Nash desingularization for binomial varieties as Euclidean multidimensional division : low complexity in dim ≤ 2

DIMA GRIGORIEV AND PIERRE D. MILMAN

Abstract. We establish an equivalence of a multidimensional Euclidean division and of the Nash algorithm for desingularization of affine binomial varieties. A structure theorem for these varieties identifying their irreducible components as mutually isomorphic and their (local) singularities as products of manifolds with irreducible germs of their subvarieties, say Y (obtained by restricting all of their nonvanishing affine coordinates to value one), is a byproduct. Proved for the differences of monomials as binomials, it also implies that local singularities 'do not change' upon restriction of nonvanishing affine coordinates to vary in a manifold (covering the case of nonunit coefficients of defining binomial equations).

When $\dim Y = 2$ the length of desingularization by composites of normalizations with Nash blowings up is bounded by the area, say D, of a parallelogram generated by the least integral points on the extremal rays of the cone spanned by the exponents of any monomial parametrization of the torus of Y. The numbers of covering charts at each step of Nash algorithm is bounded by $D/2 + 1$, while the complexity of the algorithm along a single branch is polynomial in the binary size of the input and in D.

Contents

Key words and phrases. Nash blow ups, binomial varieties, desingularization, Euclidean multidimensional algorithm.

Research partially supported by Natural Sciences and Engineering Research Council of Canada Discovery Grant OGP 0008949 (Milman).

1. Introduction

1.1. **Nash conjecture.** For an n-dimensional algebraic variety X the Gauss map G_X is defined off singular points $\operatorname{Sing} X$ of X and sends points $\mathcal{P} \in \operatorname{Reg} X := X \setminus \operatorname{Sing} X$ to the tangent spaces $T_{\mathcal{P}}X$ (to X at $\mathcal{P}$) as points of the respective Grassmanian bundle restricted over X . (Using embeddings of affine charts of X in $\mathbb{C}^N$ consider the Grassmanian variety of n-dimensional subspaces of $\mathbb{C}^N$. The latter naturally embeds into projective space $\mathbb{P}(\wedge^n \mathbb{C}^N)$ by means of the classical Plücker coordinates, i. e. homogeneous coordinates in $\wedge^n \mathbb{C}^N$.) Nash blow up $N(X)$ of X is the closure of the graph of G_X with the natural projection $N_X : N(X) \to X$. John Nash conjectured that his algorithm of successive Nash blowings up starting with an algebraic variety X results in a desingularization of X , namely

Nash conjecture. The sequence of Nash blow ups starting with any algebraic variety stabilizes, i. e. Nash blowings up become isomorphisms, resulting in a desingularization.

Remark 1.1. Of course, if X consists of several irreducible components $X = \cup_i X_i$ then $N(X) = \cup_i N(X_i)$ and $N(X_i)$ are the irreducible components of $N(X)$. Also, when locally (in Zariski or even in the classical topology), say in U , variety X is a product of a nonsingular variety Z with a (possibly) singular one, say Y , then $N(X)$ over U is isomorphic to the product $Z \times N(Y)$ of Z with $N(Y)$. Nash blow up either separates any pair of equidimensional irreducible smooth local analytic components, or reduces *contact* between the latter pair. (With I_j , $j = 1, 2$, being the ideals of local analytic components in the local analytic ring $\mathcal{O}$ of the ambient manifold contact is the largest integer l such that $I_1 + \mathcal{M}^l = I_2 + \mathcal{M}^l$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$.) Thus the sequence of Nash blowings up of a variety with smooth local analytic irreducible components terminates separating 'Nash liftings' of these components.

Normalization $\mathcal{N}_X : \mathcal{N}(X) \to X$ of a variety X is defined (locally over affine charts) via their respective rings $\mathbb{C}[\mathcal{N}(X)]$ and $\mathbb{C}[X]$ of 'regular functions' with the former ring being the integral closure in the field of fractions of the latter. With variety X being locally a product of a nonsingular Z with Y it follows that (locally) $\mathcal{N}(X)$ is isomorphic to the product $Z \times \mathcal{N}(Y)$. Of course a single normalization separates all local analytic irreducible components. We refer to the composites of normalizations with Nash blow ups as *normalized Nash blow ups.* The following is a modified version of Nash conjecture

Conjecture 1.2. *The sequence of normalized Nash blowings up starting with any algebraic variety stabilizes, i. e. normalized Nash blowings up become isomorphisms, resulting in a desingularization.*

So far though Nash and normalized Nash desingularizations remain elusive in dimensions larger than one and two, respectively. Moreover,

Remark 1.3. In dimension larger than one an apriori estimate for the length of normalized Nash desingularization is not known (as well as in any reasonable sense for other desingularizations).

(i) If Nash blow up $N_X : N(X) \to X$ is an isomorphism then X is nonsingular, see [6] and [7].

(ii) Nash conjecture is well known to be true when $\dim X = 1$ and there is a simple apriori estimate for the length of Nash desingularization (e. g. by means of Newton-Puiseux expansion).

(iii) M. Spivakovsky proved that the sequence of normalized Nash blowings up terminates when $\dim X = 2$, see [9] and [5].

In Sections 2 and 7 we state and prove our main Theorem 2.5 on the structure of affine binomial varieties. Proved for the differences of monomials as binomials, it implies that the irreducible components of local analytic germs of singularities of any subvariety of a binomial variety X obtained by restricting nonvanishing on X affine coordinates to vary in a manifold are 'the same' as the respective local analytic components of singularities of X . This class of subvarieties, say $\mathcal{AB}$, includes varieties with not all coefficients of their defining binomial equations $= 1$, as well as varieties with the 'coefficients' of their defining binomial equations being any nonvanishing on the respective variety X elements of the subring of regular functions on X generated by all of the nonvanishing on X affine coordinates.

Theorem 2.5 also provides reduction of Nash (respectively normalized Nash) desingularizations of $\mathcal{AB}$ varieties to the respective desingularizations of irreducible binomial varieties passing through the origins of the (appropriate) ambient affine coordinate charts, while the latter are the closures in $\mathbb{C}^N$ of the tori that are the isomorphic images $\phi(\mathbb{T}^m)$ of the standard tori $\mathbb{T}^m := (\mathbb{C}^*)^m$ under monomial bijections $\phi : \mathbb{T}^m \to \phi(\mathbb{T}^m) \hookrightarrow \mathbb{T}^N$ with the convex hull in $\mathbb{R}^m$ of the exponents involved $\not\ni \mathbf{0}$, see Remark 4.3. Following the process of changes in the latter exponents under successive Nash blowings up of these varieties *essentially* provides a 'combinatorial' version of Nash algorithm and so we refer to them as *essential*.

Varieties with torus as an open dense subset are binomial, but not necessarily normal, e. g. Whitney Umbrella $\{x^2 - z \cdot y^2 = 0\} \subset \mathbb{C}^3$. Moreover, Nash blowings up of normal varieties with open dense tori may fail to be normal, e. g. Nash blow up of surface $S := \overline{\phi(\mathbb{T}^2)} \subset \mathbb{C}^3$, where $\phi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$, fails to be normal in spite that S is a normal surface. Indeed, normality of the latter is a consequence (due to a criterion in Section 2.1 of [4]) of the property of the exponents of monomial map ϕ to generate over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$ all points of its integral lattice within the (positive) cone that the respective exponents span in $\mathbb{R}^2$, see Example 5.3 for the details of the failure of normality for $N(S)$. Consequently, for the sake of convenience (though in abuse of terminology), we refer to the varieties with an open dense torus as *toric varieties* (as in [1]), while in [4] they are refered to as toric only when normal.

It turns out that Nash blow up of essential variety is a finite union of affine charts which are essential, see Claim 4.6 . The latter allows us to establish in Section 4 a 'combinatorial bookkeeping' of the progress in Nash (respectively normalized Nash) sequence of blowings up for essential varieties leading to an equivalent algorithm that in a multidimensional setting resembles classical Euclidean division algorithm. When the essential subvariety is of dimension 2 we introduce an integer, say D , which is simple to calculate in terms of exponents of a monomial parametrization of the torus of essential subvariety, state (in Section 3) and prove (in Section 5) that $D-1$ provides an elementary apriori bound on the length of the normalized Nash desingularization. In Section 6 we establish the (local) invariance of integer D with respect to local analytic isomorphisms preserving passing through the origin and invariant under the action of the torus of Y hypersurfaces. Unfortunately termination of our multidimensional and normalized multidimensional Euclidean divisions in general, in spite of their combinatorial nature and simple formulations, remain so far as elusive as their geometric counterparts.

2. Reduction from binomial to essential toric varieties

We consider algebraic varieties (so called binomial) that admit (Zariski) open coverings by 'affine binomial' varieties, i. e. closures $\hat{V}$ in $\mathbb{C}^N$ of sets $V^*(\hat{f}) := \{w \in (\mathbb{C}^*)^N : \hat{f}_j(w) = 0 \, , \, 1 \le j \le M \}$, where $(\hat{f})$ are the ideals in the ring $\mathbb{C}[w]$ of polynomials in N variables $w := (w_1, \ldots, w_N)$ generated by binomials

$$\hat{f}_j := w_1^{\hat{\alpha}_{j1}} \cdots w_N^{\hat{\alpha}_{jN}} - w_1^{\hat{\beta}_{j1}} \cdots w_N^{\hat{\beta}_{jN}} \ . \tag{2.1}$$

We call matrix $\hat{E}$ with entries $\hat{\alpha}_{ji} - \hat{\beta}_{ji}$ an *exponents matrix of* $\hat{V}$ ($\operatorname{rank} \hat{E} = N - \dim_{\mathbb{C}} V^*(\hat{f})$ and $V^*(\hat{f}) \subset \operatorname{Reg} \hat{V}$ are easy to verify).

Set $\hat{V}^* := \hat{V} \cap (\mathbb{C}^*)^N = V^*(\hat{f})$ and is a subgroup of torus $(\mathbb{C}^*)^N$ with a coordinatewise multiplication and unit $\mathbb{I}_N = (1, \ldots, 1)$. Let $\mathbf{0}$ denote the origin of $\mathbb{C}^N$, $\exp((h_1, \ldots, h_N)) := (e^{h_1}, \ldots, e^{h_N})$ and $\mathbb{R}_+ \subset \mathbb{R}$, $\mathbb{Q}_+ \subset \mathbb{Q}$ and $\mathbb{Z}_+ \subset \mathbb{Z} \setminus \{\mathbf{0}\}$ denote, respectively, the subsets of non negative real, rational and natural numbers. We placed proofs of all claims of this section in Section 7 .

Claim 2.1. *Torus of an affine m-dimensional toric variety $X \subset \mathbb{C}^N$ admits parametrization $y_j = x^{\vec{\Delta}_j}$, $\{\vec{\Delta}_j\}_{1 \le j \le L} \subset \mathbb{Z}_+^m$, iff $\mathbf{0} \in X$.*

We split all w variables into two groups $w = (y\ ,\ z)$ of y-variables, whenever $\{w_j = 0\} \cap \hat{V} \neq \emptyset$, and z-variables otherwise.

Lemma 2.2. *Variable w_j , is not a z-variable (equivalently is a y-variable) for $\hat{V}$ iff there is $\vec{\xi} \in \operatorname{Ker} \hat{E} \cap \mathbb{R}_+^N$ with $(\vec{\xi})_j > 0$.*

Corollary 2.3. *There is $\vec{\xi^+} \in \operatorname{Ker} \hat{E} \cap \mathbb{R}_+^N$ with $(\vec{\xi^+})_j > 0$ iff w_j is a y-variable. Therefore $(\mathbf{0}, \mathbb{I}_{N-L}) = \lim_{t \to -\infty} \exp(t \cdot \vec{\xi^+}) \in \hat{V}$.*

For the sake of completeness we include the following

Claim 2.4. *Polynomial $P \in \mathbb{C}[w]$ vanishes on $\hat{V}$ if and only if $(y_1 \cdot\ \ldots\ \cdot y_L)^l \cdot P \in (\hat{f})$ for some $l \in \mathbb{Z}_+$.*

Let $(j) \in \mathbb{C}^N$ denote a vector with the only nonzero j-th coordinate equal one. For a binomial variety $\hat{V} \subset \mathbb{C}^N$ let subspace $\mathbb{C}^L$ be the span of all (j) such that w_j is a y-variable and $\pi : \mathbb{C}^N \to \mathbb{C}^{N-L}$ be the linear projection to the complementary coordinate subspace with $\operatorname{Ker} \pi = \mathbb{C}^L$. For $w \in \mathbb{C}^N$ denote $|w| \in \mathbb{R}^N$ a point with coordinates being the absolute values $|w_j|$ of coordinates of w .

Theorem 2.5. *Pick the irreducible component $V \subset \hat{V}$ with $\mathbb{I}_N \in V$.*

A. Group $\Gamma := G/(G \cap V)$ is finite, where $G := \{w \in \hat{V} : |w| = \mathbb{I}_N\}$, sets $\{g \cdot V\}_{[g] \in \Gamma}$ are the distinct irreducible components of $\hat{V}$ listed by the equivalence classes $[g]$ of $g \in G$ and V^* is a torus dense in V.

B. Set $\hat{Y} := \hat{V} \cap (\mathbb{C}^L \times \mathbb{I}_{N-L})$ coincides with binomial variety $V^*(\hat{f}) \cap (\mathbb{C}^L \times \mathbb{I}_{N-L})$ (due to A. its irreducible component Y , $\mathbb{I}_N \in Y$, is toric). Also, $\pi(\hat{V}) = \pi(\hat{V}^*)$ is binomial and closed in $\mathbb{C}^{N-L}$.

C. There is a torus $Z \subset V^* = V \cap (\mathbb{C}^*)^N$ closed in $\mathbb{C}^N$ such that morphisms $\pi_{|Z} : Z \to \pi(V)$ and that of coordinatewise multiplication $\mu : Z \times (V \cap (\mathbb{C}^L \times \mathbb{I}_{N-L})) \to V$ are surjective, finite and both are local analytic isomorphisms.

Remark 2.6. Of course $\tilde{Y} := V \cap (\mathbb{C}^L \times \mathbb{I}_{N-L})$ is binomial (due to parts A. and B.), variety Y is an irreducible component of $\tilde{Y}$ as well as of $\hat{Y}$ and hence the restriction $\mu_{Z\times Y} : Z\times Y \to V$ of μ to $Z\times Y$ is finite. Also, $Z \times Y^* \hookrightarrow Z \times \tilde{Y}^* = (\mu)^{-1}(V^*) \subset \mathrm{Reg}\,(Z \times \tilde{Y}^*)$ and therefore $\mu(Z\times Y^*)$ is open and closed in V^*. It follows that $\mu_{Z\times Y}$ is also surjective, but $\mu_{Z\times Y}$ need not be a local analytic isomorphism as an example of variety $V := \{y_1^2 = z_1 \cdot y_2^2 \ , \ z_1 \cdot z_2 = 1\}$ and $Z := \{z_1 = y_1 = y_2^2 \ , \ z_1 \cdot z_2 = 1\}$ shows. Nevertheless Structure Theorem 2.5 implies that local analytic components of a toric variety V (as well as of a binomial variety $\hat{V}$) are isomorphic (by algebraic isomorphisms) to analytic germs of $Z \times Y$ at appropriate points.

We refer to $Y \hookrightarrow \hat{V}$ as *essential subvariety* and, if $Y = \hat{V}$ to $\hat{V}$ as *essential* (e. g. due to Corollary 2.3 Y is). Theorem 2.5 implies

Claim 2.7. *Assume $\tilde{V}$ is a subvariety of a binomial variety $\hat{V}$ obtained by restricting nonvanishing on $\hat{V}$ coordinates w_j to vary in a manifold. Then Cartesian products of irreducible components of local analytic germs of singularities of $\tilde{V}$ with smooth analytic germs of appropriate dimensions are isomorphic to the appropriate irreducible components of local analytic germs of singularities of $\hat{V}$. Consequently, Remark 1.1 as well as the conclusion of Corollary 2.8 apply to all $\mathcal{AB}$ varieties. Any affine variety with nonunit coefficients of defining binomial equations is in $\mathcal{AB}$ class.*

For Nash/normalized Nash blowings up Theorem 2.5 implies

Corollary 2.8. *It follows that the 'towers' of Nash (as well as normalized Nash) blowings up starting with varieties gV for $g \in \Gamma$ are mutually isomorphic and therefore it suffices to study the effect of this process on a single irreducible component V to make them all smooth. Moreover, Remark 1.1 implies that the stabilization of the sequence of Nash blowings up (respectively normalized Nash blowings up) of an affine binomial variety is equivalent to the stabilization of the respective sequence for its essential toric subvariety.*

3. Apriori bound on desingularization length: essential dim= $\mathbf{2}$

Let $\hat{V}$ be an affine binomial variety, denote $\hat{E}$ an associated with $\hat{V}$ matrix and set $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}^L$ be such that vectors $\vec{\delta}_i \times \mathbf{0} \in \mathbb{Z}^N$ generate over $\mathbb{Z}$ the integral lattice in $\mathrm{Ker}\,\hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$. Our main complexity estimate is

Theorem 3.1. Complexity bound on desingularization in dim $= 2$. *Assume that $m = 2$ and let D be the (absolute) value of the coordinate* *of* $\vec{\delta}_1 \wedge \vec{\delta}_2$ *at* $(l) \wedge (k)$, $1 \leq l$, $k \leq L$ *for which the positive cone in* $\mathbb{R}^2$ *spanned over* $\mathbb{R}_+$ *by vectors* $((\vec{\delta}_1)_l \ , \ (\vec{\delta}_2)_l)$ *and* $((\vec{\delta}_1)_k \ , \ (\vec{\delta}_2)_k)$ *contains vectors* $((\vec{\delta}_1)_j \ , \ (\vec{\delta}_2)_j)$ *for all* j , $1 \leq j \leq L$. *Then after at most* $D-1$ *normalized Nash blowings up starting with variety* $\hat{V}$ *the process stabilizes (in a nonsingular toric variety).*

Remark 3.2. Note that for any integral basis $\{ \vec{\delta}_i\}_{1\leq i\leq m}$, as considered above, the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis are unique up to a sign and can simply be found by choosing *any* $\mathbb{Q}$-basis $\{\vec{v}_i\}_{1\leq i\leq m}$ with the same $\mathbb{Q}$-span as that of the $\{\vec{\delta}_i\}_{1\leq i\leq m}$, then multiplying the respective coordinates of $\vec{v}_1 \wedge \cdots \wedge \vec{v}_m$ by their least common denominator and subsequently dividing obtained integers by their g.c.d. . For $m=2$ we may, moreover, determine the bound D of Theorem 3.1 up to a sign by detecting which $(l)\wedge(k)$ coordinate of the resulting sequence of integers to take. To that end the criterion of detecting pair (l,k) of Theorem 3.1 does not depend on the choice of a basis and can be applied as well with a basis $\{\vec{v}_i\}_{1\leq i\leq m}$. In particular, it follows by making use of Lemma 6.3 and Corollary 6.5 that integer D introduced in Theorem 3.1 is a local invariant of $\hat{V}$ at $\mathbf{0}$.

4. Reduction of Nash algorithm to a combinatorial one

4.1. **Gauss map and Nash blow up of an essential subvariety.** Monomial map $\phi : (\mathbb{C}^*)^m \to Y^* := Y \cap (\mathbb{C}^*)^N \hookrightarrow \mathbb{C}^L$, where

$$\phi_j(x) := \prod_{1\leq i\leq m} x_i^{\delta_{ji}}, \ 1 \leq j \leq L; \quad \phi_s \equiv 1, \ L < s \leq N \ , \tag{4.1}$$

induces an isomorphism, or an epimorphism with a finite kernel, iff $\{\vec{\delta}_i \times \mathbf{0}\}_{1\leq i\leq m} \subset \mathbb{Z}^N$, where $\vec{\delta}_i := (\delta_{1i},\ldots,\delta_{Li})$, generate the integral lattice of $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$ over $\mathbb{Z}$, respectively over $\mathbb{Q}$, cf. proof of Proposition 7.1 . (Moreover, since in view of Corollary 2.3 the closure $Y \hookrightarrow \mathbb{C}^L$ of $Y^* = Y \cap (\mathbb{C}^*)^L$ contains $\mathbf{0} \in \mathbb{C}^L$ Claim 2.1 implies that one may choose exponents δ_{ji} of (4.1) positive .)

Remark 4.1. Since $\phi_{|(\mathbb{R}_+ \setminus\{\mathbf{0}\})^m} : (\mathbb{R}_+ \setminus \{\mathbf{0}\})^m \to Y \cap (\mathbb{R}_+ \setminus \{\mathbf{0}\})^N$ is an isomorphism its tangent map $(\mathbb{R}^m)^{dual} \ni h \mapsto (h(\vec{\Delta}_1),...,h(\vec{\Delta}_L)) \times \mathbf{0} \in \operatorname{Ker} \hat{E} \cap (\mathbb{R}^L \times \{\mathbf{0}\})$ (at $\mathbb{I}_m \in \mathbb{R}^m$), where $\vec{\Delta}_j := (\delta_{j1},\ldots,\delta_{jm})$, $1 \leq j \leq L$, is bijective. Therefore due to the choice of vector $\vec{\xi}^+$ from Corollary 2.3 there is a functional $h^+ \in (\mathbb{R}^m)^{dual}$ such that each $h^+(\vec{\Delta}_j) = (\xi^+)_j > 0$. Hence the convex hull of $\vec{\Delta}_j$'s in $\mathbb{R}^m$ does not contain the origin. We refer to $C := \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ with the latter property as *essential. It enables recording of the process of Nash (and/or normalized Nash) blow ups as a 'combinatorial' algorithm.*

To 'control' the closure of torus Y^* we prove in Section 7

Lemma 4.2. *One can reach all points* $\mathcal{P} \in Y \setminus (\mathbb{C}^*)^N$ *by means of* $\lim_{t\to-\infty} \exp(\vec{a} + t \cdot \vec{\xi}) \times \mathbb{I}_{N-L} \in Y^*$*, with* $\vec{\xi} \times \mathbf{0}$, $\vec{a} \times \mathbf{0} \in \operatorname{Ker} \hat{E} \cap \mathbb{C}^N$ *and* $\vec{\xi} \in \mathbb{R}^L_+$ *. Moreover, coordinates* ξ_j *of* $\vec{\xi}$ *are positive or vanish depending on the respective coordinate of* $\mathcal{P}$ *being equal to zero or not.*

Remark 4.3. *Limits and criteria of being an essential variety.* Whenever there are exponents $\delta_{ji} < 0$ map ϕ would not extend to all of $\mathbb{C}^m$ and even if all $\delta_{ji} > 0$, as in Claim 2.1 , map $\phi : \mathbb{C}^m \to Y \hookrightarrow \mathbb{C}^L$ may not be surjective. Nevertheless one may reach all points $\mathcal{P} \in Y \setminus (\mathbb{C}^*)^N$ by means of $\phi(\exp(b+t\cdot h)) \in Y^*$ as $t \to -\infty$ with $b \in (\mathbb{C}^m)^{dual}$ and $h \in (\mathbb{R}^m)^{dual}$. Indeed, due to the choice of $\{\vec{\delta}_i\}_{1\le i\le m}$ and by making use of Lemma 4.2 there are unique $b \in (\mathbb{C}^m)^{dual}$ and $h \in (\mathbb{R}^m)^{dual}$ such that each $a_j = b(\vec{\Delta}_j)$, $\xi_j = h(\vec{\Delta}_j)$, $1 \le j \le L$. Note that given b and h the limit as above exists iff $h(\vec{\Delta}_j) \ge 0$ for all j . In particular, by identifying $\mathbb{C}^L$ with $\mathbb{C}^L \times \mathbb{I}_{N-L} \hookrightarrow \mathbb{C}^N$ and by making use of Corollary 2.3 , it follows that the origin of $\mathbb{C}^L$ is in Y . Equivalently, there is a point $h^+ \in (\mathbb{R}^m)^{dual}$ such that for $1 \le j \le L$ values $h^+(\vec{\Delta}_j) = (\vec{\xi}^+)_j > 0$ and is also equivalent to $C := \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^m$ being essential. *This property is proved in Claim 4.6 to be hereditary for an appropriate choice of affine charts covering Nash blow up of* Y *.*

Remark 4.4. *Gauss map in local coordinates.* Consider the composite of the Gauss map G_Y of Y on Y^* with a monomial parametrization (4.1) of Y^* and identify $G_Y(\phi(x)) \in \mathcal{G}_m(\mathbb{C}^L) \hookrightarrow \mathbb{CP}^{\binom{L}{m}-1}$, where the latter is the embedding of the Grassmanian $\mathcal{G}_m(\mathbb{C}^L)$ of the m-dimensional subspaces of $\mathbb{C}^L$ by means of Plücker coordinates, with the image $T_{\phi(x)}Y$ of $T_x\mathbb{C}^m \simeq \mathbb{C}^m$ by the tangent map to ϕ at $x \in (\mathbb{C}^*)^m$. The homogeneous (Plücker) coordinates $\tilde{w} = [\ \dots : \tilde{w}_J : \dots\ ]$ of $G_Y(\phi(x)) = \operatorname{Im} \frac{\partial\phi}{\partial x}(x)$ are the subdeterminants $\det_J(\mathcal{J}_\phi)(x)$ of the $m \times m$ size submatrices of the jacobian matrix $\mathcal{J}_\phi(x)$ of map $y = \phi(x)$ and are listed by the choices of $J = \{j_1, \dots, j_m\} \subset \{1, \dots, L\}$ of m distinct rows of the $L \times m$ matrix $\mathcal{J}_\phi$, i. e. $\tilde{w}_J = \det_J(\mathcal{J}_\phi(x)) = \det_J(\delta) \cdot x^{\sum_{j\in J} \vec{\Delta}_j} / (x_1 \cdot \ldots \cdot x_m)$, where $\det_J(\delta)$ are the respective subdeterminants of the exponents matrix δ in (4.1) . Denote $\mathcal{S} := \mathcal{S}(C) := \{J : \det_J(\delta) \ne 0\}$ and $L^* := \#\mathcal{S} - 1$ (notation $\mathcal{S}(C)$ is justified since $\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\{\vec{\Delta}_j\}_{j\in J} = m$ iff $\det_J(\delta) \ne 0$). Let $\mathbb{CP}^{L^*} := \bigcap_{\{J:\det_J(\delta)=0\}} \{\tilde{w}_J = 0\} \hookrightarrow \mathbb{CP}^{\binom{L}{m}-1}$.

Then $G_Y \circ \phi(x) \in \mathbb{CP}^{L^*}$ for all $x \in (\mathbb{C}^*)^m$. Moreover, then $G_Y \circ \phi : (\mathbb{C}^*)^m \to \cap_{J\in\mathcal{S}}\{\tilde{w}_J \neq 0\} =: T$.

Of course each $\mathcal{W}_J := \{\tilde{w}_J \neq 0\} \simeq \mathbb{C}^{L^*}$ and via this isomorphism T identifies with $(\mathbb{C}^*)^{L^*} \subset \mathbb{C}^{L^*}$. In abuse of notation let then $\mathcal{W}_J^*$ denote $T \hookrightarrow \mathcal{W}_J$. Similarly, denote $\mathcal{U}_J := \mathbb{C}^L \times \mathcal{W}_J$, $\mathcal{U}_J^* := (\mathbb{C}^*)^L \times \mathcal{W}_J^*$ and also $N(Y)_J := N(Y) \cap \mathcal{U}_J$, $N(Y)_J^* := N(Y) \cap \mathcal{U}_J^*$. Of course $N(Y)_{J_0}^* = \cap_{J\in\mathcal{S}} N(Y)_J$ for any $J_0 \in \mathcal{S}$.

Remark 4.5. *Essential affine charts of* $N(Y)$. Then $\mathcal{U}_J^* \hookrightarrow \mathcal{U}_J$ is isomorphic to $(\mathbb{C}^*)^{L+L^*} \hookrightarrow \mathbb{C}^{L+L^*}$ and affine toric variety $N(Y)_J$ is the closure of the image $N(Y)_J^*$ of torus $(\mathbb{C}^*)^m \subset \mathbb{C}^m$ under an algebraic group monomorphism $x \mapsto \psi(x) := (\phi(x)\ ,\ G_Y \circ \phi(x))$. Remark 4.3 implies that for any $J_0 \in \mathcal{S}$ one may reach all points $\tilde{\mathcal{P}} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$ by means of $\lim_{t\to-\infty} \psi(\exp(\tilde{b} + t\cdot\tilde{h}))$ with $\tilde{b} \in (\mathbb{C}^m)^{dual}$ and $\tilde{h} \in (\mathbb{R}^m)^{dual}$. The existence of the latter limit implies both that $\tilde{h}(\vec{\Delta}_j) \geq 0$ for all j and that $\tilde{h}(\vec{\Delta}_J - \vec{\Delta}_{J_0}) \geq 0$ for every $J \in \mathcal{S} \setminus \{J_0\}$, where $\vec{\Delta}_J := \sum_{j\in J} \vec{\Delta}_j$ for $J \in \mathcal{S}$. Moreover, affine chart $N(Y)_{J_0}$ contains the origin of $\mathcal{U}_{J_0} \simeq \mathbb{C}^{L+L^*}$, i. e. is essential, iff there is $\tilde{h} \in (\mathbb{R}^m)^{dual}$ such that all inequalities of the latter sentence are strict and is equivalent (Lemma 2.2) to all coordinates on $\mathcal{U}_{J_0}$ being 'y-variables' for $N(Y)_{J_0}$. Equivalently (Remark 4.3) the convex hull in $\mathbb{R}^m$ of set $C_{J_0} := \{\vec{\Delta}_j\}_{1\leq j\leq L} \cup \{\vec{\Delta}_J - \vec{\Delta}_{J_0}\}_{J_0\neq J\in\mathcal{S}} \subset \mathbb{Z}^m$ does not contain the origin.

Claim 4.6. *Assuming that toric variety* $Y \hookrightarrow \mathbb{C}^L \simeq \mathbb{C}^L \times \mathbb{I}_{N-L}$ *contains* $\mathbf{0} \in \mathbb{C}^L$ *it follows that* $N(Y) = \cup_{J\in\mathcal{S}'} N(Y)_J$ *, where* $\mathcal{S}'$ *is the subset of all* $J \in \mathcal{S}$ *such that affine charts* $N(Y)_J$ *are essential.*

Proof. With reference to Remark 4.3 our assumption is equivalent to $\mathrm{Conv}\,(C) \not\ni \mathbf{0}$. Let cone $\tilde{\mathcal{C}} := \{h \in (\mathbb{R}^m)^{dual} : h_{|C} \geq 0\}$ and, likewise, for every $J \in \mathcal{S}$ let $\tilde{\mathcal{C}}_J := \{h \in \tilde{\mathcal{C}} : h_{|C_J} \geq 0\}$. Then point h^+ from Remark 4.3 is in the interior of cone $\tilde{\mathcal{C}}$ (in particular $\dim_{\mathbb{R}} \tilde{\mathcal{C}} = m$). We refer to $h = (h_1, \ldots, h_m) \in (\mathbb{R}^m)^{dual}$ with $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{h_1, \ldots, h_m\} = m$ as an *irrational* point of $(\mathbb{R}^m)^{dual}$. For any irrational $h \in \tilde{\mathcal{C}}$ there is (and unique) $J \in \mathcal{S}$ such that h is in the interior of $\tilde{\mathcal{C}}_J$. Therefore $\dim_{\mathbb{R}} \tilde{\mathcal{C}}_J = m$ iff $\mathrm{Conv}\,(\tilde{\mathcal{C}}_J) \not\ni \mathbf{0}$. (Due to Remark 4.3 set $\mathcal{S}'$ coincides with the set of all J that appear in the previous sentence.) It follows that $\tilde{\mathcal{C}} = \cup_{J\in\mathcal{S}'} \tilde{\mathcal{C}}_J$.

Consider any $J_0 \in \mathcal{S}$. Torus $N(Y)_{J_0}^*$ coincides with the image $\psi((\mathbb{C}^*)^m) \subset \cap_{J\in\mathcal{S}'} N(Y)_J$. Let $\mathcal{P} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$. Then, as in Remark 4.3 , there are $b \in (\mathbb{C}^m)^{dual}$ and $h \in (\mathbb{R}^m)^{dual}$ such that $\mathcal{P} = \lim_{t\to-\infty} \psi(\exp(b + t\cdot h))$ and $\psi(\exp(b + t\cdot h)) \in N(Y)_{J_0}^*$. Moreover, values $h(\vec{\Delta}_j)$, $1 \le j \le L$, and all $h(\vec{\Delta}_J - \vec{\Delta}_{J_0})$, $J \in \mathcal{S} \setminus \{J_0\}$, are positive or vanish depending on the respective coordinate of $\mathcal{P}$ being equal to zero or not, see Lemma 4.2. Thus $h \in \tilde{\mathcal{C}} = \cup_{J \in \mathcal{S}'} \tilde{\mathcal{C}}_J$ and, therefore, there exists $J_1 \in \mathcal{S}'$ such that $h \in \tilde{\mathcal{C}}_{J_1}$. As a consequence $h(\vec{\Delta}_{J_0}) = h(\vec{\Delta}_{J_1})$. It follows that the ratio $\tilde{w}_{J_0}/\tilde{w}_{J_1}$ of the homogeneous coordinates of $G_Y \circ \phi(\exp(b + t \cdot h)) \in \mathbb{CP}^{L^*}$ does not depend on t , in fact is equal to $e^{b(\vec{\Delta}_{J_0}) - b(\vec{\Delta}_{J_1})}$. Therefore $\mathcal{P} \in N(Y)_{J_1} \setminus N(Y)^*_{J_1}$, which completes the proof. □

In the next two sections we summarize our 'translation' of Nash and of normalized Nash blowings up into respective combinatorial versions in terms of the smallest (in every reasonable sense) subsets of generators for additive semigroups $\mathbb{Z}_+(C)$ generated by finite sets $C \subset \mathbb{Z}^m$ with $\mathrm{Conv}\,(C) \not\ni \mathbf{0}$ and for $\mathbb{Q}_+(C)_{\mathbb{Z}} := \mathrm{Span}_{\mathbb{Z}}(C) \cap \mathrm{Span}_{\mathbb{Q}_+}(C) \setminus \{\mathbf{0}\}$.

For an additive semigroup without zero, say G_+ , we introduce a notion of the set $\mathcal{E}xt(G_+)$ of all Z_+*-extremal points* of G_+ , i. e. of all $g \in G_+$ such that $g \ne g_1 + g_2$ for any g_1 , $g_2 \in G_+$.

Denote $\nabla(J)$ the convex hull $\mathrm{Conv}\,(J \cup \{\mathbf{0}\})$ of $J \cup \{\mathbf{0}\} \subset \mathbb{R}^m$, where $\mathbf{0}$ is the origin of $\mathbb{R}^m$, and by $\mathrm{int}(\nabla(J))$ the interior of $\nabla(J)$.

Remark 4.7. Assume that set $C \subset \mathbb{Z}^m$ is finite and essential.

(i) Obviously set $\mathcal{E}xt(\mathbb{Z}_+(C))$ is finite and generates $\mathbb{Z}_+(C)$, while for $\mathbb{Q}_+(C)_{\mathbb{Z}}$ a similar claim is a consequence of Gordon's lemma (Prop.1 in 1.2 [4]) since $\mathrm{Span}_{\mathbb{Q}_+}(C)$ coincides with the dual cone $(\tilde{\mathcal{C}})^{dual}$ of its own dual cone $\tilde{\mathcal{C}}$ and $\mathbb{Q}_+(C)_{\mathbb{Z}}$ is the set of its integral points (meaning points in $\mathrm{Span}_{\mathbb{Z}}(C)$).

(ii) Note that $C' = \mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$ implies, by making use of (i), that $\mathbb{Z}_+(C') = \mathbb{Q}_+(C)_{\mathbb{Z}} \subset \mathrm{Span}_{\mathbb{Q}_+}(C) = \mathrm{Span}_{\mathbb{Q}_+}(C')$. Hence $\mathbb{Q}_+(C')_{\mathbb{Z}} = \mathbb{Q}_+(C)_{\mathbb{Z}}$ and therefore $C' = \mathcal{E}xt(\mathbb{Q}_+(C')_{\mathbb{Z}})$.

(iii) Assume $C = \mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$ and $J \in \mathcal{S}'$ (with notations from Claim 4.6). Then $\mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C)_{\mathbb{Z}} = \emptyset$. Indeed, if otherwise and $\vec{a} \in \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C)_{\mathbb{Z}}$ let us choose an irrational $h \in \tilde{\mathcal{C}}$, as in Claim 4.6, such that $h(\vec{\Delta}_J) = \min_{J' \in \mathcal{S}} h(\vec{\Delta}_{J'})$ and let $j_0 \in J$ be such that $h(\vec{\Delta}_{j_0}) = \max_{j \in J} h(\vec{\Delta}_j)$. Then, to begin with, $\vec{a} \notin C$, since otherwise collection $J_0 := (J \cup \{\vec{a}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_0}) < h(\vec{\Delta}_J)$. Then $\vec{a} \in \mathbb{Z}_+(C)$, due to assumption on C , and therefore there is a vector $\vec{b} \in C$ such that $J_1 := (J \cup \{\vec{b}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_1}) < h(\vec{\Delta}_J)$ (since if $\vec{a} \in \vec{b} + \mathbb{Z}_+(C)$ then inequalities $h(\vec{\Delta}_{j_0}) > h(\vec{a}) > h(\vec{b})$ hold), contrary to the choice of h .

4.2. **Multidimensional Euclidean division as a bookkeeping.** In this section we complete translation of the process of Nash blowings up into a *combinatorial* tree-like branching algorithm on finite essential subsets of $\mathbb{Z}^m$. To that end we choose $\{(\delta_{1i}, \ldots, \delta_{Li})\}_{1\le i\le m} \subset \mathbb{Z}^L$ as in (4.1). The input of this algorithm is collection $\mathcal{E}xt(\mathbb{Z}_+(C))$, where $C = \{\vec{\Delta}_j = (\delta_{j1}, \ldots, \delta_{jm})\}_{1\le j\le L}$ is the essential collection (see Remark 4.3) of exponents of a monomial parametrization of torus Y^* of an essential variety Y , we may assume that $C = \mathcal{E}xt(\mathbb{Z}_+(C))$.

In notations of Claim 4.6 the record of changes (derived in section 4.1) in the collections of exponents parametrizing the tori of the essential charts of Nash blowings up starting with variety Y is the **Multidimensional Euclidean algorithm on essential collections:**

with $\mathcal{S} = \mathcal{S}(C)$ *being the set of all* m*-tuples of linearly independent vectors in a finite essential (input) collection* $C = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ *we augment set* C *to a collection* C_J *by adjoining set* $\{\vec{\Delta}_{J'} - \vec{\Delta}_J\}_{J\ne J'\in\mathcal{S}}$ *provided that* $J \in \mathcal{S}' := \{J \in \mathcal{S} : C_J$ *is essential* $\}$*. Finite essential set* $N_J(C) := \mathcal{E}xt(\mathbb{Z}_+(C_J))$ *generates semigroup* $\mathbb{Z}_+(C_J)$ *and is the output of an algorithm branching according to the choices of* $J \in \mathcal{S}'$.

A branch of this algorithm terminates at a node with an associated to the node collection $C = \{\vec{a}_j\}_j \subset \mathbb{Z}^m$ whenever $\#(C) = m$.

Remark 4.8. Note that differences $\vec{\Delta}_{J'} - \vec{\Delta}_J$ with $\#(J' \setminus J) = 1$ generate over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$ all other differences in collections C_J , i. e. it suffices to include in C_J only them. Indeed, matrix $(a_{ji})_{j\in J'\ ,\ i\in J}$ transforming basis J of $\mathbb{Q}^m$ into basis J' is not degenerate implying existence of a bijection $J' \ni j \mapsto i = i(j) \in J$ with all $a_{j\ i(j)} \ne 0$ and $\vec{\Delta}_{J'} - \vec{\Delta}_J = \sum_{j\in J'}(\vec{\Delta}_j - \vec{\Delta}_{i(j)}) = \sum_{j\in J'}(\vec{\Delta}_{J\cup j\setminus i(j)} - \vec{\Delta}_J)$, as required.

Nash desingularization of essential affine toric subvariety Y of an affine binomial variety $\hat{V}$ leads to a Nash desingularization of $\hat{V}$ by making use of Theorem 2.5 A. and C. and of Remark 1.1 . Variety Y' resulting from a sequence of Nash blowings up of Y is a union of its essential affine charts $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{C}^{L'}$ due to Claim 4.6 . Every affine chart $Y' \cap \mathcal{U}'$ corresponds to a *node* of a branch of our combinatorial bookkeeping algorithm. With $\{\vec{a}_j\}_{1\le j\le L'} \subset \mathbb{Z}^m$ being the essential collection associated with the latter node it follows that the essential affine toric variety $Y' \cap \mathcal{U}'$ corresponding to the node admits a monomial parametrization of its torus by $(\mathbb{C}^*)^m$ in coordinates y'_j , $1 \le j \le L'$, on $\mathcal{U}'$ as follows: $y'_j = (\Phi)_j(x) := x^{\vec{a}_j}$, $1 \le j \le L'$. We finally show the equivalence of stabilization of the sequence of Nash blowings up of Y to the termination of our combinatorial algorithm

Claim 4.9. *A branch $\mathcal{B}$ of the multidimensional analogue of Euclidean division algorithm terminates iff the essential affine chart $Y' \cap \mathcal{U}'$ corresponding to the terminal node of $\mathcal{B}$ is nonsingular.*

Proof. Say $C' = \{\vec{a}_j\}_{1 \le j \le k}$ is the collection corresponding to a node of branch $\mathcal{B}$ and $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{C}^{L'}$ is the corresponding essential affine chart. Then exponents of monomial parametrization $y'_j = x^{\vec{a}_j}$, $1 \le j \le L'$, of torus $(Y' \cap \mathcal{U}')^* = (Y' \cap \mathcal{U}') \cap (\mathbb{C}^*)^{L'}$ include collection C' and, moreover, are in $\mathbb{Z}_+(C')$, i. e. can be expressed as nonnegative integral linear combinations $\vec{a}_j = \sum_{1 \le l \le k} n_{jl} \cdot \vec{a}_l$, $k+1 \le j \le L'$.

Therefore, if branch terminates, i. e. collection C' associated with its *terminal node* is of size m, then $Y' \cap \mathcal{U}'$ is nonsingular being the graph of map $y'_j = (y'_1)^{n_{j1}} \cdot \ldots \cdot (y'_m)^{n_{jm}}$, $m+1 \le j \le L'$.

Conversely if $Y' \cap \mathcal{U}'$ is nonsingular at the origin of $\mathcal{U}'$, it follows that it is a graph of a complex analytic map-germ $\mathcal{G}$ at the origin over a coordinate subspace $\mathbb{C}^m \subset \mathbb{C}^{L'}$. Since the closure $Y' \cap \mathcal{U}'$ of torus $(Y' \cap \mathcal{U}')^*$ contains the origin $\mathbf{0}$ of $\mathcal{U}' \simeq \mathbb{C}^{L'}$ Claim 2.1 implies that there is a monomial parametrization $y'_j = x^{\vec{\omega}_j}$, $1 \le j \le L'$, of $(Y' \cap \mathcal{U}')^*$ with $\{\vec{\omega}_j\}_{1 \le j \le L'} \subset \mathbb{Z}^m_+$. Then (uniqueness of Taylor series expansion of the composite of $\mathcal{G}$ with the components of parametrization $y'_{j_l} = x^{\vec{\omega}_{j_l}}$, $1 \le l \le m$, associated with the aforementioned coordinate subspace $\mathbb{C}^m$ implies that) map-germ $\mathcal{G}$ is monomial. We may conclude now that vectors $\vec{a}_j$, $1 \le j \le L'$, are generated over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$ by their subset (of size m) corresponding to the coordinate subspace $\mathbb{C}^m$ of the previous sentence. ☐

Remark 4.10. The proof of Claim 4.9 shows that essential toric variety is nonsingular iff it is nonsingular at the origin.

4.3. **Effect of normalization.** Normalization $\mathcal{N}(Y)$ of essential affine variety Y adjoins as regular functions on $\mathcal{N}(Y)$ all monomials $\mathcal{M}$ in coordinates y_j, $1 \le j \le L$, on $\mathbb{C}^L$ whenever $\mathcal{M}^d$ for some $d \in \mathbb{Z}_+$ coincides on Y with another monomial $\mathcal{M}'$ in y_j's with non negative integral exponents (see Section 2.1 in [4]). Since torus Y^* is parametrized by monomials $y_j = x^{\vec{\Delta}_j}$, $1 \le j \le L$,

normalization translates into a combinatorial algorithm: *augment an essential input set $C = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ to a semigroup $\mathbb{Q}_+(C)_{\mathbb{Z}}$ generated by its finite essential subset $\mathcal{N}(C) := \mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$ (Remark 4.7 (i)) - the output of combinatorial normalization.*

Of course a sequence of composites of normalized Nash blowings up followed by normalization coincides with normalization followed by the sequence of Nash blowings up composed with normalizations. For the convenience of exposition (and reflecting the latter) essential collection $\mathcal{N}(C)$, with $C = \{\vec{\Delta}_j\}_{1\le j\le L}$ from (4.1), is the input for

normalized multidimensional Euclidean division algorithm: *whose each step starts with essential set C as an input and results in essential collections $\mathcal{N}(N_J(C))$ for $J \in \mathcal{S}(C)'$ as the output.*

The latter records a sequence of normalized Nash blow ups (followed by normalization) of an essential toric variety Y . By definition a branch of this tree-like algorithm terminates at a node with an essential collection C provided that the size of C is m .

The proof of Claim 4.9 applies to show that a branch $\tilde{\mathcal{B}}$ of normalized multidimensional Euclidean division terminates iff the essential chart corresponding to the terminal node of $\tilde{\mathcal{B}}$ is nonsingular. Since normalization separates all local analytic irreducible components (and due to Theorem 2.5 A. , C. and Remark 1.1) the lengths of the normalized Nash desingularization of the essential subvariety Y of an affine binomial variety $\hat{V}$ and that of $\hat{V}$ coincide.

5. Termination of normalized Euclidean division: dim= 2

Conjecture 5.1. *Tree $\overline{\mathcal{T}}$ associated with the multidimensional Euclidean algorithm is finite for any initial data.*

By König's lemma the latter is equivalent to the property that the algorithm terminates along every branch of tree $\overline{\mathcal{T}}$. In dimension > 2 'normalized' version of 5.1 is the following

Conjecture 5.2. *Tree $\mathcal{T}$ associated with the normalized multidimensional Euclidean algorithm is finite for any initial data.*

We start with an example from Introduction of a normal toric surface in $\mathbb{C}^3$ whose Nash blow up is not normal

Example 5.3. With $\phi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$ let $S := \overline{\phi(\mathbb{T}^2)} \subset \mathbb{C}^3$. Exponents $C := \{(1,1) \ , \ (1,2) \ , \ (3,2)\} \subset \mathbb{Z}^2$ generate over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$ integral points $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(C)$ of cone $\mathrm{Span}_{\mathbb{Q}_+}(C) \subset \mathbb{Q}^2$ spanned by C , because $\det((3,2) \ , \ (1,1)) = 1 = \det((1,1) \ , \ (1,2))$ implies that cones $\mathrm{Span}_{\mathbb{Q}_+}(\{(3,2) \ , \ (1,1)\})$ and $\mathrm{Span}_{\mathbb{Q}_+}(\{(1,1) \ , \ (1,2)\})$ are, respectively, generated by pairs of vectors $(3,2)$, $(1,1)$ and $(1,1)$, $(1,2)$ and because the union of these two cones is exactly the cone generated by C . Then due to a criterion of Section 2.1 in [4] it follows that surface S is normal. Next, with reference to Section 4.2 there are exactly two elements in the set

$\mathcal{S}(C)'$, namely: $J_1 = \{(1,1)\ ;\ (1,2)\}$ and $J_2 = \{(1,1)\ ;\ (3,2)\}$, - and the Nash blow up $N(S)$ of S is covered by two respective affine charts $N(S)_{J_j}$, $j = 1, 2$, as explained in Claim 4.6 . (In the remainder we make use of notations of Remark 4.5 .) It turns out $N(S)_{J_1} \subset \mathbb{C}^5$ is not normal, i. e. collection of exponents C_{J_1} of monomial parametrization

$$\psi : (x_1, x_2) \mapsto (x_1 \cdot x_2\ ,\ x_1 \cdot x_2^2\ ,\ x_1^3 \cdot x_2^2\ ,\ x_1^2 \cdot x_2\ ,\ x_1^2)$$

of torus $N(S)^*_{J_1}$ does not generate $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(C_{J_1})$ over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$, because obviously point $(1,0) \in \mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(C_{J_1}) \setminus \mathbb{Z}_+(C_{J_1})$, but $(1,0) \notin \mathbb{Z}_+(C \cup \{(2,1)\ ,\ (2,0)\})$, implying $N(S)$ is not normal. (Note, that $\psi_3(x) = \psi_1(x) \cdot \psi_4(x)$, i. e. exponent $(3,2)$ is generated over $\mathbb{Z}_+ \cup \{\mathbf{0}\}$ by 'others', illustrating passage from C_J to $\mathcal{E}xt(\mathbb{Z}_+(C_J))$ in the combinatorial algorithm recording Nash blowing up.)

Consider a node τ of a tree $\mathcal{T}$ associated with normalized multidimensional Euclidean division for initial essential collection $\mathcal{N}(C)$ with C from (4.1). Let $C_\tau \subset \mathbb{Z}^2$ denote the associated with node τ essential collection. In abuse of notation we will not indicate the dependence of $\mathcal{S}_\tau := \mathcal{S}(C_\tau)$ and $\mathcal{S}'_\tau := \mathcal{S}(C_\tau)'$ on τ (for $\mathcal{S}(C)$ and $\mathcal{S}'$ see Remark 4.4 and Claim 4.6). Note that $\mathrm{int}(\nabla(J)) \cap \mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}}$ for $J \in \mathcal{S}_\tau$ and that $J \in \mathcal{S}'_\tau$ implies that $\mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}} = \emptyset$, see Remark 4.7 (ii) , (iii) . Of course $\mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{Span}_{\mathbb{Z}}(C)$ for any node τ . We may assume that $\mathbb{Z}^m = \mathrm{Span}_{\mathbb{Z}}(C)$, otherwise we 'rescale' replacing the latter span by $\mathbb{Z}^m$. Finally, we refer to the initial node τ_0 of $\mathcal{T}$ as its root and to the collection of 'immediate descendants' of τ in $\mathcal{T}$ as *child nodes* of τ - terms commonly used in the 'theory of trees'.

5.1. **An apriori bound in (essential) dimension $m = 2$ on the length of desingularization by normalized Nash blow ups.** Below we assume that $m = 2$, nodes τ_0 and τ are not terminal and with node τ associate an integer $\mathcal{V}(\tau) :=$ twice the area of $\mathrm{Conv}\,(C_\tau)$. We refer to vectors $\{\vec{\Delta}_{j_i}\}_{i=1,2} \subset C := \{\vec{\Delta}_j\}_{1 \le j \le L} \subset \mathbb{Z}^2$ minimal on the intersection of C with two extremal rays of the cone generated by C over $\mathbb{R}_+$ as the *extremal vectors* of C . Of course extremal vectors of the input $\mathcal{N}(C)$ for the normalized 2-dimensional Euclidean division are the same vectors. Integer D of Theorem 3.1 equals $|\det(\vec{\Delta}_{j_1}\ ,\ \vec{\Delta}_{j_2})|$. In abuse of notation we will not distinguish in this section between the subsets $J \in \mathcal{S}_\tau$ of indices of vectors in collections C_τ and the sets of the respective vectors themselves. Let $b_1,\ ,\ b_2 \in C_\tau$ be the extremal vectors of C_τ . Denote $D(\tau) := |\det(b_1\ ,\ b_2)|$ and pick a 2-tuple $J := \{u_j\}_{j=1,2} \in \mathcal{S}'$. In other words J corresponds to a child node $\overline{\tau}$ of τ and determines the branching of $\mathcal{T}$ at node τ . Of course $C_\tau = \mathcal{E}xt(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Every $J \in \mathcal{S}'$ is a *frame*, i. e. is a collection of linearly independent vectors, and moreover is a *minimal frame of* C_τ *.* By minimal we mean that for an irrational functional h positive on the convex hull of collection $C_{\overline{\tau}} \subset \mathbb{Z}^2$ the value of $h(\vec{\Delta}_J)$, where $\vec{\Delta}_J := u_1 + u_2$, is smaller than the value of $h(\vec{\Delta}_{J'})$ for any other choice of $J' \in \mathcal{S}$. This property of frames $J \in \mathcal{S}'$ does not depend on the choice of irrational h being positive on the convex hulls of collections $C_{\overline{\tau}} \subset \mathbb{Z}^2$, corresponding to $\overline{\tau}$ and provides a bijective correspondence between the minimal frames of C_τ and the child nodes $\overline{\tau}$ of τ , cf. Claim 4.6 . We identify in explicit geometric terms sets involved in the proof below of an apriori bound Theorem 3.1 in the following

Claim 5.4. *Generators* $\mathcal{E}xt(\mathbb{Q}_+(C)_\mathbb{Z})$ *of any* $C \subset \mathbb{Z}^2$ *with* $C \not\ni \mathbf{0}$ *and* $\mathrm{Span}_{\mathbb{Z}}(C) = \mathbb{Z}^2$ *are the integral points of bounded edges* Γ *of* $K := \mathrm{Conv}\,(\mathbb{Q}_+(C)_\mathbb{Z})$ *. For any node* τ *of tree* $\mathcal{T}$

$$D(\tau) - \mathcal{V}(\tau) = \#(C_\tau) - 1 \tag{5.1}$$

Proof. Inclusion of the integral points of bounded edges Γ of K in $\mathcal{E}xt(\mathbb{Q}_+(C)_\mathbb{Z})$ is obvious. To show the opposite inclusion we pick any pair J of adjacent integral points $\{u_1\ ,\ u_2\}$ on any bounded edge Γ of K . Then the only integral points of triangle $\nabla(u_1\ ,\ u_2)$ are its vertices. Therefore the only integral points in the parallelogramm $P(J)$ spanned by vectors u_1 , u_2 are its extremal points, which implies (by tiling of $\mathbb{R}^2$ by translations of $P(J)$) that $\mathrm{Span}_Z(J) = \mathbb{Z}^2$. Consequently $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(J) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(J)$ and $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap C = J$. (Equivalently $1 = |\det(u_1\ ,\ u_2)| = 2 \cdot \mathrm{area}(\nabla(u_1\ ,\ u_2))$ for any adjacent integral points u_1 , u_2 on any bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(C)_\mathbb{Z})$ and (5.1) follows for any node τ .) With J as above inclusion now is implied by $\mathrm{Span}_{\mathbb{Q}_+}(C) = \cup_J \mathrm{Span}_{\mathbb{Q}_+}(J)$. □

Remark 5.5. Any $J = \{u_1\ ,\ u_2\} \in \mathcal{S}(C)'$ must lie on a bounded edge Γ of $\mathrm{Conv}\,(\mathbb{Q}_+(C)_\mathbb{Z})$. Moreover, frame J is a minimal frame iff u_1 , u_2 $\in \Gamma$ are adjacent integral points and at least one of them is a vertex of edge Γ , since $J \in \mathcal{S}(C)'$ iff $\dim \tilde{\mathcal{C}}_J = 2$ (see proof of Claim 4.6). Of course $|\det(u_1\ ,\ u_2)| = 1$ for any pair $\{u_1\ ,\ u_2\}$ of adjacent integral points on a bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(C)_\mathbb{Z})$ is a byproduct of the proof of Claim 5.4 above.

Of course $\mathcal{V}(\tau) = 0$ for a terminal node τ and if node τ is not terminal but $\mathcal{V}(\tau) = 0$, then there are exactly two child nodes of node τ and both are terminal. We now restate and then prove Theorem 3.1

Theorem 5.6. *Assume* $\overline{\tau}$ *is not terminal. With every step of normalized* 2*-dimensional Euclidean algorithm integer* $\mathcal{V}(\tau)$ *decreases, i. e.* $\mathcal{V}(\tau) > \mathcal{V}(\overline{\tau})$.

Corollary 5.7. *Normalized* 2*-dimensional Euclidean algorithm terminates after at most* $\mathcal{V}(\tau_0) + 1 \leq D(\tau_0) - 1$ *steps.*

Proof. Fix an irrational h and by reindexing arrange that $h(b_1) < h(b_2)$. Let b'_1 , $b'_2 \in C_{\overline{\tau}}$ be the extremal vectors of $C_{\overline{\tau}}$ and $\tilde{b}'_1$, $\tilde{b}'_2 \in N_J(C_\tau)$ be the minimal vectors in the intersection of $N_J(C_\tau)$ with two extremal rays of the cone generated by $N_J(C_\tau)$ over $\mathbb{R}_+$. Of course the latter cone does not change under 'normalization', i. e. coincides with the cone generated by $C_{\overline{\tau}}$ over $\mathbb{R}_+$, see Section 4.3 . In particular, it follows that (after an appropriate choice of indices) *extremal vectors* $\tilde{b}'_1$, $\tilde{b}'_2$ *preceding normalization* are proportional to the extremal vectors b'_1 , b'_2 with coefficients from $\mathbb{Z}_+$.

Remark 5.8. Node τ is terminal iff $|\det(b_1 , b_2)| = 1$ iff $\#(C_\tau) = 2$ iff $\{b_1 , b_2\}$ is a minimal frame in C_τ . To establish the only nonobvious implication (i. e. that the last property implies the first) it suffices to apply Claim 5.4 . The latter reference and node τ not being terminal also imply that if $J \not\subset \mathrm{int}\nabla(b_1 , b_2)$ then $\#(\{b_1 , b_2\} \cap J) = 1$ and $b_2 \notin J$ (otherwise $h(b_1) < \min h_{|J} < h(b_2)$ contrary to the choice of the irrational functional $h \in \tilde{\mathcal{C}}_J$).

Plan : Our proof of decrease of $\mathcal{V}(\tau)$ splits into several cases identified below. First we consider the case that $J \subset \mathrm{int}\nabla(b_1 , b_2)$ and otherwise $b_1 \in J$, $b_2 \notin J$ (due to Remark 5.8) and, also, $b_1 \in \{b'_1 , b'_2\}$ follows by making use of $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap C_\tau = J$ established in Claim 5.4 , cf Figures 1 , 2 and 3 . Say $b'_1 = b_1$ and $u_1 = b_1$. The remaining cases are split according to either $u_2 \notin \mathrm{int}\nabla(b_1 , b_2)$ (and then $\overline{\tau}$ is terminal contrary to our assumption) or otherwise and then according to $\#(C_\tau) = 3$ (when $\#(C_\tau) = 2$ node is terminal) or $\#(C_\tau) \geq 4$. We show that in the latter case $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ for the bounded edge $\Gamma \supset J$ of $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ implies that node $\overline{\tau}$ must be terminal, i. e. is contrary to our assumption. In the previous case of $u_2 \in \mathrm{int}\nabla(b_1 , b_2)$ and $\#(C_\tau) = 3$ the arguments of our proof differ depending on $D(\tau)$ being even or odd : if $D(\tau) = 2k - 1$ is odd then it turns out that $C_{\overline{\tau}} = \{b_1 , u_2 , b_2 - (k-1) \cdot u_2 , b_2 - b_1\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, on the other hand if $D(\tau) = 2k$ is even then $C_{\overline{\tau}} = \{b_1 , u_2 , (b_2 - b_1)/2\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$. In each of the cases (with nodes τ and $\overline{\tau}$ not being terminal) we establish that (after 'normalization') integer $\mathcal{V}(\tau)$ decreases. We now start with

1. Points u_1 , u_2 in the interior of $\nabla(b_1\ ,\ b_2)$.

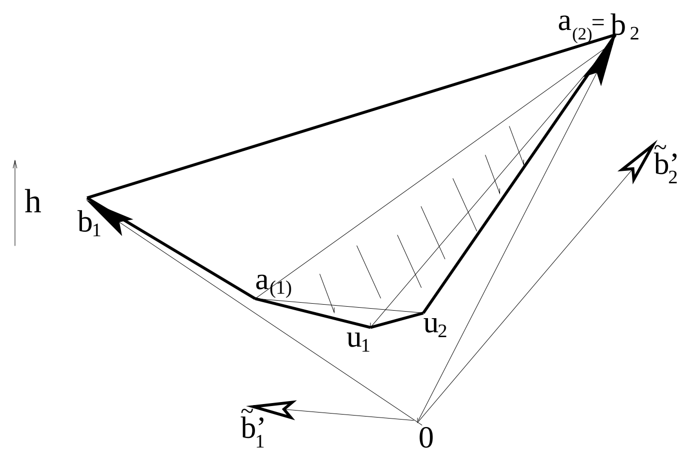

FIGURE 1. $C_\tau = \{\ b_1\ ,\ a_{(1)}\ ,\ u_1\ ,\ u_2\ ,\ b_2\ \}$.

Then after one step of 2-dimensional Euclidean division (and prior to normalization) each extremal vector $\tilde{b}'_l = a_{(l)} - u_{j_l}$ for appropriate points $a_{(l)} \in C_\tau \cap\ (\mathrm{int}(\nabla(b_1\ ,\ b_2)) \cup \{b_1\ ,\ b_2\})$, $l = 1\ ,\ 2$, $j_l \in \{1\ ,\ 2\}$, and after one step of normalized 2-dimensional Euclidean algorithm extremal vectors b'_1 , b'_2 are proportional to their respective counterparts $\tilde{b}'_1$, $\tilde{b}'_2$ with positive coefficients majorated by 1 , so that $D(\overline{\tau}) \le |\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)|$. Denote by H and $\mathcal{A}_H$ the convex hull of $\{a_{(1)}\ ,\ a_{(2)}\ ,\ u_{j_1}\ ,\ u_{j_2}\}$ and its area. Of course the areas of triangles $\nabla(b_1\ ,\ b_2)$ and $\nabla(b'_1\ ,\ b'_2)$ are $D(\tau)/2$ and, repectively, $D(\overline{\tau})/2$. Then claimed inequality follows from

$$\mathcal{V}(\overline{\tau}) < D(\overline{\tau}) \le |\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)| = 2 \cdot \mathcal{A}_H \le \mathcal{V}(\tau)\ .$$

2. Extremal vector $b_1 \in \{u_1\ ,\ u_2\}$ if it is not case 1.

Since τ is not terminal $b_2 \notin J = \{u_1\ ,\ u_2\}$ and $b_1 \in \{b'_1\ ,\ b'_2\}$ (see 'Plan'). Set both $b'_1 = b_1$, $u_1 = b_1$, i. e. $b'_1 = \tilde{b}'_1 = b_1 = u_1$ for the remainder of the proof. **Case 2.** splits into several starting with

2a. $u_2 \notin$ **int**$\nabla(b_1\ ,\ b_2)$ **.**

Then, with reference to Claim 5.4 , u_2 is in the open edge $(b_1\ ,\ b_2)$ (i. .e. excluding endpoints b_1 , b_2) of triangle $\nabla(b_1\ ,\ b_2)$ and therefore, moreover, the whole C_τ is a subset of closed edge $[b_1\ ,\ b_2]$. Then

$\tilde{b}_2' = a - u_2 \neq 0$ for the adjacent to u_2 point $a \in C_\tau \cap [u_2 , b_2]$ implying $b_2' = \tilde{b}_2' = u_2 - u_1$. Hence, with reference to Claim 5.4 , $|\det(b_1' , b_2')| = |\det(u_1 , u_2)| = 1$ and $\overline{\tau}$ is terminal (Remark 5.8).

In the remaining cases $u_2 \in \mathrm{int}\nabla(b_1 , b_2)$ and the assumptions of the next one imply that $\overline{\tau}$ is terminal.

2b. Assume $u_2 \in \mathbf{int}\nabla(b_1 , b_2)$, $\#(C_\tau) \geq 4$ and $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ for the bounded edge $\Gamma \supset J$ of $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Then, with reference to Claim 5.4 , $\tilde{b}_2' = a - u_2 \neq 0$ for the adjacent to u_2 point $a \in C_\tau \cap \Gamma \setminus \{u_1\}$ implying (as in the previous case) that $b_2' = \tilde{b}_2' = u_2 - u_1$, that $|\det(b_1' , b_2')| = |\det(u_1 , u_2)| = 1$ and, finally, that $\overline{\tau}$ is a terminal node, contrary to initial assumption.

2c. Assume $u_2 \in \mathbf{int}\nabla(b_1 , b_2)$, $\#(C_\tau) \geq 4$ and $\#(\mathbb{Z}^2 \cap \Gamma) = 2$ for the bounded edge $\Gamma \supset J$ of $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

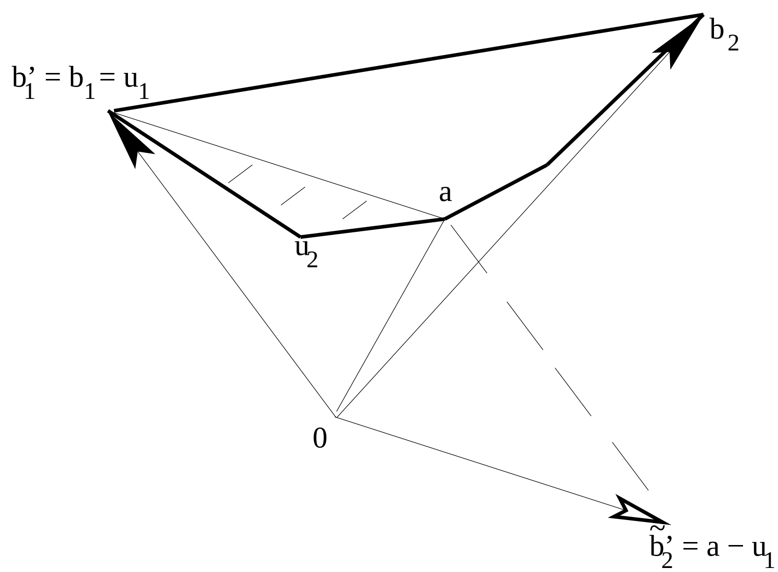

FIGURE 2. The area of $\mathrm{Conv}\,(C_\tau \setminus \{u_2\}) \geq 1$.

Then $\mathbb{Z}^2 \cap \Gamma = J$, $\#(C_\tau \setminus J) \geq 2$ and, with reference to Claim 5.4 , $\tilde{b}_2' = a - u_1$ with $a \in C_\tau \setminus \{u_1\}$ adjacent to u_2 . Therefore integer $\mathcal{V}(\tau) - 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1 , a - u_1)) > 0$ implying $|\det(\tilde{b}_1' , \tilde{b}_2')| = 2 + 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1 , a - u_1)) \leq 2 + (\mathcal{V}(\tau) - 1)$. Combining with (5.1) and Remark 5.8 proves inequality $\mathcal{V}(\overline{\tau}) < \mathcal{V}(\tau)$, as required:

$$2 + \mathcal{V}(\overline{\tau}) \leq D(\overline{\tau}) \leq |\det(\tilde{b}_1' , \tilde{b}_2')| \leq 1 + \mathcal{V}(\tau) \quad .$$

2d. Assume $u_2 \in \mathbf{int}\nabla(b_1\ ,\ b_2)$ **and** $\#(C_\tau) = 3$ **.**

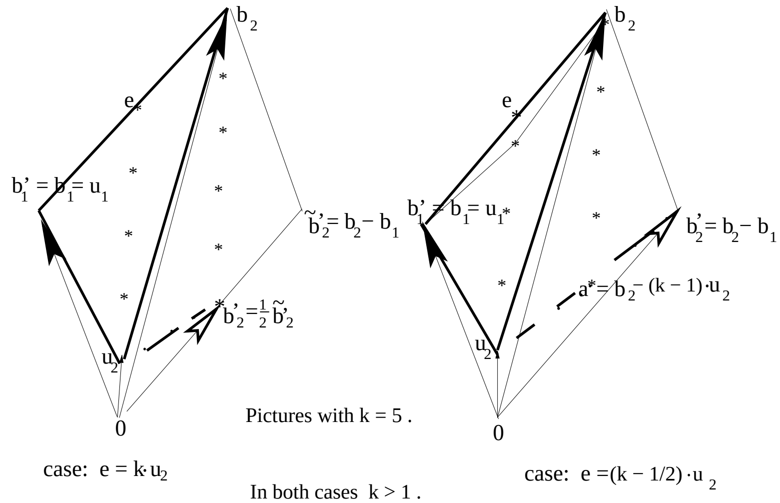

FIGURE 3. $D(\tau) = 2k$ or $2k - 1 \Rightarrow \#C_{\overline{\tau}} = 3$ or 4 respectively.

Let e be the point of intersection of edge $(b_1\ ,\ b_2)$ with ray $\mathbb{R}_+ \cdot u_2$, say $\lambda{\cdot}u_2 = e$, $\lambda > 0$. Due to Claim 5.4 $\nabla(b_1\ ,\ b_2) \cap \mathbb{Z}^2 \setminus \{\mathbf{0}\ ,\ b_1\ ,\ b_2\} \subset \mathbb{Z}_+ \cdot u_2$ and $|\det(b_2\ ,\ u_2)| = 1 = |\det(u_2\ ,\ b_1)|$ implying $\tilde{b}'_2 = b_2 - b_1$ and that the areas of triangles $\nabla(b_2\ ,\ e)$ and $\nabla(b_1\ ,\ e)$ coincide. Hence $e = (b_1 + b_2)/2$ and, also, $\lambda = |\det(e\ ,\ b_1)| = D(\tau)/2$. The arguments in the remainder depend on $D(\tau)$ being even or odd.

2d.* Assume $D(\tau)$ **is even and let** $k := D(\tau)/2$ **.**

Then $(b_2 - b_1)/2$ is the only integral point in the open 'interval' $(\mathbf{0}\ ,\ \tilde{b}'_2)$ implying that $b'_2 = \tilde{b}'_2/2$ and, since $|\det((b_2 - b_1)/2\ ,\ u_2)| = |(\det(b_2\ ,\ u_2) + \det(u_2\ ,\ b_1))/2| = 1$, that $C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ (b_2 - b_1)/2\}$ (Claim 5.4). Finally, with reference to (5.1), it follows that

$$\mathcal{V}(\overline{\tau}) + 2 = D(\overline{\tau}) = |\det(b_1\ ,\ (b_2 - b_1)/2)| = D(\tau)/2 = (\mathcal{V}(\tau) + 2)/2$$

implying that $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$, as required.

2d. Assume** $D(\tau)$ **is odd and let** $k := (D(\tau) + 1)/2$ **.**

Then there are no integral points on edge $(b_1\ ,\ b_2)$ (as well as on 'interval' $(\mathbf{0}\ ,\ \tilde{b}'_2)$) implying that $b'_2 = \tilde{b}'_2 = b_2 - b_1$. Denote point $a := b_2 - (k-1) \cdot u_2 = (u_2 + b'_2)/2$. Then, since $|\det(b'_2\ ,\ u_2)| = 2$, it follows that $|\det(b'_2\ ,\ a)| = |\det(a\ ,\ u_2)| = 1$. Now, with reference to Claim 5.4 it follows that $C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ b_2 - (k-1) \cdot u_2\ ,\ b_2 - b_1\}$ and finally, due to (5.1), that

$$\mathcal{V}(\overline{\tau}) + 3 = D(\overline{\tau}) = |\det(b'_1\ ,\ b'_2)| = D(\tau) = \mathcal{V}(\tau) + 2$$

implying that $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, which completes the proof. □

Bound on the numbers of the child nodes for the nodes of $\mathcal{T}$:

Remark 5.9. Denote $n(\tau) \in \mathbb{Z}_+$ the number of child nodes of node τ . With reference to Remark 5.5 (and by making use of Claim 5.4) if $\mathcal{V}(\tau) = 0$ then either node τ is terminal or its child nodes are terminal and $n(\tau) = 2$. Otherwise $n(\tau) \le \#(C_\tau) - 1 = D(\tau) - \mathcal{V}(\tau) < D(\tau)$ and also $n(\tau) \le \mathcal{V}(\tau) + 1$ (since any triangle with vertices in $\mathbb{Z}^2$ has area $\ge 1/2$), implying that $n(\tau) \le min\{(D(\tau)+1)/2 \ ; \ D(\tau) - 1\}$.

5.2. **Complexity issues.** We have constructed an algorithm by means of Lemma 2.2 (via linear programming) and subsequently in section 4.1, whose input is the exponents matrix $\hat{E}$ (from (2.1)) and the output is an essential collection $C = \{\vec{\Delta}_j\}_{1 \le j \le L}$ of the exponent vectors of a monomial parametrization of (4.1). Complexity of the designed algorithm is polynomial in the binary size of the input relying on the following two subroutines, namely:

(i) The first one by means of linear programming [8] separates variables w_j on $\mathbb{C}^N$ into two groups of z-variables and y-variables.

(ii) The second ([3]) yields a $\mathbb{Z}$-basis $\{(\vec{\delta}_{1i} \ , \ \dots \ , \ \vec{\delta}_{Li}) \times \mathbf{0}\}_{1 \le i \le m}$ of the integral lattice in $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \{\mathbf{0}\}) \subset \mathbb{Q}^N$ and vectors from collection C by formulae $\vec{\Delta}_j = (\delta_{j1} \ , \ \dots \ , \ \delta_{jm})$ for each j .

Combination of the latter two subroutines results in an algorithm whose input being an exponents matrix of an affine binomial variety $\hat{V} \subset \mathbb{C}^N$ provides exponents $\vec{\Delta}_j \in \mathbb{Z}^m$, $1 \le j \le \ L$, of a monomial parametrization $(\mathbb{C}^*)^m \to Y \cap (\mathbb{C}^*)^N \hookrightarrow \hat{V} \cap ((\mathbb{C}^*)^L \times \mathbb{I}_{N-L})$ of torus of the essential toric subvariety $Y \hookrightarrow \hat{V}$, defined by formulae $y_j = x^{\vec{\Delta}_j}$, $1 \le j \le \ L$. As explained in Corollary 2.8 normalized Nash desingularization of variety Y implies normalized Nash desingularization of the same length of variety $\hat{V}$. When $m = 2$ the sequence of normalizations followed by Nash blowings up stabilizes, as is proved in this section, and provides normalized Nash desingularization of Y . This process recorded by means of a combinatorial algorithm on the exponents of monomial parametrizations of the successive composites of the normalized Nash blowings up starting with essential variety Y is the normalized 2-dimensional Euclidean algorithm (described in section 4.3 and in great detail here) whose complexity is estimated below

Remark 5.10. After each step of the normalized 2-dimensional Euclidean algorithm the maximal binary size of points of the input (set

$C \subset \mathbb{Z}^2$ of the algorithm in Section 4.3) increases at most by an additive constant. Since the length of any branch of the algorithm is bounded by $D-1$ (due to Theorem 3.1) it follows that complexity of a single step of the algorithm (as well as the complexity along a single branch) is polynomial in D and in the binary size of the initial input.

6. Invariance of termination bounds

This section is entirely devoted to the issue of the invariance of the integer D introduced in Sections 3 and 5 in terms of which the termination and complexity bounds are expressed (though has no evident bearing on the problem of termination of neither normalized multidimensional Euclidean division nor of its geometric counterpart for $m > 2$). Considered in both sections in the case of dimension $m = 2$ and associated with a monomial parametrization $(\mathbb{C}^*)^m \ni x \mapsto y = \phi(x) \in Y^*$ (with components $y_j = \phi_j(x) := x^{\vec{\Delta}_j}$) of the torus Y^* of an essential toric subvariety Y of a binomial variety $\hat{V} \subset \mathbb{C}^N$ number D is expressed in terms of the exponents $C = \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^m$ of map ϕ as the area of a parallelogram generated by the extremal vectors, i. e. the least points of $\mathrm{Span}_{\mathbb{Z}}(C)$ on the (two) extremal rays of the cone spanned over $\mathbb{R}_+$ by the exponents in C , see Section 5 .

Due to Theorem 2.5 , Corollary 2.3 and Claim 2.1 we may, as well, assume all exponents to be strictly positive, i. e. that $C \subset \mathbb{Z}^m_+$. Also, we may assume without loss of generality that $\mathrm{Span}_{\mathbb{Z}}(C) = \mathbb{Z}^m$. Recall that Y is 'essential' means that $Y \ni \mathbf{0}$ and is equivalent to $\mathrm{Conv}(\mathbb{Z}_+(C)) \not\ni \mathbf{0}$, Sections 2 and 4 . By *extremal vectors* for any m we (similarly) mean the subset $\mathcal{E}(C) \subset \mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$, where $\mathbb{Q}_+(C)_{\mathbb{Z}} = \mathrm{Span}_{\mathbb{Z}}(C) \cap \mathrm{Span}_{\mathbb{Q}_+}(C) \setminus \{\mathbf{0}\}$, of all minimal in size points of $\mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$ on the extremal rays of cone $\mathrm{Span}_{\mathbb{Q}_+}(C)$ and 'normality' property of Y is equivalent in terms of exponents C to $\mathbb{Z}_+(C) = \mathbb{Q}_+(C)_{\mathbb{Z}}$ and (by construction) is valid for 'normalized' algorithms (Nash and/or 2-dimensional Euclidean) of Section 5 for which termination is proved. We may also (without loss of generality) assume that $C = \mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$ since the 'left out' exponents (and corresponding affine coordinates) are in $\mathbb{Z}_+(\mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}}))$ (and, respectively, coincide on Y with monomials in the coordinates corresponding to elements in $\mathcal{E}xt(\mathbb{Q}_+(C)_{\mathbb{Z}})$). Number D admits a natural extension for an arbitrary m in terms of set C as the smallest $D = D(C) \in \mathbb{Z}_+$ such that $D \cdot \vec{\Delta}_j \in \mathbb{Z}_+(\mathcal{E}(C))$ for all $\vec{\Delta}_j \in C$.

Next we restate the definition of *denominator* $D(C)$ as a local invariant of Y (as well as of any of the isomorphic irreducible components of $\hat{V}$, say of V) at any point $o \in Y$. Invariance we consider is with respect to the germs at o of local analytic isomorphisms preserving coordinate hyperplanes that contain o . We restrict variety $X := Y$ (or respectively $X := V$) to affine charts $\mathcal{U}_o$ obtained by exclusion of all coordinate hyperplanes off o , which we refer to as the *origin* (recall, Section 2 and Remark 7.5 , that 'y-variables' of varieties Y , V and even of $\hat{V}$ coincide). To be precise charts $\mathcal{U}_o$ are constructed by introducing a 'double' $\tilde{z}_j$ of every affine coordinate $z_j := w_j$ with $w_j(o) \neq 0$, say $j = 1, ..., \tilde{L}$, and

$$\mathcal{U}_o := \{(z, \tilde{z}) \in \mathbb{C}^{2\tilde{L}} : z_j \cdot \tilde{z}_j = 1 \ , \ 1 \leq j \leq \tilde{L}\} \times \mathbb{C}^{L_o} \hookrightarrow \mathbb{C}^{L_o + 2\tilde{L}} \ ,$$

with 'y-variables' of variety X being the remaining L_o variables induced by the original 'y-coordinates' with $y_j(o) = 0$. Then, according to Theorem 2.5 and Remark 2.6 , the germ X_o of variety X at o is isomorphic to a product of a germ Z_o of a nonsingular subvariety Z with a germ of a union of, possibly several, mutually isomorphic subvarieties (including the germ Y_o at o of the 'essential' toric subvariety of X). Moreover, $Z_o = \cap_j \{y_j = 0\} \cap X_o$ and is identified with $\pi(Z_o) = \pi(X_o) \hookrightarrow \mathbb{C}^{2\tilde{L}}$ for projections π with $\operatorname{Ker} \pi = \mathbb{C}^{L_o} := \operatorname{Span}_{\mathbb{C}}\{y\text{−variables}\}$, i. .e. via $\pi^*_{|X} \circ (\pi^{-1}_{|Z_o})^*$ there is an embedding $\mathcal{O}_{Z,o} \hookrightarrow \mathcal{O}_{X,o}$.

By attaching subscript o indicating the dependence on the new origin $o \in X$ we will assume below that all notations (and assumptions) of the second paragraph of this section (including of the sets of exponents C_o associated with essential subvariety Y_o of X_o and of the extremal vectors $\mathcal{E}(C_o) \subset C_o$, as well as of the numbers $m_o := \dim Y_o$ and $D_o := D(C_o)$ are associated with toric variety $X \hookrightarrow \mathcal{U}_o$. By reindexing y_j's we may assume that $\mathcal{E}(C_o) = \{y_j\}_{1 \leq j \leq L'_o}$. In abuse of notation we will (occasionally) write $j \in \mathcal{E}(C_o)$ instead of $y_j \in \mathcal{E}(C_o)$.

For the sake of invariance we must consider notions allowing to define denominator $D(C_o)$ in the respective local ring $\mathcal{O}_{X,o}$ (i.e. in $\mathcal{O}_{Y,o}$ and/or $\mathcal{O}_{V,o}$) while in $\mathcal{O}_{X,o}$ its 'defining equations' are no longer binomial, i. e. binomials do not generate the ideal of relations between local parameters (even though including among the latter all affine coordinates y_j with $y_j(o) = 0$ because we examine the invariance with respect to the germs of local isomorphisms preserving all germs of sets $\{y_j = 0\}$). To overcome this problem we consider local ring $\mathcal{R}_o := \mathcal{O}_{X,o}[S^{-1}]$ of quotients with numerators in $\mathcal{O}_{X,o}$ and denominators in the multiplicative collection S of nonzero germs in $\mathcal{O}_{Z,o}$. (Of course resulting rings $\mathcal{R}_o$ are isomorphic for different choices of projections π of $\mathbb{C}^{L_o + 2\tilde{L}}$ with $\operatorname{Ker} \pi = \operatorname{Span}_{\mathbb{C}}\{y\text{−variables}\}$.)

Denote $\mathcal{M}_o$ the maximal ideal of ring $\mathcal{R}_o$ and by $\overline{y}_j$ the classes in $\mathcal{R}_o$ of all affine coordinates y_j with $y_j(o) = 0$. Then, of course, collection $\mathcal{P}ar(\mathcal{R}_o) := \{\overline{y}_j\}_{1\le j\le L_o} \subset \mathcal{M}_o$ induces a basis of $\mathcal{M}_o/\mathcal{M}_o^2$ over field $\mathbb{K} := \mathcal{R}_o/\mathcal{M}_o = \mathcal{O}_{Z,o}[S^{-1}]$.

Remark 6.1. Sets $\mathcal{E}(\mathcal{P}ar(\mathcal{R}_o)) \subset \mathcal{P}ar(\mathcal{R}_o)$ can be defined in terms of collection $\mathcal{P}ar(\mathcal{R}_o) \subset \mathcal{R}_o$ as follows: $j \in \mathcal{E}(\mathcal{P}ar(\mathcal{R}_o))$ **iff**
(i) $\overline{y}_i^p = \overline{y}_j^q$, $(p,q) \in \mathbb{Z}_+^2$, $i \ne j$, implies $p < q$;
(ii) $\overline{y}_j$ is not in the integral closure in $\mathcal{R}_o$ of the subring of $\mathcal{R}_o$ generated by $\overline{y}_i$'s such that $\overline{y}_i^p \ne \overline{y}_j^q$ for any $(p,q) \in \mathbb{Z}_+^2$.

Note that
(iii) ring $\mathcal{R}_o$ is the integral closure of its subring $\mathcal{R} \hookrightarrow \mathcal{R}_o$ generated by $\overline{y}_j$'s with $j \in \mathcal{E}(\mathcal{P}ar(\mathcal{R}_o))$ (consequence of Section 2.1 of [4]).
We may therefore introduce in terms of collection $\mathcal{P}ar(\mathcal{R}_o)$ the smallest positive integer $D = D(\mathcal{P}ar(\mathcal{R}_o))$ such that for all j , $\overline{y}_j^D \in \mathcal{R}$. Obviously, the value of *denominator* D of $\mathcal{P}ar(\mathcal{R}_o)$ coincides with $D_o = D(C_o)$, where C_o is the collection of exponents $\{\vec{\Delta}_j\}_j$ of any monomial map ϕ (including with nonpositive exponents) parametrizing torus Y_o^* , i. e. $D(C_o)$ is a local invariant due to the definition of $D = D(C_o)$ being stated entirely in terms of collection $\mathcal{P}ar(\mathcal{R}_o)$.

Remark 6.2. With reference to Section 4.3 normalization $\mathcal{N}(Y)$ of Y is a toric variety in $\mathbb{C}^{L'}$ whose torus $\mathcal{N}(Y)^* := \mathcal{N}(Y) \cap (\mathbb{C}^*)^{L'}$ is parametrized by a map $\psi : (\mathbb{C}^*)^m \ni x \mapsto y = \psi(x) \in \mathcal{N}(Y)^*$ with components $y_j = \psi_j(x) := x^{\vec{\Delta}_j}$, and the collection of exponents, say $C' := \{\vec{\Delta}_j\}_{1\le j\le L'} \subset \mathrm{Span}_{\mathbb{Z}}(C) \cap \mathrm{Span}_{\mathbb{Q}_+}(C) \subset \mathbb{Z}_+^m$, augmenting set $C = \{\vec{\Delta}_j\}_{1\le j\le L}$ so that $\mathbb{Z}_+(C') = \mathrm{Span}_{\mathbb{Z}}(C) \cap \mathrm{Span}_{\mathbb{Q}_+}(C) \setminus \{\mathbf{0}\}$. It follows that $\mathbb{Z}_+(C') = \mathrm{Span}_{\mathbb{Z}}(C') \cap \mathrm{Span}_{\mathbb{Q}_+}(C') \setminus \{\mathbf{0}\}$. In short, all assumptions of the lemma following (except on the size of $\mathcal{E}(C)$ when $m > 2$) are satisfied for Y replaced by its normalization $\mathcal{N}(Y)$. Of course elements of $\mathcal{E}(C')$ and of $\mathcal{E}(C)$ span the same extremal rays with the extremal vectors of $\mathcal{E}(C')$ being (equal or) shorter than their respective counterparts in $\mathcal{E}(C)$.

For a matrix M of size $m\times m$ with entries in $\mathbb{Z}$ let $\mathrm{den}\,(M) \in \mathbb{Z}_+$ denote the least $d \in \mathbb{Z}_+$ with the entries of $d \cdot M^{-1}$ being integers. Obviously, entries of matrix $d \cdot M^{-1}$ generate a unit ideal in $\mathbb{Z}$ and if also $m = 2$ and the entries of M have no common divisor then $\mathrm{den}\,(M) = |\det(M)|$. Below we denote a matrix whose columns are elements of collection $C \subset \mathbb{C}^m$ by the same letter C .

Lemma 6.3. *If* $\mathrm{Span}_{\mathbb{Z}}(C) = \mathbb{Z}^m$, $\mathbb{Z}^m \cap \mathrm{Span}_{\mathbb{Q}_+}(C) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(C)$ *and* $\#(\mathcal{E}(C)) = m$ *it follows that* $D(C) = \mathrm{den}\,(\mathcal{E}(C))$.

Remark 6.4. Of course, if $\#(\mathcal{E}(C)) = m$ and $D(C) = 1$ affine variety Y being of dimension m must be nonsingular. Also, if $m = 2$, then obviously $\#(\mathcal{E}(C)) = m$ and $D(C) = |\det(\mathcal{E}(C))|$.

Proof. Inclusion $\operatorname{den}(\mathcal{E}(C)) \in D(C)\cdot\mathbb{Z}$ is a simple consequence of the definitions. It therefore suffices to show that for any prime number p and $s \in \mathbb{Z}_+$ it follows from $\operatorname{den}(\mathcal{E}(C)) \in p^s\cdot\mathbb{Z}$ that $D(C) \in p^s\cdot\mathbb{Z}$. Let $M := \operatorname{den}(\mathcal{E}(C))\cdot\mathcal{E}(C)^{-1}$. Then there is a column $\vec{\lambda}$ of matrix M with a nonvanishing $\mod p$ entry and modifying the latter column to $\vec{\lambda}' := \vec{\lambda} + p^s\cdot t\cdot\mathbb{I}_m$ with a sufficiently large positive $t \in \mathbb{Z}_+$ so as to make all entries of $\vec{\lambda}'$ positive it follows that $\vec{\lambda}' \neq \mathbf{0} \pmod p$. Therefore vector $\mathcal{E}(C)\cdot\vec{\lambda}' \in (p^s\cdot\mathbb{Z}^m)\cap \operatorname{Span}_{\mathbb{Q}_+}(C)\setminus\{\mathbf{0}\}$. It follows that $D(C) \in p^s\cdot\mathbb{Z}$, as required. □

Corollary 6.5. *Denominator $D(C)$ of essential subvariety of a binomial variety $\hat{V}$ is the bound D appearing in our abstract for $m = 2$ (and is a local integral invariant of $\hat{V}$).*

7. Structure of binomial varieties, proof of reduction

We consider affine binomial varieties $\hat{V} := \overline{V^*(\hat{f})}$ in $\mathbb{C}^N$ determined by a set $\hat{f} := \{\hat{f}_j\}_{1\le j\le M}$ of binomials from (2.1). Let $r := \operatorname{rank}\hat{E}$. Denote T^{tr} the transpose of matrix T and by $E = \{E_{ji}\}$ a matrix of size $r\times N$ with rows being a basis over $\mathbb{Z}$ of $(\hat{E})^{tr}(\mathbb{Q}^M)\cap\mathbb{Z}^N$. Then ideal generated in $\mathbb{Z}$ by $r\times r$ minors of matrix E is the unit ideal, which is equivalent to the following property

$$(\mathbb{Z}) \qquad \{\xi\in\mathbb{R}^N : E\xi\in\mathbb{Z}^r\} = \operatorname{Ker}E\cap\mathbb{R}^N + \mathbb{Z}^N \subset \mathbb{R}^N .$$

Let $\alpha_{ji} := \max\{E_{ji}\ ,\ 0\}$, $\beta_{ji} := -\min\{E_{ji}\ ,\ 0\}$ and denote X the closure of $V^*(f) := \{w\in(\mathbb{C}^*)^N : f_j(w) = 0\ ,\ 1\le j\le r\}$ in $\mathbb{C}^N$, where binomials

$$f_j := w_1^{\alpha_{j1}}\cdots w_N^{\alpha_{jN}} - w_1^{\beta_{j1}}\cdots w_N^{\beta_{jN}} . \tag{7.1}$$

Then $V^*(f)\subset V^*(\hat{f})$ and both are subgroups of $(\mathbb{C}^*)^N$. Also, $\operatorname{Ker}E = \operatorname{Ker}\hat{E}$ and $V^*(f)\cap\mathbb{R}^N_+ = V^*(\hat{f})\cap\mathbb{R}^N_+$. A simple calculation shows that $V^*(f)\subset\operatorname{Reg}X$, $V^*(\hat{f})\subset\operatorname{Reg}\hat{V}$ and that $\dim_{\mathbb{C}}V^*(f) = \dim_{\mathbb{C}}\operatorname{Ker}E = \dim_{\mathbb{C}}\operatorname{Ker}\hat{E} = \dim_{\mathbb{C}}V^*(\hat{f})$. Hence $V^*(f)$ is an open (and closed) subset of $V^*(\hat{f})$. But $V^*(\hat{f})$ need *not be connected*.

Let $G := \{w\in V^*(\hat{f}) : |w| = 1\}$ and $G_0 := \{w = \exp(2\pi\sqrt{-1}\cdot h) : h\in\mathbb{R}^N\ ,\ Eh = \mathbf{0}\}$. Then the latter two are subgroups of $V^*(\hat{f})$. Theorem 2.5 A. follows from

Proposition 7.1. *Set $V^*(f)$ is a torus, group $\Gamma := G/G_0$ is finite, the distinct connected components of $V^*(\hat{f})$ are $\{g \cdot V^*(f)\}_{[g]\in\Gamma}$ listed by the classes of equivalence $[g] \in \Gamma$ of $g \in G$ and $G_0 = G \cap V^*(f)$.*

Proof. Equality $V^*(\hat{f}) = \cup_{g\in G}(g \cdot V^*(f))$ is straightforward (since $g := w \cdot |w|^{-1} \in G$ and $|w| \in V^*(f)$ whenever $w \in V^*(\hat{f})$).

Connectedness of $V^*(f)$ follows from the existence of a monomial isomorphism $\psi : \mathbb{T}^{N-r} \to V^*(f)$ constructed by means of a basis $\{\vec{\lambda_j}\}_{1\le j\le N-r}$ of $\operatorname{Ker} E \cap \mathbb{Z}^N$ over $\mathbb{Z}$ as follows: denote λ_{ij} the i-th coordinate of $\vec{\lambda_j}$, $1 \le i \le N$, then property $(\mathbb{Z})$ implies that monomial map $\psi((u_1, \ldots , u_{N-r})) := (\prod_j u_j^{\lambda_{1j}}, \ldots , \prod_j u_j^{\lambda_{(N-r)j}})$ is as required. Hence sets $g \cdot V^*(f)$ are closed, open and connected.

Property ($\mathbb{Z}$) implies $G \cap V^*(f) = G_0$ and, therefore, for any $g \in G$ sets $V^*(f)$ and $g \cdot V^*(f)$ coincide iff set $V^*(f) \cap g \cdot V^*(f)$ is not empty, i. e. sets $(V^*(f))_{[g]} := g \cdot V^*(f)$ listed by the classes of equivalence $[g] \in \Gamma := G/G_0$ of $g \in G$ are distinct.

Group Γ is finite since map $\xi \to \exp(2\pi\sqrt{-1}\cdot\xi)$ provides a bijection of an additive group $\Gamma_* := \{\xi \in \mathbb{R}^N : \hat{E}(\xi) \in \mathbb{Z}^M\}/(\mathbb{Z}^N + \operatorname{Ker} E)$ onto Γ , while Γ_* is finite (since for any choice of a basis $\{\vec{h_j}\}_{1\le j\le r}$ of $\hat{E}(\mathbb{R}^N) \cap \mathbb{Z}^M$ over $\mathbb{Z}$ there is a choice of $\{\vec{\xi_j}\}_j \subset \mathbb{Q}^N$ with each $\vec{h_j} = \hat{E}(\vec{\xi_j})$ and $\operatorname{Ker} E = \operatorname{Ker} \hat{E}$). $\square$

Corollary 7.2. *It follows that X is an irreducible component of $\hat{V}$ with $\mathbb{I}_N \in X$, i. e. $X = V$. The closures $[g]V$ of connected components $g \cdot V^*$ of $V^*(\hat{f})$ for classes of equivalence $[g] \in \Gamma$ of $g \in G$ are the irreducible components of variety $\hat{V}$.*

We will make use of the following key

Claim 7.3. *Assume $\{\vec{\xi}(k)\}_{k\in\mathbb{Z}_+}$ is a sequence of vectors in a subspace S of $\mathbb{R}^N$ and $I_0 \subset \{1, \ldots, N\}$ are such that $\lim_{k\to\infty}(\vec{\xi}(k))_i = a_i \in \mathbb{R}$ for all $i \in I_0$ and for $i \in \{1, \ldots, N\} \setminus I_0$ this limit equals $-\infty$. Then there are vectors $\vec{a}$ and $\vec{h}$ in S , such that $(\vec{a})_i = a_i$, $(\vec{h})_i = 0$ for all $i \in I_0$ and $(\vec{h})_i > 0$ for all $i \in \{1, \ldots, N\} \setminus I_0$.*

Proof. Below, starting with $\vec{\xi_0}(k) := \vec{\xi}(k)$, $k \in \mathbb{Z}_+$ we inductively for $j = 0, 1, 2, \ldots$ define vectors $\vec{h}_{j+1}$ in $S\backslash\{\mathbf{0}\}$, sequences of numbers $\{t_{j+1}(k)\}_k$ in $\mathbb{R}_+$, followed by sequences of vectors $\{\vec{\xi}_{j+1}(k)\}_k$ and $\{\vec{\eta}_{j+1}(k)\}_k$ in $S \setminus \mathbf{0}$ and finally sets I_{j+1} , $I_j \subsetneq I_{j+1} \subset \{1, \ldots, N\}$. Due to the latter property our inductive process would have to terminate after $s < N$ steps. Let $\{\vec{\eta}_j(k)\}_{k\in\mathbb{Z}_+}$ denote an appropriate subsequence of $\{\vec{\xi}_j(k)\}_{k\in\mathbb{Z}_+}$ so that limit

$$\vec{h}_{j+1} := \lim_{k\to\infty} (\vec{\eta}_j(k)/||\vec{\eta}_j(k)||) \in S \setminus \{\mathbf{0}\} ,$$

exists. It follows that $(\vec{h}_{j+1})_i \le 0$ for all i and $= 0$ if $i \in I_j$. Next let

$$t_{j+1}(k) := \min_{\{i:(\vec{h}_{j+1})_i<0\}} ((\vec{\eta}_j(k))_i/(\vec{h}_{j+1})_i) \ \in \mathbb{R}_+ ,$$

then define sequence

$$\{\vec{\xi}_{j+1}(k) := \vec{\eta}_j(k) - t_{j+1}(k)\cdot \vec{h}_{j+1}\}_k \subset S .$$

Finally define I_{j+1} to be the set of all i such that the $\lim_{k\to\infty}(\vec{\xi}_{j+1}(k))_i$ is finite. (We may assume that $\{\vec{\xi}_{j+1}(k)\}_{k\in\mathbb{Z}_+} \subset S\setminus\{\mathbf{0}\}$ and that the latter limits exist for all i as $-\infty$ or finite in $\mathbb{R}_-$ after choosing appropriate successive subsequences.) It follows that for every j values $(\vec{h}_{j+1})_i < 0$ for $i \in I_{j+1}\setminus I_j$ and that the process terminates with $I_{s-1} \subsetneq I_s = \{1,\dots,N\}$. In particular, limit $\vec{a} := \lim_{k\to\infty}\vec{\xi}_s(k) \in S$ exists and satisfies the conclusion of our lemma, as required. Finally, due to the properties of our construction $\vec{h} := -\sum_{1\le j\le s}\vec{h}_j \in S$ is as required as well. □

Corollary 7.4. *Claim 7.3 implies (a) equality* $\hat{Y} = \overline{V^*(\hat{f})\cap(\mathbb{C}^L\times\mathbb{I}_{N-L})}$ *of Theorem 2.5 B., (b) Lemma 4.2 and (c) Lemma 2.2 :*

Proof. Indeed, starting with the proof of (a) let sequence $\{w(k)\}_k \subset V^*(\hat{f})$ be such that $\lim_{k\to\infty} w(k) = \tilde{w} \in \hat{V}\cap(\mathbb{C}^L\times\mathbb{I}_{N-L})$. Then there are sequences $\{\xi(k)\}_{k\in\mathbb{Z}_+} \subset \mathbb{R}^N$, $|(\xi(k))_i| \le 1/2$ for all $k \in \mathbb{Z}_+$, $1 \le i \le N$, with $\{\hat{E}(\xi(k))\}_{k\in\mathbb{Z}_+} \subset \mathbb{Z}^M$ and $\{h(k)\}_{k\in\mathbb{Z}_+} \subset \operatorname{Ker}\hat{E}\cap\mathbb{R}^N_+$, such that $\exp(h(k) + 2\pi\sqrt{-1}\cdot\xi(k)) := w(k)$ for all $k\in\mathbb{Z}_+$. It follows (choosing a subsequence if needed) that there exist limits $\xi = \lim_{k\to\infty}\xi(k) \in \mathbb{R}^N$, $a_i = \lim_{k\to\infty}(h(k))_i \in \mathbb{R}$, whenever $(\tilde{w})_i \ne 0$, and $\lim_{k\to\infty}(h(k))_i = -\infty$ for other i's , and such that for the last $N-L$ coordinates $\xi_i = 0$ and $a_i = 0$ (i. e. when $(\tilde{w})_i = 1$). Of course $\hat{E}(\xi) \in \mathbb{Z}^M$. Applying Claim 7.3 it follows that there exist $\vec{a}$ and $\vec{h}$ in $\operatorname{Ker}\hat{E}\cap\mathbb{R}^N$ such that $(\vec{a})_i = a_i$ and $(\vec{h})_i = 0$, whenever $(\tilde{w})_i \ne 0$, and with $(\vec{h})_i > 0$ for other i's . It follows that $\widetilde{w(t)} := \exp(\vec{a} + 2\pi\sqrt{-1}\cdot\vec{\xi} + t\vec{h}) \in V^*(\hat{f})\cap(\mathbb{C}^L\times\mathbb{I}_{N-L})$, for $t\in\mathbb{R}$, and $\tilde{w} = \lim_{t\to-\infty}\widetilde{w(t)}$, as required in (a).

Starting with the proof of item (b) assume, as in Lemma 4.2, that $\tilde{w} \in Y\setminus(\mathbb{C}^*)^N$, $\{w(k)\}_k \subset Y^*$ and $\tilde{w} = \lim_{k\to\infty} w(k)$. Due to part (a) $\hat{Y}$ is an affine binomial variety and therefore so is Y (by Proposition 7.1). Hence the arguments of part (a) apply with $\hat{V}$ replaced by

Y and $V^*(\hat{f})$ being replaced by $Y \cap (\mathbb{C}^*)^N$ implying the conclusion of Lemma 4.2 .

Finally, item (c). Using that $w \in \hat{V}$ implies $|w| \in \hat{V} \cap \mathbb{R}^N$, it follows that $\{w_j = 0\} \cap \hat{V} = \emptyset$ iff $\{w_j = 0\} \cap \hat{V} \cap \mathbb{R}^N = \emptyset$. Since existence of a point $(\xi_1, \dots, \xi_N) \in \operatorname{Ker} \hat{E} \cap \mathbb{Q}^N$ with nonnegative coordinates and $\xi_j > 0$ implies that $\tilde{w} := \lim_{t \to -\infty} \exp(t \cdot \xi) \in \hat{V} \cap \{w_j = 0\}$ it remains to prove vice versa that if $\{w_j = 0\} \cap \hat{V} \cap \mathbb{R}^N \neq \emptyset$ then exists $\xi \in \operatorname{Ker} \hat{E} \cap \mathbb{Q}^N$ with nonnegative coordinates and the j-th coordinate $\xi_j > 0$. The latter claim is a part of the conclusion of Claim 7.3 . $\square$

Remark 7.5. Since $\operatorname{Ker} \hat{E} = \operatorname{Ker} E$ the splitting of variables w into y and z variables for a binomial variety $\hat{V} \subset \mathbb{C}^N$ and for its irreducible component V containing $\mathbb{I}_N$ coincide.

Let matrix $(\hat{\Omega} \quad \hat{\gamma}) := \hat{E}$ with the columns of $\hat{\Omega}$ and of $\hat{\gamma}$ corresponding to y and, respectively, to z -variables. Claim following implies that $\pi(V)$ is a closed binomial variety and since $\pi(V^*)$ is connected completes the **proof of Theorem 2.5 B.** (using part A.)

Claim 7.6. $\pi(V^*(\hat{f})) = \pi(\hat{V})$ *, is closed in* $\mathbb{C}^{N-L}$ *and is binomial.*

Proof. Let matrix T of size $M' \times M$, $M' := M - \operatorname{rank}(\hat{\Omega})$, have as rows a basis over $\mathbb{Z}$ of $\operatorname{Ker}(\hat{\Omega})^{tr} \cap \mathbb{Z}^M$. With $H := T \cdot \hat{\gamma}$ it follows that $\operatorname{Ker} H = \pi(\operatorname{Ker} \hat{E})$ and that $\pi(V^*(\hat{f})) = \{z \in (\mathbb{C}^*)^{N-L} : z^H = \mathbb{I}_{M'}\}$ (to show inclusion $\pi(V^*(\hat{f})) \supset \{z \in (\mathbb{C}^*)^{N-L} : z^H = \mathbb{I}_{M'}\}$ make use of property $(\mathbb{Z})$ for matrix T). In other words $\pi(V^*(\hat{f}))$ is the vanishing set of binomials and H is a matrix associated with variety $\hat{W} = \overline{\pi(V^*(\hat{f}))}$. Finally, applying Corollary 7.4 (c) (and by making use of its vector $\vec{\xi}^+$) it follows that all variables are the 'z-variables' for $\hat{W}$ and therefore $\pi(V^*(\hat{f}))$ is a closed binomial variety and coincides with $\pi(\hat{V})$, as required. $\square$

Corollary 7.7. *It follows that* $\pi(V) = \pi(V^*(f)) = \pi(V^*) \hookrightarrow (\mathbb{C}^*)^{N-L}$ *and, being connected, is a torus (by Proposition 7.1) closed in* $\mathbb{C}^{N-L}$ *.*

Next we prove Theorem 2.5 C. *We choose as* $Z \subset V$ *an irreducible component passing through* $\mathbb{I}_N \in \mathbb{C}^N$ *of variety* $\hat{Z} := V \cap \{(y,z) \in \mathbb{C}^N : y^{\vec{\delta}_i} = 1 \ , \ 1 \le i \le m\}$, where $\vec{\delta}_i$'s are from (4.1). Inclusion $\hat{Z} \subset (\mathbb{C}^*)^N$ makes $\hat{Z}$ binomial and follows from $\prod_{1 \le i \le m} (y^{\vec{\delta}_i})^{h_i^+} = y^{\vec{\xi}^+}$ (with h^+ and $\vec{\xi}^+$ from Remark 4.1), while Z is a torus due to $Z = Z^*$ and A. of Theorem 2.5 .

Proof. Let $\hat{\mathcal{E}}$ and $\mathcal{E}$ be matrices associated with binomial and, respectively, toric varieties $\hat{Z}$ and Z (implying property $(\mathbb{Z})$ for $\mathcal{E}$) and denote Δ a matrix of size $m \times L$ whose rows are $\vec{\delta}_i$, $1 \leq i \leq m$.

First we establish properties of $\pi_{|Z} : Z \to \pi(V)$ starting with surjectivity. Since $\pi(V) = \pi(V^*)$ (Corollary 7.7), and due to the property $(\mathbb{Z})$ of $\mathcal{E}$ it suffices for any $w = exp(\xi)$ with $\xi \in \operatorname{Ker} E$ to find $h \in \operatorname{Ker} \mathcal{E} = \operatorname{Ker} \hat{\mathcal{E}} = \operatorname{Ker} E \cap \operatorname{Ker} \Delta$ such that $\pi(h) = \pi(\xi)$. Let matrix $(\Omega \quad \gamma) := E$ with the columns of Ω and of γ corresponding to y- and, respectively, to z-variables. Since $\{\vec{\delta}_i \times \mathbf{0}\}_{1 \leq i \leq m} \subset \mathbb{Z}^L \times \mathbb{Z}^{N-L}$ is a basis of $\operatorname{Ker} \hat{\Omega} \times \{\mathbf{0}\} = \operatorname{Ker} \hat{E} \cap (\mathbb{C}^L \times \{\mathbf{0}\}) = \operatorname{Ker} E \cap (\mathbb{C}^L \times \{\mathbf{0}\}) = \operatorname{Ker} \Omega \times \{\mathbf{0}\}$ (as subspaces of $\mathbb{C}^N$) it follows that $\mathbb{C}^L = \operatorname{Ker} \Omega \oplus \operatorname{Ker} \Delta$. Therefore $\operatorname{Ker} E = (\operatorname{Ker} \Omega \times \{\mathbf{0}\}) \oplus (\operatorname{Ker} E \cap \operatorname{Ker} \Delta)$ and projection $\pi : \operatorname{Ker} E \cap \operatorname{Ker} \Delta \to \pi(\operatorname{Ker} E)$ is an isomorphism, which completes the proof of surjectivity of $\pi_{|Z}$.

Since projection $\pi_{|Z} : Z \to \pi(Z)$ is a homomorphism of tori the two remaining to prove properties of $\pi_{|Z}$ would follow from the finiteness of $\#(\pi_{|Z})^{-1}(\mathbb{I}_{N-L})$. Indeed, in combination with Sard theorem it follows that $\pi_{|Z}$ has no critical points and, combined with $\#(\pi_{|Z})^{-1}(\cdot)$ being constant on $\pi(Z)$, the latter implies the properness of $\pi_{|Z}$ as well.

So, it remains to show $\#(\pi_{|Z})^{-1}(\mathbb{I}_{N-L}) < \infty$. Pick $(y, z) \in Z \subset \hat{Z}$ with $z = \mathbb{I}_{N-L}$. Then $y^{\Omega} = \mathbb{I}_M$ and $y^{\Delta} = \mathbb{I}_m$. Since the entries of matrices Ω and Δ are integers and $\operatorname{Ker} \Omega \cap \operatorname{Ker} \Delta = \{\mathbf{0}\}$ it follows that there exists $q \in \mathbb{Z}_+$ such that $y_j^q = 1$ for every j , $1 \leq j \leq L$, and therefore that $\#(\pi_{|Z})^{-1}(\mathbb{I}_{N-L}) < \infty$, as required.

To complete the proof of Theorem 2.5 C. it remains to establish three properties of $\mu : Z \times \tilde{Y} \to V$ listed in C. Surjectivity of μ is immediate from the surjectivity of $\pi_{|Z} : Z \to \pi(V)$ since Z is a group and $\tilde{Y} := V \cap (\mathbb{C}^L \times \mathbb{I}_{N-L})$. That μ is a local analytic isomorphism at the points $a \times b \in Z \times \tilde{Y}$ follows from the respective property of $\pi_{|Z} : Z \to \pi(V)$ (equivalent to an absense of critical points). Indeed, the local inverse of map μ (near (a, b)) coincides with $V_{\mu(a,b)} \ni v \mapsto ((\pi_{Z,a})^{-1}(\pi(v)) \times [(\pi_{Z,a})^{-1}(\pi(v))]^{-1} \cdot v)$, where $[g]^{-1} := g^{-1}$ is the inverse in group $Z \hookrightarrow (\mathbb{C}^*)^N$. Finally, since map μ has no critical points and $\#\mu^{-1}(v) = \#\mu^{-1}(\pi_{|Z}^{-1}(\pi(v)))$ is finite and constant on V the properness of map μ follows (similarly to that of $\pi_{|Z}$), which completes the proof of Theorem 2.5 C. . □

We now prove (in the respective order) Claims 2.7 , 2.1 and 2.4 .

Proof. **Claim 2.7 .** Binomial variety $\pi(\hat{V}) = \pi(\hat{V}^*) \subset (\mathbb{C}^*)^{N-L}$, hence is nonsingular, i. e. its irreducible components are its disjoint connected components. To prove the first statement of Claim 2.7 it suffices (due to part A. of Theorem 2.5) to consider a submanifold $\tilde{\mathcal{M}}$ of component $\pi(V)$ and a respective subvariety $\tilde{V}$ of V (obtained by restricting z- variables (using Remark 7.5) to the submanifold of the previous sentence). Similarly, let $\tilde{Z} \hookrightarrow Z$ be obtained by restricting z-variables to $\tilde{\mathcal{M}}$. Then $\tilde{Z}$ is nonsingular (due to part C. of Theorem 2.5) and, moreover, morphisms $\pi : \tilde{Z} \to \tilde{\mathcal{M}}$ and that of coordinatewise multiplication $\mu : \tilde{Z} \times \tilde{Y} \to \tilde{V}$ are surjective, finite and both are local analytic isomorphisms (again due to part C. of Theorem 2.5), which implies the first claim of Claim 2.7 .

To show that a variety, say $\tilde{X}$, with nonunit coefficients of its defining binomial equations is a special case of the preceding construction we replace these coefficients (one per each binomial equation) by a variable, say c_j , introducing simultaneously another variable $\tilde{c}_j$ and add a binomial equation $c_j \cdot \tilde{c}_j = 1$. We thus constructed a binomial variety, say X , with 'z-variables' for X being all of the just introduced new variables and also the z-variables of the initial variety $\tilde{X}$. Obviously it suffices to show that the intersection $\tilde{\mathcal{M}}$ of the projection $\pi(X)$ of binomial variety X to the affine subspace of its z-variables with the specialization of variables c_j according to the values of the corresponding (nonvanishing) coefficients is nonsingular, thus reducing to the previous case, as required. Due to a simple explicit calculation of Claim 7.6 it follows not only that $\pi(X) = \pi(X^*)$ is binomial and closed (part B. of Theorem 2.5), but also that $\tilde{\mathcal{M}}$ is a binomial variety (but with possibly nonunit coefficients involving binomial equations expressing 'specialization' of variables c_j to the original values of the corresponding coefficients in the defining binomial equations of $\tilde{X}$). Obviously $\pi(X) = \pi(X^*)$ implies $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^*$ and using $\tilde{\mathcal{M}}^* \subset \operatorname{Reg} \tilde{\mathcal{M}}$ (due to a direct straightforward calculation classified preceding Claim 2.7 as "is easy to verify") variety $\tilde{\mathcal{M}}$ is nonsingular, as required. □

Proof. **Claim 2.1 .** The 'only if' implication is obvious. Assume that $\mathbf{0} \in X$. It follows that there are no z-coordinates and Corollary 2.3 implies existence of $\vec{\xi}^+ \in \operatorname{Ker} E \cap (\mathbb{R}_+ \setminus \{\mathbf{0}\})^N$, where E is an exponents matrix of X . Obviously existence of an 'improved' $\vec{\xi}^+ \in \operatorname{Ker} E \cap (\mathbb{Z}_+^N)$ follows. Say $m := \dim X = N - \operatorname{rank} E$. To construct a monomial parametrization of the torus of X with positive integral exponents $\{\vec{\Delta}_j\}_{1 \le j \le N} \in \mathbb{Z}^m$ it suffices to find a $\mathbb{Z}$-basis $\{\vec{\delta}_i\}_{1 \le i \le m}$ of $\operatorname{Ker} E \cap \mathbb{Z}^N$ with positive coordinates, see e. g. the proof of Proposition 7.1. Construction of the latter provides lemma below. $\square$

Lemma 7.8. *For any matrix* E *of size* $M \times N$ *with entries in* $\mathbb{Q}$ *and* $m := N - \operatorname{rank} E$ *the following properties are equivalent:*

(i) $(\operatorname{Im}\ E^{tr}) \cap \mathbb{Q}_+^N = \{\mathbf{0}\}$ *;*

(ii) there is a $\mathbb{Q}$*-basis* $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}_+^N$ *of* $\operatorname{Ker} E \cap \mathbb{Q}^N$ *;*

(iii) there is a $\mathbb{Z}$*-basis* $\{\vec{\delta}_i\}_{1\le i\le m}$ *of* $\operatorname{Ker} E \cap \mathbb{Z}^N$ *with all positive coordinates (equivalently, there exists a* $\mathbb{Q}$*-basis* $\{\vec{\delta}_i\}_i \subset \mathbb{Z}_+^N$ *of* $\operatorname{Ker} E \cap \mathbb{Q}^N$ *such that* $I = \mathbb{Z}$ *, where* $I = I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m)$ *is the ideal generated in* $\mathbb{Z}$ *by all coordinates of* $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ *in the standard basis* $\{(j_1) \wedge \cdots \wedge (j_m)\}_{1\le j_1<\cdots<j_m\le N}$ *).*

Proof. Our proof is based on a simple linear algebra and a theorem due to Gordan [2] , which states that property (i) is equivalent to the existence of a vector $\vec{v} \in \operatorname{Ker} E \cap \mathbb{Z}_+^N$. To prove (ii) it remains to choose any basis $\{\vec{v}_i\}_i \subset \mathbb{Z}^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ with $\vec{v}_1 := \vec{v}$ and then letting $\vec{\delta}_1 := \vec{v}$ and $\vec{\delta}_i := t \cdot \vec{v} + \vec{v}_i$, $i > 1$, (ii) follows for a sufficiently large $t \in \mathbb{Z}_+$.

The remaining implication "(iii) follows from (ii)" is harder. Starting with a $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}_+^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ let $s \in \mathbb{Z}_+$ be the generator of ideal I , i. e. $(s \cdot \mathbb{Z}) = I$. If $s = 1$ we are done. Otherwise, we modify basis $\{\vec{\delta}_i\}_{1\le i\le m}$ reducing the size of s , which would suffice. Pick a prime factor p of s . Denote field $\mathbb{Z}/(p\cdot\mathbb{Z})$ by $\mathbb{F}_p$. Now our collection of vectors $\{\vec{\delta}_i\}_{1\le i\le m}$ considered modulo ideal $(p\cdot\mathbb{Z})$ in $(\mathbb{F}_p)^N$ is linearly dependent, i. e. $\sum_{1\le i\le m} \lambda_i \cdot \vec{\delta}_i = \mathbf{0}$ in $(\mathbb{F}_p)^N$ for a collection of coefficients $\{\lambda_i\}_{1\le i\le m} \subset (\mathbb{F}_p)^m \setminus \{\mathbf{0}\}$. Choose $\tilde{\lambda}_i \in \mathbb{Z}$ so that $\lambda_i = \tilde{\lambda}_i \ (mod\ p)$ and $0 \le \tilde{\lambda}_i < p$, $1 \le i \le m$. Then $\tilde{\lambda}_{i_0} \ne 0$ for some i_0 , $1 \le i_0 \le m$, and $\vec{\delta}_0 := (1/p) \cdot \sum_{1\le i\le m} \tilde{\lambda}_i \cdot \vec{\delta}_i \ \in \ \mathbb{Z}_+^N$. It follows that all coordinates of the modified $\mathbb{Q}$-basis of $\operatorname{Ker} E \cap \mathbb{Q}^N$ obtained by replacing vector $\vec{\delta}_{i_0}$ of $\{\vec{\delta}_i\}_{1\le i\le m}$ by vector $\vec{\delta}_0$ are positive integers and that $I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_{i_0-1} \wedge \vec{\delta}_0 \wedge \vec{\delta}_{i_0+1} \wedge \cdots \wedge \vec{\delta}_m) = \tilde{\lambda}_{i_0} \cdot (s/p) \cdot \mathbb{Z}$. Due to the choice of $\{\tilde{\lambda}_i\}_{1\le i\le m}$ in $\mathbb{Z}^m$ the size of $\tilde{\lambda}_{i_0} \cdot (s/p)$ is smaller than the size of s , which suffices. $\square$

Remark 7.9. Complexity of construction of a basis satisfying property (iii) of the algorithm '(ii) implies (iii)' is polynomial in the maxima of the absolute values of the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis for the initial $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m}$, i. e. is exponential in the binary size of the input (unlike construction of a basis $\{\vec{\delta}_j\}_{1\le j\le m}$ of (ii) which is a typical problem of linear programming and carries a polynomial cost in the binary size of the input). Of course we do not need the output with property (iii) for the algorithms of this article.

Proof. **Claim 2.4 .** The ‘if’ implication is obvious. We first prove the ‘only if’ implication in the case that there are no y-coordinates, i. e. we must show that in this case $(\hat{f})$ is a radical ideal when $\hat{V} = \hat{V} \cap (\mathbb{C}^*)^N = V^*(\hat{f})$. Of course $V^*(\hat{f}) \subset \operatorname{Reg}\hat{V}$ (as we have remarked in the first paragraph of Section 2). Therefore, assuming that polynomial $P \in \mathbb{C}[w]$ vanishes on $\hat{V}$ it follows that polynomial P belongs to the ideals $I_{\mathcal{M}}$ generated by ideal $(\hat{f})$ in the local rings $\mathcal{O}_{\mathcal{M}}$ of the localizations of the polynomial ring $\mathbb{C}[w]$ at its maximal ideals $\mathcal{M}$. The result follows by the standard ‘partition of unity’ argument of commutative algebra. (Indeed, for every $\mathcal{M}$ there is a polynomial $Q_{\mathcal{M}} \in C[w]$ with $Q_{\mathcal{M}} \notin \mathcal{M}$ such that $Q_{\mathcal{M}} \cdot P \in (\hat{f})$. Since the ideal generated by all $Q_{\mathcal{M}}$ in $\mathbb{C}[w]$ is not in any maximal ideal $\mathcal{M}$ of $\mathbb{C}[w]$ it follows that it coincides with $\mathbb{C}[w]$ and therefore there is a finite linear combination $\sum_k h_k \cdot Q_{\mathcal{M}_k} = 1$, for an appropriate choice of polynomials $h_k \in \mathbb{C}[w]$, commonly refered to as a partition of unity. Expressing inclusions $Q_{\mathcal{M}_k} \cdot P \in (\hat{f})$ as equalities $Q_{\mathcal{M}_k} \cdot P = \sum_j G_{\mathcal{M}_k,j} \cdot \hat{f}_j$ it follows that $P = \sum_k h_k \cdot Q_{\mathcal{M}_k} \cdot P = \sum_j (\sum_k h_k \cdot G_{\mathcal{M}_k,j}) \cdot \hat{f}_j$.)

Finally, we reduce to the previously considered special case. Let $v := (v_1, \dots, v_L)$ and $g_i := y_i \cdot v_i - 1$ denote auxiliary variables and polynomials. Of course $\hat{V} \cap \{(y,z) \in \mathbb{C}^N : y_1 \cdot \ldots \cdot y_L \neq 0\} = V^*(\hat{f})$ (by definition of the y-variables). Therefore assumption that $P \in \mathbb{C}[w]$ vanishes on $\hat{V}$ (and equivalently on $V^*(\hat{f})$) implies that polynomial $P \in \mathbb{C}[w] \subset \mathbb{C}[w,v]$ vanishes on $V^*(\hat{f}, g) \subset \mathbb{C}^{N+L}$. Obviously all (w,v) variables for the collection $\mathcal{F}$ of binomials $\{\hat{f}_j\}_j \cup \{g_i\}_i$ are, as we refer to them, the ‘z-variables’. Therefore the case we considered first implies that polynomial $P(w)$ is in the ideal generated by polynomials from $\mathcal{F}$ in the ring $\mathbb{C}[w,v]$. Substitution of $v_j = 1/y_j$, $1 \le j \le L$, in the equality expressing the inclusion of the previous sentence, followed by ‘clearing’ the denominators, i. e. (in our setting) by multiplying by a sufficiently high power of $y_1 \cdot \ldots \cdot y_L$, completes the proof. □

Acknowledgements. The authors are grateful to the Max-Planck Institut für Mathematik, Bonn for its hospitality and to Dima Pasechnik for sharing his expertise regarding linear programming.

References

1. E. Bierstone and P .D. Milman, *Desingularization of toric and binomial varieties*, J. Algebraic Geom., **15** (2006), no. 3, 448–486.
2. G. B. Dantzig and M. N. Thapa, *Linear Programming: 2: Theory and Extensions*, Springer, 1997.
3. M. Frumkin, *An application of modular arithmetic to the construction of algorithms for the solution of systems of linear equations*, Soviet Math. Dokl., **229** (1976), no. 5, 1067–1070.
4. W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.
5. H. Hironaka, *On Nash blowing-up*, in Arithmetic and Geometry II, Birkhäuser, 1983, 103–111.
6. J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc., **21** (1969), 422–426.
7. A. Nobile, *Some properties of Nash blowing-up*, Pacific J. Math., **60** (1975), 297–305.
8. A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.
9. M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), no. 3, 411–491.

D. Grigoriev, Laboratoire des Mathématiques , Université de Lille I, 59655, Villeneuve d'Ascq, France

E-mail address: dmitry.grigoryev@math.univ-lille1.fr

P.D. Milman, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 3G3 Canada

E-mail address: milman@math.toronto.edu