Generalized Bernstein–Reznikov integrals

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Abstract

We find a closed formula for the triple integral on spheres in $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ whose kernel is given by powers of the standard symplectic form. This gives a new proof to the Bernstein–Reznikov integral formula in the $n = 1$ case. Our method also applies for linear and conformal structures.

1 Triple product integral formula

We consider the symplectic form $[\ ,\ ]$ on $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ given by

$$[(x, \xi), (y, \eta)] := -\langle x, \eta\rangle + \langle y, \xi\rangle. \tag{1.1}$$

In this paper we prove a closed formula for the following triple integral:

Theorem 1.1. *Let $d\sigma$ be the Euclidean measure on the sphere S^{2n-1}. Then,*

$$\int_{S^{2n-1}\times S^{2n-1}\times S^{2n-1}} \big|[Y, Z]\big|^{\frac{\alpha-n}{2}} \big|[Z, X]\big|^{\frac{\beta-n}{2}} \big|[X, Y]\big|^{\frac{\gamma-n}{2}} d\sigma(X)d\sigma(Y)d\sigma(Z)$$
$$= \left(2\pi^{n-\frac{1}{2}}\right)^3 \frac{\Gamma\left(\frac{2-n+\alpha}{4}\right)\Gamma\left(\frac{2-n+\beta}{4}\right)\Gamma\left(\frac{2-n+\gamma}{4}\right)\Gamma\left(\frac{\delta+n}{4}\right)}{\Gamma(n)\Gamma\left(\frac{n-\lambda_1}{2}\right)\Gamma\left(\frac{n-\lambda_2}{2}\right)\Gamma\left(\frac{n-\lambda_3}{2}\right)}.$$

Here, $\alpha = \lambda_1-\lambda_2-\lambda_3$, $\beta = -\lambda_1+\lambda_2-\lambda_3$, $\gamma = -\lambda_1-\lambda_2+\lambda_3$, $\delta = -\lambda_1-\lambda_2-\lambda_3$.

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The integral converges for a non-empty open region of $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ (e.g. the real parts of α, β, and γ are sufficiently large), and extends as a meromorphic function of λ_1, λ_2 and λ_3 by using the regularization of this integral (see e.g. [6]). A special case ($n = 1$) of Theorem 1.1 was previously established by J. Bernstein and A. Reznikov [4]. Our approach uses the Fourier transform in the ambient space and appeals to the classical Bochner identity. It gives a new proof even when $n = 1$.

Sections 2 and 3 are devoted to the proof of Theorem 1.1. In Section 4, we discuss analogous integrals of the triple product kernels involving $|x - y|^\lambda$ or $|\langle x, y\rangle|^\lambda$ instead of $\big|[x, y]\big|^\lambda$. At the end, we explain briefly some perspectives from representation theoretic point of view.

2 Eigenvalues of integral transforms $\mathcal{T}_\mu$

We introduce a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$\mathcal{T}_\mu : C^\infty(S^{2n-1}) \to C^\infty(S^{2n-1})$$

defined by

$$(\mathcal{T}_\mu f)(\eta) := \int_{S^{2n-1}} f(\omega)\big|[\omega, \eta]\big|^{-\mu-n} d\sigma(\omega). \tag{2.1}$$

The integral (2.1) converges if $\operatorname{Re}\mu \ll 0$, and has a meromorphic continuation for $\mu \in \mathbb{C}$. If μ is real and sufficiently negative, then $\mathcal{T}_\mu$ is a self-adjoint, Hilbert–Schmidt operator on $L^2(S^{2n-1})$. In this section, we determine all the eigenvalues of $\mathcal{T}_\mu$ and the corresponding eigenspaces.

2.1 Harmonic polynomials on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$

First, let us remind the classic theory of spherical harmonics on real and complex vector spaces.

For $k \in \mathbb{N}$, we denote by $\mathcal{H}^k(\mathbb{R}^N)$ the vector space consisting of homogeneous polynomials $p(x_1, \ldots, x_N)$ of degree k such that $\displaystyle\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} p = 0$.

By restricting these harmonic polynomials to S^{N-1}, we get an injective map from

$$\mathcal{H}(\mathbb{R}^N) := \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{R}^N)$$

into a dense subspace of $C^\infty(S^{N-1})$.

Analogously, we can define the space of harmonic polynomials on $\mathbb{C}^n$. For $\alpha, \beta \in \mathbb{N}$, we denote by $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ the vector space consisting of polynomials $p(Z, \bar{Z})$ on $\mathbb{C}^n$ subject to the following two conditions:

(1) $p(Z, \bar{Z})$ is homogeneous of degree α in $Z = (z_1, \ldots, z_n)$ and of degree β in $\bar{Z} = (\bar{z}_1, \ldots, \bar{z}_n)$.

(2) $\displaystyle\sum_{j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} p(Z, \bar{Z}) = 0.$

Then, $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ is a finite dimensional vector space. It is non-zero except for the case where $n = 1$ and $\alpha, \beta \geq 1$.

By definition, we have a natural linear isomorphism:

$$\mathcal{H}^k(\mathbb{R}^{2n}) \simeq \bigoplus_{\alpha+\beta=k} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n). \tag{2.2}$$

We shall see that $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ is an eigenspace of the operator $\mathcal{T}_\mu$ for any μ and for every α and β. To be more precise, we introduce a meromorphic function of μ by

$$A_k(\mu, \mathbb{C}^n) \equiv A_k(\mu) := \begin{cases} 0 & (k : \text{odd}), \\ 2\pi^{n-\frac{1}{2}} \dfrac{\Gamma(\frac{1-n-\mu}{2})\Gamma(\frac{k+n+\mu}{2})}{\Gamma(\frac{n+\mu}{2})\Gamma(\frac{k+n-\mu}{2})} & (k : \text{even}). \end{cases} \tag{2.3}$$

We shall use the notation $A_k(\mu, \mathbb{C}^n)$ when we emphasize the ambient space $\mathbb{C}^n$ (see (4.6)).

Theorem 2.1. *For $\alpha, \beta \in \mathbb{N}$,*

$$\mathcal{T}_\mu \Big|_{\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)} = (-1)^\beta A_{\alpha+\beta}(\mu) \, \mathrm{id} \, .$$

The rest of this section is devoted to the proof of Theorem 2.1.

2.2 Application of the Bochner identity

Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbb{R}^N$. We consider the Fourier transform $\mathcal{F} \equiv \mathcal{F}_{\mathbb{R}^N}$ on $\mathbb{R}^N$ normalized by

$$(\mathcal{F}f)(Y) := \int_{\mathbb{R}^N} f(X) e^{-2\pi i \langle X,Y\rangle} dX,$$

and we extend $\mathcal{F}$ to the space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions.

For a given function $p \in C^\infty(S^{N-1})$, we define its extension into a homogeneous function of degree λ by

$$p_\lambda(r\omega) := r^\lambda p(\omega), \quad (r > 0,\ \omega \in S^{N-1}).$$

We introduce the following meromorphic function of λ by

$$B_N(\lambda, k) := \pi^{-\lambda-\frac{N}{2}} i^{-k} \frac{\Gamma(\frac{k+\lambda+N}{2})}{\Gamma(\frac{k-\lambda}{2})}.$$

Lemma 2.2. *For any $p \in \mathcal{H}^k(\mathbb{R}^N)$*

$$\mathcal{F}p_\lambda = B_N(\lambda, k) p_{-\lambda-N}.$$

Proof. Consider polar coordinates $y = s\eta$ ($s > 0$, $\eta \in S^{N-1}$), then we have

$$(\mathcal{F}p_\lambda)(s\eta) = \int_0^\infty \Big(\int_{S^{N-1}} p(\omega) e^{-2\pi i r s \langle \omega,\eta\rangle} d\omega \Big) r^{\lambda+N-1} dr.$$

Now we can use the Bochner identity for $p \in \mathcal{H}^k(\mathbb{R}^N)$:

$$\int_{S^{N-1}} p(\omega) e^{-i\nu\langle\omega,\eta\rangle} d\omega = (2\pi)^{\frac{N}{2}} i^{-k} \nu^{1-\frac{N}{2}} J_{k+\frac{N}{2}-1}(\nu) p(\eta),$$

where $J_m(\nu)$ denotes the Bessel function of the first kind. Then, we get the following formula after a change of variables $x = 2\pi r s$.

$$(\mathcal{F}p_\lambda)(s\eta) = (2\pi)^{-\lambda-\frac{N}{2}} i^{-k} s^{-\lambda-N} p(\eta) \int_0^\infty x^{\lambda+\frac{N}{2}} J_{k+\frac{N}{2}-1}(x) dx. \tag{2.4}$$

Applying the following classical formula of the Hankel transform [7, 6.561.14]

$$\int_0^\infty x^\mu J_\nu(x) dx = 2^\mu \frac{\Gamma(\frac{1+\nu+\mu}{2})}{\Gamma(\frac{1+\nu-\mu}{2})},$$

we see that the right-hand side of (2.4) equals $B_N(\lambda, k)\, p_{-\lambda-N}(s\eta)$, Whence Lemma 2.2. $\square$

2.3 Operator $\mathcal{T}_\mu$ and Symplectic Fourier transform $\mathcal{F}_J$

The key idea to find eigenvalues of the integral transform $\mathcal{T}_\mu$ on $L^2(S^{2n-1})$ is to interpret it as the restriction of the *symplectic Fourier transform*, to be denoted by $\mathcal{F}_J$, on the ambient space $\mathbb{R}^{2n}$.

For this purpose, we first consider the restriction of the (usual) Fourier transform $\mathcal{F}$ on $\mathbb{R}^N$ to the space of homogeneous functions. We introduce a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$\mathcal{Q}_\mu : C^\infty(S^{N-1}) \to C^\infty(S^{N-1})$$

defined by

$$(\mathcal{Q}_\mu h)(\eta) := \int_{S^{N-1}} |\langle \omega, \eta \rangle|^{-\mu-\frac{N}{2}} h(\omega) d\omega. \tag{2.5}$$

We set

$$\begin{aligned} V_\mu \quad &\equiv \quad V_\mu(\mathbb{R}^N) \qquad (2.6)\\ &:= \quad \{f \in C^\infty(\mathbb{R}^N \setminus \{0\}) : f(tX) = |t|^{-\mu-\frac{N}{2}} f(X) \text{ for any } t \in \mathbb{R} \setminus \{0\}\}. \end{aligned}$$

Then, V_μ is contained in the space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions. The Fourier transform $\mathcal{F}$ gives a bijection between $V_{-\mu}$ and V_μ. On the other hand, V_μ can be identified with the space of smooth even functions on S^{N-1}. We notice that the latter space is independent of μ. Thus, we have the following diagram:

$$\begin{array}{ccccc} \mathcal{F} & : & \mathcal{S}'(\mathbb{R}^N) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^N) \\ & & \cup & \circlearrowleft & \cup \\ & & V_{-\mu} & \xrightarrow{\sim} & V_\mu \\ & & \cap & & \cap \\ \mathcal{Q}_\mu & : & C^\infty(S^{N-1}) & \longrightarrow & C^\infty(S^{N-1}) \end{array}$$

The lower diagram commutes up to the scalar constant $C_N(\mu)$ defined by

$$\begin{aligned} C_N(\mu)^{-1} &:= (2\pi)^{-\mu-\frac{N}{2}} \Gamma\left(\mu + \frac{N}{2}\right) \cos \frac{\pi}{2}\left(\mu + \frac{N}{2}\right) \\ &= \frac{\Gamma(\frac{N+2\mu}{4})}{2\pi^{\mu+\frac{N-1}{2}} \Gamma(\frac{2-N-2\mu}{4})}. \end{aligned} \tag{2.7}$$

Proposition 2.3. *As operators that depend meromorphically on μ, $\mathcal{Q}_\mu$ satisfy the following identity:*

$$\mathcal{Q}_\mu = C_N(\mu)\mathcal{F}\big|_{V_{-\mu}}.$$

Proof. Suppose $f \in V_{-\mu}$ is of the form

$$f(r\omega) = r^{\mu-\frac{N}{2}}h(\omega) \quad (r > 0,\ \omega \in S^{N-1}),$$

where $h \in C^\infty(S^{N-1})$ is an even function, i.e., $h(\omega) = h(-\omega)$. By using the polar coordinates, we have

$$\begin{aligned}(\mathcal{F}f)(s\eta) &= \int_{S^{N-1}} h(\omega)\int_0^\infty r^{\mu+\frac{N}{2}-1}e^{-2\pi irs\langle\omega,\eta\rangle}drd\omega \\ &= \Gamma\left(\mu+\frac{N}{2}\right)\int_{S^{N-1}}\frac{h(\omega)e^{-\frac{\pi i}{2}(\mu+\frac{N}{2})\operatorname{sgn}\langle\omega,\eta\rangle}}{|2\pi s\langle\omega,\eta\rangle|^{\mu+\frac{N}{2}}}d\omega.\end{aligned}$$

Since h is an even function, the right-hand side amounts to

$$\begin{aligned}&(2\pi)^{-\mu-\frac{N}{2}}s^{-\mu-\frac{N}{2}}\Gamma\left(\mu+\frac{N}{2}\right)\cos\frac{\pi}{2}\left(\mu+\frac{N}{2}\right)\int_{S^{N-1}}|\langle\omega,\eta\rangle|^{-\mu-\frac{N}{2}}h(\omega)d\omega \\ &= C_N(\mu)^{-1}s^{-\mu-\frac{N}{2}}(\mathcal{Q}_\mu h)(\eta).\end{aligned}$$

Thus, Proposition 2.3 has been proved. $\square$

Suppose now $N = 2n$. We introduce the symplectic Fourier transform defined by the formula:

$$(\mathcal{F}_J f)(Y) := \int_{\mathbb{R}^{2n}} f(X)e^{-2\pi i[X,Y]}dX.$$

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by $(x,\xi) \mapsto x + i\xi$. Correspondingly, the complex structure on $\mathbb{R}^{2n}$ is given by the linear transform

$$J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad J(x,\xi) := (-\xi, x).$$

Then the formula (1.1) is equivalent to

$$[X,Y] = \langle X, JY\rangle \quad (X,Y \in \mathbb{R}^{2n}),$$

and therefore, our $\mathcal{F}_J$ and the usual Fourier transform $\mathcal{F}_{\mathbb{R}^{2n}}$ are related by the formula:

$$(\mathcal{F}_J f)(Y) = \mathcal{F}_{\mathbb{R}^{2n}}(JY). \tag{2.8}$$

Likewise, the linear operators $\mathcal{T}_\mu$ (see (2.1)) and $\mathcal{Q}_\mu$ for $N = 2n$ (see (2.5)) are related by

$$\mathcal{T}_\mu f(Y) = \mathcal{Q}_\mu(JY).$$

Therefore, Proposition 2.3 leads us to:

Proposition 2.4. *Let $C_N(\mu)$ be the constant defined in* (2.7)*. Then,*

$$\mathcal{T}_\mu = C_{2n}(\mu)\mathcal{F}_J\big|_{V_{-\mu}}.$$

Remark 2.5. Since the symplectic Fourier transform $\mathcal{F}_J$ induces a bijection $\mathcal{F}_J\big|_{V_{-\mu}} : V_{-\mu} \xrightarrow{\sim} V_\mu$ for all $\mu \in \mathbb{C}$, Proposition 2.4 implies that $\mathcal{T}_\mu$ is also bijective as far as $C_{2n}(\mu) \neq 0, \infty$.

We note that $C_{2n}(\mu)$ has simple zeros at $\mu + n = 0, -2, -4, \dots$. In this case, the kernel $\big|[X,Y]\big|^{-\mu-n}$ is a polynomial in Y of degree $-(\mu+n)$, and correspondingly, $(\mathcal{T}_\mu f)(Y)$ is also a polynomial of the same degree. Thus, $\operatorname{Image}\mathcal{T}_\mu$ is finite dimensional, and $\operatorname{Ker}\mathcal{T}_\mu$ is infinite dimensional.

On the other hand, $C_{2n}(\mu)$ has simple poles at $\mu + n = 1, 3, 5, \dots$. This corresponds to the fact that the distribution $|x|^\lambda$ of one variable has simple poles at $\lambda = -1, -3, -5, \dots$ (see [6]).

We are now ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose $p \in \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$. Since J acts on z_j $(1 \leq j \leq n)$ by $\sqrt{-1}$ and $\bar{z}_j$ by $-\sqrt{-1}$, we have

$$p(J\eta) = (-1)^{\frac{\alpha-\beta}{2}} p(\eta). \tag{2.9}$$

In view of Lemma 2.2, Proposition 2.4, and (2.8), the operator $\mathcal{T}_\mu$ acts on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ as a scalar

$$(-1)^{\frac{\alpha-\beta}{2}} C_{2n}(\mu) B_{2n}(\mu - n, \alpha + \beta).$$

This amounts to $(-1)^\beta A_{\alpha+\beta}(\mu)$, whence Theorem 2.1. $\square$

3 Proof of Theorem 1.1

3.1 Dimension formulas for spherical harmonics

This subsection summarizes some elementary results on the dimensions of harmonic polynomials in a way that we shall use later. They are more or less known, however, we give a brief account of them for the convenience of the reader.

Let $\mathcal{P}^k(\mathbb{R}^N)$ be the complex vector space of homogeneous polynomials in N variables of degree k. Its dimension is given by the binomial coefficient:

$$\dim \mathcal{P}^k(\mathbb{R}^N) = \binom{k+N-1}{k}.$$

In light of the linear bijection

$$\mathcal{H}^k(\mathbb{R}^N) \oplus \mathcal{P}^{k-2}(\mathbb{R}^N) \xrightarrow{\sim} \mathcal{P}^k(\mathbb{R}^N), \quad (p,q) \mapsto p(X) + |X|^2 q(X),$$

we get the dimension formula of $\mathcal{H}^k(\mathbb{R}^N)$:

$$\begin{aligned} \dim \mathcal{H}^k(\mathbb{R}^N) &= \dim \mathcal{P}^k(\mathbb{R}^N) - \dim \mathcal{P}^{k-2}(\mathbb{R}^N) \\ &= \frac{(k+N-3)!(2k+N-2)}{k!(N-2)!}. \end{aligned} \tag{3.1}$$

In the next subsection, we shall use the following recurrence formula:

Lemma 3.1. $\dim \mathcal{H}^k(\mathbb{R}^N) + \dim \mathcal{H}^{k-1}(\mathbb{R}^{N+1}) = \dim \mathcal{H}^k(\mathbb{R}^{N+1})$.

Proof. By the elementary combinatorial formula

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k},$$

we have

$$\dim \mathcal{P}^k(\mathbb{R}^N) + \dim \mathcal{P}^{k-1}(\mathbb{R}^{N+1}) = \dim \mathcal{P}^k(\mathbb{R}^{N+1}). \tag{3.2}$$

Taking the difference of (3.2) for k and $k-2$, and applying (3.1), we get Lemma 3.1. $\square$

To find the dimension formula of $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ one might apply the above method (see e.g. [11, Section 11.2.1]), but it would be more convenient for our purpose to use representation theory. There is a natural action of the unitary group $U(n)$ on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$. This representation is irreducible, and its highest weight is given by $(\alpha, 0, \ldots, 0, -\beta)$ in the standard coordinates of the Cartan subalgebra. By the Weyl character formula, we get

$$\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = \frac{(\alpha+\beta+n-1)}{(n-1)!(n-2)!} \prod_{i=2}^{n-1} (\alpha+i-1)(\beta+n-i).$$

If we use the Pochhammer symbol $(a)_l$ defined by

$$(a)_l := \frac{\Gamma(a+l)}{\Gamma(a)} = a(a+1)\cdots(a+l-1),$$

then we may express these dimensions as

$$\begin{aligned} \dim \mathcal{H}^k(\mathbb{R}^N) &= \frac{(k+1)_{N-3}(2k+N-2)}{\Gamma(N-1)}, \\ \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) &= \frac{(\alpha+\beta+n-1)(\alpha+1)_{n-2}(\beta+1)_{n-2}}{\Gamma(n)\Gamma(n-1)}. \end{aligned} \tag{3.3}$$

3.2 Alternating sum of $\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$

By the direct sum decomposition (2.2), the following identity is obvious:

$$\dim \mathcal{H}^k(\mathbb{R}^{2n}) = \sum_{\alpha+\beta=k} \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n).$$

However, what we need for the proof of Theorem 1.1 is an explicit formula for the alternating sum:

$$D(k) := \sum_{\alpha+\beta=k} (-1)^\beta \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n).$$

Clearly, $D(k) = 0$ for odd k because $\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = \dim \mathcal{H}^{\beta,\alpha}(\mathbb{C}^n)$.

A closed formula of $D(k)$ for even k is the main issue of this subsection, and we establish the following relation:

Proposition 3.2.

$$D(2l) = \dim \mathcal{H}^l(\mathbb{R}^{n+1}) = \frac{(n-1)_l(\frac{n+1}{2})_l}{l!(\frac{n-1}{2})_l}. \tag{3.4}$$

Remark 3.3. The Pochhammer symbol $(a)_l$ may be regarded as a meromorphic function. Thus, the right-hand side of (3.4) can be regarded as a meromorphic function of n. In this sense, the right-hand side of (3.4) still makes sense for $n = 1$.

The rest of this subsection is devoted to the proof of Proposition 3.2. For this, we set

$$X^{(l)} := x^l + \frac{1}{x^l} \quad \text{for } l = 1, 2, \ldots .$$

It is readily seen that $X^{(l)}$ is expressed as a monomial in

$$X := x + \frac{1}{x}$$

of degree l. For example,

$$X^{(1)} = X, \; X^{(2)} = X^2 - 2, \; X^{(3)} = X^3 - 3X, \ldots . \tag{3.5}$$

For an arbitrary l, we have the following formula:

Lemma 3.4.

$$X^{(l)} = \sum_{j=0}^{[\frac{l}{2}]} (-1)^j \dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) X^{l-2j}. \tag{3.6}$$

Proof. We prove Lemma 3.4 by induction on l. The equation (3.6) holds for $l = 1, 2$ by (3.5). Suppose $l \geq 2$. We shall prove the equation (3.6) for $l + 1$. We use

$$\begin{aligned} X^{(l+1)} &= \left(x + \frac{1}{x}\right)\left(x^l + \frac{1}{x^l}\right) - \left(x^{l-1} + \frac{1}{x^{l-1}}\right) \\ &= XX^{(l)} - X^{(l-1)}. \end{aligned}$$

By substituting (3.6) for l and $l - 1$ into the right-hand side, we get

$$\begin{aligned} X^{(l+1)} = &\sum_{j=0}^{[\frac{l}{2}]} (-1)^j \dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) X^{l+1-2j} \\ &- \sum_{i=0}^{[\frac{l-1}{2}]} (-1)^i \dim \mathcal{H}^i(\mathbb{R}^{l+1-2i}) X^{l-1-2i} \\ = X^{l+1} &+ \sum_{j=1}^{[\frac{l+1}{2}]} \left((-1)^j (\dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) + \dim \mathcal{H}^{j-1}(\mathbb{R}^{l+3-2j})) X^{l+1-2j}\right). \end{aligned}$$

To see the second equality for odd l, we note that $\dim \mathcal{H}^d(\mathbb{R}^1) = 0$ for $d \geq 2$, and thus

$$\dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) = 0 \quad \text{for } j = \frac{l+1}{2}. \tag{3.7}$$

Applying the recurrence formula given in Lemma 3.1, we get (3.6) for $l+1$. By induction, we have proved Lemma 3.4. $\square$

Proof of Proposition 3.2. We take a maximal torus T of $U(n)$ and its coordinate $(x_1, \ldots, x_n)$ such that

$$T = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\},$$

and that the linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is represented as $J = (\sqrt{-1}, \ldots, \sqrt{-1}) \in T$. Then the character $\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(g)$ of the representation of $O(2n)$ on $\mathcal{H}^k(\mathbb{R}^{2n})$ takes the value

$$\sum_{\alpha+\beta=k} (-1)^{\frac{\alpha-\beta}{2}} \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = (-1)^{\frac{k}{2}} D(k)$$

at $g = J$.

By using this observation, we shall analyze the character $\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(g)$ as g approaches to the singular point $J \in T$.

Let

$$X_j^{(l)} := x_j^l + \frac{1}{x_j^l} \quad (1 \leq j \leq n,\ l \in \mathbb{N}),$$

and we set

$$s_k(x) := \det \begin{pmatrix} X_1^{(k+n-1)} & X_2^{(k+n-1)} & \cdots & X_n^{(k+n-1)} \\ X_1^{(n-2)} & X_2^{(n-2)} & \cdots & X_n^{(n-2)} \\ \vdots & \vdots & & \vdots \\ X_1^{(1)} & X_2^{(1)} & & X_n^{(1)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then, by the Weyl character formula for the group $O(2n)$ and by using a trick which reduces the summation over the Weyl group for $O(2n)$ to that over the symmetric group $\mathcal{S}_n$ (see [10]), we have

$$\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(x) = \frac{s_k(x)}{s_0(x)} \quad \text{for } x \in T.$$

Since $X^{(l)} \equiv X^l \bmod \mathbb{Q}\text{-span}\langle 1, X, \dots, X^{l-1}\rangle$ an elementary property of the determinant shows:

$$s_k(x) = \det \begin{pmatrix} X_1^{(k+n-1)} & X_2^{(k+n-1)} & \cdots & X_n^{(k+n-1)} \\ X_1^{n-2} & X_2^{n-2} & \cdots & X_n^{n-2} \\ \vdots & \vdots & & \vdots \\ X_1 & X_2 & & X_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

As x_j goes to $\sqrt{-1}$, X_j tends to 0 $(1 \le j \le n)$. Therefore, we have

$$\begin{aligned} \chi^{2l}_{\mathcal{H}(\mathbb{R}^{2n})}(J) &= \lim_{X_1,\dots,X_N \to 0} \frac{s_{2l}(x)}{s_0(x)} \\ &= \text{the coefficient of } X^{n-1} \text{ in the expansion (3.6)} \\ &\quad \text{for } X^{(2l+n-1)} \\ &= (-1)^l \dim \mathcal{H}^l(\mathbb{R}^{n+1}). \end{aligned}$$

Here, we have used Lemma 3.4 for the last equality. Thus, we have proved

$$D(2l) = \dim \mathcal{H}^l(\mathbb{R}^{n+1}).$$

The second equality of (3.4) is immediate from (3.1). □

3.3 Triple integral as a Trace

We are now ready to prove Theorem 1.1. As we remarked in Introduction, the both sides of Theorem 1.1 are meromorphic functions of λ_1, λ_2, and λ_3. Therefore, it is sufficient to prove the identity in Theorem 1.1 in an open set of the parameters $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$.

By the change of variables $\mu_j := \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 - n) - \lambda_j$ $(1 \le j \le 3)$, we first consider the case when $\operatorname{Re} \mu_1 \ll 0$, $\operatorname{Re} \mu_2 \ll 0$, and $\operatorname{Re} \mu_3 \ll 0$. Then, the operators $\mathcal{T}_{\mu_1}$, $\mathcal{T}_{\mu_2}$, and $\mathcal{T}_{\mu_3}$ are Hilbert–Schmidt operators on $L^2(S^{2n-1})$. In particular, the composition $\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}$ is of trace class, and its trace is given by

$$\begin{aligned} &\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) \\ &= \int_{(S^{2n-1})^3} \big|[X,Y]\big|^{-\mu_1-n} \big|[Y,Z]\big|^{-\mu_2-n} \big|[Z,X]\big|^{-\mu_3-n} d\sigma(X) d\sigma(Y) d\sigma(Z). \end{aligned}$$

On the other hand, the trace of the operator $\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}$ can be also computed by its eigenvalues. Therefore, by using Theorem 2.1, we have

$$\begin{aligned}\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) &= \sum_{\alpha,\beta}\left(\prod_{j=1}^{3}(-1)^{\beta}A_{\alpha+\beta}(\mu_j)\right)\dim\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)\\ &= \sum_{k=0}^{\infty}\prod_{j=1}^{3}A_k(\mu_j)\left(\sum_{\alpha+\beta=k}(-1)^{3\beta}\dim\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)\right)\\ &= \sum_{l=0}^{\infty}D(2l)\prod_{j=1}^{3}A_{2l}(\mu_j).\end{aligned}$$

Applying Proposition 3.2, we get

$$\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) = \sum_{l=0}^{\infty}A_{2l}(\mu_1)A_{2l}(\mu_2)A_{2l}(\mu_3)\dim\mathcal{H}^{l}(\mathbb{R}^{n+1}).$$

In light of the recurrence relation:

$$\frac{A_{2l+2}(\mu)}{A_{2l}(\mu)} = \frac{l+\frac{n+\mu}{2}}{l+\frac{n-\mu}{2}},$$

the meromorphic function $A_{2l}(\mu)$ can be expressed in terms of Pochhammer symbols as

$$A_{2l}(\mu) = \frac{(\frac{n+\mu}{2})_l}{(\frac{n-\mu}{2})_l}A_0(\mu),$$

where

$$A_0(\mu) = 2\pi^{n-\frac{1}{2}}\frac{\Gamma(\frac{1-n-\mu}{2})}{\Gamma(\frac{n-\mu}{2})}. \tag{3.8}$$

Therefore,

$$\begin{aligned}&\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3})\\ &= A_0(\mu_1)A_0(\mu_2)A_0(\mu_3)\sum_{l=0}^{\infty}\frac{(n-1)_l(\frac{n+1}{2})_l}{l!(\frac{n-1}{2})_l}\prod_{j=1}^{3}\frac{(\frac{n+\mu_j}{2})_l}{(\frac{n-\mu_j}{2})_l}\\ &= A_0(\mu_1)A_0(\mu_2)A_0(\mu_3)\,{}_5F_4\left(\begin{matrix}n-1 & \frac{n+1}{2} & \frac{n+\mu_1}{2} & \frac{n+\mu_2}{2} & \frac{n+\mu_3}{2}\\ & \frac{n-1}{2} & \frac{n-\mu_1}{2} & \frac{n-\mu_2}{2} & \frac{n-\mu_3}{2}\end{matrix};1\right).\end{aligned}$$

Here ${}_5F_4$ is a generalized hypergeometric function.

A generalized hypergeometric function

$$ {}_pF_q\left(\begin{matrix} \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ & \beta_1 & \cdots & \beta_q \end{matrix}; z\right) $$

is called *well-poised* (see [1]) if $p = q + 1$ and

$$ 1 + \alpha_1 = \alpha_2 + \beta_1 = \cdots = \alpha_p + \beta_q. $$

In particular, our case is well-poised, and we can use the following *Dougall–Ramanujan identity* (see [loc. cit., pp. 25–26]):

$$ \begin{aligned} &{}_5F_4\left(\begin{matrix} m-1 & \frac{m+1}{2} & -x & -y & -z \\ & \frac{m-1}{2} & x+m & y+m & z+m \end{matrix}; 1\right) \\ &= \frac{\Gamma(x+m)\Gamma(y+m)\Gamma(z+m)\Gamma(x+y+z+m)}{\Gamma(m)\Gamma(x+y+m)\Gamma(y+z+m)\Gamma(x+z+m)}. \end{aligned} $$

Together with (3.8), we get

$$ \operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) = \frac{(2\pi^{n-\frac{1}{2}})^3\Gamma(\frac{1-n-\mu_1}{2})\Gamma(\frac{1-n-\mu_2}{2})\Gamma(\frac{1-n-\mu_3}{2})\Gamma(\frac{-\mu_1-\mu_2-\mu_3-n}{2})}{\Gamma(n)\Gamma(-\frac{\mu_1+\mu_2}{2})\Gamma(-\frac{\mu_2+\mu_3}{2})\Gamma(-\frac{\mu_1+\mu_3}{2})}. \tag{3.9} $$

Now, Theorem 1.1 follows by substituting $\mu_1 = -\frac{1}{2}(\alpha + n)$, $\mu_2 = -\frac{1}{2}(\beta + n)$, and $\mu_3 = -\frac{1}{2}(\gamma + n)$.

4 Other triple integral formulas

In this section, we discuss explicit formulas for the integrals of the triple product of powers of $|x - y|$ and $|\langle x, y\rangle|$ instead of those of the symplectic form $\big|[X, Y]\big|$.

4.1 Triple product of powers of $|x - y|$

In this subsection we consider a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$ \mathcal{R}_\mu : C^\infty(S^m) \to C^\infty(S^m) $$

defined by

$$(\mathcal{R}_\mu f)(\eta) = \int_{S^m} |\omega - \eta|^{-\mu-m} f(\omega) d\sigma(\omega). \tag{4.1}$$

The multiplier action of $\mathcal{R}_\mu$ on spherical harmonics is known (see e.g. [2]):

$$\mathcal{R}_\mu \Big|_{\mathcal{H}^k(\mathbb{R}^{m+1})} = \gamma_k(\mu)\,\mathrm{id}, \tag{4.2}$$

where $\gamma_k(\mu) \equiv \gamma_k(\mu, \mathbb{R}^{m+1})$ is given by

$$\gamma_k(\mu) = \frac{\Gamma(m+\frac{1}{2})\Gamma(-\frac{\mu}{2})\Gamma(k+\frac{m+\mu}{2})}{2^{\mu+1}\sqrt{\pi}\Gamma(\frac{\mu+m}{2})\Gamma(k+\frac{m-\mu}{2})}. \tag{4.3}$$

Then, by an argument parallel to Section 3.3, we can obtain a closed formula for the triple integral built on $\mathcal{R}_\mu$ (see Theorem 4.2 below). Instead of repeating similar computations, we pin down a comparison result between the two triple integral formulas by using Proposition 3.2. This comparison result explains the reason why the same method (e.g. Dougall–Ramanujan identity) is applicable, and seems interesting for its own sake.

Proposition 4.1.

$$\begin{aligned} &\mathrm{Trace}(\mathcal{R}_{\mu_1}\mathcal{R}_{\mu_2}\mathcal{R}_{\mu_3} : L^2(S^m) \to L^2(S^m)) \\ &= c\,\mathrm{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3} : L^2(S^{2m-1}) \to L^2(S^{2m-1})), \end{aligned} \tag{4.4}$$

where

$$c = \left(\frac{\Gamma(m+\frac{1}{2})}{2^2\pi^m}\right)^3 \prod_{j=1}^{3} \frac{\Gamma(-\frac{\mu_j}{2})}{2^{\mu_j}\Gamma(\frac{-\mu_j-m+1}{2})}.$$

Proof. By (4.2) the left-hand side of (4.4) equals

$$\sum_{k=0}^{\infty} \left(\prod_{j=1}^{3} \gamma_k(\mu_j, \mathbb{R}^{m+1})\right) \dim \mathcal{H}^k(\mathbb{R}^{m+1}). \tag{4.5}$$

Comparing (4.3) with Theorem 2.1 we get

$$\frac{\gamma_k(\mu, \mathbb{R}^{m+1})}{A_{2k}(\mu, \mathbb{C}^m)} = \frac{\Gamma(m+\frac{1}{2})\Gamma(-\frac{\mu}{2})}{2^{\mu+2}\pi^m\Gamma(\frac{-\mu-m+1}{2})}. \tag{4.6}$$

By (4.6) and (3.9), we see that (4.5) equals the right-hand side of (4.4). $\square$

The right-hand side in Proposition 4.1 was found in (3.9). Then, by a simple computation, we get

$$\begin{aligned}&\operatorname{Trace}(\mathcal{R}_{\mu_1}\mathcal{R}_{\mu_2}\mathcal{R}_{\mu_3})\\&=\left(\frac{\Gamma(m+\frac{1}{2})}{2\pi^{\frac{1}{2}}}\right)^3\frac{\Gamma(\frac{-\mu_1-\mu_2-\mu_3-m}{2})}{\Gamma(m)}\prod_{j=1}^{3}\frac{\Gamma(-\frac{\mu_j}{2})}{2^{\mu_j}\Gamma(\frac{\mu_j-(\mu_1+\mu_2+\mu_3)}{2})}.\end{aligned}$$

Finally, substituting $\mu_j=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3-m)-\lambda_j$ $(1\le j\le 3)$, we have proved the following:

Theorem 4.2. *Let α,β,γ, and δ be as in Theorem 1.1*

$$\begin{aligned}&\int_{S^m\times S^m\times S^m}|Y-Z|^{\frac{\alpha-m}{2}}|Z-X|^{\frac{\beta-m}{2}}|X-Y|^{\frac{\gamma-m}{2}}d\sigma(X)d\sigma(Y)d\sigma(Z)\\&=\left(\frac{\Gamma(m+\frac{1}{2})}{2^{1-\frac{m}{2}}\pi^{\frac{1}{2}}}\right)^3\frac{1}{2^{\frac{\lambda_1+\lambda_2+\lambda_3}{2}}\Gamma(m)}\frac{\Gamma(\frac{\alpha+m}{4})\Gamma(\frac{\beta+m}{4})\Gamma(\frac{\gamma+m}{4})\Gamma(\frac{\delta+m}{4})}{\Gamma(\frac{m-\lambda_1}{2})\Gamma(\frac{m-\lambda_2}{2})\Gamma(\frac{m-\lambda_3}{2})}.\end{aligned}$$

Remark 4.3. The formula in Theorem 4.2 was previously found by A. Deitmar [5] by a different method; namely it established a recurrence formula bridging $SO_o(\ell+1,1)$ to $SO_o(\ell-1,1)$ and used the Bernstein–Reznikov formula for $SO_o(2,1)$ and an analogous formula for $SO_o(3,1)$.

4.2 Triple product of powers of $|\langle x,y\rangle|$

In this subsection we consider the third case, namely, the linear operators $\mathcal{Q}_\mu: C^\infty(S^{N-1})\to C^\infty(S^{N-1})$ defined by the kernel $|\langle x,y\rangle|^{-\mu-\frac{N}{2}}$ (see (2.5)) and the corresponding triple product integrals.

Here is the counterpart of Theorem 2.1 for $\mathcal{Q}_\mu$:

Proposition 4.4. $\mathcal{Q}_\mu\Big|_{\mathcal{H}^k(\mathbb{R}^N)}=0$ *for odd k, and*

$$\mathcal{Q}_\mu\Big|_{\mathcal{H}^{2l}(\mathbb{R}^N)}=c_N(\mu,l)\,\mathrm{id},$$

where

$$c_N(\mu,l)=(-1)^l\frac{2\pi^{\frac{N-1}{2}}\Gamma(\frac{2-N-2\mu}{4})\Gamma(l+\frac{2\mu+N}{4})}{\Gamma(\frac{N+2\mu}{4})\Gamma(l+\frac{-2\mu+N}{4})}.$$

Proof. By Lemma 2.2 and Proposition 2.3, we have

$$c_N(\mu,l)=C_N(\mu)B_N(\mu-\frac{N}{2},2l).$$

□

$$\begin{aligned}&\mathrm{Trace}(\mathcal{Q}_{\mu_1}\mathcal{Q}_{\mu_2}\mathcal{Q}_{\mu_3}:L^2(S^{N-1})\to L^2(S^{N-1}))\\&=\sum_{l=0}^{\infty}\left(\prod_{j=1}^{3}c_N(\mu_j,l)\right)\dim\mathcal{H}^{2l}(\mathbb{R}^N)\end{aligned}\tag{4.7}$$

By substituting

$$\begin{aligned}c_N(\mu,l)&=(-1)^l c_N(\mu,0)\frac{(\frac{N+2\mu}{4})_l}{(\frac{N-2\mu}{4})_l},\\ \dim\mathcal{H}^{2l}(\mathbb{R}^N)&=\frac{(\frac{N}{2}-1)_l(\frac{N-1}{2})_l(\frac{N+2}{4})_l}{l!(\frac{1}{2})_l(\frac{N-2}{4})_l}\end{aligned}$$

into the right-hand side of (4.7), we see that (4.7) equals

$$\begin{aligned}&\left(\prod_{j=0}^{3}c_N(\mu_j,0)\right)\sum_{j=0}^{\infty}(-1)^l\prod_{j=1}^{3}\frac{(\frac{N+2\mu_j}{4})_l}{(\frac{N-2\mu_j}{4})_l}\,\frac{(\frac{N}{2}-1)_l(\frac{N-1}{2})_l(\frac{N+2}{4})_l}{l!(\frac{1}{2})_l(\frac{N-2}{4})_l}\\&=\prod_{j=0}^{3}c_N(\mu_j,0)_6F_5\left(\begin{matrix}\frac{N}{2}-1 & \frac{N+2}{4} & \frac{N-1}{2} & \frac{N+2\mu_1}{4} & \frac{N+2\mu_2}{4} & \frac{N+2\mu_3}{4} \\ & \frac{N-2}{4} & \frac{1}{2} & \frac{N-2\mu_1}{4} & \frac{N-2\mu_2}{4} & \frac{N-2\mu_3}{4}\end{matrix};1\right).\end{aligned}$$

By using Whipple's transformation ([1, p.28]):

$$\begin{aligned}&{}_6F_5\left(\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix};-1\right)\\&=\frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\\&\quad\times{}_3F_2\left(\begin{matrix}1+a-b-c, & d, & e \\ & 1+a-b, & 1+a-c\end{matrix};1\right),\end{aligned}$$

we get

$$\operatorname{Trace}(\mathcal{Q}_{\mu_1}\mathcal{Q}_{\mu_2}\mathcal{Q}_{\mu_3}) = \frac{(2\pi^{\frac{N-3}{2}})^3 \prod_{j=1}^3 \Gamma(\frac{2-N-2\mu_j}{4})}{\Gamma(\frac{N}{2})\Gamma(-\frac{\mu_2+\mu_3}{2})\Gamma(\frac{N-2\mu_1}{4})} \times {}_3F_2\left(\begin{matrix} \frac{2-N-2\mu_1}{4} & \frac{N+2\mu_2}{4} & \frac{N+2\mu_3}{4} \\ & \frac{1}{2} & \frac{N-2\mu_1}{4} \end{matrix} ;1\right). \tag{4.8}$$

Hence we have proved:

Theorem 4.5.

$$\int_{S^{N-1}\times S^{N-1}\times S^{N-1}} |\langle y,z\rangle|^{-2\nu_1}|\langle z,x\rangle|^{-2\nu_2}|\langle x,y\rangle|^{-2\nu_3} d\sigma(x)d\sigma(y)d\sigma(z)$$
$$= \frac{(2\pi^{\frac{N-3}{2}})^3 \prod_{j=1}^3 \Gamma(\frac{1}{2}-\nu_j)}{\Gamma(\frac{N}{2})\Gamma(-\nu_2-\nu_3+\frac{N}{2})\Gamma(-\nu_1+\frac{N}{2})} \times {}_3F_2\left(\begin{matrix} \frac{1}{2}-\nu_1 & \nu_2 & \nu_3 \\ & \frac{1}{2} & -\nu_1+\frac{N}{2} \end{matrix} ;1\right).$$

5 Concluding remarks

In this paper we have focused on closed formulas for the triple integrals (see e.g. Theorem 1.1), based on a combination of methods from classical harmonic analysis. These methods allow us to establish explicit formulas for symplectic groups of any rank, and even in rank one case it gives a new proof of the original results due to Bernstein and Reznikov [4] and Deitmar [5].

Moreover, there are a number of interesting perspectives of this formula, and also of the steps in its proof, that deserve comments.

So far we have avoided representation theory but one aspect of Theorem 1.1 is that the triple integral considered therein arises from a particular series of representations π_μ of the symplectic group $G = Sp(n,\mathbb{R})$ of rank n induced from a maximal parabolic subgroup $P \subset G$ and depending on a complex parameter μ. Section 5.1 highlights this point.

Another aspect is that of analytic number theory, which was the main theme of [3, 4]. Motivated by the classical Rankin–Selberg method, authors considered a cocompact discrete subgroup of the rank one symplectic group and automorphic functions on the associated locally symmetric space. The product of two such functions may be decomposed in terms of a basis of automorphic functions and the corresponding coefficients are related to automorphic L-functions. The closed formula ($n = 1$ in Theorem 1.1) gave an estimate of their decay [4].

The above mentioned triple integral arises also in pseudo-differential analysis of the phase space $\mathbb{R}^{2n}$. This phenomenon was treated in [9], where the symmetries of the Weyl operator calculus on the Hilbert space $L^2(\mathbb{R}^{2n})$ were considered.

5.1 Invariant trilinear forms

Now we focus on some links between the triple integrals discussed in Sections 1–4 and representation theory of semisimple Lie groups.

We begin with a construction of an invariant trilinear form based on the Knapp–Stein intertwining operators. Let G be a connected real semisimple Lie group and P an arbitrary parabolic subgroup. Let $P = MAN$ be a Langlands decomposition, $\mathfrak{a}$ and $\mathfrak{n}$ the Lie algebras of A and N respectively, and 2ρ the sum of roots of $\mathfrak{n}$ with respect to $\mathfrak{a}$. Take a Cartan involution θ of G stabilizing MA and set $K = \{g \in G : \theta(g) = g\}$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we define (possibly degenerate) principal series representations of G, to be denoted by π_λ, on the space of smooth sections for the G-equivariant line bundle $\mathcal{L}_{\lambda+\rho} = G \times_P \mathbb{C}_{\lambda+\rho}$ over the real flag variety G/P, equivalently on the vector space

$$V_\lambda^\infty \equiv V_\lambda := \{f \in C^\infty(G) : f(gman) = a^{-\lambda-\rho} f(g),\ \forall man \in P\}.$$

Similarly, the space of distribution sections for $\mathcal{L}_{\lambda+\rho}$ will be denoted by $V_\lambda^{-\infty}$. These representations are called *spherical* because V_λ contains a K-fixed vector $\mathbb{1}_\lambda$ which is defined by the formula: $\mathbb{1}_\lambda(kman) := a^{-\lambda-\rho}$ for $kman \in KP$.

Denote by $\overline{P} = MA\overline{N}$ the opposite parabolic subgroup to P. Assume that it satisfies the condition:

C1. P and $\overline{P}$ are conjugate in G.

Then there exists the G-intertwining operators $\mathcal{T}_\lambda : V_{-\lambda} \to V_\lambda$, referred to as the *Knapp–Stein intertwining operators* [8], that depend meromorphically on λ. They are given by the distribution-valued kernels $K_\lambda(x,y) \in V_\lambda^{-\infty} \otimes V_\lambda^{-\infty}$ such that $(\mathcal{T}_\lambda f)(x) = \langle f(y),\, K_\lambda(x,y)\rangle \in V_\lambda$ for $f \in V_{-\lambda}$.

For $f_j \in V_{\lambda_j}$ $(j = 1, 2, 3)$, we set

$$\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(f_1, f_2, f_3) := \langle K_{\frac{1}{2}(\alpha-\rho)}(y,z) K_{\frac{1}{2}(\beta-\rho)}(z,x) K_{\frac{1}{2}(\gamma-\rho)}(x,y),\, f_1(x) f_2(y) f_3(z)\rangle, \tag{5.1}$$

where $\alpha = \lambda_1 - \lambda_2 - \lambda_3$, $\beta = -\lambda_1 + \lambda_2 - \lambda_3$, $\gamma = -\lambda_1 - \lambda_2 + \lambda_3 \in \mathfrak{a}_{\mathbb{C}}^*$.

We have the following:

Proposition 5.1. *Assume P and $\overline{P}$ are conjugate in G. Then there exists a non–empty open region of $(\lambda_1, \lambda_2, \lambda_3) \in (\mathfrak{a}_{\mathbb{C}}^*)^3$ for which the integral (5.1) converges. It extends as a meromorphic function of λ_1, λ_2 and λ_3. Then, the resulting continuous trilinear form*

$$\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3} : V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \longrightarrow \mathbb{C} \tag{5.2}$$

is invariant with respect to the diagonal action of G:

$$\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(\pi_{\lambda_1}(g)\, f_1,\ \pi_{\lambda_2}(g)\, f_2,\ \pi_{\lambda_3}(g)\, f_3) = \mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(f_1,\ f_2,\ f_3).$$

Proof. The meromorphic continuation can be justified by the Atiyah–Bernstein–Gelfand regularization of the integral (5.1) (see e.g. [6]). Parameters α, β and γ are chosen in such a way that the integrand in (5.1) is a section of the volume bundle of $(G/P)^3$. Whence the invariance follows. $\square$

The case when P is a minimal parabolic subgroup was considered in [5]. We note that in this situation $\overline{P}$ is automatically conjugate to P.

Returning to our settings, we have an isomorphism of Lie algebras:

$$\mathfrak{sp}(1, \mathbb{R}) \simeq \mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R}),$$

each of which is the 'bottom' of different series of Lie algebras, namely $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}(n, 1)$, and $\mathfrak{sl}(n, \mathbb{R})$. Bearing this in mind, we list the following three cases:

Case **Sp**. Theorem 1.1 corresponds to the evaluation of the trilinear form (5.2) on the K-fixed vector $\mathbb{1}_{\lambda_1} \otimes \mathbb{1}_{\lambda_2} \otimes \mathbb{1}_{\lambda_3}$ for the following particular pair: $G = Sp(n, \mathbb{R})$ and $P = MAN$ a maximal parabolic subgroup such that $M \simeq \mathbb{Z}/2\mathbb{Z} \times Sp(n-1, \mathbb{R})$ and N is the Heisenberg group in $2n-1$ variables. Notice that S^{2n-1} is a double covering of G/P. The representation space V_μ can be identified with $V_\mu(\mathbb{R}^{2n})$ introduced in (2.6). Then the kernel of the operator $\mathcal{T}_\mu$ introduced in (2.1) is $K_\mu(X, Y) = |[X, Y]|^{-\mu-n} \in V_\mu^{-\infty} \otimes V_\mu^{-\infty}$ which gives rise to the Knapp–Stein intertwining operator.

Case **SO**. Theorem 4.2 corresponds to the case where $G = SO_o(m+1, 1)$ and P is a minimal parabolic subgroup. Through the identification $G/P \simeq S^m$ the Knapp–Stein intertwining operator is given by $\mathcal{R}_\mu$ (see (4.1)), and the triple integral in Theorem 4.2 corresponds to the evaluation of the trilinear form (5.1) on the K-fixed vector.

Case: GL. Yet another expression of the sphere S^{N-1} as a homogeneous space is given by G/P, where $= GL(N,\mathbb{R})$ and P is a maximal parabolic subgroup corresponding to the partition $N = 1+(N-1)$. The operators $\mathcal{Q}_\mu$ introduced in (2.5) and involved in the Theorem 4.5 can also be interpreted as the Knapp–Stein integrals for representations induced from P and its opposite parabolic $\overline{P}$. Notice that the condition C1 fails for $N > 2$ and Proposition 5.1 does not apply.

What we have found in particular is the eigenvalues of operators $\mathcal{T}_\mu$, $\mathcal{Q}_\mu$ and $\mathcal{R}_\mu$ in terms of Gamma functions. The corresponding eigenspaces are irreducible representation spaces of the maximal compact subgroup K. Indeed, in all three cases the following condition holds:

C2. The space $K/(K\cap M)$ is a multiplicity–free space, in other words, $(K, K\cap M)$ is a Gelfand pair.

This implies that the representation space V_μ contains an algebraic direct sum of pairwise inequivalent irreducible representations of K as its dense subspace. Therefore the action of the operators $\mathcal{T}_\mu$ on each K-representation space is automatically a scalar multiple of the identity by Schur's lemma. For example in Case **Sp**, $K \simeq U(n)$, the corresponding restriction $\pi_\mu\big|_K$ is given by $\bigoplus_{\alpha,\beta\in\mathbb{N}} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$, and the eigenvalues are described in Theorem 2.1.

In Cases **SO** and **GL** the condition C2 is also satisfied. We can see this by a direct computation but also by the general observation that the unipotent radical N is abelian and consequently $(K,\, M\cap K)$ is a symmetric pair.

Another feature of our settings is the following condition:

C3. The diagonal action of G on $(G/P)^3$ admits an open orbit.

(In fact, there is only one such an open dense orbit except the case of $SL(2,\mathbb{R})$, where there are two open orbits.)

The condition C3 is connected to the upper bound of the number of linearly independent trilinear forms for generic λ_1, λ_2 and λ_3. If this number equals one then such an invariant trilinear form is proportional to the one constructed in Proposition 5.1 under the condition C1.

Case **Sp** ($n \geq 2$) is of a particular interest: the group G is of arbitrarily high rank, N is non-abelian, and $(K, M \cap K)$ is a non-symmetric pair. Nevertheless all the conditions C1, C2 and C3 are fulfilled. The corresponding trilinear form $\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}$ has recently arisen in a different context, namely in pseudo-differential analysis. More precisely, a new (non-perturbative) composition formula based on this trilinear form is established for the Weyl operator calculus on $L^2(\mathbb{R}^{2n})$ in [9], where a slightly different notation is adopted: $\mathbf{T}_{\lambda_1,\lambda_2,\,\lambda_3}(f_1, f_2, f_3) = \mathbf{J}^{0,\,0;\,0}_{-\lambda_1,-\lambda_2;\,\lambda_3}(f_1, f_2, f_3)$.

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