are

1. for every closed halfspace $H \subset E^3$ the preimage of H in M is connected (two-piece-property), or
2. every strict local supporting plane is a global supporting plane.

For other characterizations and general results about tightness see the survey articles [11], [12].

A *compact polyhedral surface* in E^3 is a finite complex consisting of vertices, straight edges and planar (but not necessarily convex) faces such that every edge is contained in exactly two faces and the faces around each vertex form locally a cone over a simply closed spherical polygon. All edges are assumed to be proper meaning that the adjacent faces are not coplanar. Such terms as "locally" or "ε-neighborhood" will always refer to the inner geometry of the complex. In particular there may occur selfintersections (like in the case of smooth immersions). A vertex is called *convex* if the corresponding cone is convex, otherwise it is called nonconvex. The number of edges meeting at a vertex v is called the *valence* of v.

We call v a *standard saddle vertex* if

(i) the valence of v is four,

(ii) all angles of the faces at v are strictly smaller than π,

(iii) there is no local supporting plane through v .

<u>THEOREM 1</u> : *Assume that M is a tight polyhedral surface in E^3 such that all of its nonconvex vertices have valence 3 . Then for every $\varepsilon>0$ there exists a tight smooth surface $M(\varepsilon)$ of the same topological type which coincides with M except in the ε-neighborhood of the union of all edges of M .*

<u>THEOREM 2</u> : *Assume that M is a tight polyhedral surface in E^3 whose non-convex vertices are either 3-valent or standard saddle vertices. Then for every $\varepsilon>0$ there is a tight polyhedral surface M' whose faces are in 1-1-correspondence with the faces of M such that corresponding faces are parallel in distance less than ε and such that all vertices are of valence 3 .*

Combining the two theorems we see that <u>every tight polyhedral surface</u> whose nonconvex vertices are either 3-valent or standard saddle vertices can be <u>approximated</u> (in a certain sense) by <u>tight smooth surfaces</u> of the same topological type.
The proof of these theorems will be given in section 2 below.
In section 3 we use theorem 1 to give an improved version of a statement originally due to N.H.Kuiper ([9]):

<u>THEOREM 3</u> : *For any odd integer $\chi = -3$ there exists a tight smooth immersion of the surface with Euler characteristic χ into E^3. It may be chosen to have a three-fold symmetry.*

In section 4 we discuss further applications and some open questions.

<u>2. PROOF OF THEOREMS 1 AND 2 : TIGHT SMOOTHING</u>

<u>Proof of theorem 1</u>: First of all we replace each edge by a piece of a cylinder over a certain convex smooth curve. For given $\delta>0$ let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying

(i) f is convex and C^∞ ,

(ii) f is strictly convex in the interval $(-\delta,\delta)$,

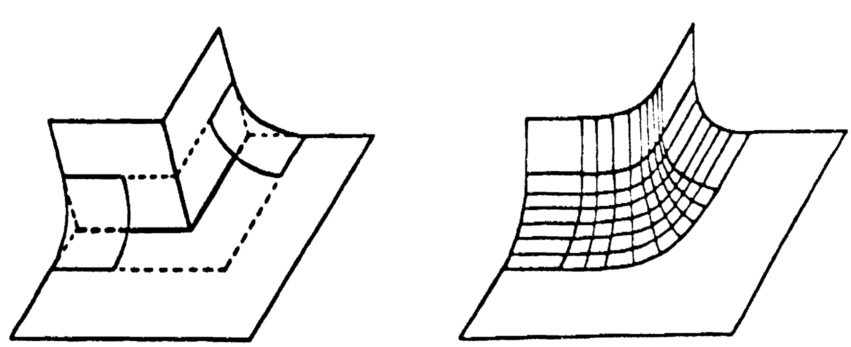
(iii) $f(x) = x$ for $x \geq \delta$,

(iv) $f(x) = f(-x)$ for all x .

The orthogonal cylinder over the graph of this function will be a suitable smoothing of an edge between two orthogonal planes P_1, P_2. Let $g_1 \subset P_1$ and $g_2 \subset P_2$ denote the generators of the cylinder through the points $(\delta,f(\delta))$ and $(-\delta,f(-\delta))$. The distance of g_1 from P_2

(and of g_2 from P_1) will be $\sqrt{2}\cdot\delta$. Now for every angle between two planes we take an affine image of this smoothed orthogonal angle. We can do this in such a way that the distances of g_1 from P_2 (and of g_2 from P_1) after the affine transformation will be exactly this same value $\sqrt{2}\cdot\delta$. Furthermore by compactness for an arbitrary given $\varepsilon>0$ one can find a $\delta>0$ such that all these cylindrical pieces will lie in the ε-neighborhood of the corresponding edge. The smoothing is now clear everywhere except in the ε-neighborhood of the vertices.

We first consider the nonconvex vertices of valence three. It is easy to see that exactly one of the three angles between edges is greater than π. The plane of this angle will be labeled as "horizontal", the direction of the remaining edge as "vertical" (see fig. 1). For both horizontal cylinders (coming from the smoothing of the horizontal edges) the upper boundary is a horizontal straight line lying in one of the vertical faces. Our assumption that both of these lines have distance $\sqrt{2}\cdot\delta$ from the horizontal plane implies that these two lines meet on the vertical edge. Using an affine transformation of E^3 we can assume without loss of generality that all these planes are orthogonal and that the two horizontal cylinders are congruent. Let $t \mapsto g_t$, $0 \leq t \leq 1$ be a smooth parametrization of the family of generators of the vertical cylinder C_v, N_t the normal plane to C_v through the generator g_t. Let γ_o be the curve of intersection of the first cylinder with the plane N_o, γ_1 the intersection of the second cylinder with N_1. Then the orthogonal trajectories of the family N_t through the points of γ_o intersect each plane N_t in a congruent copy $\tilde{\gamma}_t$ of γ_o. Because of $\tilde{\gamma}_1 = \gamma_1$ the curves $\tilde{\gamma}_t$ provide a smooth interpolation between γ_o and γ_1 (see fig. 1). The surface swept out by the $\tilde{\gamma}_t$ is usually called a "molding surface" or "Gesimsfläche" (see [6]). The $\tilde{\gamma}_t$ are lines of curvature on this surface and it is easy to check that its Gaussian curvature is nonpositive everywhere. Clearly the molding surface fits smoothly the cylinders and the planes.



Now for a convex vertex we can proceed similarly. In order to preserve a possible rotational symmetry of the vertex cone we suggest the following: truncate the vertex by a plane (preserving the symmetry, if there is any). Then the resulting vertices will have valence three. Now smooth the new edges by cylinders as described above (but with essentially smaller δ). Then we can apply the same kind of construction near the vertices. The only difference is that this time the molding surface will have nonnegative Gaussian curvature.

Altogether we get a smooth approximation of the given polyhedral surface M which coincides with M in the ε-neighborhood of the union of the edges and such that positive Gaussian curvature occurs only in the ε-neighborhood of the convex vertices. If the polyhedral surface is tight these vertices lie on the boundary of the convex hull and consequently the smoothed surface will be tight as well.

Proof of theorem 2: The idea behind theorem 2 is that generically all vertices are 3-valent: if the planes spanned by the various faces lie in general position then no more than three planes can meet at a vertex. On the other hand the tightness itself is a very special situation. Therefore we have to be very careful when changing our given tight surface.

Lemma: The vertex cone of a standard saddle vertex is the boundary of the union of two "roofs" where we mean by a roof the intersection of two closed halfspaces. The ridge of each roof will hit the interior of the other roof.

Proof of the lemma: Locally there is an orientation of the surface. This enables us to talk about "convex" and "concave" edges meeting at a standard saddle vertex v . If there are two subsequent edges (in cyclic order) of the same kind then the plane spanned by those two edges will be a supporting plane. This implies that the edges of a standard saddle vertex have to be creased in an alternating manner: convex, concave, convex, concave. Let us take the two convex edges as ridges. Then the two roofs are built up by the two pairs of planes adjacent to the two convex edges. The condition that all interior angles of the faces at v are smaller than implies that each convex edge (the ridge) hits the interior of the other roof.

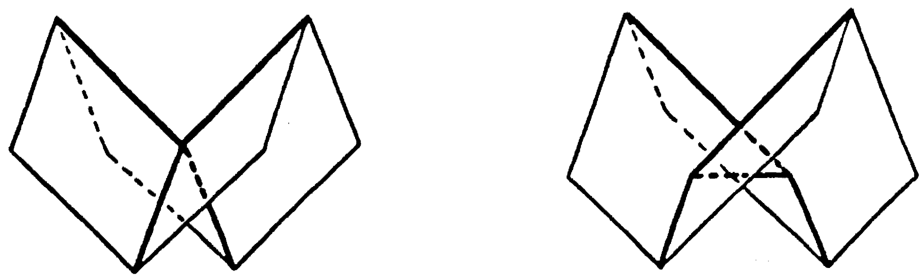


figure 2

To finish the proof of theorem 2 we observe what happens if we move one of the four planes a little bit in the normal direction. This is the same as saying that one of the roofs is moved against the other one. First of all we see that the boundary of the union of the two roofs will still be a surface. Hence the topology does not change by this process. Secondly we see that the saddle vertex splits into two vertices, each being the intersection point of a ridge with the other roof. The vertex cone of each of the new vertices will be the boundary of the union of a roof and a closed halfspace. Therefore these new vertices are nonconvex and of valence three with exactly one local supporting plane (see figure 2 for the situation before and after this process). After iterated application of this process the surface finally will have only 3-valent vertices. The convex vertices will split into convex vertices of valence 3 and the former nonconvex 3-valent vertices will keep their geometry: they will just be moved by a euclidean translation. The important point is that by this process no open sets of local supporting planes are created. Therefore if the old surface was tight the new surface will be tight as well.

3. A TIGHT POLYHEDRAL PROJECTIVE PLANE WITH TWO HANDLES

The main part of the surface to be described is a polyhedral Boy surface with a three-fold symmetry (a modified version of the one given in [1]). This has the property that all its strict local supporting planes concentrate at the vertices of three convex faces on the boundary of the convex hull. Then one has to cut out these three faces and to join the resulting surface with an exterior tetrahedral surface.

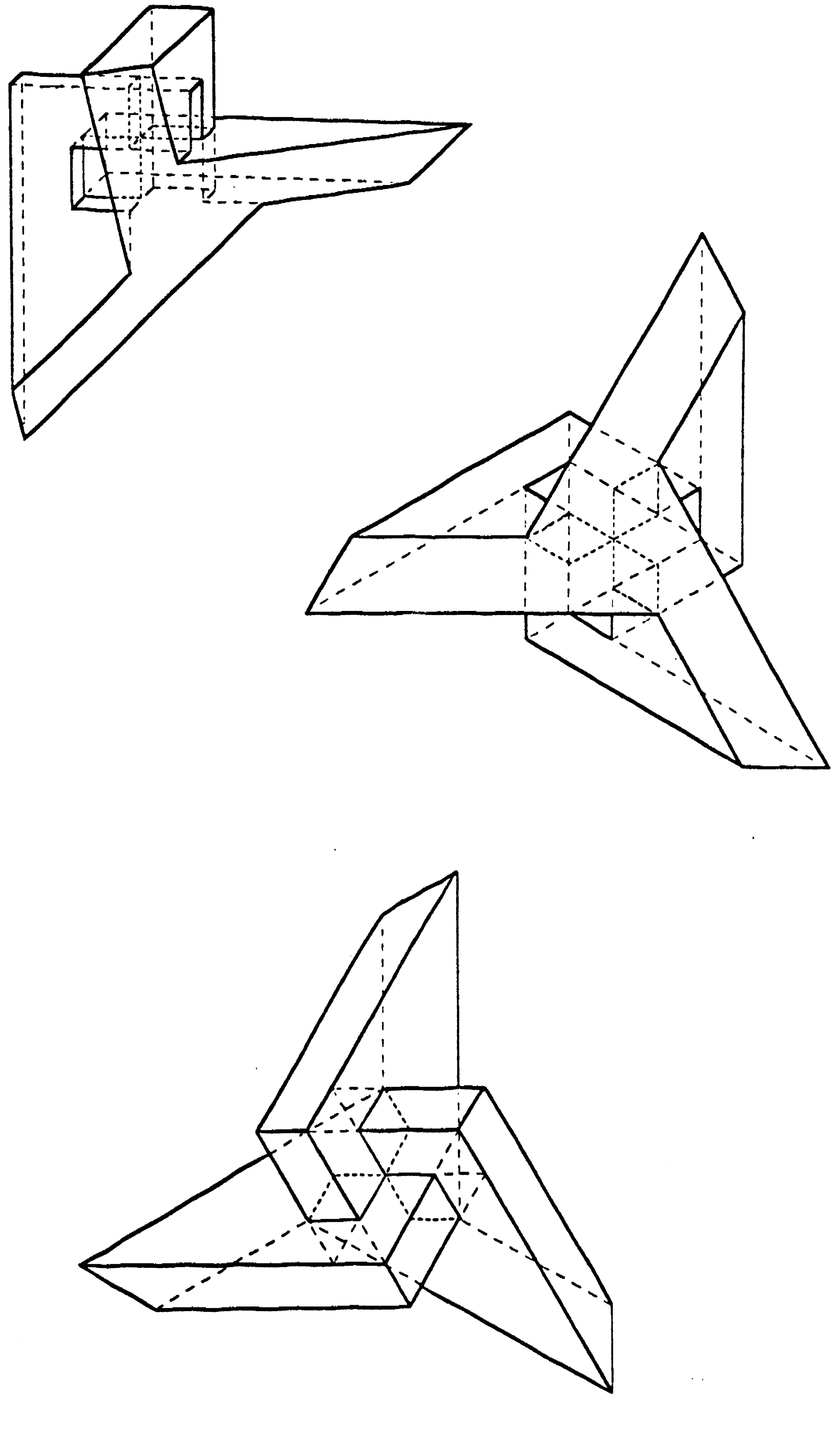
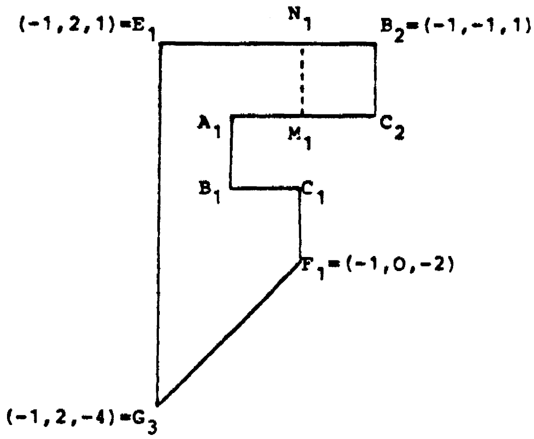
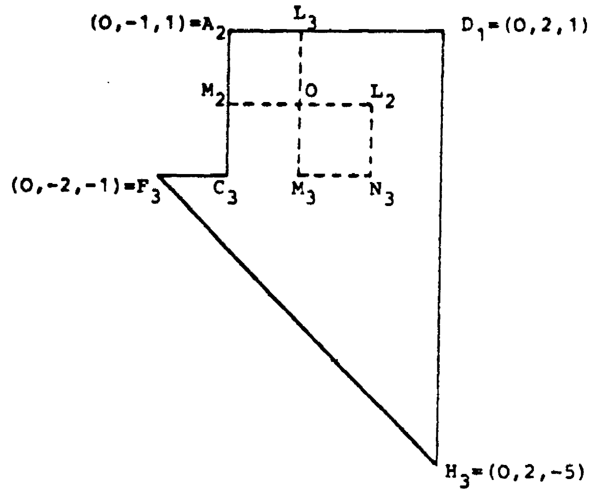


figure 3

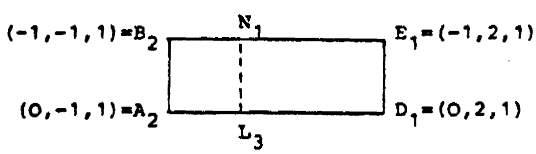
<u>type 1</u> (lying in the plane $x=-1$)



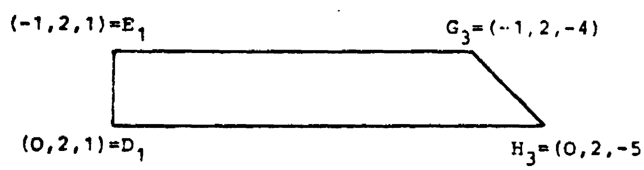
<u>type 2</u> (in the plane $x=0$)



<u>type 3</u> (in the plane $z=1$)



<u>type 4</u> (in the plane $y=2$)



<u>type 5</u> (in the plane $x+y+z=-3$)

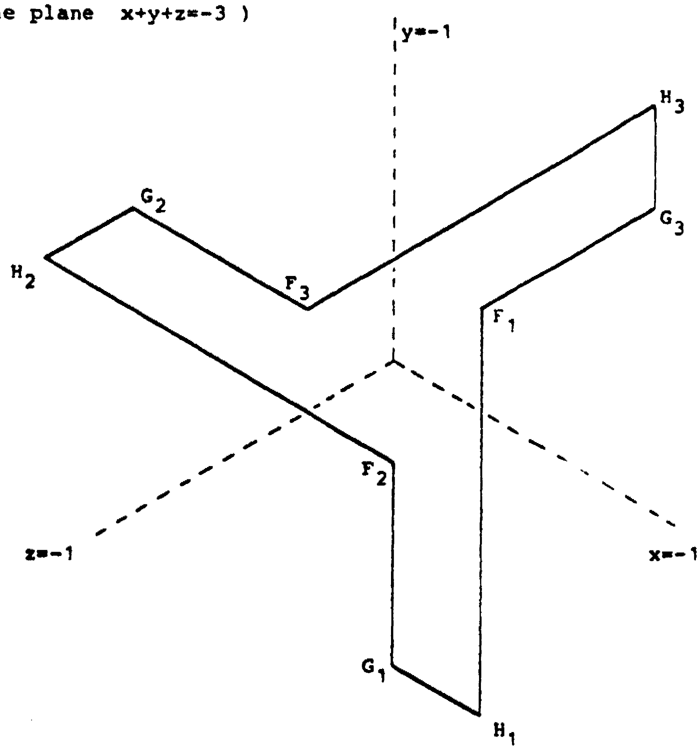
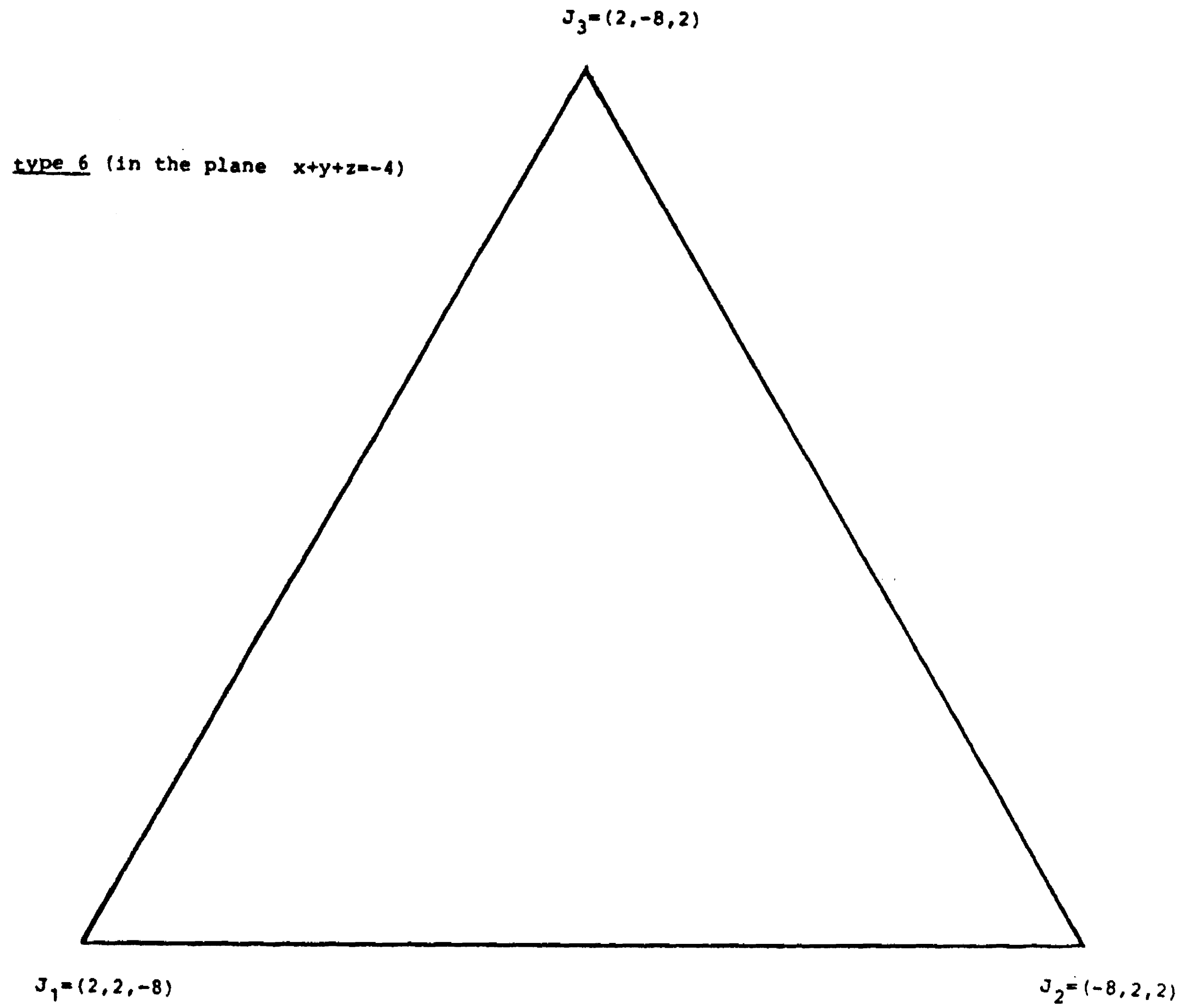
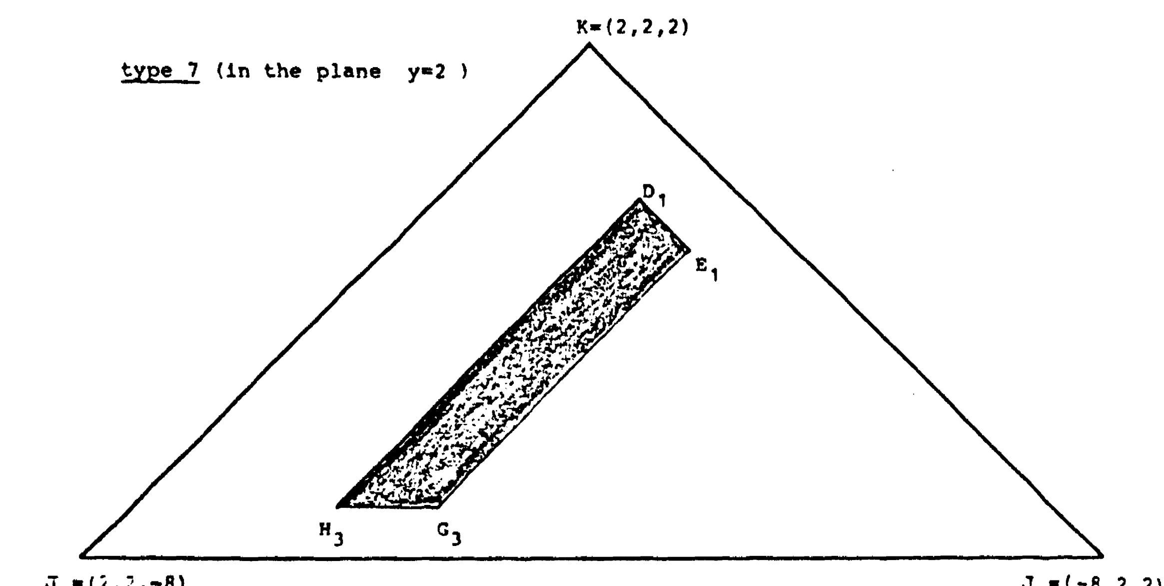


figure 4



$J_3=(2,-8,2)$
type 6 (in the plane $x+y+z=-4$)
$J_1=(2,2,-8)$
$J_2=(-8,2,2)$



$K=(2,2,2)$
type 7 (in the plane $y=2$)
D_1
E_1
H_3
G_3
$J_1=(2,2,-8)$
$J_2=(-8,2,2)$

figure 5

All the strict local supporting planes of this polyhedral Boy surface with two handles will concentrate at the vertices of the tetrahedron which implies that the surface will be tight. Every vertex lies in the cubic lattice $\mathbb{Z}^3 \subset E^3$ where the origin is the triple point of the Boy surface (which is not a vertex of the surface). All of the nonconvex vertices are in fact of valence three. The figure 3 shows three different views onto the polyhedral Boy surface (the tetrahedron does not show up there). Note one can distinguish the curve of selfintersections from the edges by a different drawing.

We describe the surface explicitly by the coordinates of the vertices and the shape of the faces. The three-fold symmetry appears as cyclic shift of the three coordinates $x \to y \to z \to x$ corresponding to the cyclic shift of the indices $1 \to 2 \to 3 \to 1$ of the following vertices:

$A_1 = (-1,\ 1,\ 0)$	$A_2 = (\ 0,-1,\ 1)$	$A_3 = (\ 1,\ 0,-1)$
$B_1 = (-1,\ 1,-1)$	$B_2 = (-1,-1,\ 1)$	$B_3 = (\ 1,-1,-1)$
$C_1 = (-1,\ 0,-1)$	$C_2 = (-1,-1,\ 0)$	$C_3 = (\ 0,-1,-1)$
$D_1 = (\ 0,\ 2,\ 1)$	$D_2 = (\ 1,\ 0,\ 2)$	$D_3 = (\ 2,\ 1,\ 0)$
$E_1 = (-1,\ 2,\ 1)$	$E_2 = (\ 1,-1,\ 2)$	$E_3 = (\ 2,\ 1,-1)$
$F_1 = (-1,\ 0,-2)$	$F_2 = (-2,-1,\ 0)$	$F_3 = (\ 0,-2,-1)$
$G_1 = (-4,-1,\ 2)$	$G_2 = (\ 2,-4,-1)$	$G_3 = (-1,\ 2,-4)$
$H_1 = (-5,\ 0,\ 2)$	$H_2 = (\ 2,-5,\ 0)$	$H_3 = (\ 0,\ 2,-5)$
$J_1 = (\ 2,\ 2,-8)$	$J_2 = (-8,\ 2,\ 2)$	$J_3 = (\ 2,-8,\ 2)$
	$K = (\ 2,\ 2,\ 2)$	

The vertices J_1, J_2, J_3, K span a tetrahedron which contains a polyhedral Boy surface spanned by the 24 vertices $A_i, B_i, \ldots, H_i$, $i = 1,2,3$. The 13 faces of this Boy surface split into 5 different types under the three-fold symmetry. Figure 4 shows these 5 types where type 5 is invariant. The dashed lines indicate the curve of selfintersections. This curve is the closed space polygon

$$(O\ L_1\ N_2\ M_2\ O\ L_2\ N_3\ M_3\ O\ L_3\ N_1\ M_1\ O)$$

where the points are the following:

$L_1 = (\ 1,\ 0,\ 0)$	$L_2 = (\ 0,\ 1,\ 0)$	$L_3 = (\ 0,\ 0,\ 1)$
$M_1 = (-1,\ 0,\ 0)$	$M_2 = (\ 0,-1,\ 0)$	$M_3 = (\ 0,\ 0,-1)$
$N_1 = (-1,\ 0,\ 1)$	$N_2 = (\ 1,-1,\ 0)$	$N_3 = (\ 0,\ 1,-1)$
	$O = (\ 0,\ 0,\ 0)$	

The faces of the tetrahedron are shown in figure 5 . In order to join the Boy surface with the tetrahedron we have to cut out the three faces of type 4 from both of them. This is already indicated in the drawing of type 7 .

The Euler characteristic of the Boy surface is easily computed to be $\chi = 24 - 36 + 13 = 1$. After gluing it together with the tetrahedron we get $\chi = 28 - 42 + 11 = -3$ (Note that the three noncontractible faces of the tetrahedron don't make a contribution to the Euler characteristic). It is not hard to see that all the vertices of this surface are in fact of valence three.

In order to complete the proof of theorem 3 one has to attach an arbitrary number of handles to this tight polyhedral surface such that the threefold symmetry is preserved. This is easy to do if this number has a residue 0 or 1 modulo 3. In the case of a residue 2 modulo 3 one can use the boundary of a hexagonal truncated pyramid where three quadrilaterals have been removed which are not adjacent to each other (see figure 6).

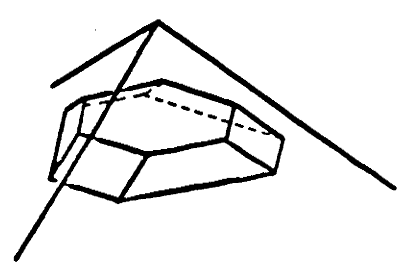


figure 6

4. CONCLUDING REMARKS

1. Many examples of tight polyhedral surfaces in the literature are of the kind to which our theorems apply. In fact such a smoothing process has already been used implicitly in Kuiper's example of a tight Klein bottle with handle (see [9]). Banchoff's tight square tori (see [4]) satisfy the assumptions of our theorem 2 if the meridian curve has a horizontal top and bottom edge. If there is a top or bottom vertex one has to truncate it horizontally a little bit.

Other examples are the boundaries of the difference set of two convex 3-polytopes. Such a construction has been used in [5] to get tight polyhedral surfaces with geometrical degree four. Assume that P and P' are 3-polytopes which are dual to each other and such that the vertices of each lie outside of the other. Then the boundary of each difference set $P \setminus P'$, $P' \setminus P$ will be a tight polyhedral surface satisfying the assumptions of our theorem 1. Quite symmetrical examples can be constructed using the Platonic or Archimedean solids and their duals (see figure 7 which shows the difference of a dodecahedron and an icosahedron). In the case of an octahedron minus a cube the resulting manifold of genus 7 happens to be a so-called "Platonic manifold" $\{5,4;7\}$ (see [14]).

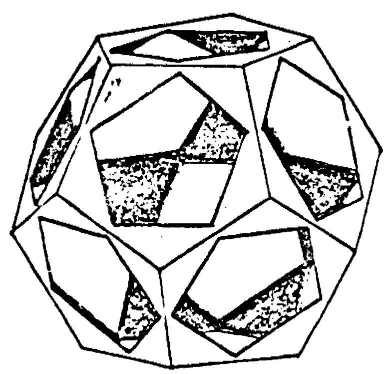


figure 7

2. However, the question is what happens for tight polyhedral surfaces in general. The 7-vertex Császár torus (cf. [8]) and Brehms's flat torus are tight and contain vertices of other types, in particular the "mixed curvature type" (see figure 8) where a nonconvex vertex admits an open set of local supporting planes.

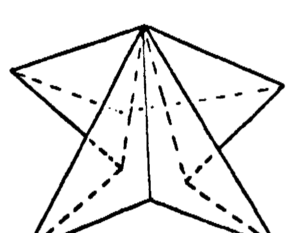


figure 8

There is the notion of a polyhedral curvature $K = K^+ - K^-$ which corresponds to the Gaussian curvature, and an absolute curvature $K = K^+ + K^-$ which corresponds to the absolute Gaussian curvature of a smooth surface (see [3],[7]). In terms of those a tight smoothing has to preserve the positive part K^+ and the negative part K^- simultaneously. In [7] it is shown that one can approximate an arbitrary polyhedral surface in such a way that the various curvatures converge. This however does not imply that a tight polyhedral surface will be smoothed tightly because additional open sets of local supporting planes will be created.

We would like to ask the following

Question: Does an arbitrary tight polyhedral surface in E^3 admit a tight smoothing?

The corresponding question for tight surfaces with boundary is known to be false because there is a tight polyhedral Möbius band in E^3 but not tight smooth one (see [10]). It is also known to be false for surfaces in high dimensional euclidean space because there are no tight smooth surfaces in E^n for $n \geq 6$ whereas there are such tight polyhedral surfaces (cf. [2]).

3. Finally we want to mention another application of our theorem. In [13] there will be studied the following problem: which regular homotopy classes of surfaces in E^3 contain a tight smooth one? In particular it will be shown that there are tight smooth surfaces of non-standard regular homotopy types. Certain tightly smoothed tight polyhedral surfaces will be used in this construction, in particular the tight Boy surface with two handles described in section 3.

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