DOUBLE SHORT EXACT SEQUENCES PRODUCE ALL ELEMENTS OF QUILLEN'S K_1

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ABSTRACT. We prove that for any exact category $\mathfrak{A}$ every element of Quillen's $K_1\mathfrak{A}$ corresponds to a double short exact sequence, i.e., any loop in the G-construction of $\mathfrak{A}$ is homotopic to a 3-edge loop. This is a strengthening of a result of Sherman and is the simplest possible description of the elements of K_1 in terms of the G-construction.

Acknowledgements. Dan Grayson attracted my attention to double short exact sequences during the K-theory conference in Paris in July 1994. Thanks to Chuck Weibel I knew about the results of Sherman. Then Clayton Sherman gave me his preprint and we had numerous stimulating discussions. The last portion of those conversations took place in Poznan' in September 1995 during the K-theory conference that was perfectly organized by Grzegorz Banaszak, Wojciech Gajda, and Piotr Krason. I am grateful to all of them. I would like to express my distinguished gratitude to Max-Planck-Institut für Mathematik in Bonn for its hospitality during my stay in 1995.

1. PRELIMINARIES

For a definition of Quillen's K-groups of an exact category $\mathfrak{A}$ we take

$$K_m\mathfrak{A} = \pi_m(G.\mathfrak{A}), \quad m \geq 0$$

and begin with a breaf review of the G-construction [GG].

An n-simplex in the simplicial set $G.\mathfrak{A}$ is a pair of triangular diagrams in $\mathfrak{A}$ of the form

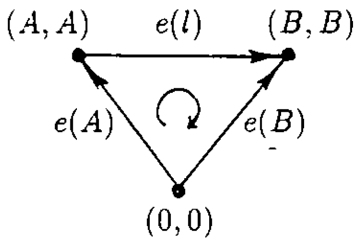
$$\begin{array}{ccccccc}
 & & & & & & P_{n/n-1} \\
 & & & & & & \uparrow \\
 & & & & & & \cdots \\
 & & & & P_{2/1} & \to \cdots \to & P_{n/1} \\
 & & & & \uparrow & & \uparrow \\
 & & P_{1/0} & \to & P_{2/0} & \to \cdots \to & P_{n/0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
P_0 & \to & P_1 & \to & P_2 & \to \cdots \to & P_n
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & & & & P_{n/n-1} \\
 & & & & & & \uparrow \\
 & & & & & & \cdots \\
 & & & & P_{2/1} & \to \cdots \to & P_{n/1} \\
 & & & & \uparrow & & \uparrow \\
 & & P_{1/0} & \to & P_{2/0} & \to \cdots \to & P_{n/0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
P'_0 & \to & P'_1 & \to & P'_2 & \to \cdots \to & P'_n
\end{array}
\qquad (1.1)$$

Key words and phrases. Exact category, Quillen's K_1, G-construction, double short exact sequence.

with equal quotient index subtriangles, where all squares commute and all the sequences of the form $P_j \to P_k \to P_{k/j}$, $P'_j \to P'_k \to P_{k/j}$, and $P_{j/i} \to P_{k/i} \to P_{k/j}$ are short exact sequences (s.e.s. for short). In particular, a vertex in $G.\mathfrak{A}$ is a pair of objects (P, P'), and an edge connecting (P_0, P'_0) to (P_1, P'_1) is a pair of s.e.s. $(P_0 \to P_1 \to P_{1/0}, P'_0 \to P'_1 \to P_{1/0})$ with equal cokernels. We will also write (s, s') for an edge, where s and s' denote s.e.s. with equal cokernels.
Let 0 denote a distinguished zero object in $\mathfrak{A}$, then we let $(0,0)$ be the base point of $G.\mathfrak{A}$. Given $A \in \mathfrak{A}$, the standard edge $e(A)$ from $(0,0)$ to (A, A) is given by $e(A) = (0 \to A \xrightarrow{1} A, 0 \to A \xrightarrow{1} A)$.
A *double short exact sequence* in $\mathfrak{A}$ (a d.s.e.s. for short) is a pair of s.e.s. $A \xrightarrow{f_1} B \xrightarrow{g_1} C$ and $A \xrightarrow{f_2} B \xrightarrow{g_2} C$ on the same objects. Given such a d.s.e.s. we will write

$$l = (A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C) \tag{1.2}$$

and let $e(l)$ denote the edge from (A, A) to (B, B) in $G.\mathfrak{A}$ given by l. Let $\mu(l)$ denote the loop



and let $m(l)$ be its class in $K_1\mathfrak{A} = \pi_1(G.\mathfrak{A})$.

Given $A \in \mathfrak{A}$ and $\alpha \in \operatorname{Aut} A$, we put

$$l(\alpha) = (0 \rightrightarrows A \underset{\alpha}{\overset{1}{\rightrightarrows}} A).$$

Thus one can regard an automorphism in $\mathfrak{A}$ as a particular case of a d.s.e.s., and the assignment $m(l(\alpha))$ to α is one of various equivalent ways to attach an element of K_1 to an automorphism. The loop $\mu(l(\alpha))$ is actually a 2-edge loop of the form

for the edge $e(0)$ is degenerate. One checks that every 2-edge loop of this form is homotopic to $\mu(l(\alpha))$ for some α, thus the elements of $K_1\mathfrak{A}$ representable by automorphisms are precisely those representable by 2-edge loops in $G.\mathfrak{A}$.

It is known that not every element of $K_1\mathfrak{A}$ can be represented in general by an automorphism. Moreover, $K_1\mathfrak{A}$ is not generated by such elements for some $\mathfrak{A}$ (Proposition 5.1 in [Ge]). However, we prove that every element of $K_1\mathfrak{A}$ can be represented by a loop of the type $\mu(l)$.

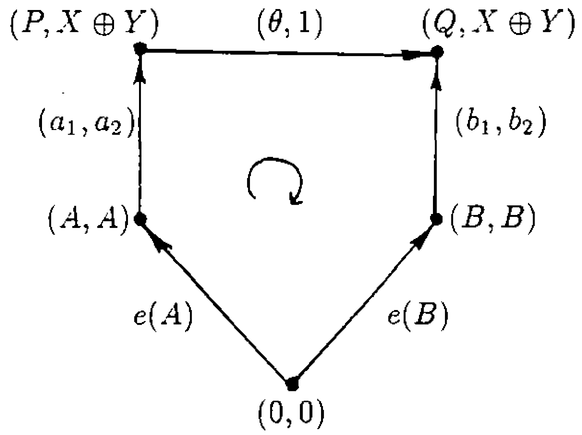
2. The main result

Theorem 2.1. For any element $x \in K_1\mathfrak{A}$ there exists a double short exact sequence l such that $x = m(l)$.

We will deduce this assertion from a result of Sherman. Consider data of the form

$$j = (A \xrightarrow{\alpha} X \xrightarrow{\gamma} C,\ B \xrightarrow{\beta} Y \xrightarrow{\delta} D;\quad \theta : A \oplus Y \oplus C \xrightarrow{\sim} X \oplus B \oplus D) \tag{2.1}$$

where $A \to X \to C$ and $B \to Y \to D$ are short exact sequences in $\mathfrak{A}$ and θ is an isomorphism. Given such data we will sometimes denote $A \oplus Y \oplus C$ by P and $X \oplus B \oplus D$ by Q for short. Sherman associates to j a loop $\nu(j)$ in $G.\mathfrak{A}$ of the form



where $(\theta, 1)$ denotes the edge $(P \xrightarrow{\theta} Q \to 0,\ X \oplus Y \xrightarrow{1} X \oplus Y \to 0)$ and the s.e.s. yielding the vertical edges are given by

$$a_1 = (A \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} A \oplus Y \oplus C \xrightarrow{\begin{pmatrix}0&0&1\\0&1&0\end{pmatrix}} C \oplus Y)$$

$$a_2 = (A \xrightarrow{\begin{pmatrix}\alpha\\0\end{pmatrix}} X \oplus Y \xrightarrow{\gamma \oplus 1_Y} C \oplus Y)$$

$$b_1 = (B \xrightarrow{\begin{pmatrix}0\\1\\0\end{pmatrix}} X \oplus B \oplus D \xrightarrow{\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}} X \oplus D)$$

$$b_2 = (B \xrightarrow{\begin{pmatrix}0\\\beta\end{pmatrix}} X \oplus Y \xrightarrow{1_X \oplus \delta} X \oplus D).$$

Let $n(j)$ denote the corresponding element in $K_1\mathfrak{A}$. The following assertion was proved by Sherman in [Sh1] for abelian categories and then in [Sh2] for arbitrary exact categories.

Theorem 2.2. For any $x \in K_1\mathfrak{A}$ there exists j of the form (2.1) such that $x = n(j)$.

Proof of Theorem 2.1. We associate to j a pair of short exact sequences

$$s_1 = (A \oplus B \xrightarrow{f} A \oplus Y \oplus C \xrightarrow{p} C \oplus D)$$
$$s_2 = (A \oplus B \xrightarrow{g} X \oplus B \oplus D \xrightarrow{q} C \oplus D)$$

where

$$f = \begin{pmatrix} 1 & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Replacing $A \oplus Y \oplus C$ in s_1 by $X \oplus B \oplus D$ via θ we obtain a double short exact sequence

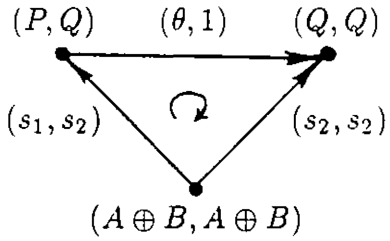
$$l(j) = (A \oplus B \underset{g}{\overset{\theta \circ f}{\rightrightarrows}} X \oplus B \oplus D \underset{q}{\overset{p \circ \theta^{-1}}{\rightrightarrows}} C \oplus D)$$

It now suffices to prove

Proposition 2.3. $m(l(j)) = n(j)$.

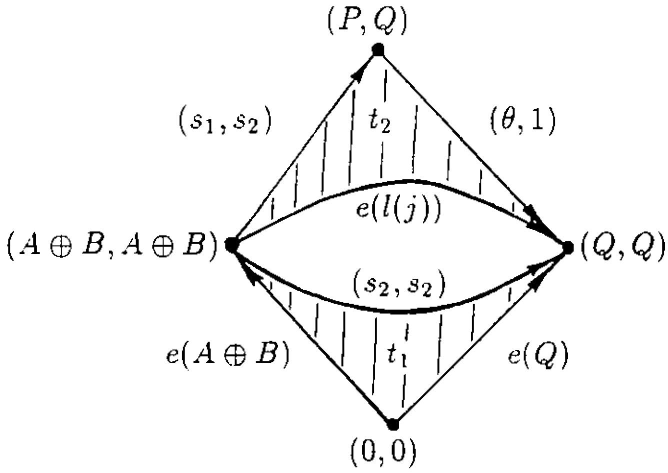
Proof. We must check that the loops $\mu(l(j))$ and $\nu(j)$ are homotopic. It suffices to show that they are freely homotopic since the group $\pi_1(G.\mathfrak{A}) = K_1\mathfrak{A}$ is abelian.

Lemma 2.4. The loop $\mu(l(j))$ is freely homotopic to the loop



(2.2)

Proof. This follows from the picture

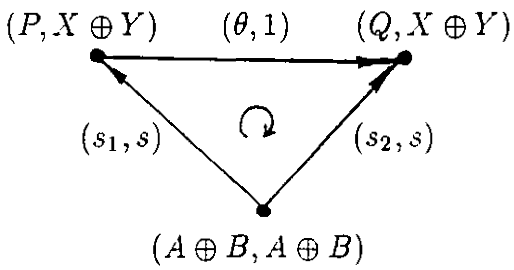


where the shaded 2-simplex t_1 is given by obvious data and t_2 is given by

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \uparrow \\ & & C\oplus D & \xrightarrow{1} & C\oplus D \\ & & {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle p\theta^{-1}} \\ A\oplus B & \xrightarrow{f} & P & \xrightarrow{\theta} & Q \end{array} \qquad \begin{array}{ccccc} & & & & 0 \\ & & & & \uparrow \\ & & C\oplus D & \xrightarrow{1} & C\oplus D \\ & & {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle q} \\ A\oplus B & \xrightarrow{g} & Q & \xrightarrow{1} & Q \end{array}$$

■

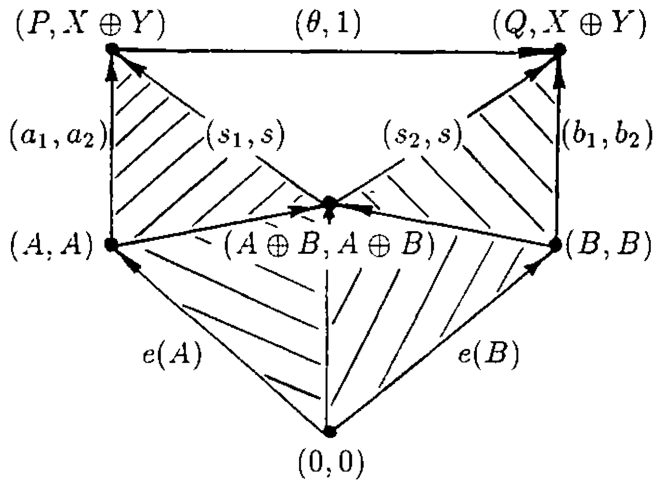
Lemma 2.5. The loop $\nu(j)$ is freely homotopic to the loop



(2.3)

where $s = (A \oplus B \overset{\alpha\oplus\beta}{\longrightarrow} X \oplus Y \overset{\gamma\oplus\delta}{\longrightarrow} C \oplus D)$.

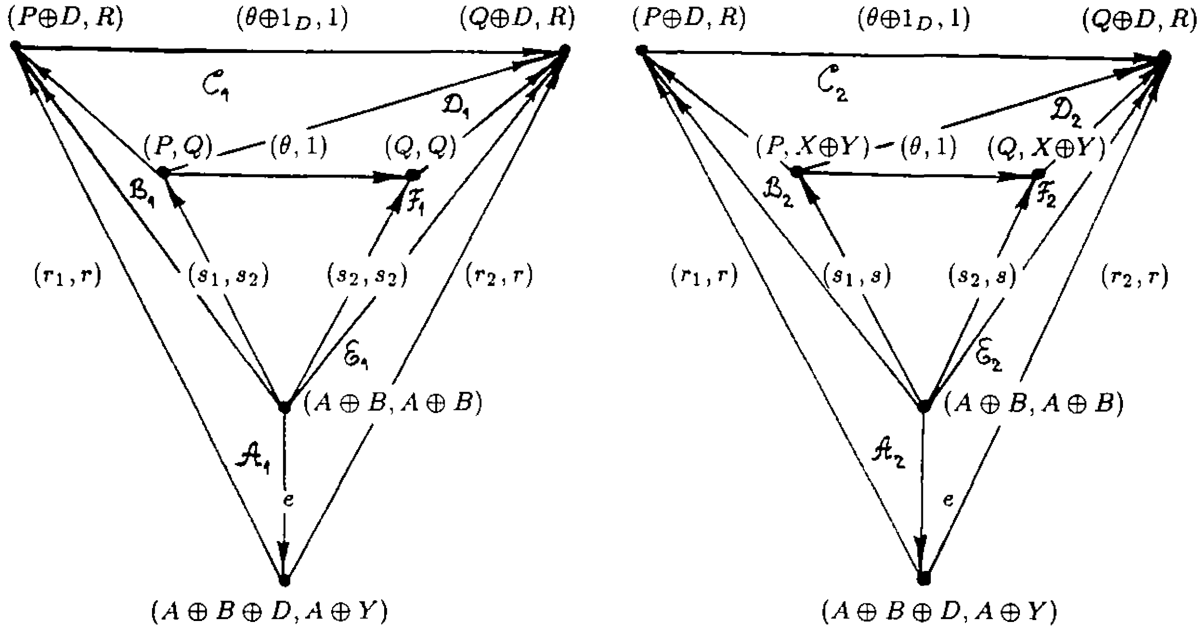
Proof. The four shaded 2-simplices in the picture



are given by obvious data of the form (1.1), and we are done. ■

Lemma 2.6. The loops (2.2) and (2.3) are freely homotopic.

Proof. We will show that both loops are freely homotopic to a third loop, namely to the common outer loop on the pictures



where R denotes $X \oplus Y \oplus D$ for short, the inner loops are given by (2.2) and (2.3) respectively, and we put

$$r_1 = (A \oplus B \oplus D \xrightarrow{f \oplus 1_D} A \oplus Y \oplus C \oplus D \xrightarrow{(p,0)} C \oplus D)$$

$$r_2 = (A \oplus B \oplus D \xrightarrow{g \oplus 1_D} X \oplus B \oplus D \oplus D \xrightarrow{(q,0)} C \oplus D)$$

$$r = (A \oplus Y \xrightarrow{\binom{\alpha \oplus 1_Y}{0}} X \oplus Y \oplus D \xrightarrow{(\gamma,0) \oplus 1_D} C \oplus D).$$

We obtain the first picture by applying the push-out procedure of [GG], sect.7, to the loop (2.2) and the edge

$$e = (A \oplus B \xrightarrow{\binom{1_{A \oplus B}}{0}} A \oplus B \oplus D \xrightarrow{(0,0,1)} D, \quad A \oplus B \xrightarrow{1_A \oplus \beta} A \oplus Y \xrightarrow{(0,\delta)} D).$$

All of the six 2-simplices $\mathcal{A}_1, \ldots, \mathcal{F}_1$ are given by obvious data of the form (1.1) since all the push-outs here amount to adding direct summands. One checks easily that the composite push-out of the edge e along (s_1, s_2) and $(\theta, 1)$ coincides with the push-out along (s_2, s_2), i.e., the corresponding edges of the 2-simplices $\mathcal{D}_1$ and $\mathcal{F}_1$ really coincide.

The second picture is the push-out of the loop (2.3) along the edge e in which the object $(X \oplus Y) \coprod_{A \oplus B} (A \oplus Y) \cong X \oplus (Y \coprod_B Y)$ is replaced by $X \oplus Y \oplus D$ by means of the isomorphism $Y \coprod_B Y \cong Y \oplus D$. One should be careful about this change of objects since there are two natural isomorphisms $Y \coprod_B Y \cong Y \oplus D$ that differ by the natural involution of $Y\coprod_B Y$. We write down the resulting 2-simplices $\mathcal{A}_2, \ldots \mathcal{F}_2$ explicitly in order to make sure that this really works.

$\mathcal{A}_2$.

$$\begin{array}{ccccc}
 & & & & C\oplus D \\
 & & & & \big\uparrow {\scriptstyle (1_{C\oplus D},0)} \\
 & & D & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & C\oplus D\oplus D \\
 & & \big\uparrow {\scriptstyle (0,0,1)} & & \big\uparrow {\scriptstyle p\oplus 1_D} \\
A\oplus B & \xrightarrow{\begin{pmatrix}1_{A\oplus B}\\0\end{pmatrix}} & A\oplus B\oplus D & \xrightarrow{f\oplus 1_D} & A\oplus Y\oplus C\oplus D
\end{array}$$

$$\begin{array}{ccccc}
 & & & & C\oplus D \\
 & & & & \big\uparrow {\scriptstyle (1_{C\oplus D},0)} \\
 & & D & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & C\oplus D\oplus D \\
 & & \big\uparrow {\scriptstyle (0,\delta)} & & \big\uparrow {\scriptstyle \gamma\oplus\begin{pmatrix}0 & 1\\ \delta & -1\end{pmatrix}} \\
A\oplus B & \xrightarrow{1_A\oplus\beta} & A\oplus Y & \xrightarrow{\begin{pmatrix}\alpha\oplus 1_Y\\0\end{pmatrix}} & X\oplus Y\oplus D
\end{array}$$

....

$\mathcal{B}_2$.

$$\begin{array}{ccccc}
 & & & & D \\
 & & & & \big\uparrow {\scriptstyle (0,0,1)} \\
 & & C\oplus D & \xrightarrow{\begin{pmatrix}1_{C\oplus D}\\0\end{pmatrix}} & C\oplus D\oplus D \\
 & & \big\uparrow {\scriptstyle p} & & \big\uparrow {\scriptstyle p\oplus 1_D} \\
A\oplus B & \xrightarrow{f} & A\oplus Y\oplus C & \xrightarrow{\begin{pmatrix}1_{A\oplus Y\oplus C}\\0\end{pmatrix}} & A\oplus Y\oplus C\oplus D
\end{array}$$

$$\begin{array}{ccccc}
 & & & & D \\
 & & & & \big\uparrow {\scriptstyle (0,0,1)} \\
 & & C\oplus D & \xrightarrow{\begin{pmatrix}1_{C\oplus D}\\0\end{pmatrix}} & C\oplus D\oplus D \\
 & & \big\uparrow {\scriptstyle \gamma\oplus\delta} & & \big\uparrow {\scriptstyle \gamma\oplus\begin{pmatrix}0 & 1\\ \delta & -1\end{pmatrix}} \\
A\oplus B & \xrightarrow{\alpha\oplus\beta} & X\oplus Y & \xrightarrow{1_X\oplus\begin{pmatrix}1\\ \delta\end{pmatrix}} & X\oplus Y\oplus D
\end{array}$$

....

$\mathcal{C}_2$.

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
 & & D & \xrightarrow{1} & D \\
 & & \uparrow{\scriptstyle (0,0,0,1)} & & \uparrow{\scriptstyle (0,0,0,1)} \\
A\oplus Y\oplus C & \xrightarrow{\binom{1_{A\oplus Y\oplus C}}{0}} & A\oplus Y\oplus C\oplus D & \xrightarrow{\theta\oplus 1_D} & X\oplus B\oplus D\oplus D
\end{array}$$

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
 & & D & \xrightarrow{1} & D \\
 & & \uparrow{\scriptstyle (0,\delta,-1)} & & \uparrow{\scriptstyle (0,\delta,-1)} \\
X\oplus Y & \xrightarrow{1_X\oplus\binom{1}{\delta}} & X\oplus Y\oplus D & \xrightarrow{1} & X\oplus Y\oplus D
\end{array}$$

$\mathcal{D}_2$.

$$\begin{array}{ccccc}
 & & & & D \\
 & & & & \uparrow{\scriptstyle 1} \\
 & & 0 & \longrightarrow & D \\
 & & \uparrow & & \uparrow{\scriptstyle (0,0,0,1)} \\
A\oplus Y\oplus C & \xrightarrow{\theta} & X\oplus B\oplus D & \xrightarrow{\binom{1_{X\oplus B\oplus D}}{0}} & X\oplus B\oplus D\oplus D
\end{array}$$

$$\begin{array}{ccccc}
 & & & & D \\
 & & & & \uparrow{\scriptstyle 1} \\
 & & 0 & \longrightarrow & D \\
 & & \uparrow & & \uparrow{\scriptstyle (0,\delta,-1)} \\
X\oplus Y & \xrightarrow{1} & X\oplus Y & \xrightarrow{1_X\oplus\binom{1}{\delta}} & X\oplus Y\oplus D
\end{array}$$

$\mathcal{E}_2$.

$$
\begin{array}{ccccc}
 & & & & C\oplus D \\
 & & & & \Big\uparrow{\scriptstyle (1_{C\oplus D},0)} \\
 & & D & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & C\oplus D\oplus D \\
 & & \Big\uparrow{\scriptstyle (0,0,1)} & & \Big\uparrow{\scriptstyle q\oplus 1_D} \\
A\oplus B & \xrightarrow{\begin{pmatrix}1_{A\oplus B}\\0\end{pmatrix}} & A\oplus B\oplus D & \xrightarrow{g\oplus 1_D} & X\oplus B\oplus D\oplus D
\end{array}
$$

$$
\begin{array}{ccccc}
 & & & & C\oplus D \\
 & & & & \Big\uparrow{\scriptstyle (1_{C\oplus D},0)} \\
 & & D & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & C\oplus D\oplus D \\
 & & \Big\uparrow{\scriptstyle (0,\delta)} & & \Big\uparrow{\scriptstyle \gamma\oplus\begin{pmatrix}0 & 1\\ \delta & -1\end{pmatrix}} \\
A\oplus B & \xrightarrow{1_A\oplus\beta} & A\oplus Y & \xrightarrow{\begin{pmatrix}\alpha\oplus 1_Y\\0\end{pmatrix}} & X\oplus Y\oplus D
\end{array}
$$

....

$\mathcal{F}_2$.

$$
\begin{array}{ccccc}
 & & & & D \\
 & & & & \Big\uparrow{\scriptstyle (0,0,1)} \\
 & & C\oplus D & \xrightarrow{\begin{pmatrix}1_{C\oplus D}\\0\end{pmatrix}} & C\oplus D\oplus D \\
 & & \Big\uparrow{\scriptstyle q} & & \Big\uparrow{\scriptstyle q\oplus 1_D} \\
A\oplus B & \xrightarrow{g} & X\oplus B\oplus D & \xrightarrow{\begin{pmatrix}1_{X\oplus B\oplus D}\\0\end{pmatrix}} & X\oplus B\oplus D\oplus D
\end{array}
$$

$$
\begin{array}{ccccc}
 & & & & D \\
 & & & & \Big\uparrow{\scriptstyle (0,0,1)} \\
 & & C\oplus D & \xrightarrow{\begin{pmatrix}1_{C\oplus D}\\0\end{pmatrix}} & C\oplus D\oplus D \\
 & & \Big\uparrow{\scriptstyle \gamma\oplus\delta} & & \Big\uparrow{\scriptstyle \gamma\oplus\begin{pmatrix}0 & 1\\ \delta & -1\end{pmatrix}} \\
A\oplus B & \xrightarrow{\alpha\oplus\beta} & X\oplus Y & \xrightarrow{1_X\oplus\begin{pmatrix}1\\ \delta\end{pmatrix}} & X\oplus Y\oplus D.
\end{array}
$$

One checks that these 2-simplices form the required configuration, in which the inner loop is (2.3) and the outer edges are given by (r_1, r), (r_2, r), and $(\theta\oplus 1_D, 1)$. Lemma 2.6, Proposition 2.3, and Theorem 2.1 are proved. ∎

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Bitte ca. 30 Exemplare meiner Doktorarbeit drucken

Wenn möglich wieder in dem dunkelgrünen Umschlag, leimen und mit schwarzem Rückenband.

Ich habe ungerade und gerade Seiten zentriert ausgedruckt. Es ist evtl. besser beide Seiten 0,5 – 1,0 cm nach außen zu verschieben

Mit freundlichem Gruß

Gerald [illegible]

(evtl. die Seitennummerierung auf Seite (i) mit Tippex überdecken, habe hier leider keines)