# Noncommutative smooth spaces

**Maxim Kontsevich, Alexander Rosenberg**

## 0. CONVENTIONS AND NOTATIONS

We will work in the category $Alg_k$ of associative unital algebras over a fixed base field $k$. If $A \in Ob(Alg_k)$, we denote by $1_A \in A$ the unit in $A$ and by $m_A : A \otimes A \longrightarrow A$ the product. For an algebra $A$, we denote by $A^{opp}$ the opposite algebra, i.e. the same vector space as $A$ endowed with the multiplication $m_{A^{opp}}(a \otimes b) := m_A(b \otimes a)$. If $A$ and $B$ are two algebras, then $A \otimes_k B$ is again an algebra. Also, $A \star B$ denotes the free product of $A$ and $B$ over $k$, the coproduct in the category $Alg_k$. By $A$-*mod* we denote the abelian category of left $A$-modules. Analogously, *mod*-$A$ are right modules (the same as $A^{opp}$-modules) and $A$-*mod*-$A$ are bimodules over $k$, or, equivalently, $A \otimes_k A^{opp}$-modules. We shall write $\otimes$ instead of $\otimes_k$. For a vector space $V$, we denote by $Sym^*(V)$ and $\otimes^*(V)$ resp. the free commutative associative (polynomial) and free associative (tensor) $k$-algebra respectively, generated by $V$.

## 1. SMOOTH ALGEBRAS

A typical example of an algebra for this text is a free finitely generated algebra $k\langle x_1, \ldots, x_d\rangle$, in contrast to usual noncommutative algebras close to commutative ones, like universal enveloping algebras, algebras of algebraic differential operators etc. We consider the free algebra as "the algebra of functions on the non-commutative affine space" which we denote $N\mathbb{A}^d$. Recently there were several attempts to understand algebraic geometry of this space. Gelfand and Retakh started the study of basic identities in the free skew field with $d$ generators (see [GR]), which is can be considered as the "generic point" in $N\mathbb{A}^d$. One of us (see [K]) tried to develop "differential geometry" in the algebra of noncommutative formal power series, i.e. at the formal neighborhood of $N\mathbb{A}^d$ at zero. Kapranov (see [Ka]) described a differential-geometric picture related with the completion of the free algebra with respect to the commutator filtration, i.e. with the algebra of functions on the infinitesimal neighborhood of the usual affine space $\mathbb{A}^d$ in $N\mathbb{A}^d$. Here we would like to study $N\mathbb{A}^d$ as the whole space, without completions and localizations. The basic intrinsic property of the free algebra is *smoothness.*

### 1.1. Equivalent definitions

Few years ago J. Cuntz and D. Quillen gave the following

**Definition** ([CQ1], Definition 3.3 and Proposition 6.1). *An algebra $A$ is quasi-free (= formally smooth) iff it satisfies one of the following equivalent properties:*

*1) (Lifting property for nilpotent extensions) for any algebra $B$, a two-sided nilpotent ideal $I \subset B$ ($I = BIB$, $I^n = 0$ for $n \gg 0$), and for any algebra homomorphism $f$ :*

*$A \longrightarrow B/I$, there exists an algebra homomorphism $\tilde{f} : A \longrightarrow B$ such that $f = pr_{B \longrightarrow B/I} \circ \tilde{f}$, where $pr_{B \longrightarrow B/I} : B \longrightarrow B/I$ is the natural projection.*

*2) $Ext^2_{A-mod-A}(A, M) = 0$ for any bimodule $M \in Ob(A\text{-}mod\text{-}A)$.*

*3) The $A$-bimodule $\Omega^1_A := Ker(m_A : A \otimes A \longrightarrow A)$ (see 1.1.2) is projective.*

The definition of formal smoothness via the lifting property 1) is analogous to the Grothendieck's (actually Quillen's) definition of formally smooth algebras in the commutative case. The equivalence of properties 1)-2)-3) is an easy exercise in homological algebra.

Properties 1),2) are not constructive: one can not check them directly. The property 3) looks better, although it is still not clear *a priori* how to work with it in complicated infinite-dimensional examples. J. Cuntz and D. Quillen have found another characterization of smooth algebras which is convenient for calculations in practice. In order to give this characterization we need an auxiliary definition.

Let $TA$ be the algebra generated by symbols $\underline{a}, \underline{Da}$, where $a \in A$, subject to the following relations:

a) the map $a \mapsto \underline{a}$ is a homomorphism of unital $k$-algebras,

b) the map $a \mapsto \underline{Da}$ is $k$-linear,

c) $\underline{D(a \cdot b)} = \underline{a} \cdot \underline{Db} + \underline{Da} \cdot \underline{b}$, (the Leibnitz rule).

Algebra $TA$ is naturally $\mathbb{Z}_{\geq 0}$-graded: $deg(\underline{a}) = 0$, $deg(\underline{Da}) = +1$. We identify $A$ with the subalgebra of $TA$ consisting of elements of degree zero.

### 1.1.2. Differential envelope

Before going further on we would like to notice that $TA$ is isomorphic as an abstract $\mathbb{Z}$-graded algebra to the so-called *differential envelope* $\Omega A$, the universal differential $\mathbb{Z}$-graded super-algebra containing $A$.

By definition, $\Omega A$ is the universal $\mathbb{Z}$-graded super-algebra containing $A$ and endowed with an odd differential $d$ of degree $+1$ (satisfying the Leibnitz rule in super sense $d(a \cdot b) = d(a) \cdot b + (-1)^{deg(a)} a \cdot d(b)$) and such that $d^2 = 0$. One can see immediately that $d1 = 0$. As a vector space, the $n$-th graded component $(\Omega A)^n$ for $n \geq 0$ is isomorphic to $A \otimes (A/k \cdot 1_A)^{\otimes n}$. The isomorphism is given by the following map

$$a_0 \otimes a_1 \otimes \ldots \otimes a_n \longmapsto a_0 \cdot da_1 \cdot \ldots \cdot da_n \in (\Omega A)^n \ .$$

The space of 1-differentials $\Omega^1_A := (\Omega A)^1$ can be furthermore identified via the map

$$A^{\otimes 3} \longrightarrow \Omega^1_A, \ \ a_1 \otimes a_2 \otimes a_3 \longmapsto a_a \cdot da_2 \cdot a_3$$

with the quotient space $A^{\otimes 3} / \partial_4 (A^{\otimes 4})$, where

$$\partial_4(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_1 a_2 \otimes a_3 \otimes a_4 - a_1 \otimes a_2 a_3 \otimes a_4 + a_1 \otimes a_2 \otimes a_3 a_4) \ .$$

The map $\partial_4$ can be, evidently, included into one of standard complexes in homological algebra

$$\ldots \longrightarrow A^{\otimes 4} \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow A \longrightarrow 0$$

This complex can be contracted using the unit element $1_A \in A$, and thus has vanishing cohomology. Therefore $Cokernel(\partial_4) = Kernel(\partial_2)$ and we get an equivalent description of $\Omega^1_A$ as the kernel of the multiplication map $A \otimes A \longrightarrow A$.

If one considers analogous definitions in the purely commutative case, one can see immediately the difference between $TA$ and $\Omega A$. In the case of $A$ equal to the algebra of functions on a smooth affine variety $X$, the commutative analogue of the algebra $TA$ is the algebra of functions on the total space $TX$ of the tangent bundle to $X$. On the contrary, the commutative analogue of $\Omega A$ is the algebra of differential forms on $X$, which is the same as the algebra of functions on supermanifold $\Pi TX$, the total space of the odd tangent bundle to $Z$ (as usual in super-mathematics, letter $\Pi$ denotes the functor of changing the parity).

### 1.1.3. Another criterion of formal smoothness

**Theorem.** *An algebra $A$ is formally smooth iff it satisfies the property*

*4) there exists a derivation $\underline{D} : TA \longrightarrow TA$ of degree $+1$ such that $\underline{D}(\underline{a}) = \underline{Da}$ for all $a \in A$.*

If $A$ is an algebra generated by elements $(a_i)_{i \in I}$, then in order to define $\underline{D}$ one should define only elements $\underline{D}(\underline{Da_i}) \in TA$ of degree $+2$ satisfying certain relations. In the case when the generating set $I$ is finite and all relations between $(a_i)$ follow from a finite number of relations we say that $A$ is finitely presented. It is easy to see that in this case algebra $TA$ is also finitely presented and in order to check that some candidates for $\underline{D}(\underline{Da_i})$ satisfy needed relations one should make only a finite calculation.

The theorem analogous to the one above holds in the commutative case (if the characteristic of the ground field $k$ is zero). The derivation $\underline{D}$ in commutative case is a vector field on the total space of the tangent bundle. One can easily see that choices of $\underline{D}$ correspond to symmetric connections (i.e. connections with vanishing torsion) on the tangent bundle to $Spec(A)$.

### 1.1.4. Homological properties of modules over smooth algebras

**Theorem.** ([CQ1], Proposition 5.1) *If $A$ is a formally smooth algebra, then $A$ is hereditary, i.e. the category $A$-mod has homological dimension $\leq 1$. In other words, any submodule of a projective module is projective. Equivalently, every $A$-module admits a projective resolution $P_{-1} \longrightarrow P_0$ of length 2.*

The proof is immediate, because

$$Ext^n_{A-mod}(\mathcal{E}, \mathcal{F}) = Ext^n_{A-mod-A}(A, Hom_{k-mod}(\mathcal{E}, \mathcal{F})) \ \ \forall \mathcal{E}, \mathcal{F} \in Ob(A - mod) \ .$$

**Definition.** *We call an algebra $A$ smooth, if it is formally smooth and finitely generated.*

**Theorem.** *If $A$ is a smooth algebra and $\mathcal{E}, \mathcal{F}$ are $A$-modules finite-dimensional over $k$, then the vector spaces $Ext^0_{A-mod}(\mathcal{E}, \mathcal{F}) = Hom_{A-mod}(\mathcal{E}, \mathcal{F})$ and $Ext^1_{A-mod}(\mathcal{E}, \mathcal{F})$ are both finite-dimensional.*

The statement of the theorem is evident for $Ext^0$. The space $Ext^1$ coincides with the set of equivalence classes of structures of an $A$-module on $\mathcal{E} \oplus \mathcal{F}$ such that natural maps

$$\mathcal{F} \longrightarrow \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{E}$$

are morphisms of $A$-modules, modulo the action of the vector space $Hom_{k-mod}(\mathcal{E},\mathcal{F})$ considered as an abelian group. If $A$ is finitely generated, then the set of $A$-module structures on $\mathcal{E}\oplus\mathcal{F}$ as above is a finite-dimensional vector space over $k$, and we get the statement of the theorem.

### 1.2. Examples of smooth algebras

E1) The free algebra $k\langle x_1,\dots,x_d\rangle=\otimes^*(k^d)\;\; d\geq 0$. We think of this algebra as the one corresponding to a noncommutative affine space $N\mathbb{A}_k^d$ (see Sect. 3 where we define noncommutative spaces in general).

E2) The matrix algebra $Mat(n\times n,k)$.

E3) The algebra of upper-triangular matrices

$$A=\{(a_{ij})_{1\leq i,j\leq n}|\; a_{ij}\in k,\;\; a_{ij}=0 \quad for \quad i>j\}\subset Mat(n\times n,k) \;\;,$$

E4) $\mathcal{O}(C)$, the algebra of functions on a smooth affine curve $C$ over $k$.

E5) $Paths(\Gamma)$, the algebra of finite paths in a finite oriented graph $\Gamma$. The basis of $Paths(\Gamma)$ consists of sequences $(v_0,e_{0,1},v_1,e_{1,2},\dots,e_{n-1,n},v_n),\;\; n\geq 0$ of vertices $v_i$ of $\Gamma$ and oriented edges $e_{i,i+1}$ connecting $v_i$ with $v_{i+1}$ (there could be several edges connecting two given vertices). The unit element in $Paths(\Gamma)$ is equal to the sum over vertices $v$ of $\Gamma$ of paths of zero length $(v)$. The multiplication in $Paths(\Gamma)$ is given by concatenation of paths:

$$(v_0,e_{0,1},\dots,v_n)\cdot(v_0',e_{0,1}',\dots,v_{n'}')=$$

$$(v_0,e_{0,1},\dots,v_n,e_{0,1}',\dots,v_{n'}')$$

if $v_n=v_0'$, and 0 otherwise.

The last example E5) contains E1) and E3) as particular cases. In E1) the corresponding graph has one vertex and $d$ loops, and in E3) it is the chain of $n-1$ consecutive oriented edges.

There are several constructions using which one can construct new smooth algebras from the old ones. In the following list $A,A_1,A_2$ denote smooth algebras, and the result of the constructions is automatically a smooth algebra:

C1) $A_1\oplus A_2$, the direct sum of two smooth algebras (or an arbitrary finite number of smooth algebras),

C2) $A_1\star A_2$, the free product of smooth algebras.

C3) (Localization) If $S$ is a subset of a formally smooth algebra $A$, then the algebra $A[S^{-1}]$ obtained by formally adjoining the inverses of elements of $S$ is formally smooth. In particular, if $A$ is smooth and $S\subset A$ is an arbitrary finitely generated multiplicative subset of $A$, then $S^{-1}A$ is smooth. More generally, one can invert not only individual elements of $A$ but square matrices with coefficients in $A$ (i.e. $S$ could be a subset of $\coprod_{n\geq 1}Mat(n\times n,A)$), or even rectangular matrices.

C4) $End_{A-mod}(P)$, where $P$ is a finitely-generated projective $A$-module.

C5) (The total space of the tangent bundle) $TA$, the algebra defined in 1.1.

### 1.3. Representation spaces of smooth algebras

Let $A$ be a smooth $k$-algebra. We associate with $A$ an infinite sequence $(Repr_n^A,\ n \geq 1)$ of smooth affine varieties over $k$. The $n$-th variety $Repr_n^A$ parameterizes homomorphisms from $A$ to the standard matrix algebra $Mat(n \times n, k)$. The set of $B$-points of it, where $B$ is a *commutative* algebra over $k$, is defined as

$$Repr_n^A(B) := Hom_{Alg_k}(A, Mat(n \times n, B)) \ .$$

It is clear that since $A$ is finitely-generated, the scheme $Repr_n^A$ is an affine scheme of finite type over $k$. Moreover, from the definitions of smoothness in both commutative and noncommutative case, it follows that the scheme $Repr_n^A$ is smooth. The affine algebraic group $PGL_k(n)$ acts via conjugations on $Repr_n^A$.

Every element $a \in A$ gives tautologically a matrix-valued function $\hat{a} \in \mathcal{O}(Repr_n^A) \otimes Mat(n \times n, k)$ on $Repr_n^A$.

#### 1.3.1. Examples

For $A = k\langle x_1, \ldots, x_d\rangle$, the representation scheme $Repr_n^A$ coincides with the affine space $\mathbb{A}^{dn^2}$ over $k$.

For $A = k \oplus k = k\langle p\rangle/(p^2 = p)$, the scheme $Repr_n^A$ is the disjoint union over all integers $m,\ 0 \leq m \leq n$ of certain bundles over Grassmanians $Gr(m, n)$. Fibers of these bundles are affine spaces parallel to the fibers of the cotangent bundles. The component of $Repr_n^A$ corresponding to $m$ (— the rank of the image of $p \in A$) has dimension $2m(n - m)$.

The example of $A = k \oplus k$ has an interesting feature. One can see that $Repr_n^A$ carries a natural non-degenerate closed 2-form

$$\omega = Trace(\hat{p}\, d\hat{p} \wedge d\hat{p})$$

where $\hat{p}$ is the matrix-valued function on $Repr_n^A$ corresponding to the generator $p$ of $A$.

#### 1.3.2. Noncommutative analogues of differential-geometric structures

We would like to propose the following general principle which could help to look for noncommutative versions of usual geometric notions (see also [K]).

*A noncommutative structure of some kind on $A$ should give an analogous "commutative" structure on all schemes $Repr_n^A,\ n \geq 1$.*

Here we make a list of several natural candidates:

**Functions**. Elements of the vector space $A/[A, A]$ give rise to *functions* on $Repr_n^A$. Namely, any $a \in A$ modulo commutators $[A, A]$ gives the function $Trace(\hat{a})$. Thus, we get a homomorphism of vector spaces

$$A/[A, A] \longrightarrow \mathcal{O}(Repr_n^A) \ .$$

It has an obvious extension to the homomorphism of algebras

$$Sym^*(A/[A, A]) \longrightarrow \mathcal{O}(Repr_n^A) \ .$$

We will denote the element of $\mathcal{O}(Repr_n^A)$ corresponding to $\phi \in Sym^*(A/[A, A])$ by $Trace(\phi)$.

In the following examples we can multiply our differential-geometric objects by an arbitrary element of the algebra $Sym^*(A/[A,A])$.

**Vector fields**. Any derivation $\xi \in Der(A)$ of $A$ gives a vector field $\tilde{\xi}$ on $Repr_n^A$.

**Differential forms**. Any element of $\Omega A$ (see 1.1.2) gives a matrix-valued differential form on $Repr_n^A$, and thus a scalar-valued differential form after taking the trace in the matrix algebra. Again, using the vanishing of the trace of commutators one can see that the map $\Omega A \longrightarrow \Omega^*(Repr_n^A)$ goes through the quotient space

$$\Omega A/[\Omega A, \Omega A]_{super} \longrightarrow \Omega^*(Repr_n^A)$$

where the commutator in superalgebra $\Omega A$ is understood in super sense. Also, one can easily see that the differential $d$ in $\Omega_A$ maps to the de Rham differential in usual forms $\Omega^*(Repr_n^A)$. For example, algebra $A = k \oplus k$ has a closed 2-form. The definition of "noncommutative differential forms" as elements of $\Omega A/[\Omega A, \Omega A]_{super}$ is due to M. Karoubi.

**De Rham cohomology**. It is natural in the light of the previous discussion to define the de Rham cohomology of $A$ as the cohomology of the complex $\Omega A/[\Omega A, \Omega A]$. Using results of Cuntz and Quillen [CQ2], one can show that $H^n_{dR}(A)$ defined in this way coincides with the $Z/2\mathbb{Z}$-graded periodic cyclic homology $HP_n(A)$ for $n = 0, 1$, and with the reduced periodic cyclic homology

$$\widetilde{HP}_{n(mod\ 2)}(A) := HP_{n(mod\ 2)}(A)/HP_{n(mod\ 2)}(k)$$

for $n \geq 2$. In other words, $H^n_{dR}(A)$ coincides with $HP_{n(mod\ 2)}(A)$ for $n = 0, 1, 3, 5, \ldots$ and has one dimension less than $HP_{n(mod\ 2)}(A)$ for $n = 2, 4, 6, \ldots$.

**Volume element, polyvector fields**. It looks plausible from examples that one can define noncommutative analogues of polyvector fields and volume elements, but we would like to postpone the discussion of these notions to the future. It seems that the divergence of a derivation of $A$ with respect to a volume element belongs to the symmetric square of the vector space $A/[A, A]$. As an exercise to the reader we leave the following simple

**Lemma.** *If $A = k\langle x_1, \ldots x_d\rangle$ then there exists a linear map*

$$div : Der(A) \longrightarrow Sym^2(A/[A, A])$$

*such that for any $\xi \in Der(A)$ the divergence of the corresponding vector field $\tilde{\xi}$ on $\mathbb{A}^{dn^2} = Repr_n^A$ with respect to the standard volume element, is equal to $Trace(div(\xi))$.*

The last remark which we would like to make is that the correspondence (*Smooth algebras*) $\longrightarrow$ (*Representation schemes*) gives a "justification" of the notation $TA$ in 1.1. Namely, there is a natural isomorphism between $Repr_n^{TA}$ and the total space of the tangent bundle $TRepr_n^A$.

### 1.3.3. Possible relations to matrix integrals and to M-theory

One can imagine that in the case $k = \mathbb{C}$, for a given "volume element" $vol$ on $A$, a "function" $f \in Sym^*(A/[A, A])$ and for a the set of real points $\gamma_n$ in $Hom(A, Mat(n \times$

$n, \mathbb{C})$) under some anti-holomorphic involution, one can take the integral of $exp(Trace(f)) \times vol$ and get a "matrix model", an infinite sequence of numbers

$$I_n := \int_{\gamma_n} e^{Trace(f)} vol$$

parameterized by the dimension $n = 1, 2, \ldots$. In mathematical physics such integrals were extensively studied. Typically, one integrates over the space of hermitean $n \times n$ matrices, or over real, or unitary matrices. In multi-matrix models the integration is taken over the set of, say, $d$-tuples $(X_1, \ldots, X_d)$ of hermitean matrices. It is believed that the asymptotic behavior of $I_n$ as $n \longrightarrow \infty$ is related with some kind of string theory.

Also, one of recently proposed matrix models, so called $M$-theory, is formulated in the same fashion. Roughly speaking, $M$-theory on the space-time manifold $X = \mathbb{R}^d$ is a matrix theory corresponding to the free algebra with $d$ generators, the noncommutative affine space. $M$-theory on curved spaces should correspond to nontrivial smooth noncommutative algebras.

### 1.3.4. Double tangent space and formal noncommutative structure

Here we would like to give some examples of natural non-classical structures on manifolds $Repr_n^A$. First of all, there is a natural vector bundle $T_{(2)}$ on the square $Repr_n^A \times_k Repr_n^A$ together with the identification of its pullback via the diagonal embedding $\Delta : Repr_n^A \longrightarrow Repr_n^A \times_k Repr_n^A$ with the tangent bundle $T_{Repr_n^A}$.

In what follows we will describe the bundle $T_{(2)}$. First of all, $k$-points of $Repr_n(A)$ can be identified with equivalence classes of pairs $(\mathcal{E}, pr_{\mathcal{E}})$ where $\mathcal{E}$ is an $A$-module finite-dimensional over $k$, and $pr_{\mathcal{E}} : A^n \longrightarrow \mathcal{E}$ is an epimorphism to $\mathcal{E}$ from the standard $n$-dimensional free $A$-module such that the set of $n$ canonical generators of $A^n$ maps to a basis of $\mathcal{E}$ over $k$. The fiber of $T_{(2)}$ at the pair $\big((\mathcal{E}, pr_{\mathcal{E}}), (\mathcal{F}, pr_{\mathcal{F}})\big)$ is defined as

$$Hom_{A-mod}(Ker(pr_{\mathcal{E}}), \mathcal{F}) \ .$$

Let us prove that it is a finite-dimensional vector space. It is easy to see that it fits into an exact sequence

$$0 \longrightarrow Hom_{A-mod}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{F}^n \longrightarrow Hom_{A-mod}(Ker(pr_{\mathcal{E}}), \mathcal{F}) \longrightarrow Ext^1_{A-mod}(E, F) \longrightarrow 0$$

All spaces except the one in question are finite-dimensional by the theorem 1.1.4. Also, it is easy to see that on the diagonal $\mathcal{E} = \mathcal{F}$, $pr_{\mathcal{E}} = pr_{\mathcal{F}}$, the space $Hom_{A-mod}(Ker(pr_{\mathcal{E}}), \mathcal{F})$ coincides with the tangent space to $Repr_n^A$ at the point $(\mathcal{E}, pr_{\mathcal{E}})$.

Another structure on $Repr_n^A$ is the *formal noncommutative structure* in the sense of Kapranov. There is a canonical sheaf $\mathcal{O}^{noncomm}_{Repr_n^A}$ of formal noncommutative functions on $Repr_n^A$ with the quotient $\mathcal{O}_{Repr_n^A}$. For example, if $A = k\langle x_1, \ldots, x_d \rangle$ and $n = 1$ then the global sections of the sheaf $\mathcal{O}^{noncomm}_{Repr_n^A}$ is the projective limit $lim(A/I_n)$ where $I_n, n \geq 1$ is a decreasing sequence of two-sided ideals

$$I_n = \sum_{l \geq 1, (m_1, \ldots, m_l) : \sum m_i = n} A \cdot A^{[m_1]} \cdot A \cdot A^{[m_2]} \cdot \ldots \cdot A^{[m_l]} \cdot A \ .$$

Here $A^{[m]}$ for $m \geq 1$ is defined as the linear span of the set of commutators of depth $m$:

$$[a_0, [a_1, [\ldots [a_{m-1}, a_m] \ldots] \in A, \;\; a_0, a_1, \ldots, a_m \in A \;\; .$$

We refer the reader to [Ka] for the definition of the formal noncommutative structure and its differential-geometric meaning. We would like only to mention that Kapranov proposed a general construction of formal noncommutative structure on moduli spaces of objects in abelian categories, like the category of coherent sheaves on algebraic varieties etc. His construction admits a useful extension. Namely, if $\mathcal{C}$ is a $k$-linear triangulated category and $\mathcal{E}_0$ is a fixed object, then one can try to consider the "moduli space" of pairs $(\mathcal{E}, p)$, where $\mathcal{E}$ is an object of $\mathcal{C}$ and $p : \mathcal{E}_0 \longrightarrow \mathcal{E}$ is a morphism. The tangent complex for such a pair is $RHom_{\mathcal{C}}(\mathcal{E}, Cone(p))$. It carries a natural structure of an associative non-unital algebra, and the arguments from [Ka] are applicable there.

### 1.4. What we would like to do?

It is clear from all the previous discussion that the subject of smooth noncommutative geometry merits further development. All our previous constructions and examples are *affine*, in particular they give affine representation schemes. It would be desirable to give a definition of "smooth noncommutative schemes", and also of "quasi-projective noncommutative schemes", such that it contains finitely generated (or maybe finitely presented) smooth algebras as an affine case, and give rise to smooth schemes via certain functor of "representations to $Mat(n \times n, k)$". We expect that representation schemes carry bitangent bundles, Kapranov's formal noncommutative structure, etc.

In the second part of this text we describe a general approach to noncommutative algebraic geometry based on flat topology. We choose (temporarily) the name "spaces" (or "noncommutative spaces") instead of "schemes" for several reasons. Noncommutative schemes were introduced in earlier works of one of us (see [R2]), where analogues of Zariski topology were studied. Also, our formalism describes not only usual schemes but algebraic spaces too. As the reader will see, an amazing variety of basic constructions in usual algebraic geometry can be extended to the noncommutative setting. Our category of noncommutative spaces could be used in other situations. For example, quantum projective spaces arising from Sklyanin algebras also fit into our definition.

In the third part we study one particular space which we call "noncommutative projective space" and denote by $N\mathbb{P}^n_k$. We beleive that $N\mathbb{P}^n_k$ is one of the principal examples of what should be called "smooth projective noncommutative variety".

## 2. NONCOMMUTATIVE SPACES

### 2.1. Covers and refinements

Here we describe intermediate objects which are not yet spaces (there are no structure sheaves on them). These objects are machines producing abelian categories. Essentially all definitions can be made in general monoidal categories (even in non-additive categories), instead of the category of vector spaces over $k$ with the monoidal structure given by the tensor product.

#### 2.1.1. Category of finite covers

**Definition.** *Objects of category $Covers_k$ are given by the following*

*DATA:*
*1) an associative algebra $B \in Ob(Alg_k)$,*
*2) $M \in Ob(B - mod - B)$, a bimodule over $B$,*
*3) $m_M : M \longrightarrow M \otimes_B M$, a homomorphism of bimodules,*
*4) $e_M : M \longrightarrow B$, also a homomorphism of bimodules,*
*satisfying the following*
*AXIOMS:*
*1) $M$ is faithfully flat as the right $B$-module, i.e. the functor $M \otimes_B : B - mod \longrightarrow B - mod$ is exact and does not kill any non-zero morphism,*
*2) $M$ with $m_M$ and $e_M$ is a coassociative coalgebra with counit in the monoidal category of $B \otimes B^{opp}$-modules.*

We will usually denote objects of the category $Covers_k$ as pairs $(B, M)$ skipping data $m_M$ and $e_M$. Morally, we consider pairs $(B, M)$ as "noncommutative stacks" together with a finite affine cover with affine pairwise intersections of members of the cover (see examples in the next subsection).

For every object $(B, M)$ of $Covers_k$ we have an abelian category $QCoh(B, M)$ whose objects are pairs $(\mathcal{E}, m_{\mathcal{E}})$ where $\mathcal{E}$ is a $B$-module and $m_{\mathcal{E}} : \mathcal{E} \longrightarrow M \otimes_B \mathcal{E}$ is a homomorphism of $B$-modules which defines a coaction of coalgebra $M$ on $\mathcal{E}$. We call objects of $QCoh(B, M)$ *quasi-coherent sheaves* on the "noncommutative stack" corresponding to $(B, M)$. One can justify the name "sheaf" introducing an appropriate Grothendieck topology.

**Definition.** *Morphisms $f$ in the category $Covers_k$ from $(B_1, M_1)$ to $(B_2, M_2)$ are given by the following*

*DATA:*
*1) $f_B : B_2 \longrightarrow B_1$, a morphism of algebras,*
*2) $f_M : M_2 \longrightarrow M_1$, a morphism of $k$-vector spaces*
*satisfying the following*
*AXIOMS:*
*1) the diagram where vertical arrows are structure morphisms of $M_i$ as $B_i$-bimodules $(i = 1, 2)$ is commutative:*

$$\begin{array}{ccc} B_2 \otimes M_2 \otimes B_2 & \xrightarrow{f_B \otimes f_M \otimes f_B} & B_1 \otimes M_1 \otimes B_1 \\ \downarrow & & \downarrow \\ M_2 & \xrightarrow{f_M} & M_1 \end{array}$$

*2),3) two analogous diagrams including coproduct morphisms $m_{M_i}$ and counit morphisms $e_{M_i}$ respectively, are commutative.*

Notice that the direction of the morphism $f = (f_B, f_M)$ is *opposite* to the direction of the pullback morphisms of algebras and bimodules.

Every morphism $f = (f_B, f_M) : (B_1, M_1) \longrightarrow (B_2, M_2)$ of covers defines a functor $f^* : QCoh(B_2, M_2) \longrightarrow QCoh(B_1, M_1)$, the pullback of quasi-coherent sheaves. This functor maps an object $(\mathcal{E}, m_{\mathcal{E}})$ to the object $(B_1 \otimes_{B_2} \mathcal{E}, m'_{\mathcal{E}})$ where the coaction morphism

$$m'_{\mathcal{E}} : B_1 \otimes_{B_2} \mathcal{E} \longrightarrow M_1 \otimes_{B_1} B_1 \otimes_{B_2} \mathcal{E} = M_1 \otimes_{B_2} \mathcal{E}$$

is defined in a natural way using $f_B$, $f_M$ and $m_{\mathcal{E}}$.

**2.1.2. Examples of finite covers**

FC1) Let $S/Spec(k)$ be a separated quasi-compact scheme, e.g. a quasi-projective scheme. We choose a finite cover $(\mathcal{U}_i)_{i \in I}$ of $S$ in Zariski topology by affine schemes. It follows from separatedness that pairwise intersections $\mathcal{U}_i \cap \mathcal{U}_j$ are again affine. The algebra $B$ and the bimodule $M$ are defined as

$$B := \mathcal{O}(\sqcup_i \mathcal{U}_i), \quad M := \mathcal{O}(\sqcup_{i,j} (\mathcal{U}_i \cap \mathcal{U}_j))$$

and the structure of coalgebra on $M$ is the natural one. It follows from the usual descent theory that the category $QCoh(B, M)$ is equivalent to the category of quasi-coherent sheaves on $S$.

FC1') The same statement holds for affine covers of separated quasi-compact schemes in fpqc topology, for algebraic stacks etc.

FC2) Let $A$ be an associative algebra and $M := A$ with the natural structure of an $A$-bimodule, and a coalgebra in the category $A$-*mod*-$A$. Then $QCoh(A, M)$ is equivalent via the tautological functor to the category $A$-*mod*.

FC3) Let $A$ be an associative algebra, and $B \supset A$ be a larger algebra such that $B$ is faithfully flat as the right $A$-module. Then we define $M$ as $B \otimes_A B$ with the natural structure of a coalgebra in the category $B$-*mod*-$B$. We have a natural morphism of covers

$$f : (A, A) \longrightarrow (B, M)$$

**Theorem.** *The morphism $f$ as above defines an equivalence $f^*$ between the categories $QCoh(B, M)$ and $QCoh(A, A) = A$-mod.*

This theorem follows from the general Barr-Beck theorem. Recall that a *comonad* (or *cotriple*) in the category $C$ consists of a functor $T : C \to C$ and two functor morphisms $\delta : T \to T^2$ and $\epsilon : T \to Id_C$ such that $T\delta \circ \delta = \delta T \circ \delta$ and $\epsilon T \circ \delta = id_T = T\epsilon \circ \delta$. A *coaction* of the comonad $T$ is a morphism $c : X \to T(X)$ such that $Tc \circ c = \delta(X) \circ c$ and $\epsilon(X) \circ c = id_X$. We denote by $T$-*comod* the category of $T$-*comodules*, i.e. objects of $C$ endowed with a coaction of $T$, with naturally defined morphisms: a morphism $(X, c) \to (X', c')$ is a morphism $f : X \to X'$ such that $Tf \circ c = c' \circ f$. An arbitrary pair of adjoint functors $F : C_1 \longrightarrow C_2$, $G : C_2 \longrightarrow C_1$ with adjunction morhisms $\epsilon : FG \to Id_{C_2}$ and $\eta : Id_{C_1} \to GF$ determines a comonad $(T, \delta, \epsilon)$, where $T = FG$, $\delta = F\eta G$, and there is an obvious functor $C_1 \longrightarrow T$-*comod*, $, Y \mapsto (F(Y), F\eta(Y))$. The following version of Barr-Beck's theorem is sufficient for our needs.

**Theorem.** *Let $F : C_1 \longrightarrow C_2$ be a functor between two categories having a right adjoint functor $G : C_2 \longrightarrow C_1$. Assume that the following conditions hold:*

*(a) a pair of arrows $f, g : X \rightrightarrows Y$ has a kernel if its image by $F$ has a kernel;*

*(b) if $h : Z \longrightarrow X$ is such that $f \circ h = g \circ h$ and $Fh$ is the kernel of the pair $(Ff, Fg)$, then $h$ is a kernel of $(f, g)$.*

*Then the canonical functor $C_1 \longrightarrow T$-comod is an equivalence of categories.*

We apply this theorem to the case $C_1 = A$-*mod*, $C_2 = B$-*mod* and $F = B \otimes_A$.

Note that the arbitrary pairs of morphisms in Barr-Beck's theorem can be replaced by so called *coreflexive pairs*, i.e. the pairs $f, g : X \rightrightarrows Y$ such that there exists a morphism $e : Y \to X$ such that $e \circ f = id_X = e \circ g$. And even this condition can be weakened (cf. [MLM], IV.4).

### 2.1.3. Refinements of covers

Let $(B_1, M_1)$ be a cover and $i : B_1 \longrightarrow B_2$ be an inclusion of algebras such that $B_2$ is faithfully flat as a right $B_1$-module. We define $M_2$ as $B_2 \otimes_{B_1} M_1 \otimes_{B_1} B_2$ with the natural structure of a coalgebra in the category $B_2$-*mod*-$B_2$. It is easy to see that $(B_2, M_2)$ is again a cover and we have a natural morphism of covers $f : (B_2, M_2) \longrightarrow (B_1, M_1)$.

**Definition.** *A morphism of covers isomorphic to $f$ as above is called a refinement morphism.*

We denote by $Ref$ the class of refinement morphisms. Analogously to the example FC3) from the previous subsection, the functors $f^*$ for $f \in Ref$ are equivalences of categories of quasi-coherent sheaves. Here we apply Barr-Beck theorem to the functor which acts from $QCoh(B_1, M_1)$ to $B_2$-*mod* and is given by formula $F(\mathcal{E}, m_{\mathcal{E}}) = B_2 \otimes_{B_1} \mathcal{E}$.

The class of refinements is closed under compositions. Also, any pullback of a refinement morphism is again a refinement morphism. Thus, we can easily describe the localization of the category of covers with respect to refinements.

## 2.2. Noncommutative spaces

In this section we give the definition of noncommutative spaces (in several steps). Morally one should think about noncommutative space $X$ as about an abelian category $QCoh(X)$ and two adjoint functors $\pi_* : QCoh(X) \longrightarrow k$-*mod*, $\pi^* : k$-*mod*$\longrightarrow QCoh(X)$. The pullback $\pi^*(k^1)$ of the standard 1-dimensional $k$-module is the "structure sheaf" $\mathcal{O}_X$.

### 2.2.1. Covers of noncommutative spaces

We define covers of noncommutative spaces adding some data and an axiom to the definition of the category of covers:

**Definition.** *Objects of category $Covers_k^{sp}$ are pairs $(C, s_C)$ where $C = (B, M)$ is a cover and $s_C : B \otimes B \longrightarrow M$ a homomorphism of bimodules which is an epimorphism and also a morphism of coalgebras. Morphisms in $Covers_k^{sp}$ are morphisms of covers compatible with structure epimorphisms from $B \otimes B$.*

In other words, we demand that $M$ should be a cyclic bimodule generated by an element $m := s_C(1_B \otimes 1_B)$ which behaves well with respect to the coalgebra structure. Such $M$ is given by a left ideal $Ker(s_C)$ in $B \otimes B^{opp}$ satisfying a complicated system of axioms.

We will call objects of $Covers_k^{sp}$ *space covers.* Notice that all our examples of covers (except stacks) are automatically space covers.

The morphism $s_C$ give rise to a canonical morphism from $(C, s_C)$ to the refinement $(B, B \otimes B)$ of the "point" object $Spec(k) := ((k,k), id)$ (the final object in $Covers_k^{sp}$). The pullback of the standard 1-dimensional module $k^1$ under this morphism we call the structure sheaf $\mathcal{O}_C$. It is represented by 1-dimensional free $B$-module $B^1 = B$ with the coaction of $M$ arising from $s_C$.

Also, for any refinement $f : (B_2, M_2) \longrightarrow (B_1, M_1)$ from cover to a space cover there exists a unique structure of space cover on $(B_2, M_2)$ such that $f$ became a morphism of space covers.

We define *equivalence classes of noncommutative spaces* as equivalence classes of space covers modulo the relation generated by refinements. In order define the "right" *category* of noncommutative spaces we need to make an additional work.

### 2.2.2. Equivalence relation between morphisms of space covers

Two different morphisms of space covers could give isomorphic functors of pullbacks on categories of quasi-coherent sheaves. We would like to identify such morphisms.

Let $(C, s_C)$ and $(C', s_{C'})$ be two space covers, and $B$ resp. $B'$ denote corresponding algebras.

**Definition.** *Two morphisms $f, g : (C, s_C) \longrightarrow (C', s_{C'})$ are equivalent iff for any element $\sum_\alpha x_\alpha \otimes y_\alpha \in Ker(s_{C'}) \subset B' \otimes B'$ the following equations hold:*

$$\sum_\alpha f(x_\alpha) \cdot g(y_\alpha) = \sum_\alpha g(x_\alpha) \cdot f(y_\alpha) = 0 \in B$$

**Theorem.** *The relation described above is an equivalence relation. There is a canonical identification of pullback functors for equivalent morphisms.*

We will not give here the proof of this theorem. It follows from general categorical considerations and from an explicit calculation of certain products in the category of space covers (see theorem in 2.3).

The equivalence of morphisms is compatible with composition. Thus, we can form a quotient category $\widetilde{Covers}_k^{sp}$.

### 2.2.3. The category of noncommutative spaces

**Definition.** *The category $Spaces_k$ of noncommutative spaces over $k$ is defined as the localization $\widetilde{Covers}_k^{sp}[Ref^{-1}]$ of the category of space covers with respect to the class of refinements.*

We call morphisms of the category $Spaces_k$ "maps". Let us spell explicitly how maps look like in terms of covers. A map from $(B, M)$ to $(B_1, M_1)$ is given by a refinement $(B', M') \longrightarrow (B_1, M_1)$ and a morphism of space covers $f' : (B', M') \longrightarrow (B_1, M_1)$. Two such data $\big((B', M'), f'\big)$ and $\big((B'', M''), f''\big)$ give the same map iff for some (equivalently, for any) given common refinement $(B''', M''')$ (e.g. the cartesian product of $(B', M')$ and $(B'', M'')$ over $(B, M)$) two corresponding cover morphisms from $(B''', M''')$ to $(B_1, M_1)$ are equivalent.

**Theorem.** *The category of separated quasi-compact schemes over $k$ is equivalent to a full subcategory of $Spaces_k$. The category $Alg_k^{opp}$ is also equivalent to a full subcategory of $Spaces_k$. Category $Spaces_k$ has finite limits.*

**Theorem.** *Construction $(B, M) \longrightarrow QCoh(B, M)$ extends to a functor from the category $Spaces_k$ to the category of abelian $k$-linear categories. For separated quasi-compact schemes, it gives usual quasi-coherent sheaves of commutative algebraic geometry. For associative algebras, it gives categories of left modules.*

One can consider noncommutative stacks by eliminating the condition that the structure homomorphism $s_S : B \otimes B \longrightarrow M$ is surjective. We will not develop here the more general theory of stacks.

### 2.3. Affine covers

We call *affine spaces* noncommutative spaces corresponding to the cover of the type $(B, B)$ where $B$ is an algebra. Abusing notations, we will denote the space $(B, B)$ by $Spec(B)$, although here we do not use points of the spectrum ([R3], [R1], [R4]) at all. The algebra $B$ can be identified with

$$Hom_{QCoh(Spec(B))}(\mathcal{O}_{Spec(B)}, \mathcal{O}_{Spec(B)}) \ .$$

Also, $B$ coincides as a set with the set of maps from $(B, B)$ to the (non)-commutative affine line $\mathbb{A}_k^1 = N\mathbb{A}_k^1 := Spec(k[t])$. In general, $d$-dimensional noncommutative affine space $N\mathbb{A}_k^d$ is defined as $Spec(k\langle x_1, \ldots, x_d \rangle)$. The algebra structure on $\mathcal{O}(S) = Maps(S, N\mathbb{A}_k^1)$ is induced by the algebra structure on $N\mathbb{A}_k^1$ considered as an object of $Spaces_k$.

Let $\pi : Spec(B) \longrightarrow S$ be a cover of a space by an affine space. The category of spaces admits finite limits. Thus, we can form an infinite sequence of spaces $S^{(n)}$, $n \geq 1$ taking the cartesian product over $S$ of $n$ copies of $Spec(B)$. It is easy to see that all spaces $S^{(n)}$ are affine. The algebra of functions on $S^{(n)}$ is generated by $n$ copies $i_1(B), \ldots, i_n(B)$ of the algebra $B$ subject to the set of relations which we will describe now.

**Relations:** *Let $z = \sum_\alpha a_\alpha \otimes b_\alpha \in B \otimes B$ be an arbitrary element of the kernel of the structure map $s_S : B \otimes B \longrightarrow M$. Then for any $k \neq, l$, $1 \leq k, l \leq n$ the element $\sum_\alpha i_k(a_\alpha) \cdot i_l(a_\alpha)$ is equal to zero in $\mathcal{O}(S^{(n)})$.*

One can also give a description of spaces $S^{(n)}$ in terms of categories of quasi-coherent sheaves.

**Theorem.** *The category $QCoh(S^{(n)})$ is canonically equivalent to the category of $n$-tuples $(\mathcal{E}_i)_{i=1,\ldots,n}$ of objects of $QCoh(Spec(B)) \simeq B$-mod endowed with an identfication of $n$ objects $\pi_*(\mathcal{E}_i), i = 1, \ldots, n$ in the category $QCoh(S)$.*

It is easy to see that the collection of spaces $S^{(n)}$ (and natural maps between them) form a contravariant functor from the category of finite non-empty sets to the category of affine spaces. For any space $U$, the set of maps $Maps(U, S^{(2)})$ is a subset of the square of the set $Maps(U, S^{(1)})$ and forms there a graph of an equivalence relation. A map from $U$ to $S$ can be described locally (i.e. after passing to a cover) as a map to $S^{(1)}$ modulo the equivalence relation coming from maps to $S^{(2)}$.

## 2.4. Cohomology of quasi-coherent sheaves

As we mentioned in the previous section, for every space $S$, there exists a distinguished object $\mathcal{O}_S$ in the category $QCoh(S)$. It is easy to see that the functor $\mathcal{E} \mapsto Hom(\mathcal{O}_S, \mathcal{E})$ is the right adjoint to the inverse image functor $f_S^*; QCoh(point) \longrightarrow QCoh(S)$ for the canonical map $f_S : S \longrightarrow point = Spec(k)$. We denote the vector space $Hom(\mathcal{O}_S, \mathcal{E})$ as $\Gamma(\mathcal{E})$ (the space of global sections of $\mathcal{E}$). The functor $\Gamma$ is left exact.

We would like to define now the derived functor for $\Gamma$. First of all, it makes sense because of the following

**Theorem.** *Category $QCoh(S)$ has enough injective objects.*

The proof is the following. Let us chose an affine cover $\pi : Spec(B) \longrightarrow S$. The functor $\pi^*$ has a right adjoint $\pi_*$. The category $B$-*mod* (canonically $\simeq QCoh(Spec(B))$) has enough injective objects. Now, for any quasi-coherent sheaf $\mathcal{E}$ on $S$ let us chose an embedding of $\pi^*\mathcal{E}$ into an injective $B$-module $I$. Due to the fact that the inverse image functor $\pi^*$ is exact, $\pi_* I$ is again an injective object. Since the functor $\pi^*$ is faithful, the natural homomorphisms $\mathcal{E} \longrightarrow \pi_*\pi^*\mathcal{E}$ is a monomorphism. And the image $\pi_*\pi^*\mathcal{E} \longrightarrow \pi_* I$ of the embedding $\pi^*\mathcal{E} \longrightarrow I$ is a monomorphism too, because the direct image functor $\pi_*$ is left exact (as any functor having a left adjoint).

Thus, one can proceed and define derived functor $R\Gamma$ (and more generally $RHom$, more generally derived functor of any left exact functor) using injective resolutions.

**Theorem.** *For any cover $(B, M)$ of $S$ and for any object $(\mathcal{E}, m_{\mathcal{E}}) \in QCoh(B, M)$, the cohomology of $S$ with coefficients in $\mathcal{E}$ can be calculated via the Čech complex*

$$\mathcal{E} \longrightarrow M \otimes_B \mathcal{E} \longrightarrow M \otimes_B M \otimes_B \mathcal{E} \longrightarrow \dots$$

One can extend the previous result to the relative case. For every morphism $f : (C_1, s_{C_1}) \longrightarrow (C_2, s_{C_2})$ in category $Covers_k^{sp}$ the functor $f^*$ admits a right adjoint $f_*$. Since *the direct image functor* $f_*$ is left exact, there is a derived functor for $f_*$.

## 2.5. The definition of coherent sheaves

In commutative algebraic geometry the abelian category of coherent modules are usually considered in the case of noetherian rings. In the noncommutative setting, our principal examples are free algebras in several indeterminates which are far from being noetherian. Nevertheless, one can define a reasonable abelian category of modules over smooth algebras as well.

**Definition.** *A module $M$ over an algebra $A$ is called coherent iff it is finitely presented, i.e. there exists an exact sequence*

$$F_{-1} \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

*where $F_{-1}$, $F_0$ are free finitely generated $A$-modules.*

We denote $Coh(A)$ the full subcategory of $A - mod$ objects of which are coherent modules.

**Lemma.** *For any hereditary algebra $A$ (e.g. for a formally smooth $A$), the category $Coh(A)$ is an abelian category.*

It is enough to check that the kernel of any morphism $\phi : M \longrightarrow N$ of coherent $A$-modules is coherent. The proof is the following. In hereditary abelian categories any object in the bounded derived category is isomorphic to its cohomology. Thus, $Ker(\phi)$ is a direct summand in $D^b(A - mod)$ of a perfect complex. Using telescope construction we see that $Ker(\phi)$ is quasi-isomorphic in an infinite complex bounded from above of finitely generated projective modules. From this it follows that $Ker(\phi)$ is coherent.

The property of a module over arbitrary algebra to be (or not to be) finitely presented is preserved under faithfully flat extensions of the algebra. This implies that for spaces one can define the notion of a finitely presented quasi-coherent sheaf passing to an arbitrary affine cover. The property of being finitely presented is independent of the cover.

For any noncommutative space $S$ admitting a cover by a smooth or noetherian affine space, the category of finitely-presented quasi-coherent sheaves is abelian. In such a case we call this the category $Coh(S)$ of coherent sheaves.

**2.6. Formally smooth, formally non-ramified and formally étale morphisms.**

We call an affine scheme morphism $Spec(A) \to Spec(B)$ a *thickenning* if the corresponding algebra morphism $B \to A$ is an epimorphism and its kernel is a nilpotent ideal.

Following the pattern of commutative algebraic geometry [EGA, IV.17], we give the following

**Definitions.** *1) Let $f : X \to Y$ be a morphism. We call $f$ formally smooth (resp. formally non-ramified, resp. formally étale) if, for any affine scheme $Spec(A)$, any thickenning $Spec(T) \to Spec(A)$, and any morphism $Spec(A) \to Y$, the canonical morphism*

$$Hom_Y(Spec(A), X) \to Hom_Y(Spec(T), X)$$

*defined by the immersion $Spec(T) \to Spec(A)$ is surjective (resp. injective, resp. bijective).*

*2) A $k$-space $X$ will be called formally smooth if the canonical morphism $X \to Spec(k)$ is formally smooth.*

**Remarks.** (i) If $X$ is affine, then the definition of the formal smoothness given here is in accordance with the one in 1.1.

(ii) To check that $f$ is formally smooth (resp. formally non-ramified, resp. formally étale), it suffices to do it in the case when the square of the kernel $J$ of the algebra epimorphism $A \to T$ is zero.

(iii) The properties of $f$ defined in 2.6 are the properties of the represented by $f$ functor $Y' \mapsto Hom_Y(Y', X)$ from the category dual to the category of $Y$-spaces to the category $Sets$. They make sense for any functor $(Spaces_k/Y)^{op} \to Sets$, representable or not.

(iv) It follows from the definition that $f$ is formally étale iff it is both formally smooth and formally non-ramified. One can show that $Spec(A)$ is étale (i.e. $Spec(A) \to Spec(k)$ is étale) iff the algebra $A$ is *separable*. The latter means that $A$ is a projective $A$-bimodule, or equivalently, $A$ has dimension zero with respect to Hochschild cohomology. An example of a separable algebra is the algebra $Mat(n \times n, k)$ of $n \times n$ matrices.

## 3. NONCOMMUTATIVE PROJECTIVE SPACE

### 3.1. Definition á la Grothendieck.

For any integer $d \geq 1$, we define the *noncommutative projective space* $N\mathbb{P}_k^{d-1}$ as a space representing a certain contravariant functor from the category of affine spaces to the category $Sets$.

Let $A$ be an algebra. The set $Map(Spec(A), N\mathbb{P}_k^{d-1})$ should be functorially identified with the set of quotient modules $\mathcal{F}$ of the standard free $d$-dimensional $A$-module $A^d$, such that locally in flat topology $\mathcal{F}$ is isomorphic to the free 1-dimensional $A$-module. The last condition implies that $\mathcal{F}$ is projective.

From this definition, it is not clear whether $N\mathbb{P}_k^{d-1}$ exists. We shall sketch a prove of its existence by producing an explicit affine cover.

### 3.2. Explicit cover.

We will construct a cover of $N\mathbb{P}_k^{d-1}$ analogous to the Jouanolou's cover of the projective space in commutative algebraic geometry. Recall that this is a cover by the affine quadric:

$$\{(x_1, \ldots, x_d; y_1, \ldots, y_d) | \sum_{i=1}^{d} x_i y_i = 1\} \longrightarrow \{(x_1 : \ldots : x_d)\} = \mathbb{P}^d(k) \ .$$

Let $B$ be the associative algebra generated by $2d$ variables $x_1, \ldots, x_d, y_1, \ldots, y_d$ satisfying the relation $\sum_{i=1}^{d} y_i x_i = 1$. The space $Spec(B)$ represents the following functor on affine spaces: $Maps(Spec(A), Spec(B))$ is the set of quotient modules $\mathcal{F}$ of $A^d$ together with an isomorphism $i : A^1 \simeq \mathcal{F}$ of $A$-modules, and a splitting $s$ of $A^n$ into the direct sum of $\mathcal{F}$ and a complementary module $\mathcal{F}'$. Also, for each $n \geq 1$ we can consider functor whose value on $Spec(A)$ is the set of collections $(\mathcal{F}, i_1, s_1, \ldots, i_n, s_n))$, where $\mathcal{F}$ is a quotient module of $A^d$ and for each $k = 1, \ldots, n$, the triple $(\mathcal{F}, i_k, s_k)$ belongs to $Maps(Spec(A), Spec(B))$. It is easy to see that this functor is representable by an affine space whose algebra of functions we denote by $B^{(n)}$.

Thus we have constructed a contravariant functor from the category of non-empty finite sets to the category of affine spaces. We can construct (as in 2.3) a new contravariant functor from spaces to sets by passing locally to sets of equivalence classes of maps to $Spec(B)$ modulo an equivalence relation coming from maps to $Spec(B^{(2)})$. This new functor coincides, more or less by definition, with the functor described in the previous section 3.1.

One can check by direct calculations that the algebra $B^{(2)}$ is generated by two copies $i_1(B)$, $i_2(B)$ modulo relations of the type described in 2.3. The bimodule $M$ in this case is $(B \otimes B)/I$ where $I$ is the left $B \otimes B^{opp}$-ideal generated by the following set of elements:

$$e_j := -1 \otimes x_j + \sum_{i=1}^{d} x_j y_i \otimes x_i, \quad j = 1, \ldots, d \ .$$

This shows that the functor defined in 3.1 is representable by $(B, M)$.

Analogously, one can imitate the Grothendieck's definition of projective space for coherent sheaves and introduce relative projective space $\mathbb{P}(\mathcal{E})$ for any noncommutative space $S$ and a finitely generated quasi-coherent sheaf $\mathcal{E}$ on $S$.

### 3.3. Derived category of quasi-coherent sheaves

Here we will only state the results without proofs (proofs will appear among other things in [KR]). We denote by $Q_d$ the quiver which has two vertices $\{v_0, v_1\}$ and $d$ oriented edges all going from $v_0$ to $v_1$.

**Theorem.** *For any $d \geq 1$ the category $QCoh(N\mathbb{P}_k^{d-1})$ has cohomological dimension 1. The bounded derived category $D^b(QCoh((N\mathbb{P}_k^{d-1}))$ is equivalent to the bounded derived category of representations of the quiver $Q_d$.*

**Theorem.** *The algebra $B$ described in 3.2 and the projective space $N\mathbb{P}_k^{d-1}$ are formally smooth. The bounded derived category of coherent sheaves on $N\mathbb{P}_k^{d-1}$ is of finite type (i.e. for any two objects $\mathcal{E}, \mathcal{F}$ we have $\sum_i rk(RHom^i(\mathcal{E}, \mathcal{F})) < +\infty$), and it is equivalent to the bounded derived category of finite-dimensional representations of $Q_d$. The group $K_0(Coh(N\mathbb{P}_k^{d-1}))$ is free abelian group with 2 generators.*

The main ingredient in the proof of both theorems is a noncommutative analogue of Beilinson's resolution of the sheaf of functions on the diagonal in $\mathbb{P}_k^{d-1} \times \mathbb{P}_k^{d-1}$. The noncommutative resolution is much shorter than the commutative one, and has length 2. There are two coherent sheaves $\mathcal{O}$ and $\mathcal{O}(1)$ (the universal quotient module $\mathcal{F}$ from 3.1) which generate in an appropriate sense the whole derived category on $N\mathbb{P}_k^{d-1}$.

There are two surprising particular cases: $d = 1$ for which the noncommutative projective space is more complicated than the usual one (which is a point), and the case $d = 2$. It is well-known that the category $D^b(Coh(\mathbb{P}_k^1))$ is equivalent to $D^b(Q_2 - mod_{finite})$. Thus, in the case $d = 2$ we have three different abelian categories with the same derived category: the noncommutative projective line, the commutative projective line, and the quiver $Q_2$.

### 3.4. Grassmanians

One can define Grassmanians $NGr_k(d', d)$ for $d', d \geq 1$ in the same fashion as the projective space, changing the requirement that the quotient module $\mathcal{E}$ of $\mathcal{O}^d$ should be locally isomorphic to a free $d'$-dimensional module, instead of a free 1-dimensional module. We claim that in contrast to the commutative case we do not get a new space. It follows from the fact that there exists a *non-zero* algebra $A$ such that $A^1$ is isomorphic $A^2$ (and thus to $A^{d'}$) as $A$-module. For example, one can take $A$ equal to $End(V)$ where $V$ is an infinite-dimensional vector space. On the faithfully flat extension which is obtained by taking free product with such algebra $A$, we identify conditions on $\mathcal{E}$ for all Grassmanians. Thus, $NGr_k(d', d)$ coincides with $N\mathbb{P}_k^{d-1} = NGr_k(1, d)$.

We expect that various remarkable identities in noncommutative linear algebra in free skew field discovered by Gelfand and Retakh (see [GR]) can be interpreted as identities between morphisms of coherent sheaves on $N\mathbb{P}_k^{d-1}$ and on similar spaces.

### 3.5. Representation spaces

For a noncommutative space $S$ we can try to define representation schemes $Repr_n^S$. At least, it is clear how sets of $k$-points should look like:

$$Reps_n^S(k) = Maps(Spec(Mat(n \times n, k)), S) \ .$$

It follows from the definition of the projective space in 3.1 and from the discussion in 3.4 that the set of $k$-points of $N\mathbb{P}_k^{d-1}$ is equal to the set of *all* non-zero quotient spaces of $k^d$. We leave to the reader as an exercise the description of maps from $Spec(Mat(n \times n, k))$ to $N\mathbb{P}_k^{d-1}$ for the case $n \geq 2$.

Let us express again our hopes: representation schemes $Repr_n^S$ for (formally) smooth non-affine spaces $S$ should carry all structures described in Part 1 (the double tangent bundle, Kapranov's formal noncommutative functions etc.).

**References.**

[CQ1] J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, Journal of AMS, v.8, no. 2, 1995, 251–289

[CQ2] J. Cuntz, D. Quillen, Cyclic homology and nonsingularity, Journal of AMS, v.8, no. 2, 1995, 373–442

[EGA] A. Grothendieck, J.A. Dieudonné, Eléments de Géométrie Algébrique, Springer Verlag, New York - Heidelberg - Berlin, 1971

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967

[GR] I. Gelfand and V. Retakh, Quasideterminants, I, Selecta Math., v.3, 1997, 517–546

[K] M. Kontsevich, Formal non-commutative symplectic geometry, in "The Gelfand Mathematical Seminars, 1990-1992", Birkhäuser, Boston-Basel-Berlin (1993), pp. 173–187

[Ka] M. Kapranov, Noncommutative geometry based on commutator expansions, math.AG/9802041 (1998), 48 pp.

[Kn] D. Knutson, Algebraic spaces, LNM 203, Springer-Verlag, 1971

[KR] M. Kontsevich, A. Rosenberg, Noncommutative separated spaces, in preparation

[ML] S. Mac-Lane, Categories for the working mathematicians, Springer - Verlag; New York - Heidelberg - Berlin (1971)

[MLM] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic, Springer-Verlag, New York - Heidelberg - Berlin-London-Paris (1992)

[R1] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.

[R2] A.L. Rosenberg, Noncommutative schemes, Compositio Mathematica 112 (1998), 93-125

[R3] A.L. Rosenberg, Noncommutative local algebra, Geometric and Functional Analysis (GAFA), v.4, no.5 (1994), 545-585

[R4] A.L. Rosenberg, The spectrum of abelian categories and reconstruction of schemes, in "Algebraic and Geometric Methods in Ring Theory", Marcel Dekker, Inc., New York, (1998), 255-274