THREE POINT COVERS WITH BAD REDUCTION

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Abstract. We study Galois covers of the projective line branched at three points with bad reduction to characteristic p, under the condition that p strictly divides the order of the Galois group. As an application of our results, we prove that the field of moduli of such a cover is at most tamely ramified at p.

Introduction

Ramified primes in the field of moduli. Let $f : Y \to X = \mathbb{P}^1_{\mathbb{C}}$ be a three point cover, i.e. a finite cover of the Riemann sphere branched at 0, 1 and ∞. The 'obvious direction' of Belyi's Theorem states that f can be defined over a number field. Therefore, the absolute Galois group of $\mathbb{Q}$ acts on the set of isomorphism classes of all three point covers, and we can associate to f the number field K such that $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ is the stabilizer of the isomorphism class of f. The number field K is called the field of moduli of f. Very little is known about the correspondence between three point covers and their associated field of moduli. One of the most general things that we know about K is given by the following theorem of Beckmann [Bec89]. Let G be the monodromy group of f. Then the extension $K/\mathbb{Q}$ is unramified outside the set of primes diving the order of G. This result is related to the fact that f has good reduction at each prime ideal of K dividing such primes p.

A recent theorem of Raynaud [Ray99] gives a partial converse to Beckmann's Theorem. Suppose that the prime p *strictly* divides the order of G, i.e. that p^2 does not divide $|G|$. Let $\mathfrak{p}$ be a prime ideal of K dividing p and suppose that the ramification index $e(\mathfrak{p}/p)$ is strictly smaller than the number of conjugacy classes in G of elements of order p. Then f has potentially good reduction at $\mathfrak{p}$.

The proof of this theorem depends on an analysis of the *stable reduction* of $f : Y \to X$ at $\mathfrak{p}$. If f does not have potentially good reduction at $\mathfrak{p}$ then this analysis yields a certain lower bound for the ramification index $e(\mathfrak{p}/p)$. The theorem follows from this bound. We remark that this sort of result is not restricted to three point covers, but extends, under certain conditions, to covers of more general curves X. The essential condition here is that p strictly divides the order of G.

In [Wew] the author has continued Raynaud's study of the stable reduction, and the present paper is a further continuation of this work. Our results yield an *upper* bound for the ramification index of p in the field of moduli of certain three point covers. For instance, we prove the following theorem.

2000 *Mathematics Subject Classification.* Primary 14H30, 11G20.

Key words and phrases. Three point cover, stable reduction, field of moduli.

Theorem 1. *Let $f : Y \to \mathbb{P}^1$ be a three point cover, with monodromy group G and field of moduli K. Let p be a prime which* strictly *divides the order of G. Then p is at most tamely ramified in the extension $K/\mathbb{Q}$.*

In fact we obtain a formula for the ramification index $e(\mathfrak{p}/p)$ in terms of the stable reduction of $f : Y \to X$ at $\mathfrak{p}$. This formula shows that $e(\mathfrak{p}/p)$ is prime to p and gives an upper bound.

In contrast to the theorem of Raynaud mentioned above, our results are very particular to three point covers. They depend in an essential way on the fact that three point covers are 'rigid' objects, i.e. do not admit any nontrivial deformation.

Stable reduction. Let R be a complete discrete valuation ring, with residue field k of characteristic $p > 0$ and fraction field K of characteristic zero. Let $f_K : Y_K \to X_K := \mathbb{P}^1_K$ be a three point cover defined over K. We assume that f_K is Galois, with Galois group G. In order to prove results such as Theorem 1, this is no restriction. After replacing the field K by a finite extension, we may assume that Y_K has a semistable model Y_R over R such that the ramification points of f_K specialize to pairwise distinct smooth points on the special fiber. Among all semistable models with this property, we let Y_R be the minimal one. The action of G extends to Y_R; let $X_R := Y_R/G$ denote the quotient. This is a semistable model of $\mathbb{P}^1_K$ over R. Therefore, the special fiber $\bar{X} := X_R \otimes_R k$ is a tree of curves of genus zero. The map $f_R : Y_R \to X_R$ is called the *stable model* of the Galois cover f_K. Its restriction to the special fiber $\bar{f} : \bar{Y} \to \bar{X}$ is called the *stable reduction* of f_K.

We say that the cover $f_K : Y_K \to X_K$ has good reduction if the curves X_R and Y_R are smooth over R. If this is the case, then the stable reduction $\bar{f} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ is a three point Galois cover in characteristic p. If f_K does not have good reduction we say that it has *bad reduction*. Our first main result is a structure theorem for the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ in the case of bad reduction.

Theorem 2. *Suppose that p strictly divides the order of G and that the cover $f_K : Y_K \to X_K$ has bad reduction. Then the following holds (compare with Figure 1).*

(1) *The curve $\bar{X}$ consists of one central component $\bar{X}_0$, canonically isomorphic to $\mathbb{P}^1_k$, and $r \geq 1$ tails $\bar{X}_1, \ldots, \bar{X}_r$, each of which intersects $\bar{X}_0$ in one point.*
(2) *Let $\bar{Y}_0$ be a irreducible component of $\bar{Y}$ lying above the central component $\bar{X}_0$. Then the map $\bar{Y}_0 \to \bar{X}_0$ induced from $\bar{f}$ is the composition of a purely inseparable map $\bar{Y}_0 \to \bar{Z}_0$ of degree p and a Galois cover $\bar{Z}_0 \to \bar{X}_0$.*
(3) *For $j = 1, \ldots, r$, let $\bar{Y}_j$ be an irreducible component of $\bar{Y}$ lying over the tail $\bar{X}_j$. Then the map $\bar{Y}_j \to \bar{X}_j$ induced by $\bar{f}$ is a Galois cover, ramified at most at two points. More precisely, $\bar{Y}_j \to \bar{X}_j$ is wildly ramified at the point where $\bar{X}_j$ intersects $\bar{X}_0$ and is tamely ramified at any point which is the specialization of one of the K-points 0, 1 or ∞.*

Part (1) is the essential statement and depends on the assumption that f_K is a three point cover. Part (2) and (3) follow from (1), using the results of [Ray99] and the assumption that p strictly divides the order of G.

Under the additional assumption that all the ramification indices of f_K are prime to p, Theorem 2 (1) can be deduced from the results of [Wew], via Raynaud's construction of the *auxiliary cover* (see the introduction of [Wew]). The proof of the general case given in the present paper is based on a generalization of the

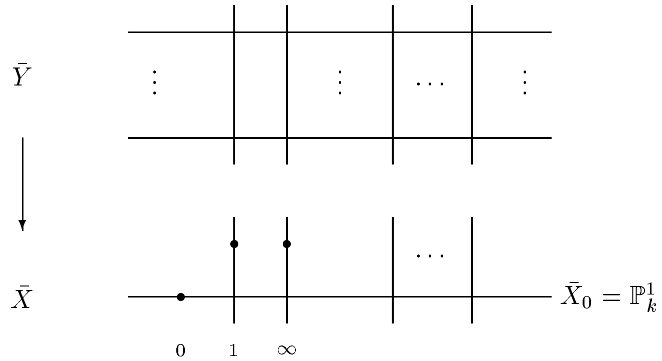

FIGURE 1. The stable reduction of a three point cover

methods of [Wew]. This generalization avoids the assumption on the ramification indices and the use of the auxiliary cover.

Lifting of special G-maps. The stable reduction of a three point Galois cover $f_K : Y_K \to X_K := \mathbb{P}^1_K$ is, by definition, a finite map $\bar{f} : \bar{Y} \to \bar{X}$ between semistable curves over the residue field k, together with an action of a finite group G on $\bar{Y}$ which commutes with $\bar{f}$. In the case of bad reduction, the curves $\bar{X}$ and $\bar{Y}$ are both singular, and the map $\bar{f}$ is inseparable over some of the irreducible components of $\bar{X}$. This suggests the following question. Given a map $\bar{f} : \bar{Y} \to \bar{X}$ of the sort we have just described, does it occur as the stable reduction of a three point Galois cover $f_K : Y_K \to X_K$ in characteristic zero? If this is the case then we say that $f_K : Y_K \to X_K$ is a *lift* of $\bar{f} : \bar{Y} \to \bar{X}$.

Theorem 2 gives a list of necessary conditions on $\bar{f}$ for the existence of a lift (under the condition that p strictly divides the order of the group G). There is one extra condition on $\bar{f}$ which we have omitted from the statement of Theorem 2, but which plays a central role in this paper. This additional condition is the existence of a certain differential form ω_0 on the curve $\bar{Z}_0$ which is 'compatible' with $\bar{f}$ (see Theorem 2 (2) for the definition of $\bar{Z}_0$). For instance, ω_0 is *logarithmic*, i.e. of the form $\mathrm{d}u/u$, and it is an eigenvector under the action of the Galois group of the cover $\bar{Z}_0 \to \bar{X}_0$. Moreover, the divisor of zeroes and poles of ω_0 is related to and determined by the map $\bar{f}$ and the action of G on $\bar{Y}$. The existence and the basic properties of ω_0 follow from work of Green–Matignon [GM99] and Henrio [Hen00]. In some sense, ω_0 carries infinitesimal information about the action of G on $\bar{Y}$ which 'got lost' by reduction to characteristic p. Following [Wew], we call $(\bar{Z}_0, \omega_0)$ a *special deformation datum.* (Actually, in [Wew] we assume that ω_0 has no poles. This is true if and only if all the ramification indices of the three point cover $f_K : Y_K \to X_K$ are prime to p.)

In the present paper we define the notion of a *special G-map.* This is a map $\bar{f} : \bar{Y} \to \bar{X}$ between semistable curves in characteristic p together with an action of a finite group G on $\bar{Y}$ satisfying certain conditions. Essentially, these conditions say that $\bar{f}$ satisfies the conclusion of Theorem 2 and that there exists a special deformation datum $(\bar{Z}_0, \omega)$ which is 'compatibel' with $\bar{f} : \bar{Y} \to \bar{X}$ and the action of G on $\bar{Y}$. Our second main result can be stated as follows.

Theorem 3. *Let $\bar{f} : \bar{Y} \to \bar{X}$ be a special G-map, defined over an algebraically closed field k of characteristic $p > 0$. Let K_0 denote the fraction field of the ring of Witt vectors over k. Then the following holds.*

(1) *There exists a lift of $\bar{f}$, i.e. a three point Galois cover $f_K : Y_K \to X_K$ whose stable reduction is isomorphic to $\bar{f} : \bar{Y} \to \bar{X}$.*
(2) *Each lift of $\bar{f} : \bar{Y} \to \bar{X}$ can be defined over a finite extension K/K_0 which is at most tamely ramified.*

We actually prove more, namely we determine the set of isomorphism classes of all lifts of $\bar{f}$, together with the action of $\mathrm{Gal}(\bar{K}_0/K_0)$ on this set, in terms of certain numerical invariants attached to $\bar{f}$. From this more precise result one can deduce an upper bound for the degree of the minimal extension K/K_0 over which a given lift of $\bar{f}$ can be defined. In any case, Theorem 1 follows from Theorem 2 and Theorem 3 by a straightforward argument.

Part (1) of Theorem 3, i.e. the mere existence of a lift, follows already from the results of [Wew] (assuming that ω_0 has no poles). Part (2) is more difficult and needs new methods. The key step in the proof of (2) is the study of the deformation theory of a certain curve with group scheme action associated to the special deformation datum $(\bar{Z}_0, \omega_0)$. A detailed exposition of this deformation theory can be found in [Wew02].

This paper is divided into four sections. The first section contains the general theory of stable reduction of Galois covers, the monodromy action and deformation data. In §2 we apply this theory to the case of three point cover and prove our first main result, which essentially corresponds to Theorem 1 above. We also define the notion of a special G-map. In §3 we apply the deformation theory of [Wew02] in order to lift a special deformation datum to an auxiliary cover with certain good properties. This is preparatory work for §4, where we prove the lifting result corresponding to Theorem 3. At the end of §4 we draw some conclusions from this result, concerning the field of moduli, and we give some examples.

Acknowledgments. This work grew out of discussions with Michel Raynaud on my previous paper [Wew]. I would like to thank him heartily for his critical remarks and encouragement. The final version of this paper was written while I was a guest of the Max–Planck–Institut in Bonn.

1. The stable reduction of Galois covers

Let K be a field of characteristic zero equipped with a complete discrete valuation whose residue field k is algebraically closed and has characteristic $p > 0$. Let $f : Y \to X$ be a Galois cover of smooth projective curves over K. In this section we define and study the *stable reduction* of f. We make two crucial assumptions on the reduction of f. First, we assume that the marked curve (X, S) (where S is the branch locus of f) has good reduction. Second, we assume that f has *mild reduction*, a property which is automatically verified if p strictly divides the order of the Galois group of f.

We start by recalling some results of Raynaud [Ray99] on the structure of $\bar{f} : \bar{Y} \to \bar{X}$. Then we attach to each irreducible component of $\bar{X}$ where the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ is inseparable a so-called *deformation datum*. This datum encodes infinitesimal information on the action of the Galois group of $f : Y \to X$. The deformation data attached to the individual components of $\bar{X}$ are related by certain compatibility conditions. The existence of a set of compatible deformation data imposes strong restrictions on the map $\bar{f} : \bar{Y} \to \bar{X}$. In this section, we content ourself with working out those restrictions which are of a purely combinatorial nature. For instance, we obtain a new proof of Raynaud's *vanishing cycle formula* [Ray99].

Most of the results of this section are already present in [Wew], in a special case.

1.1. **The stable model and the monodromy action.**

1.1.1. The following notation is fixed throughout §1 and §2. Let R_0 be a complete discrete valuation ring, with quotient field K_0 of characteristic 0 and residue field k of characteristic $p > 0$. We assume that k is algebraically closed. We fix an algebraic closure $\bar{K}$ of K_0.

Let (X_0, S_0) be a *smooth* stably marked curve over R_0. Since R_0 is strictly henselian this implies that $S_0 = \{\, x_{R_0,j} \mid j \in B_0 \}$, where $x_{R_0,j} : \operatorname{Spec} R_0 \to X_0$ are pairwise distinct sections, indexed by a finite set B_0. If K/K_0 is a field extension then we write X_K and S_K instead of $X_0 \otimes_{R_0} K$ and $S_0 \otimes_{R_0} K$. If $K = \bar{K}$ then we simply write X and S instead of $X_{\bar{K}}$ or $S_{\bar{K}}$. The elements of S are denoted by x_j, $j \in B_0$. In §2 we will consider the case $X_0 = \mathbb{P}^1_{R_0}$ and $S_0 = \{0, 1, \infty\}$.

Let G be a finite group and $f : Y \to X$ a G-cover of X with branch locus S. We consider the set of ramification points of the cover f as a marking on Y; this makes Y a stably marked curve. Let K be a finite extension of K_0 and R the ring of integers of K. A *model* of f over K is a G-cover $f_K : Y_K \to X_K$ such that $f = f_K \otimes_K \bar{K}$. A *good model* of f over K is a model which extends to a finite and tame cover $f_R : Y_R \to X_0 \otimes_{R_0} R$, ramified along $S_0 \otimes_{R_0} R$ and étale everywhere else. We say that f has *good reduction* if there exists a good model over some extension K/K_0. If f does not have good reduction, we say that it has *bad reduction*.

The following is well known, see [Bec89]. If f has good reduction then there exists a good model $f_{K_0} : Y_{K_0} \to X_{K_0}$ of f over K_0, unique up to isomorphism. Moreover, if p does not divide the order of the Galois group G, then $f : Y \to X$ has good reduction.

1.1.2. From now on, we assume that $f : Y \to X$ has bad reduction. Note that this implies that p divides the order of G. If K/K_0 is a sufficiently large finite extension, then there exists a model $f_K : Y_K \to X_K$ of f over K such that Y_K extends to a stably marked curve Y_R over R. In particular, Y_R is a semistable model of Y_K over R such that the ramification points of the cover f_K specialize to pairwise disjoint, smooth points on the special fiber $\bar{Y} := Y_R \otimes k$.

Since the stably marked model is unique, the action of the group G on Y_K extends to Y_R. Let $X_R := Y_R/G$ be the quotient; then X_R is a semistable curve over R with generic fiber X_K, see [Ray90], Appendix. Moreover, the branch points x_j of f_K specialize to pairwise distinct smooth points $\bar{x}_j$ on the special fiber $\bar{X} := X_R \otimes k$. We consider $\bar{X}$ as a marked semistable curve (note, however, that $\bar{X}$ is *not* stably marked). By [Knu83], there exists a canonical contraction morphism $q_R : X_R \to X_0 \otimes_{R_0} R$. Let $\bar{X}$ denote the special fiber of X_R. There is a unique irreducible component of $\bar{X}$ on which the induced map $\bar{q} : \bar{X} \to \bar{X}_0 := X_0 \otimes_{R_0} k$ is an isomorphism. We may and will identify this component with $\bar{X}_0$. All other components of $\bar{X}$ are of genus 0 and are contracted by $\bar{q}$ to a closed point of $\bar{X}_0$.

Definition 1.1. The natural map $f_R : Y_R \to X_R$ is called the *stable model* of the Galois cover f_K. The induced map $\bar{f} : \bar{Y} \to \bar{X}$ on the special fibers is called the *stable reduction* of f_K. The component $\bar{X}_0$ of $\bar{X}$ is called the *original component*.

It is clear that the stable model is 'stable' under extension of the field K. In particular, the stable reduction does not depend on the choice of K. We remark that the stable model defined above differs slightly from the model used in [Ray99]. This difference is not essential, but certain results of [Ray99] have to be reformulated. This is done in Lemma 1.4 below.

1.1.3. For an element $\sigma \in \mathrm{Gal}(\bar{K}/K_0)$ we denote by ${}^\sigma f : {}^\sigma Y \to {}^\sigma X = X$ the conjugate G-cover. Let Γ^{in} be the subgroup of $\mathrm{Gal}(\bar{K}/K_0)$ consisting of all elements σ such that $f \cong {}^\sigma f$, as G-covers. Similarly, let Γ^{ab} be the subgroup of $\mathrm{Gal}(\bar{K}/K_0)$ consisting of elements σ such that ${}^\sigma f \cong f$ as mere covers. Set

$$K^{\mathrm{in}} := \bar{K}^{\Gamma^{\mathrm{in}}}, \qquad K^{\mathrm{ab}} := \bar{K}^{\Gamma^{\mathrm{ab}}}.$$

The field K^{in} (resp. K^{ab}) is the *field of moduli* of the G-cover f (resp. of the mere cover f), relative to the extension $\bar{K}/K_0$. See [DD97] for more details.

Let K/K_0 be a Galois extension such that f has stable reduction over K, and let $f_R : Y_R \to X_R$ be the stable model. Also, let $\sigma \in \Gamma^{\mathrm{ab}}$. By definition of Γ^{ab}, there exists a σ-linear automorphism $\kappa_{K,\sigma} : Y_K \xrightarrow{\sim} Y_K$ which normalizes the action of G and induces the canonical σ-linear automorphism $X_K \xrightarrow{\sim} X_K$. The automorphism $\kappa_{K,\sigma}$ is unique up to composition with an element of G. It extends to a σ-linear automorphism $\kappa_{R,\sigma} : Y_R \xrightarrow{\sim} Y_R$. Let $\bar{\kappa}_\sigma : \bar{Y} \xrightarrow{\sim} \bar{Y}$ be the restriction of $\kappa_{R,\sigma}$ to the special fiber. This is a k-linear automorphism which normalizes the action of G. It is uniquely determined by $\sigma \in \Gamma^{\mathrm{ab}}$, up to composition with an element of G. We obtain a continuous homomorphism

$$\bar{\kappa}^{\mathrm{ab}} : \Gamma^{\mathrm{ab}} \longrightarrow \mathrm{Aut}_k(\bar{Y})/G,$$

which we shall call the (absolute) *monodromy action*. Note that $\bar{\kappa}_\sigma$ induces an automorphism of $\bar{X}$ which is the identity on the original component. It follows from [DD97] that the set of K^{ab}-models of the mere cover f is in bijection with the set of homomorphisms $\Gamma^{\mathrm{ab}} \to \mathrm{Aut}_k(\bar{Y})$ which lift $\bar{\kappa}^{\mathrm{ab}}$. If a model $f_{K^{\mathrm{ab}}}$ of f is specified, we will sometimes consider $\bar{\kappa}^{\mathrm{ab}}$ as a homomorphism $\Gamma^{\mathrm{ab}} \to \mathrm{Aut}_k(\bar{Y})$.

Let $\mathrm{Aut}_G(\bar{Y})$ denote the group of k-linear automorphisms of $\bar{Y}$ which commute with the G-action, and let C_G denote the center of G. The restriction of $\bar{\kappa}^{\mathrm{ab}}$ to Γ^{in} yields a homomorphism

$$\bar{\kappa}^{\mathrm{in}} : \Gamma^{\mathrm{in}} \longrightarrow \mathrm{Aut}_G(\bar{Y})/C_G,$$

called the inner monodromy action. Similar to what we stated above, K^{in}-models of the G-cover f correspond one-to-one to lifts of $\bar{\kappa}^{\mathrm{in}}$ to a homomorphism $\Gamma^{\mathrm{in}} \to \mathrm{Aut}_G(\bar{Y})$.

Let Γ^{st} denote the subgroup of Γ^{ab} of elements σ such that $\bar{\kappa}^{\mathrm{ab}}_\sigma \in G$, and set $K^{\mathrm{st}} := \bar{K}^{\Gamma^{\mathrm{st}}}$. One shows that K^{st}/K_0 is the smallest extension over which f has stable reduction. Compare with [Ray99], §2.2.

1.2. **Mild reduction.**

1.2.1. Let $f : Y \to X$ be a G-cover as in §1.1. We assume that f has bad reduction and denote by $\bar{f} : \bar{Y} \to \bar{X}$ its stable reduction. If η is a point on $\bar{Y}$ (closed or generic), we denote by G_η (resp. I_η) the corresponding decomposition (resp. inertia) group. By definition, G_η is the set of elements of G which stabilize η, and I_η is the normal subgroup of G_η consisting of elements which act trivially on the residue field $k(\eta)$.

Definition 1.2. We say that $f : Y \to \mathbb{P}^1$ has *mild reduction* it has bad reduction and if the following two conditions hold for all points $\eta \in \bar{Y}$. Firstly, the p-part of I_η has at most order p. Secondly, if $I_\eta \neq 1$ then the order of G_η/I_η is prime to p.

If $f : Y \to X$ has bad reduction and p strictly divides the order of the Galois group G, then the reduction of f is automatically mild. From now on, let us assume that $f : Y \to X$ has mild reduction.

1.2.2. Let us denote by $(\bar{X}_v)$ the family of irreducible components of the curve $\bar{X}$.

Definition 1.3. A component $\bar{X}_v$ of $\bar{X}$ which is not the original component and which is connected to the rest of $\bar{X}$ in a single point is called a *tail*. The point where a tail $\bar{X}_v$ meets the rest of $\bar{X}$ is denoted by ∞_v. If a tail $\bar{X}_v$ contains one of the points $\bar{x}_j$, $j \in B_0$, then $\bar{X}_v$ is called *primitive*. Otherwise, the tail $\bar{X}_v$ is called *new*. A component $\bar{X}_v$ which is not a tail is called an *interior component*.

For each component $\bar{X}_v$ of $\bar{X}$, we choose an irreducible component $\bar{Y}_v$ of $\bar{Y}$ above $\bar{X}_v$. We write G_v for the decomposition group of $\bar{Y}_v$ and I_v for the inertia group of $\bar{Y}_v$.

Lemma 1.4 (Raynaud). *Suppose that $f : Y \to \mathbb{P}^1$ has mild reduction. Then the following holds.*

(1) *If $\bar{X}_v$ is a tail, then the inertia group I_v is trivial. In other words, the map $\bar{Y}_v \to \bar{X}_v$ is a Galois cover, with Galois group G_v. If $\bar{X}_v$ is a primitive tail and contains the point $\bar{x}_j$, $j \in B_0$, then the cover $\bar{Y}_v \to \bar{X}_v$ is wildly ramified at ∞_v, tamely ramified at $\bar{x}_j$ and étale everywhere else. If $\bar{X}_v$ is a new tail, then $\bar{Y}_v \to \bar{X}_v$ is wildly ramified at ∞_v and étale everywhere else.*
(2) *If $\bar{X}_v$ is not a tail, then the inertia group I_v is cyclic of order p. The cover $\bar{Y}_v \to \bar{X}_v$ factors through the k-linear Frobenius map $F : \bar{Y}_v \to \bar{Z}_v := \bar{Y}_v^{(p)}$. The resulting map $\bar{Z}_v \to \bar{X}_v$ is a Galois cover, with Galois group $H_v := G_v/I_v$. It is étale outside the set of points which are either singular points of $\bar{X}$ or belong to the set $\{\bar{x}_j \mid j \in B_0\}$.*
(3) *For $j \in B_0$, let n_j be the ramification index of $f : Y \to X$ in x_j. Suppose that n_j is prime to p. Then the specialization $\bar{x}_j$ of x_j lies on a primitive tail $\bar{X}_v$, and the ramification index of $\bar{Y}_v \to \bar{X}_v$ at $\bar{x}_j$ is n_j.*
(4) *With the same notation as in (3), suppose that $n_j = pn'_j$. Then $\bar{x}_j$ lies on a component $\bar{X}_v$ which is not a tail. Moreover, the Galois cover $\bar{Z}_v \to \bar{X}_v$ has ramification index n'_j at $\bar{x}_j$.*

Proof. This is essentially proved in [Ray99], §2 and §3. There are two points where our assumptions differ from the assumptions made in [Ray99]. Firstly, it is assumed in [Ray99] that p exactly divides the order of G. However, one only needs that f has mild reduction. The second point is that (3) becomes false if we replace our definition of the stable reduction with the definition used in [Ray99]. It is proved in [BW00], Lemma 2.1.1 that (3) holds if we define $\bar{f}$ as in Definition 1.1. □

1.3. Deformation data.

1.3.1. A priori, our definition of a deformation datum is independent of the study of stable reduction. So the following notation replaces, during §1.3.1, the notation introduced so far. Let p be a prime number, k an algebraically closed field of characteristic p and X a smooth projective and irreducible curve over k.

Definition 1.5. A *deformation datum* over X is a pair (Z,ω), where $Z\to X$ is a tame (connected) Galois cover and ω is a meromorphic differential form on Z such that the following holds.

(1) The order of the Galois group $H=\mathrm{Gal}(Z/X)$ is prime to p.
(2) We have
$$\beta^*\omega \;=\; \chi(\beta)\cdot\omega, \qquad \text{for all } \beta\in H,$$
where $\chi: H\to\mathbb{F}_p^\times$ is a character with values in the prime field $\mathbb{F}_p$.
(3) The differential ω is either logarithmic (i.e. of the form $\omega=\mathrm{d}u/u$) or exact (i.e. $\omega=\mathrm{d}u$). In the first case, the deformation datum (Z,ω) is called *multiplicative*, in the second case it is called *additive*.

The pair (H,χ) is called the *type* of the deformation datum (Z,ω).

Let $\xi\in Z$ be a closed point and $\tau\in X$ its image. Denote the stabilizer of ξ by $H_\xi\subset H$. We attach to the triple (Z,ω,τ) the following three invariants:

$$m_\tau \;:=\; |H_\xi|, \qquad h_\tau \;:=\; \mathrm{ord}_\xi(\omega)+1, \qquad \sigma_\tau \;:=\; h_\tau/m_\tau. \tag{1}$$

It is clear that m_τ, h_τ and σ_τ depend only on τ but not on ξ, and that $(m_\tau,h_\tau)=(1,1)$ for all but a finite number of points $\tau\in X$. If $(m_\tau,h_\tau)\neq(1,1)$ then τ is called a *critical point* for the deformation datum (Z,ω).

Given a deformation datum (Z,ω), we often use the following or a similar notation. Let $(\tau_j)_{j\in J}$ be the family of critical points for (Z,ω), indexed by the finite set J. For $j\in J$, we write m_j, h_j and σ_j instead of m_{τ_j}, h_{τ_j} and σ_{τ_j}. The data $(m_j;h_j)_{j\in B}$ is called the *signature* of the deformation datum (Z,ω). A straightforward computation shows that

$$\sum_{j\in J}(\sigma_j-1) \;=\; 2g_X-2. \tag{2}$$

(where g_X denotes the genus of the curve X).

Remark 1.6. Let $\tilde Z:=Z/\mathrm{Ker}(\chi)$. It is easy to see that ω descents to a differential $\tilde\omega$ on $\tilde Z$ and that $(\tilde Z,\tilde\omega)$ is again a deformation datum, with critical points τ_j and invariants

$$\tilde m_j \;=\; \frac{m_j}{|\mathrm{Ker}(\chi)|}, \qquad \tilde h_j \;=\; \frac{h_j}{|\mathrm{Ker}(\chi)|}, \qquad \tilde\sigma_j \;=\; \frac{\tilde h_j}{\tilde m_j} \;=\; \sigma_j.$$

1.3.2. Let us now return to our original notation, concerning the Galois cover $f:Y\to X$ and its stable reduction $\bar f:\bar Y\to\bar X$. Let $\bar X_v$ be an interior component of $\bar X$ (see Definition 1.3). Following [Hen00] and [Saï01], we define a deformation datum $(\bar Z_v,\omega_v)$ over $\bar X_v$, as follows.

Choose an irreducible component $\bar Y_v$ of $\bar Y$ above $\bar X_v$. By Lemma 1.4 (2), the inertia group I_v of $\bar Y_v$ is cyclic of order p and the quotient group $H_v:=G_v/I_v$ is of prime-to-p order. Moreover, the cover $\bar Y_v\to\bar X_v$ is the composition of a purely inseparable cover $\bar Y_v\to\bar Z_v$ of degree p and an H_v-Galois cover $\bar Z_v\to\bar X_v$. Let $\hat Y_v:=\mathrm{Spec}\,\hat{\mathcal{O}}_{Y_R,v}$ be the completion of Y_R at the generic point of the component

$\bar{Y}_v$; thus $\hat{\mathcal{O}}_{Y_R,v}$ is a complete discrete valuation ring with residue field $k(\bar{Y}_v)$. The action of I_v on Y_R induces an action on $\hat{Y}_v$. Let $\hat{Z}_v := \hat{Y}_v/I_v$ be the quotient; we have $\hat{Z}_v = \operatorname{Spec} A$ for a complete discrete valuation ring A with residue field $k(\bar{Z}_v)$. We may assume that the field K contains the pth roots of unity. Let us choose an isomorphism $I_v \cong \boldsymbol{\mu}_p(K)$ between I_v and the group of pth roots of unity. By [Ray99], §1.2.1, the map $\hat{Y}_v \to \hat{Z}_v$ is a torsor under a finite flat R-group scheme $\mathcal{G}$ of rank p, such that $\mathcal{G} \otimes_R K = \boldsymbol{\mu}_p$ acts on $\hat{Y}_v \otimes_R K$ via the isomorphism $I_v \cong \boldsymbol{\mu}_p(K)$. By Kummer theory, the $\boldsymbol{\mu}_p$-torsor $\hat{Y}_v \otimes_R K \to \hat{Z}_v \otimes_R K$ is given by a Kummer equation $y^p = u$, with $u \in A$.

To define the differential ω_v, we have to distinguish two cases. First, let us assume that $\mathcal{G} \otimes_R k \cong \boldsymbol{\mu}_p$. This happens if and only if we may choose the Kummer equation $y^p = u$ such that the residue $\bar{u}$ of u is not a pth power. In this case, we set $\omega_v := \mathrm{d}\bar{u}/\bar{u}$ and we say that the cover f has *multiplicative reduction* at $\bar{Y}_v$. Otherwise $\mathcal{G} \otimes_R k \cong \boldsymbol{\alpha}_p$ and we say that f has *additive reduction* at $\bar{Y}_v$. In this case we can choose the equation $y^p = u$ such that $u = 1 + \pi^p w$, with $\pi \in R$, $w \in A$ and $0 < v_R(\pi) < v_R(p)/(p-1)$, and such that the residue $\bar{w}$ of w is not a pth power. We set $\omega_v := \mathrm{d}\bar{w}$. It is clear that ω_v depends on the choice of the isomorphism $I_v \cong \boldsymbol{\mu}_p(K)$. Indeed, composing this isomorphism with the automorphism $\zeta_p \mapsto \zeta_p^a$ (where $a \in \mathbb{F}_p^\times$) amounts to multiplying ω_v by a. Let $\chi_v : H_v \to \mathbb{F}_p^\times$ be the character describing the action of H_v on I_v by conjugation. It is clear that $\beta^*\omega_v = \chi_v(\beta)\cdot\omega_v$ for all $\beta \in H_v$.

We have defined the differential ω_v over the generic point of $\bar{Z}_v$. However, we can extend this definition to a neighborhood of any point of $\bar{Z}$ which lies over a smooth point of $\bar{X}$. To be more precise, let $\tau \in \bar{X}_v$ be a closed point which is smooth in $\bar{X}$ and choose points $\eta \in \bar{Y}_v$ and $\xi \in \bar{Z}_v$ above τ. Let $\hat{Y}_\eta$ be the completion of Y_R at η and $\hat{Z}_\xi = \hat{Y}_\eta/I_v$. Then $\hat{Y}_\eta \otimes_R k = \operatorname{Spec}\hat{\mathcal{O}}_{\bar{Y}_v,\eta} \cong k[[y]]$ and $\hat{Z}_\xi = \operatorname{Spec}\hat{\mathcal{O}}_{\bar{Z}_v,\xi} \cong k[[z]]$. As above, the action of I_v on $\hat{Y}_\eta$ gives rise to a differential ω_ξ of the field $\operatorname{Frac}(\hat{\mathcal{O}}_{\bar{Z}_v,\xi}) \cong k((z))$. One checks easily that ω_ξ is the pullback of ω_v to the completion of $\bar{Z}_v$ at ξ. Now the local definition of ω_ξ shows the following. The differential ω_v does not vanish at ξ. Moreover, if ω_v has a pole at ξ, then this pole is simple, ω_ξ is logarithmic and the point $\tau \in \bar{X}_v$ is the specialization of a branch point of f with ramification index divisible by p. We summarize the discussion in the next proposition.

Proposition 1.7. *Over each interior component $\bar{X}_v$ we obtain a deformation datum $(\bar{Z}_v, \omega_v)$, with the following properties. If $\tau \in \bar{X}_v$ is a critical point for $(\bar{Z}_v, \omega_v)$ then one of the following two statements holds.*

(1) *The point $\tau \in \bar{X}_v$ is a singular point of $\bar{X}$.*
(2) *The point τ is the specialization of a point $x_j \in S_0$ at which the cover f has ramification of order divisible by p.*

In Case (1), we have $\sigma_\tau \notin \{0, 1\}$. In Case (2), we have $\sigma_\tau = 0$ and $(\bar{Z}_v, \omega_v)$ is multiplicative.

1.3.3. By Proposition 1.7, the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ of a G-cover $f : Y \to X$ with mild reduction is naturally equipped with a collection of deformation data $(\bar{Z}_v, \omega_v)$. These deformation data are related among each other by certain compatibility conditions. To formulate these conditions, let τ be a singular point of

$\bar{X}$, and let $\bar{X}_{v_1}, \bar{X}_{v_2}$ be the two components of $\bar{X}$ which intersect in τ. For $i = 1, 2$, we define a pair (m_i, h_i) as follows.

- If $\bar{X}_{v_i}$ is an interior component, we define (m_i, h_i) as the pair of invariants attached to $(\bar{Z}_{v_i}, \omega_{v_i}, \tau)$.
- Suppose that $\bar{X}_{v_i} = \bar{X}_j$ is a tail of $\bar{X}$. Let $\eta \in \bar{Y}_j$ be a point lying above τ and let $I_\eta \subset G$ be the inertia group at η. We define m_i as the order of the maximal prime-to-p quotient of I_η. Furthermore, we let h_i be the conductor of I_η, i.e. the unique integer such that $|I_{\eta,h_i}| = p$ and $I_{\eta,h_i+1} = \{1\}$. (Here $(I_{\eta,\nu})_{\nu \geq 0}$ denotes the filtration of higher ramification groups.)

Proposition 1.8. *We have $m_1 = m_2$ and $h_1 = -h_2$.*

Proof. The equality $m_1 = m_2$ is trivial. Let $\hat{Y} := \operatorname{Spec} \hat{\mathcal{O}}_{Y_R,\eta}$ be the completion of Y_R at η. The R-scheme $\hat{Y}$ is a formal annulus and is acted upon by I_η. The invariants h_1 and h_2 can be defined purely in terms of this local action, and the equality $h_1 = -h_2$ is proved in [Hen00], Proposition 1.10. (In [Hen00], the invariant h is called $-m$.) □

Remark 1.9. It seems plausible that the existence of a set of deformation data satisfying these compatibility conditions gives a necessary and sufficient condition for a map $\bar{f} : \bar{Y} \to \bar{X}$ between semistable curves to occur as the stable reduction of a Galois cover between smooth curves in characteristic 0. That this is true for three point covers is proved in §4. In the case of p-cyclic covers, results in this direction are obtained in [Hen00] and [Saï01].

1.4. **The vanishing cycle formula.** The vanishing cycle formula, proved by Raynaud in [Ray99], is the key formula which controls the (local) invariants describing the bad reduction occuring over the tails by a global expression. In this subsection we give another proof. The main observation is that (2) gives a 'local' vanishing cycle formula for each interior component of $\bar{X}$. The global vanishing cycle formula is then easily obtained from its local version using the compatibility relation of Proposition 1.8 and induction over the tree of components of $\bar{X}$. We use this occasion to set up some more notation which may seem at bit technical at first but will pay off in §2.1.

1.4.1. Recall that an *unordered graph* is given by a quintupel $(V, E, s, t, ^-)$. Here V is the set of *vertices*, E is the set of *(ordered) edges*, $s, t : E \to V$ are the *source* and the *target* map, and $E \xrightarrow{\sim} E$, $e \mapsto \bar{e}$, is a fixed point free involution of E such that $s(\bar{e}) = t(e)$ for all $e \in E$.

Let $T' = (V', E', s, t, ^-)$ be the dual graph of the semistable curve $\bar{X}$. Thus, a vertex $v \in V'$ corresponds to an irreducible component $\bar{X}_v$ of $\bar{X}$, and an edge $e \in E'$ corresponds to a triple $(\bar{x}, v_1, v_2)$ such that the (distinct) components $\bar{X}_{v_1}$ and $\bar{X}_{v_2}$ intersect in the (singular) point $\bar{x}$. Let $B_{\rm wild} \subset B_0$ be the set of indices j such that ramification index of the Galois cover $f : Y \to X$ at x_j is divisible by p. We define $T = (V, E, s, t, ^-)$ as the unordered graph obtained from T' by adding a vertex for each point x_j with $j \in B_{\rm wild}$. More precisely, $V := V' \cup B_{\rm wild}$ and a vertex $j \in B_{\rm wild}$ is connected to the vertex $v \in V'$ by an edge $e \in E$ if and only if the point $x_j \in S_0$ specializes to the component $\bar{X}_v$. Note that T is a connected tree. Also, to each edge e of T corresponds a closed point $\bar{x}_e \in \bar{X}$, as follows. If the edge e joins two vertices $v_2, v_2 \in V'$ then $\bar{x}_e$ is the point where $\bar{X}_{v_1}$ intersects

$\bar{X}_{v_2}$. If e joins two vertices $j \in B_{\rm wild}$ and $v \in V'$ then $\bar{x}_e := \bar{x}_j \in \bar{X}_v$ is the point to which x_j specializes.

We consider the vertex v_0 corresponding to the original component $\bar{X}_{v_0}$ as the *root* of the tree T. Also, we denote by B the set of *leaves* of T, i.e. the set of vertices $v \in V - \{v_0\}$ which are adjacent to a unique vertex v'. Thus, a vertex $v \in V_{\rm int} := V - B$ corresponds to an interior component $\bar{X}_v$. By definition, $B_{\rm wild} \subset B$. An element j in $B_{\rm prim} = B_0 - B_{\rm wild}$ corresponds to a ramification point x_j of f of order prime-to-p. By Lemma 1.4, x_j specializes to a point $\bar{x}_j$ on a primitive tail $\bar{X}_v$. The resulting map $B_{\rm prim} \to B$ is clearly injective. Therefore, we may and will consider $B_0 = B_{\rm prim} \cup B_{\rm wild}$ as a subset of B. Then $B_{\rm new} := B - B_0$ is the set of leaves corresponding to the new tails of $\bar{X}$.

1.4.2. We shall now attach integers m_e, h_e and a rational number σ_e to each edge e of the tree T. Suppose first that $v := s(e)$ is an interior vertex. Let $(\bar{Z}_v, \omega_v)$ be the deformation datum over $\bar{X}_v$, as defined in §1.3.2. Note that $\bar{x}_e \in \bar{X}_v$ is a critical point for $(\bar{Z}_v, \omega_v)$, by Proposition 1.7. We define the numbers m_e, h_e and σ_e as the invariants attached to the triple $(\bar{Z}_v, \omega_v, \bar{x}_e)$. If $v := s(e)$ is a leaf of T, then we set $m_e := m_{\bar{e}}$, $h_e := -h_{\bar{e}}$ and $\sigma_e := -\sigma_{\bar{e}}$. Equation (2) and Proposition 1.8 imply the following.

Proposition 1.10. *For all interior vertices v of T we have*

$$\sum_{s(e)=v} (\sigma_e - 1) \;=\; 2g_v - 2 \tag{3}$$

(where g_v is the genus of $\bar{X}_v$). Furthermore, for all edges e of T we have

$$\sigma_e + \sigma_{\bar{e}} \;=\; 0. \tag{4}$$

For each leaf $j \in B$, let e_j be the unique edge with target j. We set $m_j := m_{e_j}$, $h_j := m_{e_j}$ and $\sigma_j := \sigma_{e_j}$. If $j \notin B_{\rm wild}$ then m_j is the prime-to-p part of the ramification index of the Galois cover $\bar{Y}_j \to \bar{X}_j$ at $\bar{x}_e = \infty_j$ and h_j is the conductor of the wild ramification in that point, see Proposition 1.8. Therefore, the rational number $\sigma_j = h_j/m_j$ is the ramification invariant introduced in [Ray99], §1.1. It is equal to the jump in the filtration of higher ramification groups for the branch point $\infty_j \in \bar{X}_j$, with respect to the upper numbering, see [Ser68]. Note that we have defined $\sigma_j = 0$ for $j \in B_{\rm wild}$. We shall call the tuple $(\sigma_j)_{j\in B}$ the *signature* of the stable reduction $\bar{f}: \bar{Y} \to \bar{X}$. It is shown in [Ray99] that $\sigma_j > 1$ for $j \in B_{\rm new}$. Moreover, we have the following.

Corollary 1.11 (The vanishing cycle formula)**.** *The signature $(\sigma_j)_{j\in B}$ of $\bar{f}$ verifies*

$$\sum_{j\in B_{\rm prim}} \sigma_j \;+\; \sum_{j\in B_{\rm new}} (\sigma_j - 1) \;=\; 2g_X - 2 + |B_0|. \tag{5}$$

Proof. This is proved in [Ray99], §3.4. Here is another proof, using Proposition 1.10. Let $T_0 \subset T_1 \subset \ldots \subset T_n$ be a chain of subtrees of T such that $T_0 = \{v_0\}$, $T_i = T_{i-1} \cup \{v_i\}$ and $T_n = T - B$. Let E_i denote the set of edges of T such that $s(e) \in T_i$ and $t(e) \notin T_i$. We claim that $\sum_{e\in E_i}(\sigma_e - 1) = 2g_X - 2$ holds for all $0 \leq i \leq n$. Indeed, Proposition 1.10 shows that the claim is true for $i = 0$ and then, by induction, for all i. For $i = n$, the claim is equivalent to (5). $\square$

2. The case of three point covers

In this section we apply the results of the last section to the case of three point covers. In §2.1 we prove our first main result, which essentially states that the stable reduction of a three point cover with mild reduction is as simple as possible. The basic idea underlying the proof of this result is that $\bar{f}:\bar{Y}\to\bar{X}$, together with the associated deformation data, should be a 'rigid' objects, i.e. should not allow any nontrivial deformation. This imposes strong restrictions on the deformation data $(\bar{Z}_v,\omega_v)$ attached to $\bar{f}$. We verify these restrictions coming from rigidity only indirectly, using the local and global vanishing cycle formulas. A key lemma from [Wew] shows that $\bar{f}$ can support no additive deformation datum, and only a single multiplicative one. To make this proof work one has to exclude two exceptional cases, see §2.1.1. This is somewhat annoying but not serious, because all our final results apply to these exceptional cases as well.

In §2.2 we define the notion of a *special G-deformation datum*. To such an object we can associate a map $\bar{f}:\bar{Y}\to\bar{X}$ which looks as if it were the stable reduction of a three point G-cover, and which we call a *special G-map*.

2.1. The structure of $\bar{f}$.

2.1.1. We continue with the notation introduced in §1. In addition, we assume that $X_0=\mathbb{P}^1_{R_0}$ and $S_0=\{0,1,\infty\}$. Note that the right hand side of (5) is equal to 1.

Definition 2.1. We say that the stable reduction of f is *exceptional* if one of the following two cases occurs.

(1) We have $B_{\rm prim}=\{j_0\}$, $B_{\rm new}=\emptyset$ and $\sigma_{j_0}=1$.
(2) We have $B_{\rm prim}=\emptyset$, $B_{\rm new}=\{j_0\}$ and $\sigma_{j_0}=2$.

Proposition 2.2. *Let $f:Y\to X=\mathbb{P}^1_K$ be a Galois cover ramified at $\{0,1,\infty\}$, with exceptional reduction. Then the curve $\bar{X}$ consists only of the original component $\bar{X}_0$ and the unique tail $\bar{X}_{j_0}$.*

Proof. We will prove the proposition only for Case (2) of Definition 2.1 and leave Case (1) to the reader. Let T' be the dual graph of the curve $\bar{X}$, considered as a subtree of T. By assumption, T' has a unique leaf j_0. Therefore, T' is simply a chain going from the root v_0 to j_0. Let v be the unique vertex of T' adjacent to the leaf j_0 and let e be the edge with source v and target j_0. This means that the interior component $\bar{X}_v$ intersects the unique (new) tail $\bar{X}_{j_0}$ in the point $\bar{x}_e$. We have to show that $v=v_0$, i.e. that $\bar{X}_v$ is the original component $\bar{X}_0$. So, let us assume that $v\neq v_0$. Then there exists a unique edge $e'\neq e$ with source v, corresponding to a point $\bar{x}_{e'}$ where $\bar{X}_v$ intersects another component $\bar{X}_{v'}$. Let us first suppose that there exists another edge e'' (of the tree T) with source v besides e and e'. Then $t(e'')\in B_{\rm wild}$, i.e. the point $\bar{x}_{e''}$ is the specialization of a branch point of f with ramification index divisible by p. Since $\bar{X}_v$ is not the original component, there can be at most one branch point specializing to $\bar{X}_v$. Therefore, e, e', e'' are the only edges with source v. Since $\sigma_e=2$ and $\sigma_{e''}=0$, (3) shows that $\sigma_{e'}=-1$. This means that the differential ω_v has a pole of order ≥ 2 at each point of $\bar{Z}_v$ lying above $\bar{x}_{e'}$. This shows that ω_v cannot be logarithmic. On the other hand, ω_v has a simple pole at each point above $\bar{x}_{e''}$, so ω_v cannot be exact. This is a contradiction. We may therefore assume that e and e' are the only edges with source v. It follows that the cover $\bar{Z}_v\to\bar{X}_v$ is ramified at most at two points; in particular, the genus of $\bar{Z}_v$ and therefore the genus of $\bar{Y}_v$ as well, is zero. Moreover, $\bar{Y}_v$ intersects the rest of $\bar{Y}$ in two points and no ramification point specializes to $\bar{Y}_v$. Thus, the 'three point condition' does not hold for $\bar{Y}$, in contradiction with the definition of the stable model in §1.1. The proposition follows. □

Remark 2.3. It may happen that the Galois cover $f : Y \to X$ has bad reduction but that the curve Y has good reduction. If this is the case then the stable reduction of f is exceptional.

2.1.2. For the rest of §2.1 we assume that the reduction of f is not exceptional. The vanishing cycle formula (5), together with the inequalities $\sigma_j > 0$ for $j \in B_{\rm prim}$ and $\sigma_j > 1$ for $j \in B_{\rm new}$, immediately imply the inequalities

$$0 < \sigma_j < 1 \quad \text{for } j \in B_{\rm prim}, \qquad 1 < \sigma_j < 2 \quad \text{for } j \in B_{\rm new}. \tag{6}$$

Moreover, $|B_{\rm prim} \cup B_{\rm new}| \geq 2$. For an edge e of T we define $\nu_e := \lfloor \sigma_e \rfloor$ to be the largest integer less than or equal to the rational number σ_e. Note that we have $\sigma_e = \nu_e + \langle \sigma_e \rangle$, where $\langle \sigma_e \rangle \geq 0$ denotes the fractional part of σ_e. For a leaf $j \in B$, we write $\nu_j := \lfloor \sigma_j \rfloor$. By (6) we have $\nu_j = 0$ for $j \in B_0$ and $\nu_j = 1$ for $j \in B_{\rm new}$. Since $|B_0| = 3$, the vanishing cycle formula (5) implies

$$\sum_{j \in B} \langle \sigma_j \rangle \;=\; 1. \tag{7}$$

Recall that T' denotes the dual graph of the curve $\bar{X}$, considered as subtree of T. Let E' denote the set of edges of T'. Moreover, let $E'_{\rm int} \subset E'$ be the set of edges connecting two interior vertices.

Proposition 2.4.

(1) *For all edges $e \in E'$ we have $\nu_e + \nu_{\bar{e}} = -1$.*
(2) *For all interior vertices $v \in V_{\rm int}$ we have*

$$\sum_{s(e)=v} \langle \sigma_e \rangle \;=\; 1 \qquad \textit{and} \qquad \sum_{s(e)=v} (\nu_e - 1) \;=\; -3.$$

Proof. For $e \in E'$ define

$$d_e \;:=\; \langle \sigma_e \rangle + \langle \sigma_{\bar{e}} \rangle - 1 \;=\; -\nu_e - \nu_{\bar{e}} - 1.$$

The last equality follows from (4). Clearly, $d_e = d_{\bar{e}}$, $d_e \in \{-1, 0\}$ and $d_e = -1$ if and only if $\sigma_e \in \mathbb{Z}$. For $v \in V_{\rm int}$ define

$$c_v \;:=\; -1 + \sum_{s(e)=v} \langle \sigma_e \rangle \;=\; -3 + \sum_{s(e)=v} (\nu_e - 1).$$

The last equality is a consequence of (3). It is easy to see that $c_v \in \{-1, 0, 1, \ldots\}$ and that $c_v = -1$ if and only if $d_e = -1$ for all $e \in E'$ such that $s(e) = v$. We have to show that $d_e = c_v = 0$.

It is convenient to introduce the following terminology. A vertex v of T is said to *precede* an edge e (notation: $v \prec e$) if v lies in the connected component of $T - \{e\}$ which contains $s(e)$. An edge e' is said to precede another edge e (notation: $e' \prec e$) if $t(e') \prec e$. Note that '$\prec$' is a partial order on the set E of edges.

Suppose that there exists an edge $e \in E'$ with $d_e \neq 0$. We may suppose that e is minimal with this property, i.e. $d_{e'} = 0$ for all edges $e' \in E'$ which precede e. We claim that $c_v \geq 0$ for all $v \in V_{\rm int}$ which preceed e. Indeed, since $\bar{X}$ has at least two tails, there exist at least two (distinct) edges in E' with target v. At least one of them precedes e. Hence there exists an edge $e' \in E'$ with $d_{e'} = 0$ and $t(e) = v$. This proves the claim. As a consequence, we obtain the inequality

$$
\begin{aligned}
0 \le \sum_{\substack{v \in V_{\rm int} \\ v \prec e}} c_v &= \langle\sigma_e\rangle + \Big(\sum_{\substack{e' \in E'_{\rm int} \\ e' \prec e}} d_{e'}\Big) + \Big(\sum_{\substack{j \in B \\ j \prec e}} \langle\sigma_j\rangle\Big) - 1 \\
&= \langle\sigma_e\rangle + \Big(\sum_{\substack{j \in B \\ j \prec e}} \langle\sigma_j\rangle\Big) - 1.
\end{aligned} \tag{8}
$$

Since the curve $\bar{X}$ has at least two tails, there exists $j \in B_{\rm prim} \cup B_{\rm new}$ such that j does not precede e. Moreover $\sigma_j > 0$. Therefore, (8) implies $\langle\sigma_e\rangle > 0$. But this is a contradiction to the assumption $d_e \neq 0$. We conclude that $d_e = 0$ for all $e \in E'$. It remains to prove $c_v = 0$ for all $v \in V_{\rm int}$. The same argument we used above together with $d_e = 0$ for all $e \in E'$ shows that $c_v \ge 0$ for all $v \in V_{\rm int}$. Now the proposition follows from

$$\sum_{v \in V_{\rm int}} c_v = \Big(\sum_{e \in E'_{\rm int}} d_e\Big) + \Big(\sum_{j \in B} \langle\sigma_j\rangle\Big) - 1 = 0.$$

□

Lemma 2.5. *Suppose that $\bar{f}$ is not exceptional, and let e be an edge of T. Then*

$$-2 \le \nu_e \le 1.$$

Moreover, $\nu_e \ge 0$ if and only if the root v_0 precedes e. In particular, if v is an interior vertex unequal to v_0 then there exists a unique edge e with source v such that $\nu_e < 0$.

Proof. (Compare to [Wew], Lemma 2.6.) Using Proposition 2.4 and induction in the tree T one shows that

$$\nu_e = 1 - |\{ j \in B_0 \mid j > e \}|.$$

Note that $|B_0| = 3$ and that the root v_0 is the median of the set of leaves B_0. The lemma follows immediately. □

2.1.3. We can now prove our first main result. It applies in particular if p strictly divides the order of G.

Theorem 2.6. *Let $f : Y \to X = \mathbb{P}^1_{\bar{K}}$ be a Galois cover ramified at $\{0, 1, \infty\}$, with bad reduction. If f has mild reduction (Definition 1.2) then the curve $\bar{X}$ consists only of the original component $\bar{X}_0$ and the tails $\bar{X}_j$, $j \in B_{\rm prim} \cup B_{\rm new}$.*

Proof. If $B_{\rm wild} = \emptyset$ then this follows from [Wew], Theorem 2.1, using Raynaud's construction of the *auxiliary cover*. The proof we give below follows the same lines, but avoids the construction of the auxiliary cover.

We prove the theorem by contradiction. Assume that there exists an interior vertex v of T other than the root v_0. By Proposition 2.2, the reduction of f is not exceptional. We may assume that v is adjacent to the root v_0, i.e. there exists an edge e_0 with source v and target v_0. This means that the component $\bar{X}_v$ intersects the original component $\bar{X}_0$ in the point $\bar{x}_{e_0}$. Let $(\bar{Z}_v, \omega_v)$ be the deformation datum over $\bar{X}_v$ as defined in §1.3.2. For the rest of this proof, e runs over the set of edges with source v. By Proposition 2.4 we have $\sum_e \langle\sigma_e\rangle = 1$ and $\sum_e(\nu_e - 1) = -3$. Also, Lemma 2.5 implies that $-2 \leq \nu_e \leq 1$ for all e and $\nu_e < 0$ if and only if $e = e_0$. In this situation, [Wew], Lemma 2.8 says that ω_v cannot be exact (actually, in [Wew] one assumes that the type of $(\bar{Z}_v, \omega_v)$ has an injective character χ_v; the general case follows from Remark 1.6). Therefore, ω_v is logarithmic. On the other hand, since $\nu_{e_0} < 0$, the differential has a pole of order ≥ 2 at each point of $\bar{Z}_v$ lying above $\bar{x}_{e_0}$. Therefore, ω_v cannot be logarithmic. This gives the desired contradiction and proves the theorem. □

2.2. **Special G-deformation data.** In this section we define special G-deformation data. To a special G-deformation datum we then associate a special G-map $\bar{f} : \bar{Y} \to \bar{X}$. Briefly, a special G-map is a map between semistable curves over k which looks like the stable reduction of a three point G-cover with mild reduction.

2.2.1. Let k be an algebraically closed field of characteristic $p > 0$ and $\bar{X}$ a smooth projective curve over k of genus 0. Let $(\bar{Z}, \omega)$ be a deformation datum over $\bar{X}$, of type (H, χ). As usual, we denote by $(\tau_j)_{j \in B}$ the set of critical points for $(\bar{Z}, \omega)$. Also, for $j \in B$ we denote by m_j, h_j and σ_j the invariants attached to $(\bar{Z}, \omega, \tau_j)$. See §1.3.

Definition 2.7. The deformation datum $(\bar{Z}, \omega)$ is called *special* if for all $j \in B$ we have $\sigma_j < 2$ and $\sigma_j \neq 1$ and if there exists a subset $B_0 \subset B$ with exacly three elements such that $\sigma_j < 1$ if and only if $j \in B_0$.

Remark 2.8. In case the character χ is injective and $\sigma_j > 0$ we recover the definition of a special deformation datum given in [Wew].

In the following we assume that $(\bar{Z}, \omega)$ is special. Let $B_{\rm wild}$ (resp. $B_{\rm prim}$, resp. $B_{\rm new}$) denote the set of all indices $j \in B$ such that $\sigma_j = 0$ (resp. $0 < \sigma_j < 1$, resp. $1 < \sigma_j < 2$). Then $B_0 = B_{\rm wild} \cup B_{\rm prim}$, and it follows from (3) that

$$\sum_{j \in B} \langle \sigma_j \rangle \;=\; 1. \tag{9}$$

We call the special deformation datum $(\bar{Z}, \omega)$ *normalized* if $\bar{X} = \mathbb{P}^1_k$ and $\{\tau_j \mid j \in B_0\} = \{0, 1, \infty\}$.

Their connection to three point covers suggests that special deformation data are 'rigid' objects, i.e. there should not exist any nontrivial families of special deformation data. This is indeed true; a precise statement is formulated and proved in [Wew02], §5.4. See also §3 below.

2.2.2. In this subsection, $\bar{X}$ will denote a smooth projective curve over k of genus 0, together with a distinguished closed point $\bar{\infty} \in \bar{X}$. We also fix a finite group G.

Definition 2.9. A *G-tail cover* is a (not necessarily connected) G-cover $\bar{f} : \bar{Y} \to \bar{X}$ such that the following holds.

(1) The cover $\bar{f}$ is wildly ramified above $\bar{\infty}$, of order pm with $(p, m) = 1$.
(2) The restriction of $\bar{f}$ to $\bar{X} - \{\bar{\infty}\}$ is tame and branched at most at one point.

A *pointed G-tail cover* is a G-tail cover together with a point $\eta \in \bar{Y}$ above $\bar{\infty}$ and a subgroup H of the inertia group I_η such that $I_\eta = I_{\eta,1} \rtimes H$. An (outer) automorphism of a pointed G-tail cover is an automorphism $\gamma : \bar{Y} \xrightarrow{\sim} \bar{Y}$ which normalizes the action of G (and hence induces an automorphism of $\bar{X}$), centralizes the action of $H \subset G$ and fixes the point η. Such an automorphism is called *inner* if it centralizes the action of G.

Let $(\bar f:\bar Y\to\bar X,\eta,H)$ be a pointed G-tail cover. By Condition (1), $I_{\eta,1}$ is cyclic of order p and H cyclic of order prime-to-p. Let m denote the order of H and h the conductor of $I_{\eta,1}$. The pair (m_j,h_j) is called the *ramification type* and the rational number $\sigma:=h/m$ the *ramification invariant* of $\bar f$. Note that $h>0$ and $(h,p)=1$. Moreover, $m|(p-1)h$, by the Hasse–Arf Theorem (see [Ser68], IV §3).

Definition 2.10. A tail cover $\bar f:\bar Y\to\bar X$ is called *special* if the following two conditions hold.

(1) We have $\sigma<2$ and $\sigma\neq 1$.
(2) If $\sigma>1$ then the restriction of $\bar f$ to $\bar X-\{\bar\infty\}$ is étale.

If $\sigma<1$ then we say that the tail cover $\bar f$ is *primitive*, otherwise we say that it is *new*.

By [Ray99], Proposition 1.1.6, the converse of Condition (2) holds automatically: if $\sigma<1$ then there exists a unique point $\bar 0\in\bar X-\{\bar\infty\}$ above which $\bar f$ is tamely ramified. The reason is this: if $\bar f$ were étale over $\bar X-\{\bar\infty\}$ and $\sigma<1$, then the Riemann–Hurwitz formula would produce a negative number for the genus of $\bar Y$.

It is clear from the results of §2.1 that the tail covers obtained from the stable reduction of a three point Galois cover are special (provided that the reduction is not exceptional). We can therefore expect that a special tail cover $\bar f:\bar Y\to\bar X$ is 'rigid'. In fact, using the methods of [BM00] one can show that the curve $\bar Y$ does not admit any nontrivial G-equivariant deformations. What is more important for us is that all special G-tail covers with a given ramification type (m,h) are 'locally isomorphic', in the following sense.

Definition 2.11. Let $(\bar f_1:\bar Y_1\to\bar X,\eta_1,H)$ and $(\bar f_2:\bar Y_2\to\bar X,\eta_2,H)$ be two pointed G-tail covers with the same subgroup $H\subset G$ and the same ramification type (m,h). Let $\bar X_{\bar\infty}$ denote the generic point of the completion of $\bar X$ at $\bar\infty$ and set $\bar Y_{i,\bar\infty}:=\bar Y_i\times_{\bar X}\bar X_{\bar\infty}$. A *local isomorphism* between $\bar f_1$ and $\bar f_2$ is a G-equivariant isomorphism

$$\varphi_{\bar\infty}:\bar Y_{1,\bar\infty}\ \stackrel{\sim}{\longrightarrow}\ \bar Y_{2,\bar\infty}$$

with the following properties.

(1) The automorphism $\psi_{\bar\infty}:\bar X_{\bar\infty}\stackrel{\sim}{\to}\bar X_{\bar\infty}$ induced by $\varphi_{\bar\infty}$ extends to a global automorphism $\psi:\bar X\stackrel{\sim}{\to}\bar X$ (which necessarily fixes $\bar\infty$).
(2) We have $\varphi_{\bar\infty}(\eta_1)=\eta_2$. Moreover, if $\bar f_1$ is ramified at $\bar 0_1\in\bar X-\{\bar\infty\}$ then $\bar f_2$ is also ramified at some point $\bar 0_2\in\bar X-\{\bar\infty\}$, and the automorphism ψ in (1) can be chosen in such a way that $\psi(\bar 0_1)=\bar 0_2$.

Lemma 2.12. *All special pointed G-tail covers with given ramification type (m,h) and subgroup $H\subset G$ are locally isomorphic.*

Proof. This is a special case of the results of [Pri02], §4. For the convenience of the reader, we sketch a proof.

Let $(\bar f:\bar Y\to\bar X,\eta,H)$ be a special pointed G-tail cover, of ramification type (m,h). We will show that $\bar f:\bar Y\to\bar X$ is uniquely determined, up to local isomorphism, by the datum (G,H,m,h). First of all, by the definition of local isomorphism it suffices to look at tail covers $\bar f$ which are totally ramified above $\bar\infty$. In other words, we may assume that $G=I\rtimes H$, where $I\subset G$ is cyclic of order p and H is cyclic of order m. Second, we shall assume that the tail cover $\bar f$ is new. Hence we can write $h=m+a$, with $0<a<m$. Note that $m|a(p-1)$ and that a is prime to p. We will indicate later how to modify the proof in the case that $\bar f$ is primitive.

Let x be a global coordinate for $\bar{X} \cong \mathbb{P}^1_k$ such that $\bar{X}_\infty = \operatorname{Spec} k((x^{-1}))$. By elementary Galois theory, the localized cover $\bar{Y}_\infty \to \bar{X}_\infty$ is given by two equations

$$z^m \;=\; x, \qquad y^p - y \;=\; z^a\,(b_0 x + b_1 + b_2 x^{-1} + \ldots), \tag{10}$$

with $b_i \in k$ and $b_0 \neq 0$. The action of G on $\bar{Y}_\infty$ is given as follows. We may choose a generator α of $I \cong \mathbb{Z}/p$ such that $\alpha^* z = z$ and $\alpha^* y = y + 1$. Furthermore, for any $\beta \in H$ we have $\beta^* z = \psi(\beta)\cdot z$ and $\beta^* y = \chi(\beta)\cdot y$, where $\psi : H \to \bar{\mathbb{F}}_p^\times$ is a character of order m such that $\chi = \psi^a$. See e.g. [Pri02], Lemma 1.4.1.

A local isomorphism amounts to a change of coordinates

$$x \;\mapsto\; c(x+d), \qquad z \;\mapsto\; c^{1/m} z(1 + dx^{-1})^{1/m}, \qquad y \;\mapsto\; y + g, \tag{11}$$

where $c, d \in k$, $c \neq 0$ and $g = z^a(e_1 x^{-1} + e_2 x^{-2} + \ldots) \in k[[z^{-1}]]$. An easy computation shows that for a suitable choice of the constants c and d, and after transforming the equations (10) by the change of coordinate (11), we may assume that $b_0 = 1$ and $b_1 = 0$. It is also easy to see that we may choose $e_1, e_2, \ldots$ in such a way that

$$g^p - g \;=\; -z^a\,(b_2 x^{-1} + b_3 x^{-2} + \ldots) \tag{12}$$

(here we use that $m | a(p-1)$). With this choice, equations (10) becomes

$$z^m \;=\; x, \qquad y^p - y \;=\; z^h,$$

which visibly depends only on the ramification type (m, h). This proves the lemma for a new tail cover $\bar{f}$. The proof for a primitive tail cover is similar. The main difference is that we are only allowed to do a homothety $x \mapsto cx$ instead of a general affine transformation $x \mapsto c(x+d)$. However, since $0 < h < m$ the proof goes through. □

2.2.3. As before, k denotes an algebraically closed field and G a finite group. Suppose we are given the following data.

- A subgroup $G_0 \subset G$, of the form $G_0 = I_0 \rtimes H_0$, where $I_0 \subset G_0$ is cyclic of order p and $H_0 \subset G_0$ is of prime-to-p order. Let $\chi_0 : H_0 \to \mathbb{F}_p^\times$ denote the character defined by the rule $\beta\alpha\beta^{-1} = \alpha^{\chi_0(\beta)}$, where $\alpha \in I_0$, $\beta \in H_0$.
- A normalized special deformation datum $(\bar{Z}_0, \omega_0)$ over $\bar{X}_0 := \mathbb{P}^1_k$ of type (H_0, χ_0). Let $(\tau_j)_{j\in B}$ denote the set of critical points, m_j, h_j and σ_j the invariants attached to $(\bar{Z}_0, \omega_0, \tau_j)$.
- For all $j \in B - B_{\text{wild}}$ a point $\xi_j \in \bar{Z}_0$ above $\tau_j \in \bar{X}_0$.
- For all $j \in B - B_{\text{wild}}$ a subgroup $G_j \subset G$ and a connected pointed G_j-tail cover $(\bar{f}_j : \bar{Y}_j \to \bar{X}_j, \eta_j, H_j)$.

By abuse of notation, we shall write $(\bar{Z}_0, \omega_0; \bar{f}_j)$ to denote all of the data introduced above.

Definition 2.13. The datum $(\bar{Z}_0, \omega_0; \bar{f}_j)$ is called a *special G-deformation datum* if the following conditions hold.

1. The group G is generated by its subgroups G_0 and G_j.
2. The tail cover $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ has ramification of type (m_j, h_j). Moreover,

$$I_{\eta_j} \;=\; I_0 \rtimes H_j, \quad \text{and} \quad H_j \;=\; H_0 \cap I_{\eta_j}.$$

It is clear from the results of §2.1 that the stable reduction of a three point cover with mild reduction gives rise to a special G-deformation datum. To make this statement more precise, we introduce the following terminology.

Definition 2.14. A *stable G-map* over a scheme S is a homomorphism $f : Y \to X$ of marked semistable curves over S together with an S-linear action of G on Y such that Y is stably marked and f commutes with the G-action. Two stable G-maps $f_i : Y_i \to X_i$ are said to be isomorphic if there exist isomorphisms $\varphi : Y_1 \xrightarrow{\sim} Y_2$ and $\psi : X_1 \to X_2$ of marked curves such that $f_2 \circ \varphi = \psi \circ f_1$ and such that φ is G-equivariant.

If $f : Y \to X$ is a G-cover over $\bar{K}$, as in §1.1, then we may consider the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ of f as a stable G-map over k.

Fix a special G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$. We will show that $(\bar{Z}_0, \omega_0; \bar{Y}_j)$ gives rise to a stable G-map $\bar{f} : \bar{Y} \to \bar{X}$ over k. Define

$$\bar{X} \;:=\; (\, \bar{X}_0 \;\coprod\; \coprod_j \bar{X}_j \,)/\sim.$$

Here the equivalence relation '$\sim$' identifies the critical point $\tau_j \in \bar{X}_0$ with the distinguished point $\bar{\infty}_j \in \bar{X}_j$. The topological space $\bar{X}$ carries a unique structure of a semistable k-curve such that $\bar{X}_0$ and $\bar{X}_j$ are the irreducible components of $\bar{X}$. We define marked points $\bar{x}_j$ on $\bar{X}$ (indexed by the set $B_0 = B_{\rm prim} \cup B_{\rm wild}$), as follows. For $j \in B_{\rm prim}$ we let $\bar{x}_j \in \bar{X}_j$ be the unique tame branch point of the tail cover $\bar{f}_j$ (see the remark after Definition 2.10). For $j \in B_{\rm wild}$ we set $\bar{x}_j := \tau_j$.

To define the curve $\bar{Y}$, let $\bar{Y}_0$ be the smooth projective curve with function field $k(\bar{Z}_0)^{1/p}$. The action of H_0 on $\bar{Z}_0$ extends uniquely to an action on $\bar{Y}_0$. We let G_0 act on $\bar{Y}_0$ via the quotient map $G_0 \to H_0$. Furthermore, we define

$$\bar{Y} \;:=\; (\,{\rm Ind}_{G_0}^G(\bar{Y}_0) \;\coprod\; \coprod_j {\rm Ind}_{G_j}^G(\bar{Y}_j)\,)/\sim.$$

Here the equivalence relation '$\sim$' is defined as follows: for $\eta' \in {\rm Ind}_{G_0}^G(\bar{Y}_0)$ and $\eta'' \in {\rm Ind}_{G_j}^G(\bar{Y}_j)$ we have $\eta' \sim \eta''$ if and only if there exists an element $\alpha \in G$ such that $\alpha(\eta') \in \bar{Y}_0$ lies above $\xi_j \in \bar{Z}_0$ and $\alpha(\eta'') = \eta_j \in \bar{Y}_j$. (This is an equivalence relation because of Condition (2) of Definition 2.13.) Again, $\bar{Y}$ carries a canonical structure of a semistable curve over k. It follows from Condition (1) of Definition 2.13 that $\bar{Y}$ is connected. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the unique map which commutes with the natural G-action on $\bar{Y}$ and whose restriction to $\bar{Y}_0$ (resp. to $\bar{Y}_j$) is equal to $\bar{f}_0$ (resp. to $\bar{f}_j$). The marked points of $\bar{Y}$ are, by definition, those which lie over a marked point of $\bar{X}$. Thus, a point on $\bar{Y}$ is marked either if it is the G-translate of a ramification point of a tail cover $\bar{f}_j$ or if it is a G-translate of a point on $\bar{Y}_0$ which lies above a wild critical point. One checks that $\bar{f} : \bar{Y} \to \bar{X}$ is a stable G-map. We call $\bar{X}_0 = \mathbb{P}^1_k$ the original component of $\bar{f}$.

Definition 2.15. A stable G-map $\bar{f} : \bar{Y} \to \bar{X}$ over k is called *special* if there exists a special G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$ such that $\bar{f}$ is isomorphic to the stable G-map constructed above.

We can summarize the results of §1 and §2.1 as follows.

Proposition 2.16. *Let $f : Y \to X = \mathbb{P}^1_{\bar{K}}$ be a G-Galois cover branched at 0, 1 and $\bar{\infty}$, with mild reduction. Then the stable reduction of f is either exceptional or special.*

In §4.1 we will prove the 'converse' of this proposition: every special G-map arises as the stable reduction of a three point cover $f : Y \to X = \mathbb{P}^1_{\bar{K}}$ with mild reduction.

2.2.4. Let us fix a special G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$ and let $\bar{f}: \bar{Y} \to \bar{X}$ be the associated special G-map. Let us denote by $\mathrm{Aut}^0_G(\bar{f})$ the group of automorphisms of the special G-map $\bar{f}$ which induce the identity on the original component $\bar{X}_0$. If $\bar{f}$ is the stable reduction of a three point G-cover $f: Y \to X$ then the image of the inner monodromy action $\bar{\kappa}^{\rm in}$ associated to f is contained in $\mathrm{Aut}^0_G(\bar{f})/C_G$. We are interested in a description of $\mathrm{Aut}^0_G(\bar{f})$ in terms of the special deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$.

Lemma 2.17. *There is a natural bijection between the group $\mathrm{Aut}^0_G(\bar{f})$ and the set of tuples $(\gamma_0; \gamma_j)$, where $\gamma_0 \in G$ and $\gamma_j : \bar{Y}_j \xrightarrow{\sim} \bar{Y}_j$ is an (outer) automorphism of the pointed G_j-tail cover $(\bar{f}_j, \eta_j, H_j)$ (see Definition 2.9) such that the following holds.*

1. *The element $\gamma_0 \in G$ normalizes I_0 and centralizes H_0.*
2. *The equation*
$$\gamma_0^{-1} \circ \alpha \circ \gamma_0 \;=\; \gamma_j \circ \alpha \circ \gamma_j^{-1}$$
holds for all $\alpha \in G_j$ and $j \notin B_{\rm wild}$.

Proof. Let $\gamma : \bar{Y} \xrightarrow{\sim} \bar{Y}$ be an element of $\mathrm{Aut}^0_G(\bar{f})$. It is clear that we can find an element $\gamma_0 \in G$ such that $\gamma_0^{-1} \circ \gamma$ induces the identity on $\bar{Y}_0$. It is also clear that γ_0 normalizes G_0 and that conjugation by γ_0 induces the identity on $G_0/I_0 \cong H_0$. Since the order of H_0 is prime to p, the Schur–Zassenhaus Theorem implies that the subgroup $\gamma_0 H_0 \gamma_0^{-1} \subset G_0$ is conjugate to H_0 by an element of G_0. Using the fact that the action of H_0 on I_0 is irreducible, one shows that there exists a unique element $\alpha \in I_0$ such that
$$\gamma_0 \circ \beta \circ \gamma_0^{-1} \;=\; \alpha^{-1} \circ \beta \circ \alpha$$
for all $\beta \in H_0$. After replacing γ_0 by $\alpha \circ \gamma_0$, we may thus assume that γ_0 centralizes H_0. For all $j \notin B_{\rm wild}$, set $\gamma_j := \gamma_0^{-1} \circ \gamma|_{\bar{Y}_j}$. One easily checks that γ_j is an (outer) automorphism of the pointed G_j-tail cover $(\bar{f}_j, \eta_j, H_j)$ and that the tuple $(\gamma_0; \gamma_j)$ satisfies Condition (1) and (2).

Conversely, given a tuple $(\gamma_0; \gamma_j)$ such that (1) and (2) hold, we define an automorphism $\gamma : \bar{Y} \xrightarrow{\sim} \bar{Y}$ of $\bar{f}$, as follows. If y is a point on $\bar{Y}_0$ and α is an element of G, we set
$$\gamma(\alpha(y)) \;:=\; \alpha \circ \gamma_0(y).$$
This is well defined because of Condition (1). If y is a point on $\bar{Y}_j$ and $\alpha \in G$, we set
$$\gamma(\alpha(y)) \;:=\; \alpha \circ \gamma_0 \circ \gamma_j(y).$$
This is well defined because $\gamma \circ \gamma_j$ commutes with G_j, by Condition (2). Moreover, the two definitions respect the equivalence relation '$\sim$' used in the definition of $\bar{Y}$ in §2.2.3. Hence we obtain an automorphism $\gamma : \bar{Y} \xrightarrow{\sim} \bar{Y}$, which is easily seen to be an element of $\mathrm{Aut}^0_G(\bar{f})$. This proves the lemma. □

The association $\gamma \mapsto \gamma_0$ is a group homomorphism $\mathrm{Aut}^0_G(\bar{f}) \to G$. Let us denote by n the order of the image of this homomorphism in G/C_G. Note that $n|(p-1)$. Let us denote by $\mathrm{Aut}_G(\bar{f}_j)$ the group of inner automorphisms of the pointed G_j-tail covers $(\bar{f}_j, \eta_j, H_j)$. The lemma shows that
$$|\,\mathrm{Aut}^0_G(\bar{f})| \;=\; n \cdot |C_G| \cdot \prod_{j \notin B_{\rm wild}} |\,\mathrm{Aut}_G(\bar{f}_j)|.$$

Remark 2.18. Suppose that the character $\chi : H_0 \to \mathbb{F}_p^\times$ is injective. (In particular, H_0 is cyclic). Then the integer n defined above is a multiple of

$$n' \;:=\; \gcd_j(m_j).$$

In fact, let γ be an element of H_0 which is contained in the stabilizer H_j of ξ_j, for all $j \in B_{\rm prim} \cup B_{\rm new}$. Then the tuple $(\gamma_0;\gamma_j)$ with $\gamma_j := \gamma_0^{-1}|_{\bar Y_j}$ represents an element of $\mathrm{Aut}^0_G(\bar f)$ which does not lie in C_G and which acts trivially on the vertical components of $\bar Y$.

3. The auxiliary cover

Let $(\bar Z,\omega)$ be a special deformation datum of type (H,χ). By a *lift* of $(\bar Z,\omega)$ we mean a Galois cover $f : Y \to X = \mathbb{P}^1$ with Galois group $G := \mathbb{Z}/p \rtimes_\chi H$ such that the deformation datum occuring over the original component of the stable reduction of f is equal to $(\bar Z,\omega)$. As we will see below, such a lift is uniquely determined by the deformation datum $(\bar Z,\omega)$ and the set of branch points (x_j) of f (which lift the critical points (τ_j) of $(\bar Z,\omega)$).

Following [Wew], we say that the lift $f : Y \to X$ is *special* if the stable reduction of f is as simple as possible, i.e. if the curve $\bar X$ is a comb. For instance, if $f : Y \to X$ is the *auxiliary cover* associated to a three point cover with mild reduction, then f is special by Theorem 2.6 (note that f is in general not a three point cover itself).

In [Wew] we showed that a lift f of $(\bar Z,\omega)$ is special if each branch point of f is 'sufficiently closed' to a K_0-rational point. The main result of this section – Theorem 3.9 – says that the converse is true as well. This theorem will play a crucial role in §4.

The proof of Theorem 3.9 relies on the results of [Wew02], which concern the deformation theory of a certain curve with group scheme action associated to $(\bar Z,\omega)$. The connection is as follows. If $f : Y \to X$ is a lift of the deformation datum $(\bar Z,\omega)$ then there exist a certain natural R-model $f_R : Y_R \to X_R = \mathbb{P}^1_R$ (which is *not* the stable model) such that the G-action on Y extends to an action of the group scheme $\mathcal{G} := \boldsymbol{\mu}_p \rtimes H$ on Y_R. The special fiber $\bar Y$ of Y_R is a singular curve with an action of $\mathcal{G}$; it depends on $(\bar Z,\omega)$ but not on the lift $f : Y \to X$. Therefore, a lift of $(\bar Z,\omega)$ is the same thing as an equivariant deformation of $\bar Y$.

3.1. The $\mathcal{G}$-cover associated to a multiplicative deformation datum.

3.1.1. Throughout this section, we fix an odd prime p and an algebraically closed field k of characteristic p. We denote by $W(k)$ the ring of Witt vectors over k.

Let $\bar X$ be a smooth projective curve over k and let $(\bar Z,\omega)$ be a multiplicative deformation datum over $\bar X$ of type (H,χ), see Definition 1.5. In this section we show that $(\bar Z,\omega)$ corresponds naturally to a (not necessarily smooth) curve $\bar Y$ over k together with an action of a certain group scheme $\mathcal{G}$ such that $\bar X = \bar Y/\mathcal{G}$.

We define $\mathcal{G}$ as a group scheme over $W(k)$, as follows. If R is a $W(k)$-algebra, an element of $\mathcal{G}(R)$ is given by a tuple (ζ,β), where $\zeta \in R$ satisfies $\zeta^p = 1$ and $\beta \in H$. Multiplication is defined as follows:

$$(\zeta_1,\beta_1)\cdot(\zeta_2,\beta_2) \;:=\; (\zeta_1\zeta_2^{\chi(\beta_1)},\beta_1\beta_2).$$

In other words: $\mathcal{G}$ is the semidirect product $\boldsymbol{\mu}_p \rtimes H$ determined by the action of H on $\boldsymbol{\mu}_p$ via χ. If R is a $W(k)$-algebra and Y a scheme over R, we say that $\mathcal{G}$ acts on Y if we are given an R-linear action of $\mathcal{G} \otimes R$ on Y.

Let $(\tau_j)_{j\in B}$ be the set of critical points for the deformation datum $(\bar{Z},\omega)$. For $j\in B$, let m_j, h_j be the invariants attached to $(\bar{Z},\omega,\tau_j)$, see §1.3. A critical point τ_j is called a *tame* (resp. *wild*) *branch point* if $m_j>1$ (resp. if $h_j=0$). We denote by $B_{\rm tame}$ and $B_{\rm wild}$ the corresponding subsets of B and set $B_{\rm ram}:=B_{\rm tame}\cup B_{\rm wild}$. In general we have $B_{\rm tame}\cap B_{\rm wild}\neq\emptyset$ and $B_{\rm ram}\neq B$.

Let $\bar{U}\subset\bar{X}$ be the complement of the wild branch points and let $\bar{V}\subset\bar{Z}$ be its inverse image. Then ω is a regular and logarithmic differential form on $\bar{V}$. Let $\bar{W}\to\bar{V}$ be the corresponding $\boldsymbol{\mu}_p$-torsor, see e.g. [Mil80], p. 129. The definition of $\bar{W}$ is roughly the following. Locally on $\bar{V}$ we can write $\omega=\mathrm{d}u/u$ for a unit u. Then $y^p=u$ is a local equation for the $\boldsymbol{\mu}_p$-torsor $\bar{W}\to\bar{V}$.

Proposition 3.1. *Let $\bar{Y}$ be the projective compactification of $\bar{W}$ such that the complement $\bar{Y}-\bar{W}$ is contained in the smooth locus of $\bar{Y}$. The action of $\boldsymbol{\mu}_p$ on $\bar{W}$ extends uniquely to an action of $\mathcal{G}$ on $\bar{Y}$ such that $\bar{Z}=\bar{Y}/\boldsymbol{\mu}_p$ and $\bar{X}=\bar{Y}/\mathcal{G}$.*

Proof. Let $\lambda:\boldsymbol{\mu}_p\times\bar{W}\to\bar{W}$ denote the action of $\boldsymbol{\mu}_p$ on $\bar{W}$. We have to show that λ extends to all of $\bar{Y}$. Let $\eta\in\bar{Y}-\bar{W}$ be a point in the complement of $\bar{W}$ and let $\xi\in\bar{Z}$ be its image. By construction, the point ξ lies above a wild critical point τ_j, so ω has a simple pole at ξ. Write $\omega=a\,\mathrm{d}u/u$, where u is a regular function in a neighborhood of ξ with a *simple* zero at ξ and $a\in\mathbb{F}_p^\times$ is the residue of ω at ξ. By definition of $\bar{Y}$ there exists a rational function y on $\bar{Y}$ such that $y^p=u$ and $\lambda^*y=\zeta^b\otimes y$, for some $b\in\mathbb{F}_p^\times$. It follows also from the definition of $\bar{Y}$ that η is a smooth point of $\bar{Y}$. Therefore, the integral equation $y^p=u$ shows that $y\in\mathcal{O}_{\bar{Y},\eta}$. Moreover, since u has a simple zero at ξ, we have $\mathcal{O}_{\bar{Y},\eta}=\mathcal{O}_{\bar{Z},\xi}[y\mid y^p=u]$. This shows that the action λ extends to $\bar{Y}$. It is obviously that this extension is unique.

It remains to show that λ extends to an action of $\mathcal{G}$ on $\bar{Y}$ which induces the canonical action of H on $\bar{Z}$. But this is clear from the rule $\beta^*\omega=\chi(\beta)\cdot\omega$ and the definition of $\bar{Y}$. □

The map $\bar{Y}\to\bar{X}$ together with the action of $\mathcal{G}$ on $\bar{Y}$ will be called the $\mathcal{G}$-cover associated to $(\bar{Z},\omega)$. Note that the curve $\bar{Y}$ is integral and has isolated, 'cusp-like' singularities precisely above the critical points which are not wild branch points, see [Ray90].

3.1.2. Let $\mathfrak{C}_k$ be the category of Noetherian Artinian local $W(k)$-algebras with residue field k. An *equivariant deformation* of $\bar{Y}$ over $R\in\mathfrak{C}_k$ is a deformation Y of the curve $\bar{Y}$ over R together with an action of $\mathcal{G}$ on Y such that the natural isomorphism $\bar{Y}\cong Y\otimes_R k$ is $\mathcal{G}$-equivariant. Let $\mathrm{Def}\,(\bar{Y},\mathcal{G})$ denote the functor which associates to a ring $R\in\mathfrak{C}_k$ the set of isomorphism classes of equivariant deformations of $\bar{Y}$ over R.

Given an equivariant deformation Y_R of $\bar{Y}$, we denote by $Z_R:=Y_R/\boldsymbol{\mu}_p$ and $X_R:=Y_R/\mathcal{G}$ the quotient schemes by $\boldsymbol{\mu}_p$ and $\mathcal{G}$, respectively. The R-curve X_R (resp. Z_R) is a smooth deformation of the curve $\bar{X}$ (resp. $\bar{Z}$). The association $Y_R\mapsto X_R$ gives rise to a morphism of deformation functors $\mathrm{Def}\,(\bar{Y},\mathcal{G})\to\mathrm{Def}\,(\bar{X})$. Here $\mathrm{Def}\,(\bar{X})$ denotes the functor which classifies deformations of $\bar{X}$. Let us write $\mathrm{Def}\,(\bar{X};\tau_j\mid j\in B_{\rm ram})$ for the functor which classifies deformations of the pointed curve $(\bar{X};\tau_j\mid j\in B_{\rm ram})$. We claim that the morphism $\mathrm{Def}\,(\bar{Y},\mathcal{G})\to\mathrm{Def}\,(\bar{X})$ factors in a natural way over the forgetful morphism $\mathrm{Def}\,(\bar{X};\tau_j\mid j\in B_{\rm ram})\to\mathrm{Def}\,(\bar{X})$, thus inducing a morphism

$$\mathrm{Def}\,(\bar{Y},\mathcal{G}) \quad\longrightarrow\quad \mathrm{Def}\,(\bar{X};\tau_j \mid j\in B_{\rm ram}). \tag{13}$$

Let Y_R be an equivariant deformation of $\bar{Y}$ and set $X_R := Y_R/\mathcal{G}$, $Z_R := Y_R/\boldsymbol{\mu}_p$. To prove the claim we have to endow, in a functorial way, the curve X_R with sections $\tau_{j,R} : \mathrm{Spec}\, R \to X_R$ lifting the branch points τ_j, for all $j \in B_{\rm ram}$. It is clear that the natural map $Z_R \to X_R$ is a tame H-Galois cover lifting $\bar{Z} \to \bar{X}$. By the theory of tame ramification, the branch divisor of $Z_R \to X_R$ consists of sections $\tau_{j,R} : \mathrm{Spec}\, R \to X_R$ lifting the tame branch points $\tau_j \in \bar{X}$, for all $j \in B_{\rm tame}$. Now let $j \in B_{\rm wild}$ and choose a point $\xi \in \bar{Z}$ above τ_j. We can write $\omega = \mathrm{d}u/u$ with $\mathrm{ord}_\xi(u) = 1$. By construction, the restriction of $\bar{Y} \to \bar{Z}$ to a neighborhood of ξ is a $\boldsymbol{\mu}_p$-cover given by the Kummer equation $y^p = u$. Hence the deformation $Y_R \to Z_R$ is, locally around ξ, still a $\boldsymbol{\mu}_p$-cover given by a Kummer equation $y^p = u_R$, where u_R lifts u. The equation $u_R = 0$ defines a section $\xi_R : \mathrm{Spec}\, R \to Z_R$ lifting ξ. We define $\tau_{j,R} : \mathrm{Spec}\, R \to X_R$ as the image of ξ_R. Using the H-action, it is easy to see that our two definitions of $\tau_{j,R}$ agree if $j \in B_{\rm tame} \cap B_{\rm wild}$. This proves the claim.

We say that $Y_R \to X_R$ is a deformation of the $\mathcal{G}$-cover $\bar{Y} \to \bar{X}$, and we call the tuple $(\tau_{j,R})_{j\in B_{\rm ram}}$ the *branch locus* of $Y_R \to X_R$. It is clear that a deformation of the $\mathcal{G}$-cover $\bar{Y} \to \bar{X}$ is the same thing as an equivariant deformation of $\bar{Y}$.

3.2. **The case of a special deformation datum.**

3.2.1. We continue with the notation of §3.1. In addition, we assume from now on that the deformation datum $(\bar{Z},\omega)$ is *special*, see Definition 2.7. We use the notation $B_{\rm prim}$, $B_{\rm new}$ and B_0 as in §2.2.1. It follows immediately from the definitions that we have $B_{\rm tame} = B_{\rm prim} \cup B_{\rm new}$ and $B_{\rm wild} = B_0 - B_{\rm prim}$. In particular, we have $B_{\rm tame} \cap B_{\rm wild} = \emptyset$ and $B = B_{\rm ram}$. Furthermore, we assume that the special deformation datum $(\bar{Z},\omega)$ is normalized, i.e. we have $\bar{X} = \mathbb{P}^1_k$ and $\{\tau_j \mid j \in B_0\} = \{0,1,\infty\}$.

Let $Y_R \to X_R$ be a deformation of the $\mathcal{G}$-cover $\bar{Y} \to \bar{X}$ with branch locus $(\tau_{j,R})_{j\in B}$. There exists a unique isomorphism $X_R \cong \mathbb{P}^1_R$ which reduces to the identity on $\bar{X} = \mathbb{P}^1_k$ and such that $\{\,\tau_{j,R} \mid j \in B_0\,\} = \{0,1,\infty\}$. Hence we may and will identify X_R with $\mathbb{P}^1_R$ and the sections $\tau_{j,R}$, for $j \in B_{\rm new}$, with the corresponding elements of R. A B-tuple $(\tau_{j,R})$ of R-points of $X_R = \mathbb{P}^1_R$ such that $\tau_{j,R}$ lifts τ_j for all $j \in B$ and $\{\,\tau_{j,R} \mid j \in B_0\,\} = \{0,1,\infty\}$ is called a *normalized B-tuple over R*.

Proposition 3.2. *The morphism of deformation functors*

$$\mathrm{Def}\,(\bar{Y},\mathcal{G}) \quad\longrightarrow\quad \mathrm{Def}\,(\bar{X};\tau_j)$$

is an isomorphism. In other words: given $R \in \mathfrak{C}_k$ and a normalized B-tuple $(\tau_{j,R})$ over R, there exists a deformation $Y_R \to X_R = \mathbb{P}^1_R$ of the $\mathcal{G}$-cover $\bar{Y} \to \bar{X} = \mathbb{P}^1_k$ with branch locus $(\tau_{j,R})$. Moreover, $Y_R \to X_R$ is unique up to unique isomorphism.

Proof. This is a special case of [Wew02], Theorem 5.7. □

For $j \in B_{\rm new}$, let T_j be an indeterminate; set $\tilde{R} := W(k)[[\,T_j \mid j \in B_{\rm new}\,]]$. For $j \in B_{\rm new}$, let $[\tau_j] \in W(k)$ denote the Teichmüller lift of τ_j (in fact, any lift of τ_j to an element of $W(k)$ would do as well). Let $(\tilde{\tau}_j)$ be the normalized B-tuple over $\tilde{R}$ which has $\tilde{\tau}_j = [\tau_j]+T_j$ for $j \in B_{\rm new}$. Let $\mathcal{X}$ denote the formal completion along the special fiber of the projective line over $\tilde{R}$. It is clear that for any $R \in \mathfrak{C}_k$ there is a one-to-one correspondence between local $W(k)$-algebra homomorphism $\tilde{R} \to R$ and normalized B-tuples over R. Therefore, $(\mathcal{X};\tilde{\tau}_j)$ is the universal formal deformation of the pointed curve $(\bar{X};\tau_j)$. Hence Proposition 3.2 implies:

Corollary 3.3. *Let $\mathcal{Y}\to\mathcal{X}$ be the (unique) formal deformation of the $\mathcal{G}$-cover $\bar{Y}\to\bar{X}$ with branch locus $(\tilde{\tau}_j)$. Then $\mathcal{Y}$ is a universal formal equivariant deformation of $\bar{Y}$.*

3.2.2. For $j\in B$, we denote by $\hat{X}_j$ the completion of $\bar{X}$ at τ_j and set $\hat{Y}_j:=\bar{Y}\times_{\bar{X}}\hat{X}_j$. The action of $\mathcal{G}$ on $\bar{Y}$ induces an action on $\hat{Y}_j$ such that $\hat{X}_j=\hat{Y}_j/\mathcal{G}$. Given a deformation $Y_R\to X_R$ of the $\mathcal{G}$-cover $\bar{Y}\to\bar{X}$, we denote by $\hat{X}_{j,R}$ the completion of X_R at τ_j and set $\hat{Y}_{j,R}:=\hat{X}_j\times_{X_R}\hat{X}_{j,R}$. Then $\hat{Y}_{j,R}$ is a $\mathcal{G}$-equivariant deformation of $\hat{Y}_j$. The association $Y_R\mapsto(\hat{Y}_{j,R})_{j\in B}$ defines a morphism of deformation functors

$$\Phi:\ \mathrm{Def}\,(\bar{Y},\mathcal{G})\ \longrightarrow\ \prod_{j\in B}\mathrm{Def}\,(\hat{Y}_j,\mathcal{G}), \tag{14}$$

Here $\mathrm{Def}\,(\hat{Y}_j,\mathcal{G})$ denotes the deformation functor which classifies $\mathcal{G}$-equivariant formal deformations of $\hat{Y}_j$. Following [BM00], we call Φ the *local-global morphism.*

Proposition 3.4. (1) *The functor* $\mathrm{Def}\,(\hat{Y}_j,\mathcal{G})$ *admits a miniversal deformation over the ring*

$$\tilde{R}_j\ :=\ \begin{cases} W(k), & \text{for } j\in B_0,\\ W(k)[[t_j]], & \text{for } j\in B_{\mathrm{new}}.\end{cases}$$

Here t_j is an indeterminate.

(2) *The local-global morphism Φ is an isomorphism. Therefore,*

$$\tilde{R}\ \cong\ \widehat{\otimes}_{W(k)}\,\tilde{R}_j\ =\ W(k)[[\,t_j\mid j\in B_{\mathrm{new}}\,]].$$

(3) *For every $j\in B_{\mathrm{new}}$ there exists a unit $w_j\in\tilde{R}$ such that*

$$t_j\ \equiv\ w_jT_j \pmod{p}.$$

Proof. Assertion (1) and (2) are a special case of [Wew02], Theorem 5.11. Assertion (3) corresponds to [Wew02], Proposition 5.14. □

3.2.3. Let $\mathcal{Y}\to\mathcal{X}$ be the universal deformation over $\tilde{R}$ of the $\mathcal{G}$-cover $\bar{Y}\to\bar{X}$, as defined in §3.2.1. Fix an index $j\in B_{\mathrm{new}}$. We denote by $\hat{\mathcal{X}}_j$ the completion of $\mathcal{X}$ at τ_j and set $\hat{\mathcal{Y}}_j:=\mathcal{Y}\times_{\mathcal{X}}\hat{\mathcal{X}}_j$. Being an equivariant deformation of $\hat{Y}_j$ over $\tilde{R}$, $\hat{\mathcal{Y}}_j$ is induced from the miniversal deformation of Proposition 3.4 via a homomorphism of $W(k)$-algebras $\varphi_j:\tilde{R}_j\to\tilde{R}$. The homomorphism φ_j is unique up to an automorphism of $W(k)$-algebra $\tilde{R}_j$. Since φ_j is injective by Proposition 3.4 (2), we may identify the ring $\tilde{R}_j$ with the image of φ_j. By Proposition 3.4 (1) the ring $\tilde{R}_j$ is a formal power series over $W(k)$ in one variable t_j. It is clear that Assertion (3) of Proposition 3.4 does not depend on the choice of the parameter t_j as long as $\tilde{R}_j\cong W(k)[[t_j]]$.

The quotient $\mathcal{Z}:=\mathcal{Y}/\boldsymbol{\mu}_p$ is a formal H-equivariant deformation of $\bar{Z}$. Let us choose a point $\xi\in\bar{Z}$ above τ_j and denote by $\hat{\mathcal{Z}}_j$ the completion of $\mathcal{Z}$ at ξ_j. Since $\mathcal{Z}$ is formally smooth over $\tilde{R}$ we have $\hat{\mathcal{Z}}_j=\mathrm{Spec}\,\tilde{R}[[z]]$ for some parameter z. We may assume

$$\beta^*z\ \equiv\ \chi_j(\beta)\cdot z \pmod{p}, \tag{15}$$

for all $\beta\in H_j$ and some character $\chi_j:H_j\to k^\times$. Actually, we have $\chi_j^{h_j}=\chi|_{H_j}$. (Of course, we could have choosen z in such a way that the above congruence would become an equality in $\tilde{R}[[z]]$, but this would give a stronger restriction on the choice of z than we are are willing to make.)

The natural map $\mathcal{Y} \to \mathcal{Z}$ is a $\boldsymbol{\mu}_p$-torsor in a neighborhood of ξ. Therefore, the pullback to the completion $\hat{\mathcal{Z}}_j$ is defined by a Kummer equation

$$y^p \;=\; u \;=\; \sum_{\mu\geq 0} c_\mu\, z^\mu, \tag{16}$$

with $c_\mu \in \tilde{R}$ and $c_0 \in \tilde{R}^\times$. The unit $u \in \tilde{R}[[z]]^\times$ is unique up to a pth power. As we have already remarked above, the $\boldsymbol{\mu}_p$-torsor $\hat{\mathcal{Y}}_j \to \hat{\mathcal{Z}}_j$ is the pullback of a $\boldsymbol{\mu}_p$-torsor defined over $\tilde{R}_j$. In other words, we can choose the parameter z and the Kummer equation (16) in such a way that all coefficients c_μ lie in $\tilde{R}_j \subset \tilde{R}$. In the following, we will assume that $c_\mu \in \tilde{R}_j$ and we pretend that $\hat{\mathcal{Y}}_j \to \hat{\mathcal{Z}}_j$ is actually defined over $\tilde{R}_j$.

Lemma 3.5. *Let $a_j := h_j - m_j$. We may choose the parameter z, the Kummer equation* (16) *and a parameter t_j for $\tilde{R}_j$ such that*

$$u \;\equiv\; 1 \,+\, t_j\, z^{a_j} \;+\; z^{h_j} \quad \mathrm{mod}\; (\,p,\, t_j\, z^{a_j+1},\, z^{h_j+1}\,).$$

Proof. Given an element $c \in \tilde{R}_j$, we write c' (resp. $\bar{c}$) for the image of c in $R' := \tilde{R}_j/(p)$ (resp. in $k = \tilde{R}_j/(p,t_j)$). We claim that $c_0' \in (R'^\times)^p$. To prove this claim, let β be a generator of $H_j \cong \mathbb{Z}/m_j$ and $0 < a < p$ an integer such that $a \equiv \chi(\beta)$ mod p. The subgroup scheme $\mathcal{G}_j := \boldsymbol{\mu}_p \cdot H_j \subset \mathcal{G}$ acts on $\hat{\mathcal{Y}}_j$, extending the action of H_j on $\hat{\mathcal{Z}}_j$ given by (15). Therefore, there exists a unit $w \in \tilde{R}_j[[z]]^\times$ such that $\beta^* y = w y^a$. Taking pth powers on both sides we obtain $\beta^* u = w^p u^a$. Using (15) shows that c_0^{1-a} is congruent to a pth power modulo p. Since $\sigma_j = h_j/m_j$ is not an integer, we have $a \not\equiv 1 \mod p$. The claim follows. In particular, after multiplying u with a pth power in $\tilde{R}_j[[z]]$ we may assume that $c_0' = 1$. We may also assume that $c_\mu' = 0$ whenever $p|\mu$.

Let $\omega' := \mathrm{d}u'/u'$. This is a logarithmic differential which lifts ω from $\hat{Z}_j$ to $\hat{\mathcal{Z}}_j \otimes R'$; it classifies the $\boldsymbol{\mu}_p$-torsor $\hat{\mathcal{Y}}_j \otimes R' \to \hat{\mathcal{Z}}_j \otimes R'$ up to isomorphism. Since the action of $\boldsymbol{\mu}_p$ on $\hat{\mathcal{Y}}_j$ extends to an action of $\mathcal{G}_j$, we have $\beta^*\omega' = \chi(\beta)\cdot\omega'$ for all $\beta \in H_j$. This and (15) shows that

$$\omega' \;=\; \sum_{\mu\geq 0} b_\mu'\, z^{\mu m_j + a_j - 1}\,\mathrm{d}z$$

with $b_\mu' \in R'$ (note that $0 < a_j < m_j$ and $\chi_j^{a_j} = \chi|_{H_j}$). Since ω has a zero of exact order $h_j - 1$ at ξ_j, we have $\bar{b}_0 = 0$ and $\bar{b}_1 \neq 0$. After a short computation, we conclude that $c_\mu' = 0$ for $0 < \mu < a_j$ and $\bar{c}_\mu = 0$ for $a_j < \mu < h_j$.

So far, we have shown that

$$u \;\equiv\; 1 \,+\, c t_j\, z^{a_j} \;+\; z^{h_j} \qquad \mathrm{mod}\; (\,p,\, t_j\, z^{a_j+1},\, z^{h_j+1}\,)$$

for some element $c \in \tilde{R}_j$. It remains to prove that $\bar{c} \neq 0$. For then we can replace t_j by $c^{-1}t_j$ and obtain the desired congruence. Let $k[\epsilon]$ be the ring of dual numbers over k and $\kappa : R_j \to k[\epsilon]$ the unique $W(k)$-algebra homomorphism with $\kappa(t_j) = \epsilon$. We can write

$$\kappa(u) \;=\; \bar{u} + \epsilon\cdot\bar{v} \;=\; (1 + z^{h_j} + \sum_{\mu\geq h_j} \bar{c}_\mu\, z^\mu) + \epsilon\cdot(\sum_{\mu\geq a_j} \bar{d}_\mu\, z^\mu),$$

with $\bar{c}_\mu, \bar{d}_\mu \in k$. We have to show that $\bar{d}_{a_j} \neq 0$.

By the proof of [Wew02], Lemma 5.12, there exists an H_j-invariant derivation of $k[[z]][z^{-1}]$

$$\theta_j \;=\; (\bar{e}_0 + \bar{e}_1\, x + \ldots)\,\frac{\mathrm{d}}{\mathrm{d}x} \;=\; \frac{z^{1-m_j}}{m_j}\,(\,\bar{e}_0 + \bar{e}_1\, z^{m_j} + \ldots)\,\frac{\mathrm{d}}{\mathrm{d}z}$$

(which is the same thing as a derivation of $k[[x]]$, with $x := z^{m_j}$) such that the following holds:

(a) $\bar{v} = \theta_j(\bar{u})$,

(b) we have $\bar{e}_0 = 0$ if and only if the pullback of $\hat{\mathcal{Y}}_j$ via the homomorphism $\kappa : \tilde{R}_j \to k[\epsilon]$ is the trivial deformation of $\hat{Y}_j$.

In fact, the class of θ_j modulo $k[[x]]\,x\,\frac{\mathrm{d}}{\mathrm{d}x}$ represents the isomorphism class of the equivariant deformation $\hat{\mathcal{Y}}_j \otimes k[\epsilon]$, via a certain isomorphism

$$\mathcal{E}xt^1_{\mathcal{G}}(\Omega_{\bar{Y}}, \mathcal{O}_{\bar{Y}})_{\tau_j} \;\cong\; \mathcal{T}_{\bar{X},\tau_j}/x \cdot \mathcal{T}_{\bar{X},\tau_j},$$

see [Wew02], §5.3. But since $\hat{\mathcal{Y}}_j$ is a miniversal deformation of $\hat{Y}_j$ and κ is not constant, $\hat{\mathcal{Y}}_j \otimes k[\epsilon]$ must be nontrivial. Therefore,

$$\bar{d}_{a_j} \;=\; \frac{h_j\, \bar{e}_0}{m_j} \;\neq\; 0.$$

This proves the lemma. □

Corollary 3.6. *We may choose the parameter z and the Kummer equation* (16) *in such a way that*

$$u \;\equiv\; 1 \,+\, w_j T_j\, z^{a_j} \,+\, z^{h_j} \quad \mathrm{mod}\;(\,p^2,\, T_j\, z^{a_j+1},\, z^{h_j+1}\,).$$

for some unit $w_j \in \tilde{R}^\times$.

Proof. It follows from Proposition 3.4 (3) and Lemma 3.5 that we may assume

$$u \;\equiv\; 1 \,+\, w_j T_j\, z^{a_j} \,+\, z^{h_j} + p \cdot v \quad \mathrm{mod}\;(\,p^2,\, T_j\, z^{a_j+1},\, z^{h_j+1}\,)$$

for a unit $w_j \in \tilde{R}$ and some $v \in \tilde{R}[[z]]$. Replacing the parameter z by $z' := z - pv(h_j + a_j w_j T_j)^{-1}$, we obtain the stronger congruence, modulo p^2. □

3.3. **The stable reduction of the auxiliary cover.**

3.3.1. Let k be an algebraically closed field of characteristic $p > 0$ and let $(\bar{Z}, \omega)$ be a normalized special deformation datum over $\bar{X} := \mathbb{P}^1_k$ of type (H, χ). Let $\bar{Y} \to \bar{X}$ be the $\mathcal{G}$-cover associated to $(\bar{Z}, \omega)$, as defined in §3.1.1. Let K_0 denote the fraction field of $W(k)$ and let K/K_0 be a finite extension. We assume that K contains the pth roots of unity. The integral closure R of $W(k)$ inside K is a complete mixed characteristic discrete valuation ring with residue field k.

Let $(\tau_{j,R})$ be a normalized B-tuple over R, see §3.2.1. By Corollary 3.3 there exists a unique formal deformation $Y_R \to X_R = \mathbb{P}^1_R$ of the $\mathcal{G}$-cover $\bar{Y} \to \bar{X}$ with branch locus $(\tau_{j,R})$. By Grothendieck's Existence Theorem, this formal deformation is effective, i.e. we may regard $Y_R \to X_R$ as a finite morphism of schemes over R. Write $Y_K \to X_K = \mathbb{P}^1_K$ for the generic fiber of $Y_R \to X_R$ and $x_j := \tau_{j,K}$ for the K-rational point on X_K corresponding to the section $\tau_{j,R}$, for $j \in B$. Let

$$G \;:=\; \mathcal{G}(K) \;\cong\; \mathbb{Z}/p \rtimes_\chi H$$

be the group of K-rational points of $\mathcal{G}$. The group G acts on Y_K, and it is easy to see that $Y_K \to X_K$ is a G-Galois cover with branch locus (x_j). The ramification index of this cover at the branch point x_j is equal to m_j for $j \in B_{\rm tame} = B_{\rm prim} \cup B_{\rm new}$ and equal to p for $j \in B_{\rm wild}$. Write $Y \to X := \mathbb{P}^1_{\bar{K}}$ for the pullback of $Y_K \to X_K$ to $\bar{K}$.

Definition 3.7. The G-cover $f : Y \to X$ is called the *lift* of the special deformation datum $(\bar{Z}, \omega)$ with branch locus (x_j).

This definition is justified by the next proposition.

Proposition 3.8. *Let G' be a finite group, isomorphic to the semi-direct product $\mathbb{Z}/p \rtimes H$ determined by the character χ. Let $W \to X$ be a G'-Galois cover. Then $W \to X$ and $Y \to X$ are isomorphic as mere covers if and only if the following conditions hold.*

1. *The branch locus of $W \to X$ is equal to $\{x_j \mid j \in B\}$.*
2. *The cover $W \to X$ has bad reduction and the deformation datum occuring over the original component of the stable reduction is isomorphic to $(\bar{Z}, \omega)$.*

Proof. Let $W \to X$ be a G'-Galois cover satisfying Conditions (1) and (2) of the proposition. After replacing K by a larger field, if necessary, we may suppose that $W \to X$ has a model $W_K \to X_K$ over K which extends to a stable model $W_R^{\rm st} \to X_R^{\rm st}$. Let W_R denote the normalization of $X_R = \mathbb{P}^1_R$ in the function field of W_K. We obtain a canonical map $W_R^{\rm st} \to W_R$ which is an isomorphism except in the singular points of $\bar{W}$. By Condition (2), the action of G' on $\bar{W}$ has a kernel $I' \lhd G'$, cyclic of order p. By the assumption on G' and $W \to X$, we have $G'/I' \cong H$ and there exists a finite map $\bar{W} \to \bar{Z}$, purely inseparable of degree p and H-equivariant. Clearly, the isomorphism $G'/I' \cong H$ extends to an isomorphism $G' \cong G$.

Let $\mathcal{G}'$ be the schematic closure of $G' = \mathrm{Aut}(W_K/X_K)$ inside the R-group scheme $\mathrm{Aut}(W_R/X_R)$. By [Ray74], $\mathcal{G}'$ is a finite flat group scheme over R with generic fiber G'. We claim that the isomorphism $G' \cong G$ chosen above extends to an isomorphism $\mathcal{G}' \cong \mathcal{G} \otimes_{W(k)} R$ of R-group schemes. Indeed, it suffices to show that the closure of I' inside $\mathcal{G}'$ is isomorphic to $\boldsymbol{\mu}_p$ (because then $\mathcal{G}'$ is the minimal prolongation of G' over R, see [Ray74]). Therefore, the claim follows from Condition (2).

It is no loss of generality to identify $\mathcal{G}'$ with $\mathcal{G} \otimes_{W(k)} R$. So $\mathcal{G}$ acts on $\bar{W}$ such that $\bar{Z} = \bar{W}/\boldsymbol{\mu}_p$ and $\bar{X} = \bar{W}/\mathcal{G}$. Moreover, the restriction of the map $\bar{W} \to \bar{Z}$ to a sufficiently small open subset is a $\boldsymbol{\mu}_p$-cover corresponding to the differential ω. We conclude that there exists a $\mathcal{G}$-equivariant isomorphism $\bar{W} \xrightarrow{\sim} \bar{Y}$, see the proof of Proposition 3.1. Via this isomorphism, we may regard W_R as an equivariant deformation of $\bar{Y}$. By Condition (1), the two $\mathcal{G}$-covers $Y_R \to X_R$ ad $W_R \to X_R$ have the same branch locus $(\tau_{j,R})$. Hence the proposition follows from Corollary 3.3. □

3.3.2. Let $(\tau_{j,R})$ be a normalized B-tuple over R, $x_j := \tau_{j,K} \in X := \mathbb{P}^1_{\bar{K}}$, and let $Y \to X$ be the lift of $(\bar{Z}, \omega)$ with branch locus (x_j). After replacing K by a finite extension, we may assume that the natural K-model $Y_K \to X_K$ extends to the stable model $Y_R^{\rm st} \to X_R^{\rm st}$. We denote by $\bar{Y}^{\rm st} \to \bar{X}^{\rm st}$ the stable reduction and by $\bar{x}_j \in \bar{X}^{\rm st}$ the specialization of the branch point x_j. The stable model is obtained as a blow-up of the natural R-model $Y_R \to X_R$, i.e. we have a commutative diagram

$$\begin{array}{ccc} Y_R^{\mathrm{st}} & \longrightarrow & X_R^{\mathrm{st}} \\ \downarrow & & \downarrow \\ Y_R & \longrightarrow & X_R \end{array} \tag{17}$$

in which the vertical arrows are modifications which are isomorphisms away from the critical points of $\bar{X}$ resp. the singular points of $\bar{Y}$. In particular, the map $\bar{X}^{\mathrm{st}} \to \bar{X} = \mathbb{P}^1$ induces an isomorphism between $\bar{X}$ and the original component $\bar{X}_0$ of $\bar{X}^{\mathrm{st}}$, and contracts all other components to one of the critical points τ_j, $j \in B_{\mathrm{prim}} \cup B_{\mathrm{new}}$. By construction of Y_R, the deformation datum over $\bar{X}_0 \cong \bar{X}$ associated to $\bar{Y}^{\mathrm{st}} \to \bar{X}^{\mathrm{st}}$ is isomorphic to $(\bar{Z}, \omega)$.

Here is the main result of this section.

Theorem 3.9. *The following two conditions are equivalent:*

1. *The curve $\bar{X}^{\mathrm{st}}$ consists only of the original component $\bar{X}_0$ and, for each $j \in B_{\mathrm{tame}}$, a tail $\bar{X}_j$ which contains $\bar{x}_j$ and intersects $\bar{X}_0 \cong \bar{X}$ in τ_j.*
2. *For all $j \in B_{\mathrm{new}}$ we have*

$$\mathrm{val}(\tau_{j,R} - [\tau_j]) \;\geq\; \frac{p\,m_j}{(p-1)\,h_j} \cdot \mathrm{val}(p).$$

(Recall that $[\tau_j] \in W(k) \subset R$ denotes the Teichmüller lift of τ_j.)

Remark 3.10. Under the additional assumption $B_{\mathrm{wild}} = \emptyset$, the implication '(2) $\Rightarrow$ (1)' is already proved in [Wew].

Proof. The element $\tau_{j,R} - [\tau_j] \in R$ is the image of the parameter T_j (defined in §3.2.2) under the classifying morphism $\tilde{R} \to R$ of the equivariant deformation Y_R. By abuse of notation, we write $T_j := \tau_{j,R} - [\tau_j]$ as an element of R.

Let us first prove the implication '(2) $\Rightarrow$ (1)', i.e. we assume that (1) holds. To find the modification $Y_R^{\mathrm{st}} \to Y_R$ which leads to the stably marked model of Y_R, it suffices to consider the completion of Y_R at a singular point of $\bar{Y}$. Let $\eta_j \in \bar{Y}$ be one of the singular points, lying above the critical point $\tau_j \in \bar{X}$ for $j \in B_{\mathrm{tame}}$. Let $\xi_j \in \bar{Z}$ be the image of η_j. Let $\hat{Y}_{R,j}$ denote the completion of Y_R at η_j and $\hat{Z}_{R,j}$ the completion of Z_R at ξ_j. Condition (1) of the theorem means that the germ $\hat{Y}_{R,j}$ has potentially good reduction. We can write $\hat{Z}_{R,j} = \operatorname{Spec} R[[z]]$. The $\boldsymbol{\mu}_p$-torsor $\hat{Y}_{R,j} \to \hat{Z}_{R,j}$ is given by a Kummer equation $y^p = u$, with $u \in R[[z]]^\times$.

Let us first treat the case $j \in B_{\mathrm{new}}$. By Lemma 3.5 we may assume that

$$y^p \;=\; u \;\equiv\; 1 \;+\; w_j\,T_j\,z^{a_j} \;+\; z^{h_j} \quad \bmod\,(\,p^2,\, T_j\,z^{a_j+1},\, z^{h_j+1}\,), \tag{18}$$

with $w_j \in R^\times$. We also may assume that R contains an element λ such that $\lambda^{h_j(p-1)} = -p$. Let $y = 1 + \lambda^{h_j} y'$ and $z = \lambda^p z'$. The new coordinates y' and z' give rise to a diagram

$$\begin{array}{ccc} Y'_{R,j} & \longrightarrow & Z'_{R,j} \\ \downarrow & & \downarrow \\ \hat{Y}_{R,j} & \longrightarrow & \hat{Z}_{R,j} \end{array}$$

in which the vertical arrows are blow-ups centered in the closed point of the germs $\hat{Y}_{R,j}$ and $\hat{Z}_{R,j}$, respectively. Let $\bar{Y}_j'$ (resp. $\bar{Z}_j'$) be the exceptional divisor of these blowups. The curve $\bar{Z}_j'$ is a projective line with parameter z'. To obtain the equation for the cover $\bar{Y}_j' \to \bar{Z}_j'$, rewrite (18) in terms of z' and y', subtract 1, divide by λ^{ph_j} and reduce modulo the maximal ideal of R. Using the assumption $(p-1)h_j\,\mathrm{val}(T_j) \geq pm_j\,\mathrm{val}(p)$, one obtains an Artin-Schreier equation

$$y'^p - y' \;=\; \bar{w}_j\, z'^{a_j} \;+\; z'^{h_j} \tag{19}$$

with conductor h_j. It follows that $\bar{Y}_j'$ is smooth and that $Y_{R,j}'$ is a semistable model of the germ $\hat{Y}_{R,j}$. In other words, the germ $\hat{Y}_{R,j}$ has potentially good reduction. See e.g. [Ray99], Lemme 4.3.3 or [Leh01], §3. We conclude that Condition (1) holds for the given index $j \in B_{\rm new}$.

Condition (1) holds for $j \in B_{\rm prim}$ as well, without any assumption. Indeed, if $j \in B_{\rm prim}$ then $0 < h_j < m_j$. An easy version of the proof of Lemma 3.5 shows that (for the right choice of parameter z) the $\boldsymbol{\mu}_p$-torsor $Y_{R,j} \to Z_{R,j}$ is given by a Kummer equation of the form

$$y^p \;=\; u \;\equiv\; 1 \;+\; z^{h_j} \quad \mathrm{mod}\ (p^2,\, z^{h_j+1}\,).$$

After a change of coordinate $y = 1 + \lambda^{h_j} y'$ and $z' = \lambda^p z'$ we again find an Artin–Schreier equation with conductor h_j, which shows that the germ $\hat{Y}_{R,j}$ has potentially good reduction. This finishes the proof of the implication '(2) $\Rightarrow$ (1)'.

It remains to prove the implication '(1) $\Rightarrow$ (2)'. Suppose that Condition (2) is violated for some $j \in B_{\rm new}$. We may also assume that R contains an element λ with $\lambda^{pm_j} = T_j$. After substituting $y = 1 + \lambda^{h_j} y'$ and $z = \lambda^p z'$ into (18) and reducing modulo the maximal ideal of R, we find that the cover $\bar{Y}_j' \to \bar{Z}_j'$ has equation

$$y'^p \;=\; \bar{w}_j\, z'^{a_j} \;+\; z'^{h_j}.$$

It follows that the cover $\bar{Y}_j' \to \bar{Z}_j'$ is an $\boldsymbol{\alpha}_p$-torsor (away from the point $z' = \infty$) corresponding to the exact differential $\omega' = \mathrm{d}(\bar{w}_j z'^{a_j} + z'^{h_j})$. The differential ω' has a zero of order $a_j - 1$ at $z' = 0$ and m_j further simple zeros. Above each point where ω' has a zero, $\bar{Y}_j'$ has a singularity. We conclude that the germ $\hat{Y}_{R,j}$ does not have potentially good reduction. In other words, Condition (1) of Theorem 3.9 does not hold. This finishes the proof of Theorem 3.9. □

The proof shows that, if both the conditions of the theorem hold, then the closed rigid disk

$$D_j \;:=\; \{\; x \;\mid\; \mathrm{val}(x - [\tau_j]) \geq \frac{p\,m_j}{(p-1)\,h_j} \cdot \mathrm{val}(p) \;\} \;\subset\; X_K^{\rm rig}$$

is equal to the set of points on X_K which specialize to the tail $\bar{X}_j \subset X_R^{\rm st}$. We remark at this point that, since $\frac{pm_j}{(p-1)h_j} < 1$ for $j \in B_{\rm new}$, the family of disks $(D_j)_{j \in B}$ is independent of the choice of $(\tau_{j,R})$, as long as Condition (2) of Theorem 3.9 holds.

3.3.3. Let us now assume that the branch points x_j of the lift $f : Y \to X$ of $(\bar{Z}, \omega)$ are all K_0-rational. This is equivalent to the condition that $\tau_{j,R} \in W(k)$. Then Condition (2) of Theorem 3.9 is automatically verified. Moreover, the cover $f : Y \to X$ has a natural model $f_{K_0} : Y_{K_0} \to X_{K_0} = \mathbb{P}^1_{K_0}$ over K_0 (which is *not* Galois, since K_0 does not contain the pth roots of unity). Let

$$\bar{\kappa}^{\rm ab} : \mathrm{Gal}(\bar{K}/K_0) \;\longrightarrow\; \mathrm{Aut}_k(\bar{Y}^{\rm st})$$

be the absolute monodromy action associated to the K_0-model f_{K_0} of f, see §1.1.3. The goal of this subsection is to describe this action in detail.

First some notation. Let $\bar{Y}_0$ be the unique irreducible component of $\bar{Y}^{\rm st}$ which lies above the original component $\bar{X}_0 \subset \bar{X}^{\rm st}$. We may identify $\bar{Y}_0$ with the normalization of $\bar{Y}$ or, what amounts to the same thing, the smooth projective model of the function field $k(\bar{Z})^{1/p}$. Let us choose, for all $j \in B_{\rm tame}$, a point $\eta_j \in \bar{Y}_0$ above $\tau_j \in \bar{X}$. We denote by $\xi_j \in \bar{Z}$ the image of η_j, by $H_j \subset H$ the stabilizer of ξ_j and by $I \subset G$ the cyclic subgroup of order p. Set $G_j := I \cdot H_j \subset G$. Let $\bar{Y}_j \subset \bar{Y}^{\rm st}$ (resp. $\bar{X}_j \subset \bar{X}^{\rm st}$) be the component which intersects $\bar{Y}_0$ in η_j (resp. intersects $\bar{X}$ in τ_j). By Theorem 3.9, $\bar{X}^{\rm st} = \bar{X} \cup \cup_j \bar{X}_j$. Furthermore, the proof of the theorem shows that $(\bar{Y}_j \to \bar{X}_j, \eta_j, H_j)$ is a pointed G_j-tail cover with ramification type (m_j, h_j) which is totally ramified above $\tau_j \in \bar{X}_j$, see §2.2.2.

Let us denote by $\mathrm{Aut}^0_k(\bar{Y}^{\rm st})$ the group of k-linear automorphisms of $\bar{Y}^{\rm st}$ which normalize the action of G, commute with the action of $H \subset G$ and induce the identity on $\bar{Y}_0$. Similarly, let $\mathrm{Aut}^0_k(\bar{Y}_j)$ denote the group of (outer) automorphisms of the pointed G_j-tail cover $\bar{f}_j$. Recall that these are k-linear automorphisms of $\bar{Y}_j$ which normalize the action of G_j, commute with the action of $H_j \subset G_j$ and fix the point η_j. For each j we have a natural homomorphism

$$\mathrm{Aut}^0_k(\bar{Y}^{\rm st}) \;\longrightarrow\; \mathrm{Aut}^0_k(\bar{Y}_j). \tag{20}$$

Let α be a generator of $I \lhd G$. For $\varphi \in \mathrm{Aut}^0_k(\bar{Y})$ we have $\varphi \circ \alpha \circ \varphi^{-1} = \alpha^\rho$ for some $\rho \in \mathbb{F}_p^\times$; this defines a character

$$\rho : \mathrm{Aut}^0_k(\bar{Y}) \;\longrightarrow\; \mathbb{F}_p^\times. \tag{21}$$

Similarly, one defines characters $\rho_j : \mathrm{Aut}^0_k(\bar{Y}_j) \to \mathbb{F}_p^\times$ for all $j \in B_{\rm tame}$. The composition of ρ with $\bar{\kappa}^{\rm ab}$ is the cyclotomic character modulo p. The kernel of the character ρ is the subgroup $\mathrm{Aut}^0_G(\bar{Y}^{\rm st}) \subset \mathrm{Aut}^0_k(\bar{Y}^{\rm st})$ of automorphisms which commute with the G-action. Compare with §2.2.4.

Proposition 3.11. (1) *We have a canonical isomorphism*

$$\mathrm{Aut}^0_k(\bar{Y}^{\rm st}) \;\cong\; \{\, (\rho; \varphi_j) \in \mathbb{F}_p^\times \times \prod_j \mathrm{Aut}^0_k(\bar{Y}_j) \;\mid\; \rho = \rho_j(\varphi_j)\ \forall j \,\}.$$

(2) *The group* $\mathrm{Aut}^0_k(\bar{Y}_j)$ *is cyclic of order* $(p-1)h_j$. *Therefore, the group* $\mathrm{Aut}^0_k(\bar{Y}^{\rm st})$ *has order* $(p-1)\prod_j h_j$.

(3) *The restriction of* $\bar{\kappa}^{\rm ab}$ *to the component* $\bar{Y}_j$ *is a surjective homomorphism*

$$\bar{\kappa}^{\rm ab}_j : \mathrm{Gal}(\bar{K}/K_0) \;\longrightarrow\; \mathrm{Aut}^0_k(\bar{Y}_j).$$

(4) *The image of* $\bar{\kappa}^{\rm ab}$ *is a cyclic subgroup of* $\mathrm{Aut}^0_k(\bar{Y}^{\rm st})$, *of order*

$$N \;:=\; (p-1)\cdot \mathrm{lcm}_j(h_j).$$

Proof. Part (1) is left to the reader. We also leave it to the reader to check that the image of the monodromy action $\bar{\kappa}^{\rm ab}$ is contained in $\mathrm{Aut}^0_k(\bar{Y}^{\rm st})$.

The explicit description of the change of coordinates $y = 1 + \lambda^{h_j} y'$, $z = \lambda^p z'$ in the proof of Theorem 3.9 shows that the restriction of $\bar{\kappa}^{\rm ab}$ to $\bar{Y}_j$ is cyclic of order $(p-1)h_j$ (it corresponds to the cyclic Galois extension $K_0(\lambda)/K_0$ where $\lambda^{(p-1)h_j} = -p$). In particular, the group $\mathrm{Aut}^0_k(\bar{Y}_j)$ has a cyclic subgroup of order $(p-1)h_j$. Therefore, in order to prove the proposition it suffices to show that the group $\mathrm{Aut}^0_k(\bar{Y}_j)$ has exact order $(p-1)h_j$.

First, we claim that $\mathrm{Aut}^0_k(\bar{Y}_j)$ is cyclic. Indeed, the natural action of $\mathrm{Aut}^0_k(\bar{Y}_j)$ on $\bar{Y}_j$ descents to a faithful action on the quotient $\bar{X}_j = \bar{Y}_j/G_j$ which fixes the two points τ_j and $\bar{x}_j$. But $\bar{X}_j$ has genus 0, so the claim follows. Now assume that $\mathrm{Aut}^0_k(\bar{Y}_j)$ has order $(p-1)h_jn$. Let $\varphi \in \mathrm{Aut}^0_k(\bar{Y}_j)$ be an element of order h_jn. Since φ lies in the kernel of the character ρ_j it commutes with the action of G_j on $\bar{Y}_j$. Let $\bar{Y}'_j := \bar{Y}_j/\langle\varphi\rangle \to \bar{X}'_j := \bar{X}_j/\langle\varphi\rangle$ be the induced G_j-cover. Recall that the action of $I \cong \mathbb{Z}/p$ on $\bar{Y}_j$ has conductor h_j at the point η_j. Let η'_j denote the image of η_j on $\bar{Y}'_j$. A classical calculation (see e.g. [Ray99], Lemme 1.1.2) shows that the action of I on $\bar{Y}'_j$ has conductor $1/n$ at η'_j. We conclude that $n = 1$. This finishes the proof of the proposition. □

The proposition implies that the G-cover $f : Y \to X$ has stable reduction over the unique tame extension K^{st}/K_0 of degree N.

4. Construction of three point covers by lifting

Let $\bar{f} : \bar{Y} \to \bar{X}$ be a special G-map, see Definition 2.15. A *lift* of $\bar{f}$ is a three point G-cover with stable reduction $\bar{f}$. Let $L(\bar{f})$ denote the set of isomorphism classes of all lifts of $\bar{f}$. The Galois group $\mathrm{Gal}(\bar{K}/K_0)$ acts on $L(\bar{f})$ in a natural way. In this section we determine the structure of $L(\bar{f})$ as a $\mathrm{Gal}(\bar{K}/K_0)$-set. In particular, we prove that $L(\bar{f})$ is nonempty and that the action of $\mathrm{Gal}(\bar{K}/K_0)$ is tamely ramified.

At the end, we derive various consequences of the above result, concerning the field of moduli of three point covers, and we discuss a aprticular class of examples, called *genus zero dessins of prime degree*.

4.1. The set of lifts of $\bar{f}$.

4.1.1. Let k be an algebraic closure of the finite field $\mathbb{F}_p$, $W(k)$ the ring of Witt vectors over k and K_0 the fraction field of $W(k)$. We fix an algebraic closure $\bar{K}$ of K_0. In this section, K/K_0 always denotes a finite field extension contained in $\bar{K}$ and R denotes the ring of integers of K. Let $(\bar{Z}_0, \omega_0; \bar{Y}_j)$ be a special G-deformation datum and let $\bar{f} : \bar{Y} \to \bar{X}$ be the associated special G-map, see §2.2.

Definition 4.1. A *lift* of $\bar{f}$ over R is a stable G-map $f_R : Y_R \to X_R$ over R (see Definition 2.14), together with an isomorphism $\bar{f} \cong f_R \otimes_R k$ of stable G-maps and an isomorphism $X_R \otimes_R K \stackrel{\sim}{\to} (\mathbb{P}^1_K, \{0, 1, \infty\})$ of marked curves. Given a lift $f_R : Y_R \to X_R$, we will usually identify $\bar{f}$ with the special fiber of f_R and $\mathbb{P}^1_K$ with the general fiber of X_R. Two lifts $f_R : Y_R \to X_R$ and $f'_R : Y'_R \to X'_R$ over R are said to be isomorphic if there exists an isomorphism $f_R \stackrel{\sim}{\to} f'_R$ of stable G-maps which induces the identity on the special fiber and on $X_K = X'_K = \mathbb{P}^1_K$.

If $f_R : Y_R \to X_R$ is a lift of $\bar{f}$, then the general fiber $f_K : Y_K \to X_K = \mathbb{P}^1_K$ of f_R is a three point G-cover and f_R is its stable model. In particular, f_K has mild reduction and $\bar{f}$ is the stable reduction of f_K. If f_R and f'_R are isomorphic lifts of $\bar{f}$ then the isomorphism $f_R \stackrel{\sim}{\to} f'_R$ is unique. (This is a general property of deformations of stably marked curves, see [Knu83].)

4.1.2. Let $L(\bar{f}, K)$ denote the set of isomorphism classes of lifts of $\bar{f}$ over R, the ring of integers of K. If K'/K is a finite extension then we have a natural inclusion $L(\bar{f}, K) \hookrightarrow L(\bar{f}, K')$. Let

$$L(\bar{f}) := \cup_K L(\bar{f}, K)$$

be the union, taken over all finite extensions K/K_0 contained in $\bar{K}$. It is clear that $L(\bar{f})$ is a finite set. A priori, $L(\bar{f})$ may be empty; we will see later that it is not.

There is a natural continuous action of $\mathrm{Gal}(\bar{K}/K_0)$ on $L(\bar{f})$ from the left, defined as follows. Let $f_R : Y_R \to X_R$ be a lift of $\bar{f}$ over R, the ring of integers of K, and let $\sigma \in \mathrm{Gal}(\bar{K}/K_0)$. We may assume that K/K_0 is Galois. Let ${}^\sigma f_R : {}^\sigma Y_R \to {}^\sigma X_R$ be the pullback of f_R by the automorphism $\tilde{\sigma} : \mathrm{Spec}\, R \overset{\sim}{\to} \mathrm{Spec}\, R$ induced by σ. Since σ induces the identity on the residue field k, we have a canonical isomorphism $f_R \otimes_R k \overset{\sim}{\to} {}^\sigma f_R \otimes_R k$. We may therefore consider ${}^\sigma f_R$ as a deformation of $\bar{f}$. This defines the action of $\mathrm{Gal}(\bar{K}/K_0)$ on $L(\bar{f})$.

Recall that $\mathrm{Aut}^0_G(\bar{f})$ was defined in §2.2.4 as the group of automorphisms of $\bar{f}$ which induce the identity on the original component. There is a natural right action of $\mathrm{Aut}^0_G(\bar{f})$ on $L(\bar{f})$, defined as follows. If $\gamma \in \mathrm{Aut}^0_G(\bar{f})$ and $f_R : Y_R \to X_R$ is a lift of $\bar{f}$, then we define f_R^γ as the stable G-map $f_R : Y_R \to X_R$ together with the isomorphism $\bar{Y} \overset{\sim}{\to} Y_R \otimes_R k$ which is the composition with $\gamma : \bar{Y} \overset{\sim}{\to} \bar{Y}$ with the canonical isomorphism $\bar{Y} \overset{\sim}{\to} Y_R \otimes_R k$. This defines an action of $\mathrm{Aut}^0_G(\bar{f})$ on $L(\bar{f})$ from the right which commutes with the action of $\mathrm{Gal}(\bar{K}/K_0)$. We define

$$\tilde{L}(\bar{f}) \;:=\; L(\bar{f})/\,\mathrm{Aut}^0_G(\bar{f}).$$

It is clear that elements of $\tilde{L}(\bar{f})$ correspond naturally to isomorphism classes of three point G-covers $f : Y \to X$ whose stable reduction is isomorphic to $\bar{f}$. By definition we have:

Proposition 4.2. *Let $f_R : Y_R \to X_R$ be a lift of $\bar{f}$ and denote by $f : Y \to X$ the geometric generic fiber of f_R.*

(1) *The stabilizer in* $\mathrm{Gal}(\bar{K}/K_0)$ *of f_R, considered as an element of $L(\bar{f})$, is equal to Γ^{st}_f, as defined in §1.1.3.*
(2) *The stabilizer in* $\mathrm{Gal}(\bar{K}/K_0)$ *of f_R, considered as an element of $\tilde{L}(\bar{f})$, is equal to Γ^{in}_f, as defined in §1.1.3.*

4.1.3. *The auxiliary cover.* Let $f_R : Y_R \to X_R$ be a lift of $\bar{f}$. For $j \in B_{\mathrm{prim}} \cup B_{\mathrm{new}}$, let $D_j \subset X_K^{\mathrm{rig}}$ denote the closed rigid disk corresponding to the tail $\bar{X}_j$. By definition, a K-rational point $x \in D_j$ specializes to a point $\bar{x} \in \bar{X}_j$. In particular, for $j \in B_{\mathrm{prim}}$ the branch point $x_j \in \{0, 1, \infty\}$ of f lies on D_j. Note that the model X_R of $X_K = \mathbb{P}^1_K$ is uniquely determined by the family of disks $(D_j)_j$. A priori, the disks D_j (and therefore the model X_R) may depend on the lift f_R. We are now going to show that they are in fact independent of f_R. This is an important steps towards our main result.

Let us choose, for all $j \in B_{\mathrm{new}}$, a K-rational point $x_j \in D_j$. We call these points the *auxiliary branch points.* By [Ray99], §3.2, the choice of the auxiliary branch points yields a certain G_0-cover $f^{\mathrm{aux}} : Y^{\mathrm{aux}} \to X$, called the *auxiliary cover* of f. By definition, f^{aux} has branch locus $\{x_j \mid j \in B\}$ and the same type of bad reduction as f. To be more precise, let $\hat{X}_R$ denote the formal completion of X_R along $\bar{X}$ and let $\mathcal{D}_j \subset \hat{X}_R$, for $j \in B_{\mathrm{prim}} \cup B_{\mathrm{new}}$, denote the open subset corresponding to $\bar{X}_j - \{\tau_j\} \subset \bar{X}$, i.e. $\mathcal{D}_j$ is the closed formal disk whose generic fiber is D_j. Let $\mathcal{U}^\circ \subset \hat{X}_R$ denote the open subset corresponding to $\bar{X} - \cup_j \bar{X}_j \subset \bar{X}$. Finally, let $\mathcal{X}_j$ denote the completion of X_R at $\tau_j \in \bar{X}$ and set $\mathcal{U} := \mathcal{U}^\circ \cup \cup_j \mathcal{X}_j$. Recall that part of our special G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$ is a subgroup $G_0 = I_0 \cdot H_0$. The auxiliary cover of the lift $f_R : Y_R \to X_R$ is a G_0-cover $f^{\mathrm{aux}} : Y^{\mathrm{aux}} \to X$, characterized by the following properties, see [Ray99], §3.2.

(a) The cover f^{aux} has a stable model $f_R^{\text{aux}} : Y_R^{\text{aux}} \to X_R^{\text{aux}}$ over R. Moreover, we have a (unique) isomorphism $X_R \cong X_R^{\text{aux}}$ extending the identity on the generic fiber.

(b) For $j \notin B_{\text{wild}}$, the restriction of f_R^{aux} to the formal disk $\mathcal{D}_j$ is a tame cover of formal schemes, ramified along the closure of x_j in $\mathcal{D}_j$.

(c) There exists a G-equivariant isomorphism

$$\hat{Y}_R \times_{\hat{X}_R} \mathcal{U} \;\cong\; \mathrm{Ind}_{G_0}^G(\hat{Y}_R^{\text{aux}} \times_{\hat{X}_R} \mathcal{U}) \tag{22}$$

of formal $\mathcal{U}$-schemes.

It follows that the branch locus of $f^{\text{aux}} : Y^{\text{aux}} \to X$ is precisely the set of points $\{x_j \mid j \in B\}$.

Proposition 4.3. (1) *The cover f^{aux} is the (unique) lift of $(\bar{Z}_0, \omega_0)$ with branch points (x_j), see §3.3.1. In particular, as a mere cover, f^{aux} does not depend on the lift f_R of $\bar{f}$.*

(2) *The disks D_j are independent of the lift $f_R : Y_R \to X_R$. Moreover, each disk D_j contains a K_0-rational point.*

Proof. It follows immediately from Property (c) that the deformation datum over the original component associated to the stable reduction of f^{aux} can be identified with $(\bar{Z}_0, \omega_0)$. Hence (1) follows from Proposition 3.8. By Property (a), the cover f^{aux} satisfies Condition (1) of Theorem 3.9. Therefore, (2) follows from Theorem 3.9. □

We remark that the proof of Proposition 4.3 is the only place where we use the subtle implication '(1)⇒(2)' of Theorem 3.9.

4.2. **Patching data.** Lifting the special G-map $\bar{f}$ to a three point G-cover $f : Y \to X$ can be done in three steps. The first step consists in lifting the tail covers $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ to Galois covers $\mathcal{E}_j \to \mathcal{D}_j$ of the formal disks $\mathcal{D}_j$ corresponding to $D_j \subset X$. Since the restriction of $\bar{f}_j$ to $\bar{X}_j - \{\tau_j\}$ is tame, such a lift exists and is unique. The second step consists in lifting the (inseparable) cover $\bar{Y}_0 \to \bar{X}_0$ to a Galois cover $\mathcal{V} \to \mathcal{U}$, where $\mathcal{U}$ is the 'formal neighborhood' of $\hat{X}_R - \cup_j \mathcal{D}_j$ defined in §4.1.3. The existence and uniqueness of such a lift follows from the results of §3.1. In the third step we patch the covers $\mathcal{E}_j \to \mathcal{D}_j$ and $\mathcal{V} \to \mathcal{U}$ together along the boundaries $\mathcal{B}_j$ of the disks $\mathcal{D}_j$ and obtain the desired G-cover $f : Y \to X$. Unlike the first two steps, the third step is not unique; it depends on the choice of what we call a *patching datum* for $\bar{f}$. The construction we have just sketched amounts to the existence of a bijection

$$L(\bar{f}) \;\xrightarrow{\sim}\; P(\bar{f})$$

between the set of lifts of $\bar{f}$ and the set $P(\bar{f})$ of all patching data. Essentially by construction, this bijection is $\mathrm{Gal}(\bar{K}/K_0)$-invariant. Moreover, the results of §3.3.3 can be used to determine the complete structure of $P(\bar{f})$ as $\mathrm{Gal}(\bar{K}/K_0)$-set. Intuitively, the Galois group $\mathrm{Gal}(\bar{K}/K_0)$ acts on $P(\bar{f})$ by 'Dehn-twists' along the boundaries $\mathcal{B}_j$.

4.2.1. For each index $j \in B$, let us now choose, once and for all, a K_0-rational point $x_j \in X$ such that the closure of x_j in $X_0 = \mathbb{P}^1_{W(k)}$ intersects the special fiber $\bar{X}_0 = \mathbb{P}^1_k$ in the critical point τ_j. As usual, we assume that $x_j \in \{0, 1, \infty\}$ for $j \in B_0$. Let $f^\dagger : Y^\dagger \to X = \mathbb{P}^1_{\bar{K}}$ be the lift of the special deformation datum

$(\bar{Z}_0, \omega_0)$ with branch points $(x_j)_{j\in B}$, see §3.3. By definition, $f^\dagger$ is a $G^\dagger$-cover, where $G^\dagger := \boldsymbol{\mu}_p(\bar{K}) \rtimes H_0$. If f_R is a lift of $\bar{f}$, then Proposition 4.3 shows that $f^\dagger$ is isomorphic, as a mere cover, to the auxiliary cover $f^{\rm aux}$ associated to f_R. In the following, we will not identify $f^\dagger$ with $f^{\rm aux}$, because we want to emphasize the fact that such an identification is not canonical.

Let K/K_0 be a finite extension over which $f^\dagger$ has stable reduction and let $f_R^\dagger : Y_R^\dagger \to X_R^\dagger$ (resp. $\bar{f}^\dagger : \bar{Y}^\dagger \to \bar{X}^\dagger$) denote the stable model (resp. the stable reduction) of $f^\dagger$. By Theorem 3.9, the curve $\bar{X}^\dagger$ consists of the original component $\bar{X}_0 = \mathbb{P}^1_k$ and the tails $\bar{X}_j^\dagger$, where j runs over the set $B_{\rm prim} \cup B_{\rm new}$. Moreover, the deformation datum over the original component associated to $\bar{f}^\dagger$ is canonically isomorphic to $(\bar{Z}_0, \omega_0)$. We may and will identify the (unique) component $\bar{Y}_0^\dagger$ of $\bar{Y}^\dagger$ over the original component with the distinguished component $\bar{Y}_0$ of $\bar{Y}$ (by definition, $\bar{Y}_0 = \bar{Z}_0^{1/p}$, see §2.2.3). Recall that we have distinguished points $\eta_j \in \bar{Y}_0$ above $\tau_j \in \bar{X}_0$; they are part of the G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$. For $j \notin B_{\rm wild}$ we let $\bar{Y}_j^\dagger$ denote the irreducible component of $\bar{Y}^\dagger$ which intersects $\bar{Y}_0$ in η_j. Write $\eta_j^\dagger \in \bar{Y}_j^\dagger$ for the point where $\bar{Y}_j^\dagger$ intersects $\bar{Y}_0$. Let $G_j^\dagger \subset G^\dagger$ be the subgroup generated by $\boldsymbol{\mu}_p(\bar{K})$ and H_j. One checks that $(\bar{f}_j^\dagger : \bar{Y}_j^\dagger \to \bar{X}_j^\dagger, \eta_j^\dagger, H_j)$ is a pointed $G_j^\dagger$-tail cover with ramification type (m_j, h_j), totally ramified above τ_j. See §2.2.2.

Let $\mathrm{Aut}_k^0(\bar{Y}^\dagger)$ denote the group of k-linear automorphisms of $\bar{Y}^\dagger$ which normalize the action of $G^\dagger$, commute with the action of $H_0 \subset G^\dagger$ and restrict to the identity on $\bar{Y}_0$. We have seen in §3.3.3 that the absolute monodromy action on $\bar{f}^\dagger$ is a homomorphism

$$\bar{\kappa}^\dagger : \mathrm{Gal}(\bar{K}/K_0) \quad\longrightarrow\quad \mathrm{Aut}_k^0(\bar{Y}^\dagger).$$

Similarly, let $\mathrm{Aut}_k^0(\bar{Y}_j^\dagger)$ denote the group of k-linear automorphisms of $\bar{Y}_j^\dagger$ which normalize the action of $G_j^\dagger$, commute with the action of $H_j \subset G_j^\dagger$ and fix the point η_j. By Proposition 3.11, the group $\mathrm{Aut}_k^0(\bar{Y}_j^\dagger)$ is cyclic of order $(p-1)h_j$; furthermore, restricting $\bar{\kappa}^\dagger$ to $\bar{Y}_j^\dagger$ yields a surjective homomorphism

$$\bar{\kappa}_j^\dagger : \mathrm{Gal}(\bar{K}/K_0) \quad\longrightarrow\quad \mathrm{Aut}_k^0(\bar{Y}_j^\dagger) \;\cong\; \mathbb{Z}/(p-1)h_j.$$

4.2.2. Fix $j \notin B_{\rm wild}$. We denote by $\bar{X}_{j,\infty}$ (resp. $\bar{X}_{j,\infty}^\dagger$) the generic point of the completion of the tail $\bar{X}_j$ (resp. the tail $\bar{X}_j^\dagger$) at the point τ_j. We set $\bar{Y}_{j,\infty} := \bar{Y}_j \times_{\bar{X}_j} \bar{X}_{j,\infty}$ and $\bar{Y}_{j,\infty}^\dagger := \bar{Y}_j^\dagger \times_{\bar{X}_j^\dagger} \bar{X}_{j,\infty}^\dagger$. Let $P_j(\bar{f})$ be the set of pairs $(\lambda, \varphi_\infty)$ consisting of the following objects:

- $\lambda : G^\dagger \stackrel{\sim}{\to} G_0$ is an isomorphism which induces the identity on H_0, and
- $\varphi_\infty : \mathrm{Ind}_{G_j}^G(\bar{Y}_{j,\infty}) \stackrel{\sim}{\to} \mathrm{Ind}_{G_j^\dagger,\lambda}^G(\bar{Y}_{j,\infty}^\dagger)$ is a G-equivariant isomorphism such that the following holds.
  1. We have $\varphi_\infty(\eta_j) = \eta_j^\dagger$.
  2. The induced isomorphism $\psi_\infty : \bar{X}_{j,\infty} \stackrel{\sim}{\to} \bar{X}_{j,\infty}^\dagger$ extends to an isomorphism $\psi : \bar{X}_j \stackrel{\sim}{\to} \bar{X}_j^\dagger$.
  3. If $j \in B_{\rm prim}$ then $\psi(\bar{x}_j) = \bar{x}_j^\dagger$.

  (The notation '$\mathrm{Ind}_{G_j^\dagger,\lambda}^G$' means that we embed $G_j^\dagger$ into G via $\lambda : G^\dagger \stackrel{\sim}{\to} G_0 \subset G$.)

The group $\mathrm{Aut}_k^0(\bar{Y}_j^\dagger)$ acts on the set $P_j(\bar{f})$ from the left, as follows. Let $(\lambda, \varphi_\infty)$ be an element of $P_j(\bar{f})$ and γ an element of $\mathrm{Aut}_k^0(\bar{Y}_j^\dagger)$. Let $\mathrm{inn}(\gamma^{-1}) : G^\dagger \stackrel{\sim}{\to} G^\dagger$ denote the unique automorphism of $G^\dagger$ which sends an element $\alpha \in I^\dagger$ to $\gamma^{-1}\circ\alpha\circ\gamma$ and induces the identity on $H_0 \subset G^\dagger$. Set $\lambda' := \lambda\circ \mathrm{inn}(\gamma^{-1})$. One checks that γ extends uniquely to a G-equivariant isomorphism

$$\tilde\gamma : \mathrm{Ind}_{G_j^\dagger,\lambda}^G(\bar Y_j^\dagger) \;\stackrel{\sim}{\longrightarrow}\; \mathrm{Ind}_{G_j^\dagger,\lambda'}^G(\bar Y_j^\dagger).$$

Let φ'_∞ be the composition of φ_∞ with the local isomorphism induced by $\tilde\gamma$. Then $(\lambda',\varphi'_\infty)$ is an element of $P_j(\bar f)$. This defines the action of $\mathrm{Aut}_j^0(\bar Y_j^\dagger)$ on $P_j(\bar f)$. It is easy to see that this action makes $P_j(\bar f)$ a principal homogeneous $\mathrm{Aut}_k^0(\bar Y_j^\dagger)$-space. Via the monodromy action $\bar\kappa_j^\dagger$, we obtain a transitive Galois action of $\mathrm{Gal}(\bar K/K_0)$ on $P_j(\bar f)$.

Define

$$P(\bar f) \;:=\; \{\, (\lambda;\varphi_{j,\infty}) \;\mid\; (\lambda,\varphi_{j,\infty})\in P_j(\bar f)\;\forall j \,\}.$$

An element of $P(\bar f)$ is called a *patching datum.* The actions of the groups $\mathrm{Aut}_k^0(\bar Y_j^\dagger)$ on $P_j(\bar f)$ extend uniquely to an action of $\mathrm{Aut}_k^0(\bar Y^\dagger)$ on $P(\bar f)$ (use Proposition 3.11 (1)). Again, this action is transitive and fixed point free. We consider $P(\bar f)$ as a $\mathrm{Gal}(\bar K/K_0)$-set, via the monodromy action $\bar\kappa^\dagger$.

Proposition 4.4. *The set $P(\bar f)$ of patching data has exactly $(p-1)\cdot\prod_j h_j$ elements. The $\mathrm{Gal}(\bar K/K_0)$-orbits have length*

$$N \;:=\; (p-1)\cdot \mathrm{lcm}_j(h_j).$$

In particular, the action of $\mathrm{Gal}(\bar K/K_0)$ on $P(\bar f)$ is tamely ramified.

Proof. By Proposition 3.11 and the freeness of the action of $\mathrm{Aut}_k^0(\bar Y^\dagger)$ on $P(\bar f)$, it suffices to show that $P(\bar f)$ is nonempty. But this follows from Lemma 2.12. □

4.2.3. We shall now define a $\mathrm{Gal}(\bar K/K_0)$-equivariant map

$$L(\bar f) \;\longrightarrow\; P(\bar f), \tag{23}$$

as follows. Let $f_R : Y_R \to X_R$ be a lift of $\bar f$. Let $f^{\mathrm{aux}} : Y^{\mathrm{aux}} \to X$ be the auxiliary cover associated to f and the set $(x_j)_{j\in B}$. Let $f_R^{\mathrm{aux}} : Y_R^{\mathrm{aux}} \to X_R$ denote the stable model of the auxiliary cover f^{aux}. By Proposition 4.3 (1), there exists a K-linear isomorphism $\varphi_K : Y_K^{\mathrm{aux}} \stackrel{\sim}{\to} Y_K^\dagger$ which induces the identity on X_K. By the uniqueness of the stable model, φ_K extends to an isomorphism $\varphi_R : Y_R^{\mathrm{aux}} \stackrel{\sim}{\to} Y_R^\dagger$. Let $\bar\varphi : \bar Y^{\mathrm{aux}} \stackrel{\sim}{\to} \bar Y^\dagger$ denote the restriction of φ_R to the special fiber. After composing φ_K with an element of $G^\dagger$, we may assume that φ_K commutes with the action of H_0 (which is a subgroup of G_0 and $G^\dagger$) and that $\bar\varphi$ induces the identity on $\bar Y_0$ (which is a component of $\bar Y^{\mathrm{aux}}$ and $\bar Y^\dagger$). Note that this determines φ_K uniquely. Let $\lambda : G_0 \stackrel{\sim}{\to} G^\dagger$ be the isomorphism $\alpha \mapsto \varphi_K\circ\alpha\circ\varphi_K^{-1}$. Fix $j\notin B_{\mathrm{wild}}$. Let $\bar Y_j^{\mathrm{aux}}$ denote the component of $\bar Y^{\mathrm{aux}}$ which intersects $\bar Y_0$ in η_j, and set $\bar Y_{j,\infty}^{\mathrm{aux}} := \bar Y_j^{\mathrm{aux}} \times_{\bar X_j} \bar X_{j,\infty}$. The isomorphism (22) induces a G-equivariant isomorphism

$$\mathrm{Ind}_{G_j}^G(\bar Y_{j,\infty}) \;\stackrel{\sim}{\longrightarrow}\; \mathrm{Ind}_{G_0}^G(\bar Y_{j,\infty}^{\mathrm{aux}}). \tag{24}$$

Let $\varphi_{j,\infty} : \mathrm{Ind}_{G_j}^G(\bar Y_{j,\infty}) \stackrel{\sim}{\to} \mathrm{Ind}_{G^\dagger}^G(\bar Y_{j,\infty}^\dagger)$ be the composition of this isomorphism with the isomorphism induced by $\bar\varphi$. By construction, the tuple $(\lambda;\varphi_{j,\infty})$ is a patching datum. This defines the map (23).

We claim that (23) is $\mathrm{Gal}(\bar K/K_0)$-equivariant. Indeed, for any $\sigma\in\mathrm{Gal}(\bar K/K_0)$, we may identify the auxiliary cover of the twisted cover ${}^\sigma f_R := f_R\otimes_R^\sigma R$ with the σ-twist ${}^\sigma(f_R^{\mathrm{aux}}) := f_R^{\mathrm{aux}}\otimes_R^\sigma R$ of the auxiliary cover of f_R. Let $\kappa_{R,\sigma}^\dagger : {}^\sigma(Y_R^\dagger) \stackrel{\sim}{\to} Y_R^\dagger$ be the unique extension of the canonical K-linear isomorphism ${}^{\sigma}Y_K^{\dagger} \stackrel{\sim}{\to} Y_K^{\dagger}$ induced from the K_0-model $Y_{K_0}^{\dagger}$. The restriction of $\kappa_{R,\sigma}$ to the special fiber is equal to $\bar{\kappa}^{\dagger}(\sigma)$, i.e. the monodromy action of σ on $\bar{Y}^{\dagger}$. The composition $\varphi_R' := \kappa_{R,\sigma}^{\dagger} \circ \varphi_R$ is the unique isomorphism which induces an H_0-equivariant isomorphism of mere covers $f_K^{\rm aux} \cong f_K^{\dagger}$ and whose restriction to the special fiber is the identity on $\bar{Y}_0$. It follows that the map (23) maps the class of the twisted lift ${}^{\sigma}f_R$ to the image of $(\lambda;\varphi_{j,\bar{\infty}})$ under the action of $\bar{\kappa}^{\dagger}(\sigma) \in \mathrm{Aut}_k^0(\bar{Y}^{\dagger})$. This proves the claim that (23) is $\mathrm{Gal}(\bar{K}/K_0)$-equivariant.

Theorem 4.5. *The map* (23) *is an isomorphism of* $\mathrm{Gal}(\bar{K}/K_0)$*-sets.*

Proof. We will show that (23) is an isomorphism by constructing an inverse $P(\bar{f}) \to L(\bar{f})$. Let $(\lambda;\bar{\varphi}_{j,\bar{\infty}}) \in P(\bar{f})$ be a patching datum. We have to construct a lift $f_R : Y_R \to X_R$ of $\bar{f}$ which is mapped to $(\lambda;\bar{\varphi}_{j,\bar{\infty}})$ by (23).

By definition, the local isomorphism $\bar{\psi}_{j,\bar{\infty}} : \bar{X}_{j,\bar{\infty}} \stackrel{\sim}{\to} \bar{X}_{j,\bar{\infty}}^{\dagger}$ extends to a global isomorphism $\bar{\psi}_j : \bar{X}_j \stackrel{\sim}{\to} \bar{X}_j^{\dagger}$. Moreover, if $j \in B_{\rm prim}$ then $\psi_j(\bar{x}_j) = \bar{x}_j^{\dagger}$. Let $\bar{\psi} : \bar{X} \stackrel{\sim}{\to} \bar{X}^{\dagger}$ denote the isomorphism which is equal to the identity on the original component and which is equal to $\bar{\psi}_j$ on the tails. We define $X_R := X_R^{\dagger}$ as a marked curve over R. However, we identify the special fiber of X_R with $\bar{X}$ via the isomorphism $\bar{\psi}$.

Let $\hat{X}_R$ denote the formal completion of X_R along $\bar{X}$. For $j \notin B_{\rm wild}$, let $\mathcal{D}_j \subset \hat{X}_R$ be the open formal subscheme corresponding to $\bar{X}_j - \{\tau_j\} \subset \bar{X}$ and let $\mathcal{X}_j$ be the completion of X_R at τ_j. Let $\mathcal{U}^{\circ} \subset \hat{X}_R$ be the open formal subscheme corresponding to $\bar{X}_0 - \{\tau_j \mid j \in B_{\rm prim} \cup B_{\rm new}\} \subset \bar{X}$. Let $\mathcal{U}$ denote the disjoint union of $\mathcal{U}^{\circ}$ with all the $\mathcal{X}_j$. The fiber product

$$\mathcal{B}_j \;:=\; \mathcal{U} \times_{\hat{X}_R} \mathcal{D}_j \;\cong\; \mathrm{Spec}\, R[[T]]\{T^{-1}\}$$

is the 'boundary' of the formal disk $\mathcal{D}_j$. Its special fiber $\mathcal{B}_j \otimes_R k$ is canonically isomorphic to $\bar{X}_{j,\bar{\infty}}^{\dagger}$ and isomorphic to $\bar{X}_{j,\bar{\infty}}$ via $\bar{\psi}_{j,\bar{\infty}}$. Define

$$\mathcal{V} \;:=\; \mathrm{Ind}_{G^{\dagger},\lambda}^{G}(\hat{Y}_R^{\dagger} \times_{\hat{X}_R} \mathcal{U}).$$

Furthermore, let $\mathcal{E}_j \to \mathcal{D}_j$ be the (unique) tame lift of the (not necessarily connected) G-cover

$$\bar{Y} \times_{\bar{X}} (\bar{X}_j - \{\tau_j\}) \;\longrightarrow\; \bar{X}_j - \{\tau_j\} \;\xrightarrow{\bar{\psi}_j}\; \bar{X}_j^{\dagger} - \{\tau_j\}$$

which is unramified away from the section $x_{j,R} : \mathrm{Spec}\, R \to \mathcal{D}_j$. The local isomorphism $\varphi_{j,\bar{\infty}}$ yields a G-equivariant isomorphism

$$(\mathcal{E}_j \times_{\mathcal{D}_j} \mathcal{B}_j) \otimes_R k \;\cong\; \mathrm{Ind}_{G_j}^{G}(\bar{Y}_{j,\bar{\infty}}) \;\xrightarrow{\bar{\varphi}_{j,\bar{\infty}}}\; \mathrm{Ind}_{G^{\dagger},\lambda}^{G}(\bar{Y}_{j,\bar{\infty}}^{\dagger}) \;\cong\; (\mathcal{V} \times_{\mathcal{U}} \mathcal{B}_j) \otimes_R k$$

of étale $\bar{X}_{j,\bar{\infty}}^{\dagger}$-schemes. It lifts canonically to a G-equivariant isomorphism

$$\varphi_j : \mathcal{E}_j \times_{\mathcal{D}_j} \mathcal{B}_j \;\stackrel{\sim}{\longrightarrow}\; \mathcal{V} \times_{\mathcal{U}} \mathcal{B}_j$$

of étale formal $\mathcal{B}_j$-schemes. By a theorem of Ferrand–Raynaud [FR70] (see also [Pri00]), there exists a formal $\hat{X}_R$-scheme $\mathcal{Y}$ with G-action such that $\mathcal{E}_j = \mathcal{Y} \times_{\hat{X}_R} \mathcal{D}_j$ and $\mathcal{V} = \mathcal{Y} \times_{\hat{X}_R} \mathcal{U}$. By Grothendieck's Existence Theorem, $\mathcal{Y}$ is the formal completion of a projective R-curve Y_R. By construction, the natural map $f_R : Y_R \to X_R$ is a lift of $\bar{f}$ which is mapped by (23) to the patching datum $(\lambda;\bar{\varphi}_{j,\bar{\infty}})$. This completes the proof of the theorem. □

4.3. **Some applications of Theorem 4.5.** Let k be an algebraically closed field of characteristic $p > 0$. We denote by K_0 the fraction field of $W(k)$ and by $\bar{K}$ an algebraic closure of K_0. Theorem 4.5 together with Proposition 4.4 implies:

Corollary 4.6. *Let $\bar{f} : \bar{Y} \to \bar{X}$ be a special G-map defined over k. Then there exists a three point G-cover $f : Y \to X := \mathbb{P}^1_{\bar{K}}$ whose stable reduction is isomorphic to $\bar{f}$.*

This corollary is used in [BW02] to determine the stable reduction of all three point covers with Galois group $\mathrm{PSL}_2(p)$. In particular, this gives a new proof for the result of Deligne–Rapoport on the reduction of the modular curves $X_0(p)$ and $X_1(p)$.

In addition to the existence of three point covers with given stable reduction, Theorem 4.5 and Proposition 4.4 yield interesting arithmetic information about a given three point cover.

Corollary 4.7. *Let $f : Y \to X = \mathbb{P}^1_{\bar{K}}$ be a three point Galois cover with mild reduction. Let $(m_j, h_j)_{j\in B}$ be the signature of the stable reduction of f. The cover f has stable reduction over the unique tame extension K^{st}/K_0 of degree*

$$N \;:=\; (p-1)\cdot \operatorname*{lcm}_{j\notin B_{\mathrm{wild}}}(h_j).$$

In particular, any three point Galois cover $f : Y \to X$ whose degree is not divisible by p^2 can be defined over a tame extension of K_0.

Proof. Let $f_R : Y_R \to X_R$ be a stable model of f and $\bar{f} : \bar{Y} \to \bar{X}$ the stable reduction. If $\bar{f}$ is not exceptional, then the corollary follows from Proposition 3.11 and Theorem 4.5. We may therefore assume that $\bar{f}$ is exceptional. We will also assume that we are in Case (2) of Definition 2.1, leaving Case (1) for the reader. By Proposition 2.2, the curve $\bar{X}$ consists of the original component and a single new tail $\bar{X}_{j_0}$.

Let $f_R^{\mathrm{aux}} : Y_R^{\mathrm{aux}} \to X_R$ denote the auxiliary cover associated to f_R, see §4.1.3. The Galois group of f_R^{aux} is a subgroup $G_0 \subset G$, which is a *direct* product $G_0 = I_0 \times H_0$. Here $I_0 \cong \mathbb{Z}/p$ is the inertia group of a component $\bar{Y}_0$ of $\bar{Y}$ over $\bar{X}_0$ and H_0 is the Galois group of the cover $\bar{Z}_0 = \bar{Y}_0^{(p)} \to \bar{X}_0$. Let $W_R := Y_R^{\mathrm{aux}}/H_0$. Then $W_R \to X_R$ is a stable model of the p-cyclic Galois cover $W_K \to X_K = \mathbb{P}^1_K$, which is ramified at 0, 1 and ∞. Therefore, the curve W_K has an equation of the form

$$y^p \;=\; f(x) := c\,x^a(x-1)^b,$$

with integers a, b such that $a, b, a+b \not\equiv 0 \mod p$ and a unit $c \in R^\times$. After enlarging the field K if necessary and a change of coordinates we may assume that $c \in W(k)^\times$ and $f(a/(a+b)) \in (W(k)^\times)^p$. It is known from [CM88] (see also [Leh01]) that the model X_R which leads to the stable reduction of $W_K \to X_K$ is the blowup of $\mathbb{P}^1_R$ in the disk

$$D \;:=\; \{\, x \mid \mathrm{val}(x - \frac{a}{a+b}) \;\geq\; \frac{p\,\mathrm{val}\,p}{2(p-1)} \,\}.$$

Moreover, the stable model $W_R \to X_R$ descents to a stable model over the ring

$$R' \;:=\; W(k)[\,p^{1/2(p-1)},\, f(\frac{a}{a+b})^{1/p}\,] \;=\; W(k)[\,p^{1/2(p-1)}\,].$$

Note that the (admissible) H_0-cover $\bar{Y}^{\mathrm{aux}} \to \bar{W} := W_R \otimes_R k$ is ramified at the unique singular point of $\bar{W}$ of order $m = h/2$. Therefore, the stable model $f_R^{\mathrm{aux}} : Y_R^{\mathrm{aux}} \to X_R$ descents to a stable model over the ring $R'' := W(k)[p^{1/h(p-1)}]$. Using the method of the proof of Theorem 4.5, it is not hard to show that $f_R : Y_R \to X_R$ descents to a stable model over R''. This finishes the proof of the corollary. □

The last corollary can be refined in order to obtain a formula for the ramification index of the field of moduli of a three point cover in terms of its stable reduction.

Corollary 4.8. *Let $\bar{f} : \bar{Y} \to \bar{X}$ be a special G-map, with signature $(m_j, h_j)_{j\in B}$. Let $P \subset G$ denote a p-Sylow of G and let $n := [N_G(P) : C_G(P)]$ be the index of the centralizer of P inside the normalizer of P. Recall that $\tilde{L}(\bar{f})$ denotes the set of isomorphism classes of three point G-covers defined over $\bar{K}$ whose stable reduction is isomorphic to $\bar{f}$. There exists a positive integer n' dividing n which depends only on $\bar{f}$ such that*

$$|\tilde{L}(\bar{f})| \;=\; \frac{p-1}{n'} \prod_{j\notin B_{\rm wild}} \frac{h_j}{|\operatorname{Aut}_G(\bar{f}_j)|}.$$

Let $f : Y \to X$ be a lift of $\bar{f}$. Then the field of moduli $K^{\rm in}$ of f (relative to the extension $\bar{K}/K_0$) is the unique tame extension of K_0 of degree

$$N' \;:=\; \frac{p-1}{n'} \operatorname*{lcm}_{j\notin B_{\rm wild}} \frac{h_j}{|\operatorname{Aut}_G(\bar{f}_j)|}.$$

Proof. We have seen in §4.1.2 that $\tilde{L}(\bar{f}) = L(\bar{f})/\operatorname{Aut}^0_G(\bar{f})$. Moreover, the stabilizer in $\operatorname{Aut}^0_G(\bar{f})$ of any element of $L(\bar{f})$ is equal to C_G. Let n' be the order of the image of the homomorphism $\operatorname{Aut}^0_G(\bar{f}) \to G/C_G$ which is defined by Lemma 2.17. It is easy to see that n' divides n. We see that Corollary 4.8 follows from Proposition 3.11, Theorem 4.5 and Lemma 2.17. □

4.4. **Genus zero dessins of degree p.** Let p be a prime and $G \subset S_p$ be a primitive transitive permutation group of degree p. We write $N \subset G$ for the stabilizer of 1. Let $f : Y \to X := \mathbb{P}^1_{\bar{\mathbb{Q}}}$ be a three point Galois cover, with Galois group G. Then $g : Z := Y/N \to X$ is a three point cover of degree p, and $f : Y \to X$ is the Galois closure of g. We make the following assumptions.

Assumption 4.9.
(1) The subgroup $N \subset G$ is self-normalizing and G is center-free.
(2) The curve Z has genus 0.

The first assumption is made only for simplicity. It ensures that the G-cover $f : Y \to X$ and the (non-Galois) cover $g : Z \to X$ have the same field of moduli, and that this field of moduli is also the minimal field of definition for both covers, see e.g [DD97]. The second assumption is more restrictive and implies that the stable reduction of $f : Y \to X$ to characteristic p is rather easy to describe.

Let K be the field of moduli of $f : Y \to X$, relative to the extension $\bar{\mathbb{Q}}/\mathbb{Q}$. Let $\mathfrak{p}$ be a prime ideal of K dividing p. There exists a finite extension K'/K and a prime $\mathfrak{p}'$ of K' over $\mathfrak{p}$ such that f has a K'-model $f_{K'} : Y_{K'} \to X_{K'}$ which extends to a stable model over the local ring $\mathcal{O}_{K',\mathfrak{p}'}$. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the stable reduction of f'_K at $\mathfrak{p}'$. One checks that $\bar{f} : \bar{Y} \to \bar{X}$ depends only on the field K and the prime $\mathfrak{p}$. Hence we may call $\bar{f} : \bar{Y} \to \bar{X}$ the stable reduction of $f : Y \to X$ at $\mathfrak{p}$.

Let us assume that $f : Y \to X$ has bad reduction at $\mathfrak{p}$. Note that p strictly divides the order of G. Therefore, by Theorem 2.6 the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ is a special G-map, associated to a special G-deformation datum $(\bar{Z}_0, \omega_0; \bar{Y}_j)$, see §2.2.3. In the following, we use freely the notation introduced in §2.2.3.

Lemma 4.10. (1) *The character $\chi : H_0 \to \mathbb{F}_p^\times$ is injective. In particular, $H_0 \cong \mathbb{Z}/m$ with $m|(p-1)$.*

(2) *There are no new tails.*

(3) *For $j \in B_{\rm prim}$, let $x_j \in \{0,1,\infty\}$ denote the branch point of $f : Y \to X$ which specializes to the primitive tail $\bar{X}_j$. The ramification invariant of the primitive tail cover $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ is given by the formula*

$$\sigma_j \;=\; \frac{|g^{-1}(x_j)|-1}{p-1}.$$

Proof. Part (1) is an immediate consequence of the fact that the p-cyclic subgroup I_0 of S_p is self-centralizing.

To prove (2), suppose that $f_j : \bar{Y}_j \to \bar{X}_j$ is a new tail, with Galois group G_j. Let $N_j := N \cap G_j$ and $\bar{Z}_j := \bar{Y}_j/N_j$. It follows from Assumption 4.9 (2) that the curve $\bar{Z}_j$ has genus 0. Since N has index p in G and G_j contains a p-Sylow of G, the index of N_j in G_j is also p. We see that $\bar{g}_j : \bar{Z}_j \to \bar{X}_j$ is a cover of degree p between curves of genus 0, totally branched at $\bar{\infty}_j \in \bar{X}_j$ and étale elsewhere. This means that in suitable coordinates $\bar{g}_j$ is given by a monic polynomial $q \in k[z]$ of degree p such that the derivative q' has degree 0. Hence $q = z^p + az + b$, and after a further normalization we get $q = z^p - z$. It follows that the tail cover $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ is an Artin–Schreier cover with conductor 1, i.e. $\sigma_j = 1$. This contradicts the results of §2. We conclude that $\bar{f} : \bar{Y} \to \bar{X}$ has no new tails, proving (2).

For $j \in B_{\rm prim}$, let $\bar{g}_j : \bar{Z}_j \to \bar{X}_j$ be as before. The same argument as before shows that $\bar{g}_j$ is a cover between smooth curves of genus 0 of degree p, totally ramified at $\bar{\infty}_j \in \bar{X}_j$. But now $\bar{g}_j$ is tamely ramified at $\bar{x}_j$ and étale over $\bar{X}_j - \{\bar{\infty}_j, \bar{x}_j\}$ (recall that $\bar{x}_j \in \bar{X}_j$ denotes the specialization of the branch point $x_j \in \{0,1,\infty\}$). Since the stable model of f is a tame cover in a neighborhood of $\bar{X}_j - \{\bar{\infty}_j\}$ we have $|\bar{g}_j^{-1}(\bar{x}_j)| = |g^{-1}(x_j)|$. Now (3) follows from a calculation with the Riemann–Hurwitz formula and the fact that $g(\bar{Z}_j) = 0$. See e.g. [BP], Lemma 2.3.1. □

The lemma shows that the signature of $\bar{f} : \bar{Y} \to \bar{X}$ is a tripel $(\sigma_1, \sigma_2, \sigma_3)$ of rational numbers, with $\sigma_j = h_j/m_j$, $0 \le h_j < m_j$ and $(h_j, m_j) = 1$. It satisfies the vanishing cycle formula

$$\sigma_1 + \sigma_2 + \sigma_3 \;=\; 1. \tag{25}$$

The special deformation datum $(\bar{Z}_0, \omega_0)$ is essentially determined by the signature. Indeed, the curve $\bar{Z}_0$ is isomorphic to the smooth projective model of the plane curve with equation

$$z^m \;=\; x^{h_1}(x-1)^{h_2}, \tag{26}$$

where m is the least common multiple of the m_j, and the differential ω_0 has the form

$$\omega_0 \;=\; \epsilon\,\frac{z\,\mathrm{d}x}{x(x-1)}, \tag{27}$$

for some $\epsilon \in \mathbb{F}_p^\times$. Conversely, given a tripel $(\sigma_1, \sigma_2, \sigma_3)$ satisfying all of the above conditions, one checks that (26) and (27) define a special deformation datum $(\bar{Z}_0, \omega_0)$. Therefore, the existence of a special deformation datum with a given signature imposes no restriction on the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$.

Example 4.11. Let $p = 7$ and $G := S_7$. Suppose furthermore that the branch cycle description of the cover $g : Z \to X$ is $(6, 6, 2\text{-}2)$ (each entry designates a conjugacy class of S_7, e.g. '2-2' denotes the product of two 2-cycles). From the table computed by Malle [Mal94] we see that there are exactly 4 non-isomophic three point covers with monodromy group S_7 and this branch cycle description. Moreover, the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes these four covers transitively, with permutation group S_4. Therefore, the field of moduli of the G-cover $f : Y \to X$ is a degree 4 extension $K/\mathbb{Q}$.

Let $\mathfrak{p}$ be a prime ideal of K over 7. Suppose that $f : Y \to X$ has bad reduction at $\mathfrak{p}$ and let $\bar{f} : \bar{Y} \to \bar{X}$ denote the stable reduction. It follows from Lemma 4.10 (3) that the signature of $\bar{f}$ is equal to $(1/6, 1/6, 2/3)$. For $j = 1, 2$, the Riemann–Hurwitz formula implies that the Galois group G_j of the primitive tail cover $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ is isomorphic to the semi-direct product $7 : 6$. It follows that $\bar{Y}_j \to \bar{X}_j$ is given by the two equations

$$z^6 = x, \qquad y^7 - y = z. \tag{28}$$

The third tail cover $\bar{f}_3 : \bar{Y}_3 \to \bar{X}_3$ has inertia invariant $\sigma_3 = 2/3$ and branch cycle description 2-2 at the tame branch point. We say that $\bar{f}_3$ is a *primitive tail cover of type* $(2/3, 2\text{-}2)$. The Galois group G_3 of $\bar{f}_3$ is a subgroup of A_7 containing a semidirect product $7 : 3$ and a permutation of type 2-2. By elementary group theory, G_3 is isomorphic to A_7 or to $L_2(7) := \mathrm{PSL}_2(7)$.

It is a priori not clear how many non-isomorphic primitive tail covers of type $(2/3, 2\text{-}2)$ exist, and what their Galois groups are. We claim that there exists in fact a *unique* primitive tail cover of type $(2/3, 2\text{-}2)$, with Galois group $L_2(7)$. The existence of one such cover is proved in [BW02]. (It is obtained by reducing a certain three point cover with Galois group $L_2(7)$.) To prove uniqueness, we use our lifting result, Theorem 4.5.

Given $\bar{f}_3 : \bar{Y}_3 \to \bar{X}_3$ as above, one shows that there exists, up to isomorphism, a unique special G-map $\bar{f} : \bar{Y} \to \bar{X}$ with signature $(1/6, 1/6, 2/3)$ whose third tail cover is isomorphic to $\bar{f}_3 : \bar{Y}_3 \to \bar{X}_3$. By Corollary 4.8 there exist exactly

$$N' := \frac{12}{n' \cdot |\,\mathrm{Aut}_{G_3}(\bar{f}_3)|}$$

nonisomorphic three point covers $f' : Y' \to X'$ whose stable reduction (at a given prime $\tilde{\mathfrak{p}}$ of $\bar{\mathbb{Q}}$) is isomorphic to $\bar{f}$. Moreover, all of these covers are conjugate under the action of the inertia group of $\tilde{\mathfrak{p}}$. Using Lemma 2.17, Remark 2.18 and Equation (28) one shows that $n' = 3$. Thus, N' is equal to either 2 or 4.

As we have mentioned before, all three point covers with Galois group S_7 and branch cycle description $(6, 6, 2\text{-}2)$ are conjugate. Hence we may assume that $f = f'$. We conclude that for each primitive tail cover $\bar{f}_3$ of type $(2/3, 2\text{-}2)$ there exists a prime $\mathfrak{p}$ in K above 7 with $e(\mathfrak{p}/7) \in \{2, 4\}$. This prime $\mathfrak{p}$ is uniquely determined by the property that the stable reduction of $f : Y \to X$ at $\mathfrak{p}$ has a primitive tail cover isomorphic to $\bar{f}_3$. By Malle's table, we have $7 = \mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3$ in K. Therefore, our analysis of the stable reduction implies:

- The cover $f : Y \to X$ has bad reduction at $\mathfrak{p}_1$ and good reduction at $\mathfrak{p}_2$ and $\mathfrak{p}_3$. The stable reduction of $f : Y \to X$ at $\mathfrak{p}_1$ occurs after an extension which is ramified at $\mathfrak{p}_1$ of order 2.
- There exists a unique primitive tail cover of type $(2/3, 2\text{-}2)$. Its Galois group is $L_2(7)$.

We remark that good reduction at $\mathfrak{p}_2$ and $\mathfrak{p}_3$ does not follow from the results of [Ray99], because S_7 has only one conjugacy class of elements of order 7.

Example 4.12. As before, let $p = 7$ and $G = S_7$. But now suppose that the branch cycle description of the cover $f : Y \to X$ is $(2\text{-}3, 2\text{-}3, 7)$. By Malle's table there are 9 such covers, up to isomorphism. Note that they all have bad reduction to characteristic 7, because of the ramification index 7.

It follows from Lemma 4.10 (3) that the signature of $\bar{f} : \bar{Y} \to \bar{X}$ is equal to $(1/2, 1/2, 0)$. For $j = 1, 2$, the Galois group G_j of the tail cover $\bar{f}_j : \bar{Y}_j \to \bar{X}_j$ contains a 7-cycle and a 2-cycle. Hence, by a theorem of Jordan we have $G_j = S_7$.

We claim that there exists, up to isomorphism, a unique primitive tail cover $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ of type $(1/2, 2\text{-}3)$ with Galois group S_7. We have already shown its existence. To prove uniqueness, we use again Theorem 4.5.

Let us choose a point $\eta_1 \in \bar{Y}_1$ above the wild branch point $\bar{\infty}_1 \in \bar{X}_1$. Since $\sigma_1 = 1/2$ the inertia group at η_1 is a dihedral group $G_0 = I_0 \rtimes H_0$, with $|I_0| = 7$ and $H_0 = |2|$. Recall that the triple $(\bar{f}_1, \eta_1, H_0)$ is called a pointed S_7-tail cover. Let $\bar{f}_2 : \bar{Y}_2 \to \bar{X}_2$ be a copy of the S_7-tail cover $\bar{f}_1$. It is easy to see that there are exactly three points on $\bar{Y}_2$ above $\bar{\infty}_2$ with inertia group G_0. Let η_2 be one of them. Then $(\bar{f}_2, \eta_2, H_0)$ is another pointed S_7-tail cover. It is isomorphic to $(\bar{f}_1, \eta_1, H_0)$ (by an isomorphism commuting with the action of S_7) if and only if $\eta_1 = \eta_2$. Finally, let $(\bar{Z}_0, \omega_0)$ be the (unique) special deformation datum with signature $(1/2, 1/2, 0)$. One checks that $(\bar{Z}_0, \omega_0, \bar{Y}_1, \bar{Y}_2)$ gives rise to a special G-deformation datum and hence to a special G-map $\bar{f} : \bar{Y} \to \bar{X}$, see §2.2.3. Moreover, by varying the point η_2 we obtain exactly three non-isomorphic special G-maps with signature $(1/2, 1/2, 0)$ and whose primitive tail covers are both isomorphic to $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$.

Choose one of the three special G-maps $\bar{f} : \bar{Y} \to \bar{X}$ constructed from $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$. By Corollary 4.8 there exist exactly $N' = 6/n'$ non-isomorphic three point covers $f' : Y' \to X$ with Galois group S_7 whose stable reduction (at a given place $\tilde{\mathfrak{p}}$ above 7) is isomorphic to $\bar{f}$. Moreover, they form a single orbit under the action of the inertia group at $\tilde{\mathfrak{p}}$. Using Remark 2.18 we see that $2|n'$. Using the fact that S_7 has no outer automorphism one further shows that $n' = 2$ and hence $N' = 3$. Therefore, for each primitive tail cover $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ we obtain, by our lifting result, exactly 9 non-isomorphic three point covers with Galois group S_7 and branch cycle description $(2\text{-}3, 2\text{-}3, 7)$. However, there are only 9 such covers altogether. This proves that $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ is indeed unique, up to isomorphism. As a by-product, we have also shown that the absolute ramification index of any prime ideal $\mathfrak{p}$ dividing 7 in the field of moduli of $f : Y \to X$ is equal to 3.

By [Mal94] the set of 9 isomorphism classes form two Galois orbits, of length 3 and 6 respectively, corresponding to extensions $K_1/\mathbb{Q}$ and $K_2/\mathbb{Q}$. In K_1 we have $7 = \mathfrak{p}_1^3$ and in K_2 we have $7 = \mathfrak{p}_2^3\mathfrak{p}_3^3$. This confirms our prediction.

Remark 4.13. One can use Malle's table [Mal94] to verify our results in many more examples. In particular, one gets a numerical verification of Theorem 1 of the Introduction for all genus zero dessins of degree ≤ 10. One also sees that the condition that p strictly divides the order of G is necessary for Theorem 1 to hold. For instance, one finds several three point covers of degree $n \geq 10$, with monodromy group S_n or A_n, such that 5 is wildly ramified in the field of moduli.

Remark 4.14. The relation between the existence of three point covers with bad reduction on the one hand and tail covers with low ramification invariant on the other hand is exploited more systematically in [BP]. In [Zap02] Zapponi studies the stable reduction of genus zero dessins of degree p which are totally ramified at one point, by analyzing the defining equation. His results confirm our results on dessins of genus 0.

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