Cancellation of Lattices and Finite Two-Complexes

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This is the first in a series of three papers (referred to below as [I], [II] and [III]) on certain cancellation problems which arise in algebra and topology. For example, if M, M', N are modules with $M \oplus N \cong M' \oplus N$, is $M \cong M'$? If K, K' are finite two-complexes with $K \vee rS^2 \simeq K' \vee rS^2$, is $K \simeq K'$? In [I] we consider these questions for modules over orders (e.g. integral group rings $\mathbf{Z}\pi$, π a finite group) and two-complexes with finite fundamental group. Part [II] deals with cancellation of quadratic forms and general results for 4-manifolds with finite fundamental group: when does $X \sharp (S^2 \times S^2) \approx Y \sharp (S^2 \times S^2)$ imply $X \approx Y$? In [III] we study smooth structures on elliptic surfaces, and the homeomorphism classification of 4-manifolds with certain special fundamental groups.

We now give a more detailed description of the results in the present paper. Let R be a Dedekind domain and F its field of quotients. A *lattice* over an R-order A is an A-module which is projective as an R-module. The general stable range condition for cancellation of lattices over orders is free rank ≥ 2 [**1**, (3.5), p.184]. We obtain an improvement in this stable range, assuming certain local information about the lattices. The problem is to show that certain groups of elementary automorphisms act transitively on unimodular elements in lattices, and our result suggests that an inductive procedure may be useful, to pass from transitivity over a quotient order B to transitivity over A. The arguments in §1 are modelled closely on the ones given in [**1**, Chap.IV,§3]. To obtain the geometric applications, the elementary automorphisms are shown to be realizable by (simple) homotopy equivalences.

To state our condition, let A and B be orders in separable algebras over F [**4**, 71.1, 75.1], and suppose that there is a surjective ring homomorphism $\epsilon : A \to B$. We say that a finitely generated A-module L has *(A, B)-free rank ≥ 1 at a prime* $\mathfrak{p} \in R$, if there exists an integer r such that $(B^r \oplus L)_{\mathfrak{p}}$ has free rank ≥ 1 over $A_{\mathfrak{p}}$. Here $A_{\mathfrak{p}}$ denotes the localized order $A \otimes R_{(\mathfrak{p})}$. In the extreme case $B = 0$, this is just the condition that $L_{\mathfrak{p}}$ has a free direct summand. In the other extreme case $A = B$, there is no condition on L.

Theorem A: *Let L be an A-lattice and put $M = L \oplus A$. Suppose that there exists a surjection of orders $\epsilon : A \to B$ such that L has (A, B)-free rank ≥ 1 at all but finitely many primes. If $GL_2(A)$ acts transitively on unimodular elements in $B \oplus B$, then for any A-lattice N which is locally a direct summand of M^n for some integer n, $M \oplus N \cong M' \oplus N$ implies $M \cong M'$.*

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In the classification of two-complexes with finite fundamental group we find that $(\mathbf{Z}\pi, \mathbf{Z})$-locally free modules have an important role, where $\mathbf{Z}\pi$ is the integral group ring of a finite group. This special case motivates the definition of (A, B)-locally free modules given above. We check that for $B = \mathbf{Z}$, the conditions on "transitive action" in Theorem A is satisfied (see (1.16)), hence can be omitted from the statement.

For example, consider the lattices L arising as $\pi_2(K)$, where K is a finite 2-complex with fundamental group $\pi_1(K) = \pi$. These are defined by exact sequences

$$0 \to L \to C_2 \to C_1 \to C_0 \to \mathbf{Z} \to 0 \tag{0.1}$$

with $C_i = C_i(\tilde{K})$ finitely generated free $\mathbf{Z}\pi$ modules.

More generally, any lattice L with a resolution (0.1) by finitely generated projective $\mathbf{Z}\pi$ modules C_i is unique up to direct sum with projectives. The stable class is denoted $\Omega^3\mathbf{Z}$. Such lattices with minimal Z-rank need not contain any projective direct summands over $\mathbf{Z}\pi$, but rationally contain all the representations of π except perhaps the trivial one. Then L has $(\mathbf{Z}\pi, \mathbf{Z})$-free rank ≥ 1 at all primes not dividing the order of π. The simplest case occurs for π cyclic and $L = \ker\{\epsilon : \mathbf{Z}\pi \to \mathbf{Z}\}$ the augmentation ideal.

The linear cancellation theorems in §1 have applications to the homotopy type of 2-complexes. Recall that J. H. C. Whitehead proved that any two finite 2-complexes K, K' with isomorphic fundamental groups become homotopy equivalent after wedging with a sufficiently large (finite) number of S^2's. It is well-known that in fact they become *simple* homotopy equivalent since any Whitehead torsion can be realized stably by a self-equivalence. Furthermore, if $\alpha : \pi_1(K, x_0) \to \pi_1(K', x'_0)$ is a given isomorphism and K, K' have the same Euler characteristic, then there is a simple homotopy equivalence $f : K \vee rS^2 \to K' \vee rS^2$ inducing α on the fundamental groups.

The following is our main result about finite two-complexes. The analogous result for "homotopy type" instead of "simple homotopy type" was proved by W. Browning [3, 5.4].

Theorem B: *Let K and K' be finite 2-complexes with the same Euler characteristic and finite fundamental group. Let $\alpha : \pi_1(K, x_0) \to \pi_1(K', x'_0)$ be a given isomorphism and suppose that $K \simeq K_0 \vee S^2$. Then there is a simple homotopy equivalence $f : K \to K'$ inducing α on the fundamental groups.*

This is the best possible result in general, but for special fundamental groups it can sometimes be improved (see §2):

Theorem 2.1: *Let π be a finite subgroup of $SO(3)$. If K and K' are finite 2-complexes with fundamental group π and the same Euler characteristic, and $\alpha : \pi_1(K, x_0) \to \pi_1(K', x'_0)$ a given isomorphism, then there is a simple homotopy equivalence $f : K \to K'$ inducing α on the fundamental groups.*

For π cyclic or $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$, this was proved in [5], [7].

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§1: Cancellation of lattices

By an "A-module" we will mean a finitely generated right A-module. As above we suppose that $\epsilon : A \to B$ is a surjective ring homomorphism of R-orders in (possibly different) separable F-algebras. If M is an A-lattice and $N := \epsilon_*(M) = M \otimes_A B$, we get an induced homomorphism

$$\epsilon_* : GL(M) \to GL(N).$$

If $M = M_1 \oplus M_2$ is a direct sum splitting of an A-module then $E(M_1, M_2)$ denotes the subgroup of $GL(M)$ generated by the elementary automorphisms ([**1**, p.182]). Let $E_+(M_1, M_2)$ be the subgroup of elementary automorphisms of the form $1_M \oplus f$ where $f : M_1 \to M_2$ is a homomorphism. Similarly, let $E_-(M_1, M_2)$ be the subgroup consisting of those of the form $1_M \oplus g$, where $g : M_2 \to M_1$. Then

$$E(M_1, M_2) = \langle E_+(M_1, M_2), E_-(M_1, M_2)\rangle.$$

If $\mathfrak{D}$ is a two-sided ideal in A, then let $GL(M; \mathfrak{D}) = \ker(GL(M) \to GL(M/M\mathfrak{D})$. We define

$$E_\pm(M_1, M_2; \mathfrak{D}) = E_\pm(M_1, M_2) \cap GL(M; \mathfrak{D}).$$

Finally, let $E(M_1, M_2; \mathfrak{D})$ be the *normal* subgroup of $E(M_1, M_2)$ generated by all elementary automorphisms as above with $f(M_1) \subseteq M_2\mathfrak{D}$, or $g(M_2) \subseteq M_1\mathfrak{D}$, respectively.

We will frequently use the notation $P = p_0A \oplus p_1A$ for a free A-module of rank two with the basis $\{p_0, p_1\}$. It has rank one submodules $P_i = p_iA$ for $i = 0, 1$. We define $E_\pm(P) = E_\pm(p_0A, p_1A)$ and $E_\pm(P; \mathfrak{D}) = E_\pm(p_0A, p_1A; \mathfrak{D})$ when the basis is understood from the context.

Recall that for an element $x \in M$, $O_M(x)$ is the left ideal in A generated by

$$\{f(x) \mid f \in \mathrm{Hom}_A(M,\ \mathrm{A})\}.$$

If $O_M(x) = A$ we say that x is unimodular. If $N \subseteq M$ is a submodule, then an element $x \in N$ is M-unimodular if $O_M(x) = A$.

The following result of Bass is an essential ingredient in the proofs of the cancellation theorems.

Theorem 1.1: ([**1**, (3.1), p.178; (3.2), p.181]) *Suppose that $Q = A$ and $P = p_0A \oplus p_1A$, and $\mathfrak{D}$ is a two-sided ideal in A. Let $x = (p, q) \in P \oplus Q$ be an element such that*

$x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$, and $O_{P\oplus Q}(x) + \mathfrak{a} = A$ for some left ideal $\mathfrak{a}$. Then there exists an A–homomorphism $f : Q \to P\mathfrak{O}$ such that $O_P(p + f(q)) + \mathfrak{a} = A$.

We also need two other facts.

Lemma 1.2: *Let M be a finitely generated right A-module, projective over R, and $A' = A/A\mathfrak{t}$ for an ideal $\mathfrak{t} \in R$ such that the localized order $A_\mathfrak{t}$ is maximal. Then the induced map*

$$\mathrm{Hom}_A(M,\, A) \;\to \mathrm{Hom}_{A'}(M',\, A')$$

is surjective, where $M' = M/M\mathfrak{t}$.

Proof: First note that $M_\mathfrak{t}$ is projective over $A_\mathfrak{t}$. Since $A' = A_\mathfrak{t}\,/A_\mathfrak{t}\,\mathfrak{t}$ we can lift any map $f' : M' \to A'$ to $f : M_\mathfrak{t} \to A_\mathfrak{t}$. After restricting to $M \subseteq M_\mathfrak{t}$ and multiplying by an element $r \in R$ prime to $\mathfrak{t}$, we obtain a lifting of $r'f'$. But r' (the image of r in R') is a unit in A'.

Lemma 1.3: ([**2**, (2.5.2),p.225]) *If C is a semisimple algebra, then for each a, $b \in C$ there exists $r \in C$ such that $C(a + rb) = Ca + Cb$.*

We now come to the main result of the section.

Theorem 1.4: *Let A be an R-order in a separable K-algebra and suppose that $M = P \oplus L$ is an A-lattice, where $P = p_0A \oplus p_1A$, and L has (A, B)-free rank ≥ 1 at all but finitely many primes. For any two-sided ideal $\mathfrak{O}$ in A, the subgroup of*

$$G_1(\mathfrak{O}) = \langle E(p_0A, L \oplus p_1A; \mathfrak{O}), E(p_1A, L \oplus p_0A; \mathfrak{O})\rangle \subseteq GL(M; \mathfrak{O})$$

fixing $\epsilon_*(p_0)$ acts transitively on the unimodular elements $x \in M$ such that $x \equiv p_0\,(\bmod\ \mathfrak{O})$ and $\epsilon_*(x) = \epsilon_*(p_0)$.

We divide the proof into several parts, stated as separate Lemmas for use in [**II**]. Let $x = p + v \in M$ be a unimodular element, where $p = p_0a + p_1b \in P$ and $v \in L$. Let $x = p_0a + p_1b + v \in M$ be a unimodular element, with $p = p_0a + p_1b \in P$ and $v \in L$, so that $O(x) = Aa + Ab + O(v)$. We assume that $\epsilon_*(x) = \epsilon_*(p_0)$ and $x \equiv p_0\,(\mathrm{mod}\,\mathfrak{O})$, so $a \equiv 1\,(\mathrm{mod}\,\mathfrak{O})$, $b, v \equiv 0\,(\mathrm{mod}\,\mathfrak{O})$. In the proof we use the stability assumption on L to move x so that its component in $p_0A \oplus L$ is unimodular. Then we move x to p_0 to prove the statement about unimodular elements in M. At each step we must use only elements σ of $G_1(\mathfrak{O})$ fixing $\epsilon_*(p_0)$.

Lemma 1.5: *Let S be a set of primes in R, and $\bar{A} = A/\mathfrak{g}A$ where $\mathfrak{g}$ is the ideal in R generated by all the primes $\mathfrak{p} \in S$. Then after applying an element $\tau \in E_+(P_1, P_0; \mathfrak{O})$ to x, $O(\bar{x}) = \bar{A}\bar{a} = \bar{A}$ and $\epsilon_*(x) = \epsilon_*(p_0)$.*

Proof: The semi-simple quotient ring $\bar{A}/\mathrm{Rad}\bar{A} = \bar{C} \times \bar{C}'$, where $\bar{C} = \bar{B}/\mathrm{Rad}\bar{B}$ and C' is a complementary direct factor. Therefore the $\bar{C}$ component of a is already a unit since a projects to 1 in the semisimple quotient. Over the other factor we can apply [**1**, (2.8),p.87]: there exists $u \in \mathfrak{O}$, such that the element $a + ub$ projects to a unit in $\bar{C}'$

and to 1 in $\bar{B}$. Let $g : P_1 \to p_0 A \subseteq M$ such that $g(p_1) = p_0 u$. Extend g to a map from M to M by zero on the complement. Then $\tau = 1 + g$ is an element of $E_+(P_1, P_0; \mathfrak{O})$ and $\tau(x)$ has the desired properties (1.5). ∎

We apply Lemma 1.5 to the set $\mathcal{S}$ of primes $\mathfrak{p} \in R$ at which A is not maximal, or L does not have (A, B)-free rank ≥ 1.

Lemma 1.6: *If $x = p_0 a + p_1 b + v \in M$ is a unimodular element for which $Aa + \mathfrak{g}A = A$. Let $\mathfrak{t} \subseteq R$ be the ideal which is maximal among those such that $A\mathfrak{t} \subseteq Aa$. Then $\mathfrak{t}$ is relatively prime to $\mathfrak{g}$ and $A_\mathfrak{t}$ is a maximal order. In addition, after applying an element $\tau \in E_+(P_1, L; \mathfrak{O})$ we have $x = p_0 a + p_1 b + v$ with $Aa + O(v) + A\mathfrak{t} = A$, $p_0 a + v$ unimodular, and $\epsilon_*(x) = \epsilon_*(p_0)$.*

Proof: Let $\mathfrak{t} \subseteq R$ denote the ideal, maximal among those such that $A\mathfrak{t} \subseteq Aa$. If $\mathfrak{p}$ divides $\mathfrak{t}$, for some prime dividing $\mathfrak{g}$, then $\mathfrak{t} = Aa \cap R \cdot 1$ implies that $Aa \cap R \cdot 1 \subseteq \mathfrak{p}$. But $(Aa)_\mathfrak{p} = A_\mathfrak{p}$ for all primes $\mathfrak{p}$ dividing $\mathfrak{g}$, so this is impossible. Hence $\mathfrak{g}$ is relatively prime to $\mathfrak{t}$, and in particular $\mathfrak{t} \neq 0$.

Now we project to the semilocal ring $A' = A/A\mathfrak{t}$, which is the quotient of the order $A_\mathfrak{t}$ (maximal by our choice of $\mathfrak{g}$) and so the projection $\epsilon' : A' \to B'$ splits and $A' = B' \times C'$. Since over the B' factor a projects to 1, we have $(Aa)' = A'$. Over the complementary factor C' we use a suitable $\tau \in E(p_1' C', L')$, so that after applying τ we achieve the condition

$$A'a' + O(v') = A' \tag{1.7}$$

over both factors of A'. This is an application of (1.3) to the component of x in $L' \oplus p_1' C'$ using the fact that $C' \subseteq L'$. The necessary homomorphism $g \in \mathrm{Hom}_{A'}(P_1', L'\mathfrak{O}')$, which is the identity over B', can be lifted to $\mathrm{Hom}_A(P_1, L\mathfrak{O})$ since P_1 is projective and extended to M by zero on $p_0 A \oplus L$.

We now lift the relation (1.7) to A using (1.2) and obtain

$$Aa + O(v) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa$ so $v + p_0 a$ is unimodular. ∎

We now complete the proof of Theorem 1.4 by the following:

Lemma 1.8: *Let $x = p_0 a + p_1 b + v$, with $x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$ and $\epsilon_*(x) = p_0$. Suppose that $z = p_1 a + v$ is unimodular, and write $L \oplus P_0 = zA \oplus L_0$. Then there exist elementary automorphisms $\tau_1 \in E_+(zA, P_1; \mathfrak{O})$, $\tau_2 \in E_+(P_1, P_0)$, $\tau_3 \in E_+(P_0, P_1; \mathfrak{O})$ and $\tau_4 \in E_+(P_0, L; \mathfrak{O})$ such that $\tau_4 \tau_2^{-1} \tau_3 \tau_2 \tau_1(x) = p_0$ and the product fixes $\epsilon_*(p_0)$.*

Proof: This is the argument of [1, pp. 183-184] Let $g_1(z) = p_1(1 - a - b)$, with $g_1(L_0) = 0$. Define $g_2(p_1) = p_0$, $g_3(p_0) = p_1(a - 1)$, and $g_4(p_0) = -v$, where the homomorphisms are extended to the obvious complements by zero. If $\tau_i = 1 + g_i$, then

$$\tau_4 \tau_2^{-1} \tau_3 \tau_2 \tau_1(x) = p_0.$$

The product fixes $\epsilon_*(p_0)$ and lies in $E(P_1, p_0 \oplus L; \mathfrak{O})$. ∎

We now introduce the following notation: if N is a submodule of M and $G \subseteq GL(M)$, then $G(N) = \{g \in G \,|\, g(N) = N\}$. If $M = M_1 \oplus M_2$ and $G \subseteq GL(M_1)$, then (by definition) $G(N) = \{g \in G \,|\, (g \oplus 1_{M_2})(N) = N\}$.

Definition 1.9: *Suppose that $M = P \oplus L$ is an A-lattice, where $P = p_0A \oplus p_1A$, and $N \subseteq M$ is a submodule containing p_0A as a direct summand. Let $\mathfrak{O} = \mathrm{Ann}(M/N)$, a two-sided ideal in A. A subgroup $G_0 \subseteq GL(P)$ is (N, p_0, ϵ) – transitive if*
(i) $G_0(N)$ acts transitively on the images in $N/N \cap M\mathfrak{O}$ of the elements p_0a, for any $a \in A$ representing a unit in $A/\mathfrak{O}$, and
(ii) the subgroup of $G_0(N)$ which fixes $p_0 \,(\mathrm{mod}\, \mathfrak{O})$ acts transitively on the images in $\epsilon_*(P)$ of the P-unimodular elements $x \in P \cap N$ such that $x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$.

Lemma 1.10: *Let $M = P \oplus L$ be an A-lattice, where $P = p_0A \oplus p_1A$. Let $N = p_0A \oplus N' \subset M$ and $\mathfrak{O} = \mathrm{Ann}(M/N)$.*
(i) Suppose that N' is a submodule of finite index in $p_1A \oplus L$ and that there exists a subgroup $G_0 \subseteq GL(P)$ which satisfies the condition in Definition 1.9(i). If $x \in N$ is a M-unimodular element, then there exist elementary automorphisms $\tau_1 \in E_-(p_0A, L \oplus p_1A)$, $\tau_2 \in E_+(p_0A, N')$, and $\theta_1 \in G_0(N)$ such that $x' = \theta_1\tau_2\tau_1(x)$ has $x' \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$. In addition, $\tau_i(N) = N$, for $i = 1, 2$.
(ii) Suppose that there exists a subgroup $G_0 \subseteq GL(P)$ which satisfies the condition in Definition 1.9(ii). If $x \in N$ is a M-unimodular element with $x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$, then there exist elementary automorphisms $\tau_3, \tau_4 \in E(P, L; \mathfrak{O})$ and $\theta_1 \in G_0(N)$, such that $x' = \tau_4\theta_2\tau_3(x)$ has $\epsilon_*(x') = \epsilon_*(p_0)$ and $x' \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$. In addition, $\tau_i(N) = N$, for $i = 3, 4$.

Proof: (i) By assumption, $A/\mathfrak{O}$ is a finite ring. It is convenient to describe the elements of $N \subseteq p_0A \oplus p_1A \oplus L$ in the notation used above: $x = p_0a + p_1b + v$, where $p_1b + v \in N'$ and $v \in L$.

We work over $N/N \cap M\mathfrak{O}$ and start by arranging that x has $p_1b + v \equiv 0 \,(\mathrm{mod}\, \mathfrak{O})$. To see this note that $Aa + O_M(p_1b + v) = A$, so there exists $c \in O(p_1b + v)$ such that $Aa + c$ contains 1. Apply (1.3) to $A \oplus p_0A$ and the element (c, a) to find $z \in A$ with $A(a + zc) = A \,(\mathrm{mod}\, \mathfrak{O})$. There exists $g_1 : L \oplus p_1A \to p_0A$ with $g_1(p_1b + v) = p_0zc$, since $c \in O_M(p_1b + v)$. Let $u \in A$ be a lifting of $u(a + zc) \equiv \,(\mathrm{mod}\, \mathfrak{O})$, and define $f_1 : p_0A \to L \oplus p_1A$ by $f_1(p_0) = (p_1b + v)u$. Extend by zero on the complements, and define $\tau_1 = (1 + g_1)$, $\tau_2 = (1 - f_1)$. Then $\tau_2\tau_1(x) \equiv p_0a (\,\mathrm{mod}\, \mathfrak{O})$, $\tau_1, \tau_2 \in E(p_0A, L \oplus p_1A)$, and $\tau_i(N) = N$ for $i = 1, 2$. By assumption there exists $\theta_1 \in G_0(N)$ to get $x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{O})$.
(ii) We will first make the P-component $p = p_0a + p_1b$ of x unimodular, using the fact that P satisfies the hypotheses of (1.1), with $\mathfrak{O} = \mathrm{Ann}(M/N)$ and $\mathfrak{a} = 0$. Again we start with $O(p) + O(v) = A$, so there exists $c \in O(v)$ such that $O(p) + c$ contains 1. Apply (1.1) to $A \oplus P$ and the element (c, p) to find $z \in P\mathfrak{O}$ with $O(p + zc) = A$. There exists $g_3 : L \to P$ with $g_3(v) = zc$. Extend by zero on the complement, then $\tau_3(x) = (1 + g_3)(x) = (p + zc) + v$, $\tau_3(N) = N$ and $\tau_3 \in E(P, L; \mathfrak{O})$.

Finally, note that $p \equiv x \,(\mathrm{mod}\,\mathfrak{O})$ implies that $p \in P \cap N$, so we can use our assumption that a suitable element θ_2 of $G_0(N)$ moves $\epsilon_*(p)$ to $\epsilon_*(p_0)$ and preserves the condition $p \equiv p_0 \,(\mathrm{mod}\,\mathfrak{O})$. Now let $f_4 : P_0 \to L$ be defined by $f_4(p_0) = v$ and apply $\tau_4 = (1 - f_4) \in E(P_0, L)$ to x. The result is that $\epsilon_*(x) = \epsilon_*(p_0)$ and $x \equiv p_0 \,(\mathrm{mod}\,\mathfrak{O})$. Since $v \equiv 0 \,(\mathrm{mod}\,\mathfrak{O})$, $v \in N$ and so $\tau_4(N) = N$. ∎

Corollary 1.11: *Suppose that $M = P \oplus L$ is an A-lattice, where $P = p_0A \oplus p_1A$, and L has (A,B)-free rank ≥ 1 at all but finitely many primes. Let $N \subseteq M$ be a submodule of finite index containing p_0A as a direct summand, and $\mathfrak{O} = \mathrm{Ann}(M/N)$. Suppose that there exists a subgroup $G_0 \subseteq GL(P)$ which is (N, p_0, ϵ)-transitive.*

Then the subgroup $G(N)$ stabilizing N of

$$G = \langle G_0, E(p_0A, L \oplus p_1A), E(p_1A, L \oplus p_0A)\rangle \subseteq GL(M)$$

acts transitively on the set of M–unimodular elements in N.

Proof: We apply Lemma 1.10 and then Theorem 1.4 to complete the proof. Since $\sigma \equiv 1_M \,(\mathrm{mod}\,\mathfrak{O})$ for every $\sigma \in G_1(\mathfrak{O})$, it follows that $G_1(\mathfrak{O})$ leaves N invariant. ∎

We will find it convenient to refer to the special case when $\mathfrak{O} = A$ and $N = M$.

Corollary 1.12: *Suppose that $M = P \oplus L$ is an A-lattice, where $P = p_0A \oplus p_1A$, and L has (A,B)-free rank ≥ 1 at all but finitely many primes. Let $G_0 \subseteq GL(P)$ be a subgroup such that $\epsilon_*(G_0)$ acts transitively on the images in $\epsilon_*(P)$ of the P–unimodular elements. Then the group*

$$G = \langle G_0, E(p_0A, L \oplus p_1A), E(p_1A, L \oplus p_0A)\rangle \subseteq GL(M)$$

acts transitively on the unimodular elements in M.

Remark 1.13: In some cases there may be no subgroup G_0 with the required property. For example, if $B = \mathbf{Z}\pi$ is the integral group ring of a finite group π, then $GL_2(B)$ acts transitively on unimodular elements in $B \oplus B$ if and only if the relation $\mathfrak{I} \oplus B \cong B \oplus B$ for a projective ideal $\mathfrak{I}$ implies $\mathfrak{I} \cong B$. In [**11**, Thm. 3] Swan shows that this is not true for a certain ideal in $\mathbf{Z}\pi$ where π is the generalized quaternion group of order 32. Jacobinski proved in [**6**] that cancellation in this sense holds for $\mathbf{Z}\pi$ unless π has a quotient which is binary polyhedral (in particular, those satisfying the "Eichler condition"). The converse was studied in [**13**]: Swan proved that cancellation fails for $\mathbf{Z}\pi$ if π has a binary polyhedral quotient which is not one of 7 exceptional groups.

Proof of Theorem A: By Swan's Cancellation Theorem ([**12**, 9.7] and the discussion on [**12**, p.169]), $M \oplus A \cong M' \oplus A$ since $M \oplus A$ is the direct sum of two faithful modules. We apply (1.12) following [**1**, IV,3.5] to cancel the free modules.

Remark: The method does not seem to prove either Swan's or Jacobinski's cancellation theorems independently. When the hypotheses of Corollary 1.11 apply, the same method gives other cancellation results.

For geometric applications, we will be particularly interested in the case when $B = \mathbf{Z}$. We have remarked in (1.13) that $GL_2(A)$ acts transitively on unimodular elements for $A = \mathbf{Z}$ and certain group rings of finite groups (in particular those which satisfy the Eichler condition [**12**]). For applications to finite two-complexes, the transitivity of $SL_2(A)$ is more useful.

Theorem 1.14: ([**8**, 10.6]) *Suppose that A satisfies the Eichler condition, and let B be the image of A in the product of all the commutative factors of $A \otimes_R F$. Then $SL_2(A)$ acts transitively on unimodular elements in $A \oplus A$ provided that $SK_1(B) = 0$.*

Lemma 1.15: *Let $\mathfrak{O}$ be an ideal in A, $P = p_0A \oplus p_1A$, and let $N = p_0A \oplus p_1\mathfrak{O}$. Suppose that $SL_2(A)$ acts transitively on the unimodular elements of P, and that $(A/(\mathrm{mod}\,\mathfrak{O}))^\times \to K_1(A/(\mathrm{mod}\,\mathfrak{O}))$ is injective. Then $SL_2(A;\mathfrak{O})$ stabilizes N and acts transitively on the P–unimodular elements $x \in N$ such that $x \equiv p_0\,(\mathrm{mod}\,\mathfrak{O})$.*

Proof: If $x = p_0a + p_1b$ is a P–unimodular element with $b \in \mathfrak{O}$ and $a \equiv 1\,(\mathrm{mod}\,\mathfrak{O})$, then a matrix in $SL_2(A)$ which moves p_0 to x must have the form $\sigma = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where $d \equiv 1\,(\bmod\ \mathfrak{O})$ as well. Then $\sigma' = \sigma\cdot\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$ is contained in $SL_2(A;\mathfrak{O})$ and stabilizes N. ∎

Lemma 1.16:
(i) If $\epsilon : A \to \mathbf{Z}$ is a surjection of orders, then $E(P)$ maps onto $SL_2(\mathbf{Z})$.
(ii) Let $\mathfrak{O}$ be an ideal in A, $P = p_0A \oplus p_1A$, and let $N = p_0A \oplus p_1\mathfrak{O}$. Assume that one of the following conditions holds (a) $\mathfrak{O}$ contains $\ker(A \to \mathbf{Z})$, or (b) $q \in \mathfrak{O}$ where $(q) = \epsilon_*(\mathfrak{O})$, or (c) $(\mathfrak{O}, q)$ is a principal ideal, or (d) $\mathfrak{O} \cap \mathbf{Z}\cdot 1 = (q')$ for some integer q' with the same prime divisors as q. Then the subgroup of $E(P)$ which fixes $p_0\,(\mathrm{mod}\,\mathfrak{O})$ acts transitively on the images in $\epsilon_*(P)$ of the P–unimodular elements $x \in N$ such that $x \equiv p_0\,(\mathrm{mod}\,\mathfrak{O})$.

Proof: Part (i) follows from the fact that $E_2(\mathbf{Z}) = E(p_0\mathbf{Z}, p_1\mathbf{Z}) = SL_2(\mathbf{Z})$. For part (ii) we first observe that the transitivity claimed can be carried out in $E_2(\mathbf{Z})$, and then use one of our assumptions to lift the matrices to $E(P)$. This last step is straightforward except under assumption (d). In that case, we choose an element $u \in \mathfrak{O}$ such that $\epsilon_*(u) = q$, and then act on $x = p_0a + p_1b$ by a matrix of the form $\begin{pmatrix} 1 & 0 \\ ru & 1 \end{pmatrix}$ in order to obtain the relation $\epsilon_*(b) \equiv 0\,(\mathrm{mod}\,q')$. ∎

Example 1.17: *Let $A = \mathbf{Z}\pi$, where π is the direct product of two cyclic groups of order two, generated by S, T. Let $\mathfrak{O} = \langle S-1, 2(T-1), ST-1, 1+S+T+ST\rangle$. Then $\epsilon_*(\mathfrak{O}) = (4)$, but $4 \notin \mathfrak{O}$, and $(\mathfrak{O}, 4)$ is not principal. However, $\mathfrak{O} \cap \mathbf{Z}\cdot 1 = (8)$.*

We conclude this section by giving a useful generalization of the Roiter Replacement Theorem [**9**]. An A-lattice L will be called (A,B)-faithful if $B^s \oplus L$ is a faithful A-module for some integer s. If Γ is a hereditary order containing A, then $\Gamma = \Gamma(B) \times \Gamma(C)$, where

$\Gamma(B)$ is a hereditary order containing B. The Γ-module generated by an A-module L is denoted ΓL.

Theorem 1.18: *Let L be an (A,B)-faithful lattice over an order A, with respect to $\epsilon : A \to B$. Suppose that Γ is a hereditary order containing A and $n\Gamma \subseteq A$ for some integer n. In addition, we assume that the map of units $(A/n\Gamma)^\times \to (\Gamma(B)/n\Gamma(B))^\times$ is surjective. Then for any locally-free projective A-module P of rank r with $\Gamma(B)L$ free, there exists an A-module L' in the same genus as L such that $L \oplus P \cong L' \oplus A^r$.*

Proof: We consider the pull-back square

$$\begin{array}{ccc} A & \to & \Gamma \\ \downarrow & & \downarrow \\ A/n\Gamma & \to & \Gamma/n\Gamma \end{array}$$

Let $\bar{A} = A/n\Gamma$ and $\bar{\Gamma} = \Gamma/n\Gamma$ with a similar convention for modules (e.g. $\bar{L} = L \otimes_A \bar{A}$). Since L is (A,B)-faithful, by Roiter's Theorem there exists a $\Gamma(C)$-module U in the same genus as $\Gamma(C)L$ such that $U \oplus \Gamma(C)^r \cong \Gamma(C)(L \oplus P)$. Note that U is projective of rank ≥ 1 over $\Gamma(C)$. We add $\Gamma(B)(L \oplus P)$ to both sides and use the assumption $\Gamma(B)P \cong \Gamma(B)^r$, to express our original module $L \oplus P$ as a pull-back

$$(\alpha : (\bar{L} \oplus \bar{A}^r) \otimes_A \bar{\Gamma} \to (\Gamma(B)L \oplus U \oplus \Gamma^r) \otimes_\Gamma \bar{\Gamma}).$$

The isomorphism α can be varied by self-automorphisms of the right-hand side which lift over $\bar{A}$ or Γ.

We remark that for rank ≥ 2 the action of elementary matrices over $\bar{\Gamma}$ is transitive on unimodular elements. Using this variation over the $\Gamma(C)$ component of α, we can assume that α induces the identity on the $\bar{\Gamma}(C)^r$ summand. Over the $\bar{\Gamma}(B)^r$ factor, we use the assumption on $(\bar{A})^\times$ to achieve the same result. If we denote the new patching isomorphism by α', we have the block form

$$\alpha' = \begin{pmatrix} \beta & 0 \\ \tau & id \end{pmatrix}.$$

The pullback

$$(\beta : \bar{L} \otimes_A \bar{\Gamma} \to (\Gamma(B)L \oplus U) \otimes_\Gamma \bar{\Gamma})$$

is our desired module L', and it follows that $L \oplus P \cong L' \oplus A^r$. Since P is locally free, and cancellation holds locally, we see that L' is in the same genus as L. ∎

Corollary 1.19: *Let $A = \mathbf{Z}\pi$, π a finite group and L be any $(A, \mathbf{Z})$-faithful module. Then for any projective A-module P of rank r, there exists a module L' in the same genus as L such that $L \oplus P \cong L' \oplus A^r$.*

§2: Applications to Two-Complexes

The cancellation problem for 2-complexes has been extensively investigated [**3**], [**5**], [**7**], [**10**]. In particular it is known that even for finite abelian fundamental groups, there are examples of 2-complexes which are stably simply equivalent but not homotopy equivalent [**7**]. On the other hand, for a fixed finite fundamental group and Euler characteristic, $K \vee S^2$ is homotopy equivalent to $K' \vee S^2$ [**3**].

The Proof of Theorem B: Let $h : K \vee rS^2 \to K' \vee rS^2$ be a simple homotopy equivalence as above, inducing a given isomorphism α on the fundamental groups. Let $A = \mathbf{Z}[\pi_1(K)]$, $L = \pi_2(K_0)$, and note that this module has $(A, \mathbf{Z})$-free rank ≥ 1. We may assume that $r = 1$ and set $P = \pi_2(S^2 \vee S^2) \subseteq \pi_2(K_0 \vee S^2 \vee S^2)$.

By Corollary 1.12 and Lemma 1.16 the group $G = \langle E(P_0, L \oplus P_1), E(P_1, L \oplus P_0)\rangle$ acts transitively on unimodular elements in $L \oplus P$.

To realize elements in G by simple self homotopy equivalences of $K_0 \vee 2S^2 = K \vee S^2$, inducing the identity on π_1, it is enough to do this for $E(P_1, L \oplus P_0)$. This group is generated by automorphisms of the form $1 + f$ and $1 + g$, where $f\colon L \oplus P_0 \to P_1$ and $g\colon P_1 \to L \oplus P_0$ are arbitrary A-homomorphisms. Recall that $P_1 = p_1 A$ and $L \oplus P_0 = \pi_2(K)$. Consider the map $Id \vee u\colon K \vee S^2 \to K \vee S^2$, where $u = (g(p_1), p_1) \in \pi_2(K \vee S^2) = \pi_2(K) \oplus p_1 A$. It realizes $1 + g$ and its restriction to K is the identity and it also induces the identity on $(K \vee S^2)/K = S^2$. Thus the additivity formula for the Whitehead torsion implies that the torsion of $Id \vee u$ vanishes.

To realize $1 + f$ we note that $f\colon L \oplus P_1 = \pi_2(K) = H_2(K; A) \to P_1 = A$ factors through $H_2(K, K^1; A)$, with K^1 the 1-skeleton. The reason for this is that we have an exact sequence

$$\mathrm{Hom}_A(H_2(K, K^1; A), A) \to \mathrm{Hom}_A(H_2(K; A), A) \to \mathrm{Ext}^1_A(H_1(K^1; A), A)$$

and the last group vanishes since $H_1(K^1; A)$ is $\mathbf{Z}$-torsion free. Choose a factorization map $\bar{f}\colon H_2(K, K^1; A) \to A$, where $H_2(K, K^1; A)$ is a free A-module generated by the 2-cells of K (appropriately connected to the base point). Denote this basis by $e_1, .., e_k$. Now write $K = K^1 \cup D^2 \cup ... \cup D^2$. Pinch off the 2-cells to obtain $K \vee rS^2$ and denote the projection map by $p\colon K \to K \vee kS^2$. Consider the composition map $\beta\colon K \to K \vee kS^2 \to K \vee S^2$, where the second map is $Id \vee \bar{f}(e_1) \vee ... \vee \bar{f}(e_k)$. By construction the induced map in π_2 is $1 \oplus f$ and the composition $K \to K \vee S^2 \to K$ is homotopic to Id. Finally consider $\beta \vee Id\colon K \vee S^2 \to K \vee S^2$ realizing $1 + f$. Its restriction to S^2 and the induced map on K are homotopic to the identity implying from the additivity of the Whitehead torsion that $\beta \vee Id$ has trivial torsion.

We complete the cancellation by composing h with a simple self-equivalence to obtain $h' : K \vee S^2 \to K' \vee S^2$ which fixes the S^2 factor. Now the composition of h' with the inclusion and projection gives a homotopy equivalence $f : K \to K'$ which again by the additivity formula for the Whitehead torsion is simple. ∎

Although the result of Theorem B can not be improved in general for 2-complexes with finite fundamental group, there are improvements possible for special fundamental groups. For example, there is just one homotopy type for each Euler characteristic when π_1 is finite abelian of rank less than 3 [7], [10].

We wish to describe another approach to such results. Recall that the finite subgroups G of $SO(3)$ are cyclic, dihedral, A_4, S_4 and A_5. For each of these, $\mathbf{Z}G$ satisfies the Eichler condition so Browning's results measure the number of distinct two-complexes with fundamental group G (see [3, 5.4]). As an application of the method we show:

Theorem 2.1: *Let π be a finite subgroup of $SO(3)$. If K and K' are finite 2-complexes with fundamental group π and the same Euler characteristic, and let $\alpha : \pi_1(K, x_0) \to \pi_1(K', x_0')$ be a given isomorphism, then there is a simple homotopy equivalence $f : K \to K'$ inducing α on the fundamental groups.*

The method of proof is based the following more general construction. A *based* two-complex (K, γ) is a finite 2-complex K and a *surjection* $\gamma : \pi_2(K)^* \to T$ from the dual of $\pi_2(K)$ to a finite A-module T. Two such pairs (K, γ) and (K', γ') are *stably simply equivalent* if there exists a simple homotopy equivalence $h : K \vee rS^2 \to K' \vee rS^2$, inducing the identity on π_1, and isomorphisms

$$\varphi = h^* : \pi_2(K')^* \oplus A^r \to \pi_2(K)^* \oplus A^r$$

and $u : T' \to T$ such that $\gamma \circ p_1 \circ \varphi = u \circ \gamma' \circ p_1$, where p_1 is the projection on the first summand.

Lemma 2.2: *Let (K, γ) be a based finite two-complex with $\pi_1(K) = \pi$. If K' is a two-complex which is stably simply equivalent to K, then there exists a surjection γ' to T' such that the based pairs (K, γ) and (K', γ') are stably simply equivalent.*

Proof: We choose a stable equivalence $h : K \vee rS^2 \to K' \vee rS^2$, and let $h^* = \varphi : \pi_2(K')^* \oplus A^r \to \pi_2(K)^* \oplus A^r$. We can take $T' \cong T$, so choose an isomorphism $u : T' \to T$, and denote by $e(T)$ the exponent of T as a finite abelian group.

First we observe that there exists an element $\sigma \in E(\pi_2(K)^*, A^r)$ such that $\sigma(\varphi(0 \oplus A^r)) \equiv 0(\oplus A^r) (\mathrm{mod}\ e(T))$. This follows by induction on r from Lemma 1.3. Since any such σ is realized by a simple self-equivalence of $K \vee rS^2$, we may assume that φ itself preserves the summand $(0 \oplus A^r)$ modulo $e(T)$.

Next, we define $\gamma' : \pi_2(K')^* \to T'$ to be the composite

$$\gamma' = u^{-1} \circ \gamma \circ p_1 \circ \varphi \circ i_1,$$

where $i_1 : \pi_2(K')^* \hookrightarrow \pi_2(K')^* \oplus 0$ is the inclusion onto the first summand. It follows that $\gamma \circ p_1 \circ \varphi = u \circ \gamma' \circ p_1$, and hence that (K, γ) and (K', γ') are stably simply equivalent. ∎

We now assume until further notice that π does not have periodic cohomology. This excludes cyclic groups of any order or dihedral groups of order not divisible by four. It follows that $\pi_2(K)$ is not rationally isomorphic to $\mathbf{Q}\mathfrak{I}$, where $\mathfrak{I} = \mathfrak{I}(\pi)$ denotes the augmentation ideal of $A = \mathbf{Z}\pi$. This is the case for example whenever the minimal Euler characteristic is not 1. From (0.1), there is an isomorphism $\pi_2(K) \otimes \mathbf{Q} \cong \mathbf{Q}(\mathfrak{I} \oplus A^{r+1})$.

Let L be the image of $\pi_2(K)$ under the projection to $\mathbf{Q}(\mathfrak{I} \oplus A^r)$. Then we have a short exact sequence

$$(2.3) \qquad 0 \to p_1\mathfrak{A} \to \pi_2(K) \to L \to 0,$$

where $\mathfrak{A}$ is a right ideal of finite index in $P_1 = p_1 A$. Then by push-out, $\pi_2(K) \hookrightarrow p_1 A \oplus L$ is an inclusion respecting the inclusion $p_1\mathfrak{A} \subseteq p_1 A$ and the identity on L. This construction produces a based pair (K, γ) if we take γ to be the induced projection onto the dual module $T = \mathrm{Hom}_{\mathbf{Z}}(A/\mathfrak{A}, \mathbf{Q}/\mathbf{Z})$. In the next three Lemmas we construct based pairs of this type explicitly for each of our fundamental groups π.

Lemma 2.4: *Let π be a non-periodic finite subgroup of $SO(3)$. Then there exists a representative $\mathfrak{N}$ of $\Omega^3\mathbf{Z}$ with minimal $\mathbf{Z}$-rank, and a short exact sequence*

$$0 \to \langle \mathfrak{I}(\pi), 2\rangle^* \to \mathfrak{N}^* \to \mathfrak{I}(\pi) \to 0$$

which is non-split when restricted to each cyclic subgroup of order two in π. This extension is classified by an element $\theta_{\mathfrak{N}} \in \mathrm{Ext}^1_A(\mathfrak{I}(\pi), \langle \mathfrak{I}(\pi), 2\rangle^*) \cong H^2(\pi, \mathbf{Z}/2)$.

Proof: If $\tilde{\pi} \subset SU(2)$ denotes the double cover of π, there is an exact sequence

$$0 \to \mathfrak{I}^*(\tilde{\pi}) \to \tilde{C}_2 \to \tilde{C}_1 \to \tilde{C}_0 \to \mathbf{Z} \to 0$$

where the $\tilde{C}_i$ are free $\mathbf{Z}[\tilde{\pi}]$ modules. Let $<z> = \mathbf{Z}/2$ be the kernel of the epimorphism $\tilde{\pi} \to \pi$, and tensor the above exact sequence over $\mathbf{Z} <z>$ with $\mathbf{Z}$. This produces a complex over $A = \mathbf{Z}[\pi]$

$$0 \to \mathfrak{I}^*(\pi) \to C_2 \to C_1 \to C_0 \to \mathbf{Z} \to 0$$

which is exact except at C_1, where the homology is $\mathbf{Z}/2$. We further resolve by adding A to C_2, with $1 \in A$ mapped to a lift to C_1 of the generator of the homology group $\mathbf{Z}/2$. The ideal $\langle \mathfrak{I}(\pi), 2\rangle$ fits into the exact sequence

$$0 \to \langle \mathfrak{I}(\pi), 2\rangle \to A \to \mathbf{Z}/2 \to 0.$$

Now the kernel is $\mathfrak{N} = \Omega^3\mathbf{Z}$, sitting in an exact sequence

$$(2.5) \qquad 0 \to \mathfrak{I}^*(\pi) \to \mathfrak{N} \to \langle \mathfrak{I}(\pi), 2\rangle \to 0.$$

This sequence splits over $\mathbf{Z}$ and dualizing gives

$$(2.6) \qquad 0 \to \langle \mathfrak{I}(\pi), 2\rangle^* \to \mathfrak{N}^* \to \mathfrak{I}(\pi) \to 0,$$

which as an extension, is classified by an element of $\mathrm{Ext}^1_A(\mathfrak{I}, \langle I, 2\rangle^*) \cong \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2)$. Moreover, this extension group is isomorphic to $H^2(\pi, \mathbf{Z}/2)$. Since the augmentation ideal for π restricts to the augmentation ideal plus a free module over any subgroup, it follows that (2.6) is non-split when restricted to every subgroup of order two in

π. We remark that its extension class $\theta_{\mathfrak{N}} \in H^2(\pi, \mathbf{Z}/2)$ is uniquely determined by this condition, since the 2-Sylow subgroup of π is $\mathbf{Z}/2 \times \mathbf{Z}/2$ or dihedral $D(2^{k+1})$, for $k \geq 2$. ∎

Lemma 2.7: *Let π be a non-periodic finite subgroup of $SO(3)$. If $\pi \neq S_4, A_4, A_5$ let $\mathfrak{K}(\pi) = \langle \mathfrak{I}(\pi)^2, 2\mathfrak{I}(\pi) \rangle$. If $\pi = S_4, A_4$ or A_5 let $\mathfrak{K}(\pi) = \langle \mathfrak{I}(\mathbf{Z}/2 \times \mathbf{Z}/2), 2\mathfrak{I}(\pi) \rangle$, where $\langle \mathfrak{I}(\mathbf{Z}/2 \times \mathbf{Z}/2)$ denote the right ideal in $\mathbf{Z}\pi$ generated by the augmentation ideal of $\mathbf{Z}\rho$, $\rho = Syl_2(A_4)$. Then the extension class $\theta_{\mathfrak{N}}$ for (2.6) is in the image $\mathrm{Ext}^1_A(\mathfrak{I}(\pi)/\mathfrak{K}(\pi), \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}(\pi), \mathbf{Z}/2)$. If $\pi \neq S_4, A_4, A_5$ then the quotient module $\mathfrak{I}(\pi)/\mathfrak{K}(\pi) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, with trivial π-action.*

Proof: First consider the case when $\mathfrak{K}(\pi) = \langle \mathfrak{I}(\pi)^2, 2\mathfrak{I}(\pi) \rangle$. The augmentation ideal $\mathfrak{I}(\pi) = \{g-1 \mid g \in \pi\}$ as a free abelian group. However, the quotient $\mathfrak{I}/\mathfrak{K}$ as an A-module is generated by any set $\{g-1\}$ of generators for π. In addition, if g has odd order m, then $2^i(g-1) \in \langle I^4, 2^{i+1}I \rangle$ for $i \geq 0$. To check the latter claim, pick r such that $4r \equiv 1 \,(\mathrm{mod}\, m)$ and compute $(g^r-1)^4 \equiv (g-1) (\mathrm{mod}\, 2I(g))$, where $\mathfrak{I}(g) = \langle g^i - 1 : 1 \leq i \leq m-1 \rangle$. From these two observations, we see that for π dihedral, $\mathbf{Z}/2 \times \mathbf{Z}/2$ or S_4 the quotient module $\mathfrak{I}(\pi)/\mathfrak{K}(\pi) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, with trivial π-action.

We now check the statement about $\theta_{\mathfrak{N}}$ by computing the sequence:

$$\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2) \tag{2.8}$$

First note that

$$\mathrm{Hom}_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Hom}_A(\mathfrak{I}, \mathbf{Z}/2)$$

is an isomorphism, and so the first map in (2.8) is an injection. Next to compute the group $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2)$ we can work modulo $\mathfrak{K}_0 = \langle \mathfrak{I}^3, 2\mathfrak{I}^2 \rangle$. As an A-module, the quotient $\mathfrak{K}/\mathfrak{K}_0 = (\mathbf{Z}/2)^3$ with trivial π-action. Therefore $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^3$.

Finally, we compute $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2)$. Since $\mathfrak{I}/\mathfrak{K} \cong (\mathbf{Z}/2)^2$ with trivial π-action, we just need to compute $\mathrm{Ext}^1_A(\mathbf{Z}/2, \mathbf{Z}/2)$ via the exact sequence

$$0 \to \mathrm{Hom}_A(\mathbf{Z}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathbf{Z}/2, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathbf{Z}, \mathbf{Z}/2) \to 0.$$

But

$$\mathrm{Ext}^1_A(\mathbf{Z}, \mathbf{Z}/2) \cong H^1(\pi, \mathbf{Z}/2) = (\mathbf{Z}/2)^2$$

and so we get the answer $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^6$. These values can now be substituted into (2.8) to show that $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2)$ is onto.

Next we consider the case where $\pi = S_4$ and $\mathfrak{K}(\pi) = \langle \mathfrak{I}(\mathbf{Z}/2 \times \mathbf{Z}/2), 2\mathfrak{I}(\pi) \rangle$. Here $H^2(\pi, \mathbf{Z}/2) = (\mathbf{Z}/2)^2$ and $\mathfrak{I}/\mathfrak{K} = M_2(\mathsf{F}_2) \oplus \mathsf{F}_2$. Then $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^2$ injects into $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^4$ and so $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2)$ is onto.

Finally we have the cases $\pi = A_4$ or A_5 and $\mathfrak{K}(\pi) = \langle \mathfrak{I}(\mathbf{Z}/2 \times \mathbf{Z}/2), 2\mathfrak{I}(\pi) \rangle$. Here $H^2(\pi, \mathbf{Z}/2) = \mathbf{Z}/2$, generated by our extension class $\theta_{\mathfrak{N}}$. For $\pi = A_4$, let $\omega \in \pi$ be a 3-cycle. Then $\mathfrak{I}/\mathfrak{K} = \langle \omega - 1, \omega^2 - 1 \rangle$ and this module is isomorphic to the quotient module $\mathsf{F}_2(\zeta_3)$ arising from the epimorphism $\pi \to \mathbf{Z}/3$. Then $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2) \cong \mathbf{Z}/2$ injects into $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^2$ and so $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2)$ is onto.

For $\pi = A_5$, let τ_1, τ_2 be non-conjugate 5-cycles. Then $\mathfrak{I}/\mathfrak{K} = \langle (\tau_1^i - 1), (\tau_2^i - 1) : 1 \leq i \leq 4 \rangle$. This module is isomorphic to $M_2(\mathsf{F}_4)$ where $\pi \cong SL_2(\mathsf{F}_4)$ acts through its standard representation. Again, $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2) \cong \mathbf{Z}/2$ injects into $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^2$ and so $\mathrm{Ext}^1_A(\mathfrak{I}/\mathfrak{K}, \mathbf{Z}/2) \to \mathrm{Ext}^1_A(\mathfrak{I}, \mathbf{Z}/2)$ is onto. ∎

Proposition 2.9: *Let π be a non–periodic finite subgroup of $SO(3)$ and suppose that $\mathfrak{N} = \Omega^3\mathbf{Z}$. Let $\epsilon : A = \mathbf{Z}\pi \to \mathbf{Z}$ be the augmentation map. Then there exists a module M with free A-rank 2 containing $N = \mathfrak{N} \oplus p_0 A$ as a submodule of finite index such that (i) for some subgroup $G_0 \subseteq GL(P)$, $G_0(N)$ acts transitively on the images in $N/N \cap M\mathfrak{D}$ of the elements $p_0 a$, for any $a \in A$ representing a unit in $A/\mathfrak{D}$, and*
(ii) the subgroup of $G_0(N)$ which fixes $p_0 \,(\mathrm{mod}\, \mathfrak{D})$ acts transitively on the images in $\epsilon_*(P)$ of the P–unimodular elements $x \in P \cap N$ such that $x \equiv p_0 \,(\mathrm{mod}\, \mathfrak{D})$.

Proof: From Lemma 2.7 we get an element $\hat{\theta}_{\mathfrak{N}} \in \mathrm{Ext}^1_A(\mathfrak{I}(\pi)/\mathfrak{K}(\pi), \mathbf{Z}/2)$ with image $\theta_{\mathfrak{N}} \in \mathrm{Ext}^1_A(\mathfrak{I}(\pi), \mathbf{Z}/2)$. Since $\mathrm{Hom}_A(\mathfrak{K}, \mathbf{Z}/2)$ injects into $\mathrm{Hom}_{\mathbf{Z}}(\mathfrak{K}, \mathbf{Z}/2)$ we can assume that $\hat{\theta}_{\mathfrak{N}}$ gives a short exact sequence

$$0 \to \mathbf{Z}/2 \to T \to \mathfrak{I}(\pi)/\mathfrak{K}(\pi) \to 0$$

with T of exponent two.

We identify the dual $\hat{T} \cong T$ and use (2.6) to deduce a short exact sequence

$$0 \to \mathfrak{N} \to A \oplus \bar{\mathfrak{K}} \to T \to 0, \tag{2.10}$$

where $\bar{\mathfrak{K}}$ denotes the dual left module $\mathfrak{K}^*$ made into a right A-module in the usual way. If $\pi \neq S_4, A_4, A_5$, then the induced map $A \to T$ has image $T_1 \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, with *non-trivial* π-action. When $\pi = S_4$, then $T_1 \cong M_2(\mathsf{F}_2)$ and when $\pi = A_4$ or A_5, the image $T_1 \cong I(\pi)/\mathfrak{K}(\pi)$. Let $\mathfrak{A} \subset A$ be the kernel of the projection to T_1 in the above sequence. It follows that $\mathfrak{N}$ is described by (2.3): it contains $\mathfrak{A}$ as a $\mathbf{Z}$ direct summand, and has a cokernel we denote by L. We remark that since T has exponent two, $\mathfrak{A} \subseteq \langle \mathfrak{I}, 2 \rangle$.

Define $M = p_0 A \oplus p_1 A \oplus L$, and $N = p_0 A \oplus \mathfrak{N}$. Since $\mathfrak{A}$ is a right ideal in A, we can identify $p_1 A \oplus L$ with the pushout of the sequence $0 \to p_1\mathfrak{A} \to \mathfrak{N} \to L \to 0$ using the inclusion $p_1\mathfrak{A} \subset p_1 A$. As usual $P = p_0 A \oplus p_1 A$ and $\mathfrak{D} = \mathrm{Ann}(M/N)$. When $\pi \neq S_4, A_4, A_5$, we have checked that $A/\mathfrak{A}$ is $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, where π acts through an epimorphism $\pi \to \mathbf{Z}/2$. It follows that $\mathfrak{D} = \langle \mathfrak{I}(\pi_0), 2 \rangle$, where π_0 is the kernel of the epimorphism $p : \pi \to \mathbf{Z}/2$. The exceptional cases, $\pi = S_4, A_4, A_5$ lead to $\mathfrak{D} = \ker(A \to M_2(\mathsf{F}_2))$, $\mathfrak{D} = \ker(A \to \mathsf{F}_2(\zeta_3))$, or $\mathfrak{D} = \ker(A \to M_2(\mathsf{F}_4))$ respectively.

The assertions of Proposition 2.9 are now easy to verify. In fact, part (i) is trivial since the elements of π together with ± 1 suffice to lift the units. Part (ii) follows from (1.16) once we notice that $\mathfrak{A} = \mathfrak{D}$ so that the algebraic automorphisms given there do stabilize N. Indeed, they have the effect $p_1 \mapsto p_0 c + p_1 d$ with $d \equiv 1 \,(\mathrm{mod}\, \mathfrak{D})$. Since $\mathfrak{N} \subseteq p_1 A \oplus L$ can be expressed as a pull-back $\mathfrak{N} = \{(a, v) \,|\, a \,(\mathrm{mod}\, \mathfrak{D}) \equiv v \,(\mathrm{mod}\, L_0)\}$, for some $L_0 \subseteq L$, the elements $(ad, v) \in \mathfrak{N}$ whenever $(a, v) \in \mathfrak{N}$ and $d \equiv 1 \,(\mathrm{mod}\, \mathfrak{D})$. Hence the automorphisms extend by the identity on L and preserve $\mathfrak{N}$. ∎

Lemma 2.11: *Let K be a finite two-complex with $\pi = \pi_1(K, x_0)$ finite. Suppose that $f : K \to K$ is a homotopy equivalence such that the induced map $f_* : \pi_2(K) \otimes \mathbf{Q} \to$*

$\pi_2(K) \otimes \mathbf{Q}$ has trivial reduced norm at every simple factor of $\mathbf{Q}\pi$. If $SK_1(\mathbf{Z}\pi) = 0$ and f induces the identity on $\pi_1(K, x_0)$, then f is a simple homotopy equivalence.

Proof: We consider the chain homotopy equivalence induced by f on the chain complex of K tensored over the rationals, and compute its Reidemeister torsion. Our assumption implies that the induced map $f_* : \pi_2(K) \to \pi_2(K)$ has trivial determinant in $Im(Wh(\mathbf{Z}\pi) \to Wh(\mathbf{Q}\pi)) = Wh(\mathbf{Z}\pi)/SK_1(\mathbf{Z}\pi)$, hence the Whitehead torsion of f vanishes. ∎

The Proof of Theorem 2.1: For any finite subgroup π of $SO(3)$, it is known that $SK_1(\mathbf{Z}\pi) = 0$ (see [8, 14.1, 14.5]). Let K be a finite 2-complex and let $\mathfrak{N} = \pi_2(K)$. We may assume, by Theorem A, that K has minimal Euler characteristic. Suppose first that $\pi_1(K)$ is periodic, i.e. cyclic or dihedral (of order $2m$, m odd). In this case, $\mathfrak{N} = \mathfrak{I}(\pi)^*$, a two-sided fractional ideal in $\mathbf{Q}A$. By scaling, we can embed $\mathfrak{N} \subset \mathfrak{I}(\pi)$ as a two-sided ideal in A. Then

$$N = p_0 A \oplus \mathfrak{N} \subset M = p_0 A \oplus p_1 A,$$

and by Lemma 2.2 we need to show that a suitable subgroup of $GL_2(A)$, stabilizing N, acts transitively on M-unimodular elements in N. First we apply Theorem 1.14 and then Lemma 1.15 with $A = \mathbf{Z}\pi$ and $\mathfrak{O} = \mathfrak{I}(\pi)$. We conclude that the subgroup of $SL_2(A; \mathfrak{O})$ preserving N acts transitively on M-unimodular elements in N.

The algebraic automorphisms needed for transitivity on unimodular elements preserve the k-invariant of $K \vee S^2$. To see this recall that the k-invariant is an element $k \in H^3(\pi, \mathfrak{N})$. Under the action of $SL_2(A; \mathfrak{O})$, the image of k is dk, where $d \equiv 1 \,(\mathrm{mod}\, \mathfrak{O})$. However, the elements of $\mathfrak{O}$ act as zero on this cohomology group, by dimension-shifting. It now follows that such an algebraic automorphism is induced by a homotopy self-equivalence $f : K \vee S^2 \to K \vee S^2$.

By Lemma 2.11 applied to $K \vee S^2$, f is a simple homotopy equivalence. Therefore we can cancel the final S^2, to get a simple homotopy equivalence between K and K'.

Next, suppose that π is non-periodic. The construction of $N \subset M$ in (2.9) used a surjection $\gamma : \pi_2(K)^* \to T$, giving us a based pair (K, γ). By Lemma 2.2 we need to show that a suitable subgroup of $GL(M)$, stabilizing N, acts transitively on M-unimodular elements in N. This time the necessary transitivity follows from Corollary 1.11, and we conclude that $\pi_2(K) \cong \pi_2(K')$. It is not difficult to check that the self-automorphisms used in the proof do not change the k-invariant (see Lemma 1.10 where they are given explicitly). There is an exact sequence

$$\ldots \to H^3(\pi, \mathfrak{I}^*) \to H^3(\pi, \mathfrak{N}) \to H^3(\pi, \langle \mathfrak{I}, 2 \rangle)$$

The third term is isomorphic to $H^2(\pi, \mathbf{Z}/2)$, and therefore the action of $\mathfrak{O}$ is zero. Now twice the k-invariant is in the image of $H^3(\pi, \mathfrak{I}^*)$ by construction of $\Omega^3\mathbf{Z}$. Under our embedding $\mathfrak{N} \subset A \oplus \mathfrak{K}^*$, the submodule $\mathfrak{I}^*$ is mapped into $0 \oplus \mathfrak{K}^*$. It follows as above that the self-automorphisms are realized by simple homotopy equivalences. ∎

References

[1] Bass, H., "Algebraic K-Theory", W.A. Benjamin, New York, 1968.

[2] ________, *Unitary algebraic K-theory*, in "Algebraic K-Theory III: Hermitian K-theory and Geometric Applications", Lecture Notes in Math. No. 343, Springer-Verlag, Berlin-Heidelberg-New York, 1973, 57-265.

[3] Browning, W. J., "Homotopy types of certain finite CW-complexes with finite fundamental group", Ph.D. thesis, Cornell University, 1979.

[4] Curtis, C. W. and Reiner, I., "Representation Theory of Finite Groups and Associative Algebras", John Wiley & Sons, New York, 1962.

[5] Dyer, M. L. and Sieradski, A., *Distinguishing arithmetic for certain stably isomorphic modules*, J. Pure & Appl. Alg. **15** (1979), 199-217.

[6] Jacobinski, H., *Genera and decompositions of lattices over orders*, Acta. Math. **121** (1968), 1-29.

[7] Metzler, W., *Über den Homotopietyp zwei-dimensionaler CW-Komplexe und Elementartransformationen bei Darstellungen von Gruppen durch Erzeugende und Relationen*, J. f. reine u. angew. Math. **285** (1976), 7-23.

[8] Oliver, R., "Whitehead Groups of Finite Groups", Lond. Math. Soc. Lec. Notes 132, Cambridge U. Press, Cambridge, 1988.

[9] Roiter, V., *On integral representations belonging to one genus*, Izv. Akad. Nauk. SSSR (Ser. Mat.) **30** (1966), 1315-1324.

[10] Sieradski, A., *A semi-group of simple homotopy types*, Math. Zeit. **153** (1977), 135-148.

[11] Swan, R.G., *Projective modules over group rings and maximal orders*, Annals of Mathematics **76** (1962), 55-61.

[12] ________, "K-theory of finite groups and orders", Lecture Notes in Math. No. 149, Springer-Verlag, Berlin-Heidelberg-New York, 1970, .

[13] ________, *Projective modules over binary polyhedral groups*, J. reine u. angew. Math. **342** (1983), 66-172.

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