# IDEALS IN NON-ASSOCIATIVE UNIVERSAL ENVELOPING ALGEBRAS OF LIE TRIPLE SYSTEMS

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Abstract. The notion of a non-associative universal enveloping algebra for a Lie triple system arises when Lie triple systems are considered as Bol algebras (more generally, Sabinin algebras). In this paper a new construction for these universal enveloping algebras is given, and their properties are studied.

It is shown that universal enveloping algebras of Lie triple systems have surprisingly few ideals. It is conjectured, and the conjecture is verified on several examples, that the only proper ideal of the universal enveloping algebra of a simple Lie triple system is the augmentation ideal.

## 1. Introduction.

Given a smooth manifold $M$ and a point $e \in M$, a *local multiplication* on $M$ at $e$ is a smooth map $U \times U \to M$ where $U$ is some neighbourhood of $e$ and the point $e$ is a two-sided unit, that is, $xe = ex = x$ for all $x \in U$. If $x$ is sufficiently close to $e$, both left and right multiplications by $x$ are one-to-one. Therefore, there always exists a neighbourhood $V \subset U$ where the operations of left and right division are defined by the identities $a\backslash(ab) = b$ and $(ab)/b = a$ respectively. Two local multiplications at the same point $e$ of a manifold $M$ are considered to be equivalent if they coincide when restricted to some neighbourhood of $(e, e)$ in $M \times M$. Equivalence classes of local multiplications are called *infinitesimal loops*. (Sometimes infinitesimal loops are also called *local loops*.)

The importance of infinitesimal loops lies in the fact that they are closely related to affine connections on manifolds. Namely, any affine connection on $M$ defined in some neighbourhood of $e$ determines a local multiplication at $e$. Conversely, each (not necessarily associative) local multiplication at $e$ defines an affine connection on some neighbourhood of $e$; this gives a one-to-one correspondence between germs of affine connections and infinitesimal loops. The details can be found, for example, in [9].

Local non-associative multiplications on manifolds can rarely be extended to global multiplications and, thus, cannot be studied directly by algebraic means. Nevertheless, any local multiplication gives rise to an algebraic structure on the tangent space at the unit element, consisting of an infinite number of multilinear operations. Such algebraic structures are known as *Sabinin algebras*; for associative multiplications they specialise to Lie algebras. Given a Sabinin algebra that satisfies certain convergence conditions, one can uniquely reconstruct the corresponding analytic infinitesimal loop. Therefore, Sabinin algebras may be considered as the principal algebraic tool in studying local multiplications and local affine connections.

The general theory of Sabinin algebras has so far only been developed over fields of characteristic 0. From now on we shall assume that this is the case: unless stated otherwise, all vector spaces, algebras etc will be assumed to be defined over a field $F$ of characteristic zero.

Many general properties of Sabinin algebras are similar to those of Lie algebras. In particular, any Sabinin algebra $V$ can be realised as the space of primitive elements of some "non-associative Hopf algebra" $U(V)$, called the *universal enveloping algebra* of $V$. The operations in $V$ are naturally recovered from the product in $U(V)$. Just as in the Lie algebra case, the universal enveloping algebras of Sabinin algebras have Poincaré–Birkhoff–Witt bases. If a Sabinin algebra $V$ happens to be a Lie algebra, $U(V)$ is precisely the usual universal enveloping algebra of the Lie algebra $V$.

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The definition of a Sabinin algebra involves an infinite number of multilinear operations that satisfy rather complicated identities; we refer to [7], [10] or [11] for the precise form of these. However, additional conditions imposed on a local multiplication may greatly simplify the structure of the corresponding Sabinin algebra. For example, the associativity condition implies that only one of all the multilinear operations is non-zero; the identities of a Sabinin algebra specialise to the identities defining a Lie algebra, that is, antisymmetry and the Jacobi identity. If a local multiplication satisfies the Moufang law

$$a(b(ac)) = ((ab)a)c \quad \text{and} \quad ((ca)b)a = c(a(ba)), \tag{1}$$

the corresponding Sabinin algebra is a *Malcev algebra*. A vector space with a bilinear skew-symmetric operation (bracket) is called a Malcev algebra if the bracket satisfies

$$[J(a,b,c),a] = J(a,b,[a,c])$$

where $J(a,b,c) = [[a,b],c] + [[b,c],a] + [[c,a],b]$ denotes the jacobian of $a, b$ and $c$.

Imposing the left Bol identity

$$a(b(ac)) = (a(ba))c,$$

on the local multiplication, we obtain the structure of a *left Bol algebra* on the tangent space to the unit. A left Bol algebra is a vector space with one bilinear and one trilinear operation, denoted by $[\,,\,]$ and $[\,,\,,\,]$ respectively. The ternary bracket must satisfy the following relations:

$$[a,a,b] = 0$$
$$[a,b,c] + [b,c,a] + [c,a,b] = 0$$
$$[x,y,[a,b,c]] = [[x,y,a],b,c] + [a,[x,y,b],c] + [a,b,[x,y,c]].$$

The binary bracket is required to be skew-symmetric and should satisfy

$$[a,b,[x,y]] = [[a,b,x],y] + [x,[a,b,y]] + [x,y,[a,b]] + [[a,b],[x,y]].$$

Bol algebras generalise Malcev algebras. Indeed, in any Malcev algebra a ternary bracket can be defined by

$$[a,b,c] = [[a,b],c] - \frac{1}{3}J(a,b,c).$$

With this additional operation a Malcev algebra becomes a Bol algebra.

Another important subclass of Bol algebras are *Lie triple systems*; these are the Bol algebras whose binary bracket is identically equal to zero. Lie triple systems arise as tangent spaces to smooth local *Bruck loops* (also known as *K-loops*). These loops, in addition to the left Bol identity, satisfy the identity

$$(ab)^{-1} = a^{-1}b^{-1}.$$

where $x^{-1}$ is shorthand for $e/x$; see [4]. Lie triple systems play a prominent role in the theory of symmetric spaces since a symmetric space can be given the structure of a local Bruck loop at any point.

Identities satisfied in an infinitesimal loop can be translated into identities satisfied in the universal enveloping algebra of the corresponding Sabinin algebra. In particular, the universal enveloping algebra $U(M)$ of a Malcev algebra $M$ is a non-associative bialgebra that satisfies the linearisations

$$\sum a_{(1)}(y(a_{(2)}z)) = \sum ((a_{(1)}y)a_{(2)})z$$

and

$$\sum ((ya_{(1)})z)a_{(2)} = \sum y(a_{(1)}(za_{(2)}))$$

of (1). Here we use Sweedler's notation [12] for the comultiplication: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. Since $M$ coincides with the subspace of all primitive elements of $U(M)$ we have $\Delta(a) = a \otimes 1 + 1 \otimes a$ for any $a \in M$, and, hence, $a(yz) + y(az) = (ay)z + (ya)z$ and $(ya)z + (yz)a = y(az) + y(za)$, or, equivalently

$$(a,y,z) = -(y,a,z) = (y,z,a). \tag{2}$$

Therefore, $M$ lies in the generalised alternative nucleus $\mathrm{N_{alt}}(U(M))$ of $U(M)$. (The subset $\mathrm{N_{alt}}(A)$ of an algebra $A$ consists of all $a \in A$ that satisfy (2) for any $y, z \in A$). The product on $M$ is recovered as $[a,b] = ab - ba$ in $U(M)$.

The universal enveloping algebra $U(V)$ of a Bol algebra $V$ satisfies the identity

$$\sum a_{(1)}(y(a_{(2)}z)) = \sum (a_{(1)}(ya_{(2)}))z. \tag{3}$$

Since $V$ coincides with the primitive elements of $U(V)$, for any $a \in V$ and $y, z \in U(V)$ we have that

$$(a, y, z) = -(y, a, z). \tag{4}$$

This is equivalent to saying that $V$ is contained in the left generalised alternative nucleus $\mathrm{LN}_{\mathrm{alt}}(U(V))$ of the algebra $U(V)$. The binary and the ternary products on $V$ are recovered by

$$[a, b] = ab - ba \quad \text{and} \quad [a, b, c] = a(bc) - b(ac) - c(ab) + c(ba)$$

in $U(V)$. It is known [6] that for any algebra $A$

$$\mathrm{LN}_{\mathrm{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) \ \ \forall x, y \in A\}$$

is a Lie triple system with $[a, b, c] = a(bc) - b(ac) - c(ab) + c(ba)$.

Universal enveloping algebras for Malcev, Bol and general Sabinin algebras have been introduced only recently; their properties are still waiting to be explored. It might be tempting to assume that the theory of universal enveloping algebras for Lie algebras can be extended rather painlessly to the case of general Sabinin algebras, especially since many aspects of the theory are known to generalise well. However, it turns out that some very basic properties, such as the abundance of ideals in the universal enveloping algebras of Lie algebras, fail to hold in the general non-associative case. In particular, we shall see that while the properties of Malcev and Bol algebras, discussed above, may look similar, this similarity does not extend too far.

The motivation for this paper is the following version of Ado's Theorem for Malcev algebras that appeared in [8]:

**Theorem 1.** *For any finite–dimensional Malcev algebra $M$ over a field of characteristic $\neq 2, 3$ there exists a unital finite–dimensional algebra $A$ and a monomorphism of Malcev algebras $\iota\colon M \to \mathrm{N}_{\mathrm{alt}}(A)$.*

One is prompted to ask whether a similar statement holds for other classes of Bol algebras, in particular, for Lie triple systems. Given a finite dimensional Lie triple system $V$, one could ask whether it is contained as a subsystem of $\mathrm{LN}_{\mathrm{alt}}(A)$, with $ab = ba$ for all $a, b \in V$, for some finite dimensional unital algebra $A$. It is easy to see that this happens if and only if there exists an ideal of finite codimension in $U(V)$ which intersects $V$ trivially. Our answer shows that for Lie triple systems the situation is very different from the case of Lie or Malcev algebras:

**Theorem 2.** *Let $A$ be a finite dimensional unital algebra over a field $F$ of characteristic 0 and $V$ — a Lie triple system contained as a subsystem in $\mathrm{LN}_{\mathrm{alt}}(A)$ such that $ab = ba$ for all $a, b \in V$. Assume that $A$ is generated by $V$ as a unital algebra. Then $V$ is nilpotent and $A$ decomposes (as a vector space) into a direct sum of a nilpotent ideal and a central subalgebra without nonzero nilpotent elements.*

Note that we do not claim that embeddings mentioned in Theorem 2 do exist for all nilpotent Lie triple systems.

Examples suggest that the ideals in the universal enveloping algebras of Lie triple systems are even scarcer than it is implied by Theorem 2.

**Conjecture 3.** *The only proper ideal of the universal enveloping algebra of a simple Lie triple system is its augmentation ideal.*

We shall verify the above conjecture in several cases by direct calculations in Poincaré–Birkhoff–Witt bases.

For each Lie triple system $V$ there exists a canonically defined Lie algebra $\mathcal{L}_S(V)$, called the *Lie envelope of $V$* of which $V$ is a subsystem. The Poincaré–Birkhoff–Witt Theorem allows to identify the algebra $U(V)$ with a subspace of $U(\mathcal{L}_S(V))$. Motivated by analogy with Bruck loops, we shall show how the multiplication on $U(\mathcal{L}_S(V))$ can be modified to become compatible with the non-associative multiplication on $U(V)$.

The paper is organised as follows. The next section is auxiliary; it is a loose collection of various properties of Bol algebras and Lie triple systems. Section 3 contains the proof of Theorem 2. The construction of the universal enveloping algebra of a Lie triple system via its Lie envelope is given in Section 4. Finally, in Section 5 we present some evidence for Conjecture 3.

We have made no attempt to make this paper self-contained. We refer to [8] for the properties of the universal enveloping algebras of Malcev algebras, to [6] — for Bol algebras and to [6] — for general Sabinin algebras. The paper of Lister [5] is the general reference for Lie triple systems; the questions of nilpotency are treated in [1] .

About the notation: we shall often write "L.t.s." for "Lie triple system". As usual, the true meaning of "non-associative" is "not necessarily associative"; however "non-nilpotent" stands for "not nilpotent". The notations $L_x$ and $R_x$ are used to denote the multiplication by $x$ on the left and on the right respectively; the sum $L_a + R_a$ is denoted by $T_a$. The product $a(a(\cdots(aa))$ will be written simply as $a^n$. The left, middle and right associative nuclei of an algebra $A$ are denoted by $\mathrm{N}_l(A)$, $\mathrm{N}_m(A)$ and $\mathrm{N}_r(A)$ respectively, while $\mathrm{Z}(A)$ is the notation for the center of $A$. (Recall that the left associative nucleus of $A$ is the set of all $a \in A$ such that $(a, y, z) = 0$ for arbitrary $y, z \in A$; the right and the middle associative nuclei are defined similarly.) By $\mathrm{alg}\langle X\rangle$ (or $\mathrm{alg}_1\langle X\rangle$) we denote the subalgebra (unital subalgebra, respectively) generated by the subset $X \subset A$.

## 2. Some properties of the enveloping algebras for Bol algebras and Lie triple systems.

**Lemma 4.** *Let $(V, [\,,\,,\,], [\,,\,])$ be a Bol algebra. For $a, b \in V$ such that $[a, b] = 0$, the map $[L_a, L_b]$ is a derivation of $U(V)$.*

Recall that a ternary derivation of an algebra $A$ is a triple $(d_1, d_2, d_3)$ of linear maps such that

$$d_1(xy) = d_2(x)y + xd_3(y)$$

for all $x, y \in A$. The set $\mathrm{Tder}(A)$ of all ternary derivations of $A$ is a Lie algebra with the obvious bracket. It is clear that if $d_1(1) = d_2(1) = d_3(1) = 0$ then $d_1 = d_2 = d_3$ is a derivation of $A$.

*Proof of Lemma 4.* Notice that the identity (4) can be written as $(L_a, T_a, -L_a) \in \mathrm{Tder}(U(V))$ and, as a consequence,

$$([L_a, L_b], [T_a, T_b], [L_a, L_b]) \in \mathrm{Tder}(U(V)).$$

Evaluating both commutators at 1, we observe that $[L_a, L_b](1) = [a, b] = 0 = [T_a, T_b](1)$, so $[L_a, L_b] = [T_a, T_b]$ is a derivation of $U(V)$. □

**Lemma 5.** *Let $(V, [\,,\,,\,], [\,,\,])$ be a Bol algebra. For $a, b \in V$ such that $[a, b] = 0$ and any $x \in U(V)$*

$$[L_a, L_b](x) = -2(a, b, x).$$

*Proof.* The identity (4) with $y = b$ gives $L_aL_b + L_bL_a = L_{ab+ba} = 2L_{ab}$. Therefore, $[L_a, L_b](x) = L_aL_b(x) - (2L_{ab} - L_aL_b)(x) = -2(a, b, x)$. □

**Lemma 6.** *Let $(V, [\,,\,,\,], [\,,\,])$ be a Bol algebra. For any $a \in V$*

$$L_{a^n}L_{a^m} = L_{a^{n+m}}.$$

*Proof.* See Proposition 38 in [7]. □

For any Sabinin algebra $V$, the universal enveloping algebra is an H-bialgebra. That is, $U(V)$ is a non-associative unital bialgebra equipped with two bilinear maps, $\backslash\colon U(V) \times U(V) \to U(V)$ and $/\colon U(V) \times U(V) \to U(V)$ such that

$$\begin{aligned} \sum x_{(1)}\backslash(x_{(2)}y) &= \epsilon(x)y = \sum x_{(1)}(x_{(2)}\backslash y) \text{ and}\\ \sum (yx_{(1)})/x_{(2)} &= \epsilon(x)y = \sum (y/x_{(1)})x_{(2)}. \end{aligned}$$

The behaviour of these maps with respect to the comultiplication $\Delta$ and the counit $\epsilon$ is expressed by

$$\Delta(x\backslash y) = \sum x_{(1)}\backslash y_{(1)} \otimes x_{(2)}\backslash y_{(2)}, \quad \Delta(y/x) = \sum y_{(1)}/x_{(1)} \otimes y_{(2)}/x_{(2)}$$

and

$$\epsilon(x\backslash y) = \epsilon(x)\epsilon(y), \quad \epsilon(y/x) = \epsilon(x)\epsilon(y).$$

Fix an ordered basis $\{a_i\}_{i\in\Lambda}$ of $V$, with $\Lambda$ being the index set. The algebra $U(V)$ then has the Poincaré–Birkhoff–Witt basis

$$\{a_{i_1}(a_{i_2}(\cdots(a_{i_{n-1}}a_{i_n})\cdots)) \mid i_1 \leq \cdots \leq i_n \text{ and } n \in \mathbb{N}\}.$$

The algebra $U(V)$ is filtered by $U(V) = \cup_{n\in\mathbb{N}} U(V)_n$ with

$$U(V)_n = \operatorname{span}\langle a_1(a_2(\cdots(a_{m-1}a_m)) \mid a_1,\ldots,a_m \in V,\ \ m \leq n\rangle.$$

The *degree* of an element of $U(V)$ with respect to this filtration is defined in the obvious way. The corresponding graded algebra $\operatorname{Gr} U(V)$ is isomorphic to $\operatorname{Sym}(V)$, the symmetric algebra on $V$.

Let $(V, [\,,\,,\,])$ be a Lie triple system, and $U(V)$ — its universal enveloping algebra. The automorphism $a \mapsto -a$ of $V$ extends to an automorphism $S : U(V) \to U(V)$.

**Lemma 7.** *Let $(V, [\,,\,,\,])$ be a L.t.s. and $U(V)$ — its universal enveloping algebra. Then for any $a \in V$ we have that $[a, U(V)_n] \subseteq U(V)_{n-1}$.*

*Proof.* Let $x = a_1(a_2(\cdots(a_{n-1}a_n)\ldots)) \in U(V)_n$ with $a_1,\ldots,a_n \in V$. Since $\operatorname{Gr} U(V)$ is isomorphic to $\operatorname{Sym}(V)$, $[a, x]$ belongs to $U(V)_n$. On the other hand, $S([a, x]) = [-a, (-1)^n x] = (-1)^{n-1}[a, x]$. Therefore, $[a, x] \in U(V)_{n-1}$. □

The automorphism $S$ notably simplifies the left division \ on $U(V)$.

**Proposition 8.** *Let $(V, [\,,\,,\,])$ be a L.t.s. For all $x, y \in U(V)$*

$$x\backslash y = S(x)y \quad \text{and} \quad S(x) = x\backslash 1 = 1/x.$$

*Proof.* Let us prove that $\sum S(x_{(1)})x_{(2)} = \epsilon(x)1$. To this end we observe that this is a linear relation, so we only have to verify it on a set of elements spanning the vector space $U(V)$, for instance, $\{1\} \cup \{a^n \mid a \in V\}$ with $a^n = a(\cdots(aa))$. We have $\sum S({a^n}_{(1)}){a^n}_{(2)} = \sum_{k=0}^n \binom{n}{k} S(a^k)a^{n-k} = \sum_{k=0}^n \binom{n}{k}(-1)^k a^n = 0 = \epsilon(a^n)$, as desired.

From (3) and $\sum S(x_{(1)})x_{(2)} = \epsilon(x)1$ we obtain

$$\sum x_{(1)}(S(x_{(2)})(x_{(3)}y)) = \sum (x_{(1)}(S(x_{(2)})x_{(3)}))y = \sum (x_{(1)}\epsilon(x_{(2)}))y = xy.$$

By the definition of \ we have

$$\sum S(x_{(1)})(x_{(2)}y) = \sum x_{(1)}\backslash(x_{(2)}(S(x_{(3)})(x_{(4)}y))) = \sum x_{(1)}\backslash(x_{(2)}y) = \epsilon(x)y$$

so

$$S(x)y = \sum S(x_{(1)})(x_{(2)}(x_{(3)}\backslash y)) = \sum \epsilon(x_{(1)}) \cdot x_{(2)}\backslash y = x\backslash y.$$

With $y = 1$ we get $S(x) = x\backslash 1$, and from $\sum S(x_{(1)})x_{(2)} = \epsilon(x)1$ we also get $S(x) = \sum (S(x_{(1)})x_{(2)})/x_{(3)} = \sum \epsilon(x_{(1)})1/x_{(2)} = 1/x$. □

Proposition 8 ensures that $U(V)$ satisfies the linearisation of the equations defining a Bruck loop. Therefore, the linearisation of any identity satisfied by Bruck loops will hold in $U(V)$. Consider, for instance, the so-called *precession map* $\delta_{a,b} \colon c \mapsto (ab)\backslash(a(bc))$. For a Bruck loop this map is known to be an automorphism [4]. Linearising this result we obtain

**Corollary 9.** *Let $(V, [\,,\,,\,])$ be a L.t.s. The map $\delta_{x,y} \colon U(V) \to U(V)$ given by*

$$\delta_{x,y}(z) = \sum (x_{(1)}y_{(1)})\backslash(x_{(2)}(y_{(2)}z))$$

*satisfies*

$$\delta_{x,y}(wz) = \sum \delta_{x_{(1)},y_{(1)}}(w)\delta_{x_{(2)},y_{(2)}}(z).$$

The maps $\delta_{x,y}$ reflect the lack of associativity in $U(V)$. They satisfy

$$\sum (x_{(1)}y_{(1)})\delta_{x_{(2)},y_{(2)}}(z) = x(yz). \tag{5}$$

Clearly, $\Delta(\delta_{x,y}(z)) = \sum \delta_{x_{(1)},y_{(1)}}(z_{(1)}) \otimes \delta_{x_{(2)},y_{(2)}}(z_{(2)})$. Thus,

$$\delta_{x,y}(V) \subseteq V \tag{6}$$

and in general

$$\delta_{x,y}(U(V)_n) \subseteq U(V)_n.$$

The maps $\delta_{x,a}$ and $\delta_{a,x}$ are derivations of $U(V)$ for any $a \in V$. In fact, $\delta_{a,b}(x) = -(a,b,x)$ and $\delta_{a,b}(c) = \frac{1}{2}[a,b,c]$ for any $a,b,c \in V$ and $x \in U(V)$.

The following statement is a direct analogue of the corresponding result for Bruck loops [4].

**Proposition 10.** *Let $(V, [\,,\,,\,])$ be a L.t.s. Then the left and the middle associative nuclei of $U(V)$ coincide:*

$$\mathrm{N}_l(U(V)) = \mathrm{N}_m(U(V)).$$

*Proof.* The identity (3) implies

$$\sum x_{(1)}\big((S(x_{(2)})y)(x_{(3)}z)\big) = \sum \big(x_{(1)}((S(x_{(2)})y)x_{(3)})\big)z. \tag{7}$$

If $y$ is in $\mathrm{N}_m(U(V))$, the left-hand side of (7) is equal to

$$\sum x_{(1)}\big(S(x_{(2)})(y(x_{(3)}z))\big) = y(xz).$$

On the other hand, the right-hand side of (7) can be re-written as

$$\sum \big(x_{(1)}(S(x_{(2)})(yx_{(3)}))\big)z = (yx)z$$

and, hence, $y(xz) = (yx)z$ for all $x,z \in U(V)$. Therefore, $\mathrm{N}_m(U(V)) \subseteq N_l(U(V))$.

Similarly, notice that (3) also implies

$$\sum x_{(1)}\big((yS(x_{(2)}))(x_{(3)}z)\big) = \sum \big(x_{(1)}((yS(x_{(2)}))x_{(3)})\big)z.$$

For $y \in \mathrm{N}_l(U(V))$ one concludes that $x(yz) = (xy)z$ for all $x,z \in U(V)$ and, hence, that $\mathrm{N}_l(U(V)) \subseteq \mathrm{N}_m(U(V))$. □

**Lemma 11.** *Let $(V, [\,,\,,\,])$ be a L.t.s. and $A$ — a quotient of $U(V)$. If $a \in V$ satisfies $[L_a, L_b] = 0$ for all $b \in V$, then $a \in \mathrm{Z}(A)$.*

*Proof.* For any $x \in A$ we have $L_x \in \mathrm{alg}_1\langle L_b \mid b \in V\rangle$. This can be established by induction on the degree of $x$ with respect to the PBW filtration that $A$ inherits from $U(V)$, using the fact that $L_{by+yb} = L_bL_y + L_yL_b$ for any $y \in A$ and $b \in V$.

Since $[L_a, L_b] = 0$ for all $b \in B$ we have that $[L_a, L_x] = 0$ for any $x \in A$, so $a(xy) = x(ay)$ for any $x,y \in A$. Setting $y = 1$ we get that $ax = xa$ for any $x \in A$. Therefore, $(xy)a = a(xy) = x(ay) = x(ya)$ and $a \in \mathrm{N}_r(A)$. This can also be expressed by saying that the triple $(R_a, 0, R_a)$ belongs to $\mathrm{Tder}(A)$.

The identity (4) implies that $(L_a, T_a, -L_a)$ is also in $\mathrm{Tder}(A)$. Since $R_a = L_a$, it follows that $(2L_a, 2L_a, 0) \in \mathrm{Tder}(A)$ and thus $a \in \mathrm{N}_l(A)$. Similarly, $(0, 2R_a, -2L_a) \in \mathrm{Tder}(A)$ implies that $a \in \mathrm{N}_m(A)$ and, therefore, $a \in \mathrm{Z}(A)$. □

## 3. Nonexistence of ideals of finite codimension

In this section $(V, [\,,\,,\,])$ will be a Lie triple system and $U(V)$ — the non-associative universal enveloping algebra of $V$. For any $a,b,c \in V$ we have

$$[L_a, L_b](c) = a(bc) - b(ac) = [a,b,c] \quad \text{and} \quad [a,b] = 0$$

in $U(V)$. The map $[L_a, L_b]$ is a derivation of $U(V)$ and $L_{ax+xa} = L_aL_x + L_xL_a$ for any $a \in V, x \in U(V)$ by (4).

Let $A$ be a finite-dimensional unital algebra and $\mathrm{LN_{alt}}(A)$ — its left generalized alternative nucleus. We are interested in the existence of monomorphisms of L.t.s.

$$\iota\colon V \to \mathrm{LN_{alt}}(A) \tag{8}$$

such that $\iota(a)\iota(b) = \iota(b)\iota(a)$ for any $a, b \in V$. By the universal property of $U(V)$ such a map induces a homomorphism $\varphi\colon U(V) \to A$. The kernel of $\varphi$ is an ideal of finite codimension whose intersection with $V$ is trivial.

Let $S_2$ be the two-dimensional simple L.t.s. generated by $e, f$ with

$$[e,f,e] = 2e \quad \text{and} \quad [e,f,f] = -2f. \tag{9}$$

**Lemma 12.** *With $e, f$ as above,*

$$[e^n, f] = n(n-1)e^{n-1}.$$

*holds in $U(S_2)$.*

*Proof.* Observe that $fe^n = f(ee^{n-1}) = e(fe^{n-1}) - [L_e, L_f](e^{n-1}) = e(fe^{n-1}) - 2(n-1)e^{n-1}$. Repeating with $fe^{n-1}$ we obtain

$$\begin{aligned} fe^n &= e^n f - 2((n-1) + (n-2) + \cdots + 1)e^{n-1} \\ &= e^n f - n(n-1)e^{n-1}, \end{aligned}$$

□

Any semisimple L.t.s. contains a copy of $S_2$, see [5]. (This may be compared to the fact that any semisimple Lie algebra contains a copy of $sl_2$.)

**Proposition 13.** *If $(V, [\ ,\ ,\ ])$ is a semisimple L.t.s., then the only proper ideal of $U(V)$ that has finite codimension is the augmentation ideal* $\ker \epsilon$.

*Proof.* Given a proper ideal $I$ of $U(V)$ whose codimension is finite, the set $V_0 = I \cap V$ is an ideal of the L.t.s. $V$. Therefore, there exists another ideal $V_1$ with $V = V_0 \oplus V_1$ (see [5]). Both $V_0$ and $V_1$ are semisimple L.t.s., so either $V_1 = 0$, or there exists a subsystem $\mathrm{span}\langle e, f\rangle \subseteq V_1$ with multiplication as in (9). In the first case we have that $\ker \epsilon$, the ideal generated by $V$, is contained inside $I$ and, hence, since the codimension of $\ker \epsilon$ is 1, they are equal.

Assume now that we are in the second case. Since any finite–codimensional proper ideal $I$ of $U(V)$ contains an element of the form $p(e) = \alpha_0 1 + \alpha_1 e + \cdots + \alpha_{n-1}e^{n-1} + e^n$ with $n > 1$, then, by Lemma 12, it also contains $[[p(e), f], f], \dots], f] = n!(n-1)!e$. Therefore, $e \in I$ which, by definition of $V_1$, is not possible. □

Proposition 13 shows that embeddings of the type (8) do not exist for semisimple L.t.s. Since any L.t.s. decomposes (as a vector space) as the direct sum of a semisimple subsystem and a solvable ideal (see [5]), it is clear that such embedding might only exist for solvable L.t.s. We shall prove that, in fact, $V$ must be nilpotent.

Let us denote the map $c \mapsto [a,b,c]$ by $D_{a,b}$. The vector space $\mathcal{L}_S(V) = \mathrm{span}\langle D_{a,b} \mid a, b \in V\rangle \oplus V$ is a Lie algebra (see [3]) with the bracket

$$[a,b] = D_{a,b} \quad \text{and} \quad [D_{a,b}, c] = [a,b,c]. \tag{10}$$

This Lie algebra is called sometimes *the Lie envelope* of $V$. It is $\mathbb{Z}_2$–graded with even part $\mathcal{L}_S^+(V) = \mathrm{span}\langle D_{a,b} | \, a, b \in V\rangle$ and odd part $\mathcal{L}_S^-(V) = V$.

Given any unital algebra $A$ generated, as a unital algebra, by a subsystem $V$ of $\mathrm{LN_{alt}}(A)$ with $[a,b] = 0$ for any $a, b \in V$, we shall often consider the Lie algebra $\mathcal{L}(V)$ generated by $\{L_a \mid a \in V\}$. Usually, no explicit mention of $A$ will be needed. Since $[L_a, L_b]$ is a derivation of $A$ (see the proof of Lemma 4) and $A$ is generated by $V$, it follows that $\mathcal{L}(V) = \mathrm{span}\langle [L_a, L_b] \mid a, b \in V\rangle \oplus \mathrm{span}\langle L_a \mid a \in V\rangle$. The algebra $\mathcal{L}(V)$ is isomorphic to $\mathcal{L}_S(V)$ by $a \mapsto L_a$ and $D_{a,b} \mapsto [L_a, L_b]$.

It is a simple exercise to check that over algebraically closed fields of characteristic zero, the only solvable non-nilpotent two-dimensional L.t.s. is $R_2 = Fa \oplus Fb$ with

$$[a, b, a] = -b \quad \text{and} \quad [a, b, b] = 0. \tag{11}$$

**Lemma 14.** *Let $V$ be a solvable non-nilpotent L.t.s. Then there exists a homomorphic image of $V$ which contains a subsystem isomorphic to $R_2$.*

*Proof.* For $V$ a solvable L.t.s, $\mathcal{L}_S(V)$ is a solvable Lie algebra (see [5]). The solvability of $\mathcal{L}_S(V)$ implies that there exists a non-zero $v \in \mathcal{L}_S(V)$ and a homomorphism of Lie algebras $\lambda\colon \mathcal{L}_S(V) \to F$ such that

$$[x, v] = \lambda(x)v \tag{12}$$

for any $x \in \mathcal{L}_S(V)$. Observe that $\mathcal{L}_S^+(V) \subseteq [\mathcal{L}_S(V), \mathcal{L}_S(V)]$ and hence $\lambda(\mathcal{L}_S^+(V)) = 0$.

Write $v$ as a sum of its even and odd components: $v = D + b$ with $D \in \mathcal{L}_S^+(V)$ and $b \in \mathcal{L}_S^-(V)$. The odd part of the identity (12) with $x \in \mathcal{L}_S^+(V)$ implies that $[V, V, b] = 0$. Setting $x = a \in V$ in (12) gives

$$D_{a,b} = \lambda(a)D$$

as the even part, and

$$D(a) = -\lambda(a)b$$

as the odd part.

Assume that $\lambda$ is not identically equal to zero. Then we can choose $a \in V$ with $\lambda(a) = 1$. For such $a$ we have that $D = D_{a,b}$ and $D(a) = -b$ so $[a, b, a] = -b$. Since $[V, V, b] = 0$, the subspace $\operatorname{span}\langle a, b\rangle$ is a subsystem of $V$ isomorphic to $R_2$.

Now, if $\lambda$ happens to be identically equal to zero, it follows that $D = 0$ and $b \neq 0$ (since $v$ is non-zero), and that $[V, b, V] = 0$. Hence, the one-dimensional subspace $\operatorname{span}\langle b\rangle$ is contained in the centre of $V$. The L.t.s. $V/\operatorname{span}\langle b\rangle$ is solvable non-nilpotent (see [1, 5]) and its dimension is lower than the dimension of $V$. The result in this case can be obtained by induction. □

**Proposition 15.** *Given a non-nilpotent L.t.s. $V$ and an ideal $I$ of finite codimension in $U(V)$, the intersection $I \cap V$ is non-zero.*

*Proof.* Without loss of generality we may assume that $V$ is solvable. By Lemma 14 there exists an ideal $V_0$ and elements $a, b \in V$ such that $V_0 \oplus \operatorname{span}\langle a, b\rangle$ is a subsystem of $V$ with $[a, b, a] \equiv -b \mod V_0$ and $[a, b, b] \equiv 0 \mod V_0$.

By (5), in $U(V)$ we have $x(yz) = \sum x_{(1)}y_{(1)} \cdot \delta_{x_{(2)},y_{(2)}}(z)$. With $x = a^n$, $y = c \in V$ and $z = a$ we obtain

$$a^n(ca) = \begin{cases} a^{n+1}c = a \cdot a^n c \\ \\ a^n c \cdot a + \sum (a^n)_{(1)}\delta_{(a^n)_{(2)},c}(a) \\ \quad \equiv a^n c \cdot a + n a^{n-1}\delta_{a,c}(a) \mod U(V)_{n-1} \end{cases}$$

where the last congruence follows from (6). Hence $[a^n c, a] \equiv -n a^{n-1}\delta_{a,c}(a) \equiv -\frac{n}{2}a^{n-1}[a, c, a] \mod U(V)_{n-1}$. After $n$ commutations we get

$$[\ldots[[a^n c, a], a], \ldots, a] = (-1)^n \frac{n!}{2^n}[a, [a, [\ldots, [a, c, a], \ldots], a], a]$$

where we have replaced the congruence modulo $U(V)_0 = F$ by the equality because both sides lie inside $\ker \epsilon$. In the particular case $c = b$ we have $[\ldots[[a^n b, a], a], \ldots, a] = \frac{n!}{2^n}(b + v_0)$ with $v_0 \in V_0$.

Any finite-codimensional ideal $I$ contains an element of the form $p(a) = \alpha_0 1 + \alpha_1 a + \cdots + \alpha_{n-1}a^{n-1} + a^n$. It also contains $p(a)b$ and $[\ldots[p(a)b, a], \ldots, a]$ where the commutator is taken $n$ times. Therefore, $I$ also contains the nonzero element $\frac{n!}{2^n}(b + v_0)$. □

We have seen that faithful representations of the type (8) can only exist for nilpotent L.t.s. It turns out that for nilpotent L.t.s. these representations, if exist, have very specific structure. Name, assuming that in (8) the algebra $A$ is generated by $\iota(V)$, we shall prove that there exists a nilpotent ideal $R$ such that $A/R$ is a commutative associative algebra over $F$ with no nontrivial nilpotent elements. First, we need some lemmas.

**Lemma 16.** *Let $A$ be a finite-dimensional unital algebra, $a \in \mathrm{LN}_{\mathrm{alt}}(A)$ and $L_a = (L_a)_s + (L_a)_n$ — the Jordan–Chevalley decomposition of $L_a$ in $\mathrm{End}(A)$. Then there exist $a_s, a_n \in \mathrm{LN}_{\mathrm{alt}}(A)$, the semisimple and nilpotent parts of $a$, with $(L_a)_s = L_{a_s}$ and $(L_a)_n = L_{a_n}$.*

*Proof.* Recall that given $(d, d', d'') \in \mathrm{Tder}(A)$, its semisimple and nilpotent parts can be calculated componentwise: $(d, d', d'')_s = (d_s, d'_s, d''_s)$ and $(d, d', d'')_n = (d_n, d'_n, d''_n)$, where both $(d_s, d'_s, d''_s)$ and $(d_n, d'_n, d''_n)$ are also ternary derivations. Recall also that $(d, d', -d) \in \mathrm{Tder}(A)$ if and only if $d = L_a$ and $d' = T_a$ with $a \in \mathrm{LN}_{\mathrm{alt}}(A)$. Now, for any $a \in \mathrm{LN}_{\mathrm{alt}}(A)$ we have that $((L_a)_s, (T_a)_s, -(L_a)_s)$ and $((L_a)_n, (T_a)_n, -(L_a)_n) \in \mathrm{Tder}(A)$, which implies that $(L_a)_s = L_{a_s}$ and $(L_a)_n = L_{a_n}$ for some $a_s, a_n \in \mathrm{LN}_{\mathrm{alt}}(A)$. □

Let us complete $V$ inside $A$ by adding the semisimple and nilpotent parts of all its elements; it turns out that such completion retains some fundamental properties of $V$:

**Lemma 17.** *Let $A$ be a finite dimensional unital algebra. Given any subsystem $V \leq \mathrm{LN}_{\mathrm{alt}}(A)$ such that*

i) *$V$ generates $A$ as a unital algebra,*
ii) *$[a, b] = 0$ for all $a, b \in V$,*
iii) *$V$ is nilpotent,*

*there exists in $\mathrm{LN}_{\mathrm{alt}}(A)$ a subsystem $\hat{V}$ containing $V$ and satisfying i), ii) and iii), and such that $a_s, a_n \in \hat{V}$ for any $a \in \hat{V}$. Moreover, $a_s \in \mathrm{Z}(A)$ for any $a \in \hat{V}$ and $\{a_n \mid a \in \hat{V}\}$ is an ideal of $\hat{V}$.*

*Proof.* Since $V$ generates $A$ and $[a, b] = 0$ for any $a, b \in V$, the Lie algebra $\mathcal{L}(V)$, generated by $\{L_a \mid a \in V\}$ is isomorphic to $\mathcal{L}_S(V)$. By [1] the latter algebra, and hence the former, is nilpotent.

By the properties of the Jordan–Chevalley decomposition (see [2]) $(\mathrm{ad}_{L_a})_s = \mathrm{ad}_{L_{a_s}}$ and $(\mathrm{ad}_{L_a})_n = \mathrm{ad}_{L_{a_n}}$. The operators $\mathrm{ad}_{L_{a_s}}$ and $\mathrm{ad}_{L_{a_n}}$ can be expressed as polynomials in $\mathrm{ad}_{L_a}$ with zero constant term. In particular, $\mathrm{ad}_{L_{a_s}}$ leaves $\mathcal{L}(V)$ stable with a nilpotent action. By the semisimplicity of $\mathrm{ad}_{L_{a_s}}$ this means that $[L_{a_s}, \mathcal{L}(V)] = 0$. Hence $a_s \in \mathrm{Z}(A)$ by Lemma 11.

As $\mathcal{L}(V)$ is nilpotent, there exists a basis of $A$ where $\mathcal{L}(V)$ is represented by upper triangular matrices. Hence, for any $a, b \in V$ the operator $L_{a_s+b_s}$ is semisimple, while $L_{a_n+b_n}$ is nilpotent. Moreover, $a_s + b_s \in \mathrm{Z}(A)$ implies that $[L_{a_s+b_s}, L_{a_n+b_n}] = 0$. By the uniqueness of the Jordan–Chevalley decomposition we obtain that $(L_{a+b})_s = L_{a_s} + L_{b_s}$ and $(L_{a+b})_n = L_{a_n} + L_{b_n}$. In particular, $(a + b)_s = a_s + b_s$ and $(a + b)_n = a_n + b_n$.

Let $\hat{V} = \{a_s + b_n \mid a, b \in V\}$. By the previous, $\hat{V}$ is a vector subspace of $\mathrm{LN}_{\mathrm{alt}}(A)$ and, since $(a_s + b_n)_s = a_s$ and $(a_s + b_n)_n = b_n$, $\hat{V}$ contains the semisimple and nilpotent components of its elements. We also know that $a_s \in \mathrm{Z}(A)$ for any $a \in \hat{V}$.

Given $a, a', a'', b, b', b'' \in V$ we have that

$$[a_s + b_n, a'_s + b'_n] = [b_n, b'_n] = [b_s + b_n, b'_s + b'_n] = [b, b'] = 0,$$

so $\hat{V}$ satisfies ii). Moreover,

$$\begin{aligned} [a_s + b_n, a'_s + b'_n, a''_s + b''_n] &= [L_{a_s+b_n}, L_{a'_s+b'_n}](a''_s + b''_n) \\ &= [L_{b_n}, L_{b'_n}](a''_s + b''_n) \\ &= [b_n, b'_n, b''_n] \\ &= [b, b', b''] \end{aligned}$$

implies that $\hat{V}$ is a subsystem of $\mathrm{LN}_{\mathrm{alt}}(A)$ and that $[\hat{V}, \hat{V}, \hat{V}] \subseteq [V, V, V]$. In terms of the lower central series for $\hat{V}$ and $V$ (see [1]) this says that $\hat{V}^1 \subseteq V^1$. Assuming that $\hat{V}^n \subseteq V^n$, we have $\hat{V}^{n+1} = [\hat{V}^n, \hat{V}, \hat{V}] + [\hat{V}, \hat{V}, \hat{V}^n] \subseteq [V^n, \hat{V}, \hat{V}] + [\hat{V}, \hat{V}, V^n] \subseteq [V^n, V, V] + [V, V, V^n] \subseteq V^{n+1}$. The nilpotency of $\hat{V}$ follows from this observation and the nilpotency of $V$.

Finally, the left multiplication operator by $[a, b, c]$ is obtained as the commutator $[[L_a, L_b], L_c]$; in an adequate basis of $A$ it is represented as a commutator of upper triangular matrices. Therefore, it is nilpotent and $[a, b, c] = [a, b, c]_n$. Since $[\hat{V}, \hat{V}, \hat{V}] \subseteq [V, V, V]$ it follows that $[\hat{V}, \hat{V}, \hat{V}] \subseteq \{a_n \mid a \in \hat{V}\}$. In particular, the latter set is an ideal of $\hat{V}$. □

**Lemma 18.** *Let $A$ be a finite-dimensional unital algebra and let $V$ be a subsystem of $\mathrm{LN}_{\mathrm{alt}}(A)$. Assume that*

i) $a = a_n$ *for any* $a \in V$,

ii) $[a, b] = 0$ *for all* $a, b \in V$.

*Then the subalgebra generated by* $V$ *is nilpotent.*

*Proof.* Assume, as before, that $A$ is generated by $V$ as a unital algebra.

There exists an element of $V$ that lies in the centre of $A$. Indeed, the nilpotency of $V$ implies that $\mathcal{L}(V)$ consists of nilpotent transformations [1], which, in turn, implies that the centre of $\mathcal{L}(V)$ is non-zero. Given $0 \neq D + L_a \in \mathrm{Z}(\mathcal{L}(V))$ with $D \in \mathcal{L}^+(V)$, for any $b \in V$ the equality $0 = [D + L_a, L_b] = L_{D(b)} + [L_a, L_b]$ implies that $D = 0$ and $[L_a, L_b] = 0$. Therefore $0 \neq a \in \mathrm{Z}(A)$ by Lemma 11.

We shall use induction on the dimension of $V$. The case $\dim V = 0$ is obvious. Given $V$ with $\dim V = n+1$, choose $0 \neq a \in Z(A) \cap V$ as above and consider the ideal $aA$. The quotient algebra $A/aA$ is generated, as a unital algebra, by the quotient $(V + aA)/aA$ of $V$. Thus we can apply the hypothesis of induction to conclude that $\mathrm{alg}\langle V + aA/aA\rangle = \mathrm{alg}\langle V\rangle/aA$ is nilpotent.

Let us denote the ideal $\mathrm{alg}\langle V\rangle$ by $A_0$, and the linear span of all products of $N$ elements of $A_0$, regardless of the order of the parentheses, by $A_0^N$. From the nilpotency of $\mathrm{alg}\langle V\rangle/aA$ we deduce that there exists $N$ such that $A_0^N \subseteq aA$. Moreover, any product involving $2N$ elements of $A_0$ lies in the ideal $aA_0$, since is of the form $u_1u_2$ where at least one of the factors involves at least $N$ elements and, therefore, lies in $A_0^N \subseteq aA$, and the other factor belongs to $A_0$.

Let us fix $N$ such that $A_0^N \subseteq aA_0$ and prove by induction that $A_0^{N^k} \subseteq a^kA_0$. Since $a \in V$ is nilpotent, this will imply that $A_0$ is nilpotent, as desired. Assume that $A_0^{N^{k-1}} \subseteq a^{k-1}A_0$. Any product of $N^k$ elements in $A_0$ can be written as a product of $N$ factors, each belonging to $A_0$, and at least one of them lying in $A_0^{N^{k-1}} \subseteq a^{k-1}A_0$. Since $a^{k-1}$ is in the centre of $A$, the whole product lies in $a^{k-1}A_0^N \subseteq a^kA_0$. □

Finally, we are in the position to prove Theorem 2.

*Proof of Theorem 2.* By Lemma 17 we can assume that $V$ contains the semisimple and nilpotent components of all its elements. Let $Q = \mathrm{alg}\langle a_s \mid a \in V\rangle \subseteq \mathrm{Z}(A)$ and let $R$ be the ideal generated by $\{a_n \mid a \in V\}$. Clearly $A = Q + R$.

For any nilpotent element $x \in Q$, $L_x$ belongs to $\mathrm{alg}_1\langle L_{a_s} \mid a \in V\rangle$. This algebra is abelian and all its elements are semisimple transformations. But $x \in \mathrm{Z}(A)$ implies that $L_x$ is nilpotent so $L_x = 0$ and $x$ must be zero. Hence $Q$ is a commutative associative finite dimensional algebra without nonzero nilpotent elements.

Since $a_s \in \mathrm{Z}(A)$, it follows that $A = Q\,\mathrm{alg}_1\langle a_n \mid a \in V\rangle$. We can apply Lemma 18 to the algebra $\mathrm{alg}_1\langle a_n \mid a \in V\rangle$ and the subsystem $\{a_n \mid a \in V\}$ to conclude that $\mathrm{alg}\langle a_n \mid a \in V\rangle$ is nilpotent. The ideal $R$ decomposes as $R = Q\,\mathrm{alg}\langle a_n \mid a \in V\rangle$, so it is also nilpotent. Its nilpotency implies that $Q \cap R = 0$. □

## 4. The universal enveloping algebras of a L.t.s. and its Lie envelope

The following construction is based on the known construction of a Bruck loop starting from a group whose every element has a square root. Namely, any such group with the product $g * h = g^{\frac{1}{2}}hg^{\frac{1}{2}}$ becomes a Bruck loop. Observe that the linearisation of the identity $g = r(g)r(g)$ with $r(g) = g^{\frac{1}{2}}$ in an H–bialgebra reads as $x = \sum r(x_{(1)})r(x_{(2)})$ for some map $r$.

Let $L$ be a Lie algebra over a field $F$ of characteristic $\neq 2$.

**Lemma 19.** *The linear map* $q\colon U(L) \to U(L)$ *defined by* $x \mapsto \sum x_{(1)}x_{(2)}$ *is bijective.*

*Proof.* Consider the Poincaré–Birkhoff–Witt filtration $U(L) = \bigcup_{n\geq 0} U_n$ of $U(L)$. Given $a_1, \ldots, a_n \in L$,

$$q(a_1 \cdots a_n) \equiv 2^n a_1 \cdots a_n \mod U_{n-1}.$$

Since $q$ preserves the filtration, it follows that it is bijective on each $U_n$. □

Let $r$ be the inverse of $q$. Clearly, for any $x \in U(L)$ we have that $x = \sum r(x)_{(1)}r(x)_{(2)}$. Furthermore, $q$ being a coalgebra isomorphims implies that $r$ is also a coalgebra isomorphism. Therefore,

$$x = \sum r(x_{(1)})r(x_{(2)})$$

The product on $U(L)$ can be modified with the help of the map $r$ as follows:

$$x * y = \sum r(x_{(1)}) y r(x_{(2)}).$$

With this product $U(L)$ becomes a unital non-associative algebra. In fact, since $r$ is a homomorphism of coalgebras, $U(L)$ carries the structure of an H–bialgebra.

**Lemma 20.** *For all $x, y$ in $U(L)$*

$$\sum r(x_{(1)} * (y * x_{(2)})) = \sum r(x_{(1)}) r(y) r(x_{(2)}).$$

*Proof.* Indeed,

$$\begin{aligned}
\sum x_{(1)} * (y * x_{(2)}) &= \sum r(x_{(1)}) r(y_{(1)}) x_{(3)} r(y_{(2)}) r(x_{(2)}) \\
&= \sum r(x_{(1)}) r(y_{(1)}) r(x_{(2)}) r(x_{(3)}) r(y_{(2)}) r(x_{(4)}) \\
&= \sum \big(r(x_{(1)}) r(y) r(x_{(2)})\big)_{(1)} \big(r(x_{(1)}) r(y) r(x_{(2)})\big)_{(2)}
\end{aligned}$$

which proves the lemma. □

**Proposition 21.** *The algebra $(U(L), *)$ satisfies*

i) $\sum x_{(1)} * (y * (x_{(2)} * z)) = \sum (x_{(1)} * (y * x_{(2)})) * z.$
ii) $a * b = b * a$ *for any* $a, b \in L$.
iii) $a * (b * c) - b * (a * c) = \frac{1}{4}[[a, b], c]$ *for any* $a, b, c \in L$.

*Proof.* We shall only check part i); it follows from Lemma 20 by

$$\begin{aligned}
\sum x_{(1)} * (y * (x_{(2)} * z)) &= \sum r(x_{(1)}) r(y_{(1)}) r(x_{(2)}) z r(x_{(3)}) r(y_{(2)}) r(x_{(4)}) \\
&= \sum r(x_{(1)} * (y * x_{(2)}))_{(1)} z r(x_{(1)} * (y * x_{(2)}))_{(2)} = \sum (x_{(1)} * (y * x_{(2)})) * z.
\end{aligned}$$

□

Given a L.t.s. with the product $[\ ,\ ,\ ]$ and a scalar $\mu$, the new product $[\ ,\ ,\ ]' = \mu^2 [\ ,\ ,\ ]$ also defines a L.t.s. that is isomorphic to the original L.t.s. under $x \mapsto \mu x$.

**Corollary 22.** *Let $V$ be a L.t.s. and $\mathcal{L}_S(V)$ — the Lie envelope of $V$. The unital subalgebra of $(U(\mathcal{L}_S(V)), *)$ generated by $V$ is isomorphic to the universal enveloping algebra of $V$ considered as a Bol algebra with the trivial binary product.*

*Proof.* Define $[a, b, c]' = \frac{1}{4}[a, b, c]$ and let $Q$ the subalgebra of $(U(L), *)$ generated by $V$. The universal property of $U(V, [\ ,\ ,\ ]')$ together with Proposition 21 implies that there exists an epimorphism from $U(V, [\ ,\ ,\ ]')$ to $Q$. Since $a_1 * (\cdots * (a_{n-1} * a_n)) \equiv a_1 \cdots a_n \mod U_{n-1}$ with $a_1, \ldots, a_n \in V$, it follows that $Q$ admits a PBW–type basis. The epimorphism from $U(V, [\ ,\ ,\ ]')$ to $Q$ maps the PBW basis of $U(V, [\ ,\ ,\ ]')$ to this basis, so it is an isomorphism. However, as $(V, [\ ,\ ,\ ]')$ and $(V, [\ ,\ ,\ ])$ are isomorphic, their universal enveloping algebras also are. □

## 5. Ideals in the enveloping algebras of simple L.t.s.

**Lemma 23.** *Assume that $V$ is a simple L.t.s. satisfying the following condition: all elements of the universal enveloping algebra $U(V)$ that commute with $V$, are of the form $c + x$ where $c$ is a scalar and $x$ is in $V$. Then the only proper ideal of $U(V)$ is the augmentation ideal.*

*Proof.* Suppose the conditions of the lemma are satisfied. Let $I \subset U(V)$ be an ideal, and take some $r \in I$. There exists an element $x \in V$ such that $r' = rx - xr \neq 0$. It is clear that $r' \in I$ and $\deg r' < \deg r$, where the degree is taken with respect to the PBW filtration. Hence, $I$ necessarily contains a nonzero element $u$ of degree at most 1. If $u$ is a scalar, then $I = U(V)$. If $\deg u = 1$, the space of all linear combinations of (possibly iterated) brackets containing $u$, is an ideal of $V$ and, hence, coincides with $V$. All these brackets are in $I$, therefore, $I$ contains $V$. □

If $a$ is an element of $V$ and $r \in U(V)_n$, the commutator $ar - ra$ belongs to $U(V)_{n-1}$. In fact, it is possible to write an explicit formula for the terms of degree $n-1$ in this commutator.

**Lemma 24.** *Let $\{x_k\}$ be a basis for $V$ and $r \in U(V)$ — a monomial in the $x_k$. Then*

$$ar - ra = \frac{1}{2}\sum_{i,j} [a, x_i, x_j] \cdot \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} r + \textit{lower degree terms.} \tag{13}$$

Here the partial derivative $\partial/\partial x_i$ of a non-associative monomial is defined by setting $\partial/\partial x_i(uv) = u\partial/\partial x_i(v) + \partial/\partial x_i(u)v$ with $\partial/\partial x_i(x_j) = 1$ if $i = j$ and 0 otherwise.

*Proof.* The vector space $U(V)_n/U(V)_{n-1}$ is spanned by classes of elements of the form $b^n$ with $b \in V$, so it is sufficient to verify (13) for $p = b^n$.

Modulo terms of degree $n-2$ and smaller we have

$$\begin{aligned}
ab^n - b^n a &= a(bb^{n-1}) - b^n a \\
&= [L_a, L_b](b^{n-1}) + b(ab^{n-1}) - b^n a \\
&= \sum_{i+j=n-2} b^i([L_a, L_b](b) \cdot b^j) + b(ab^{n-1}) - b^n a \\
&= (n-1)[L_a, L_b](b) \cdot b^{n-2} + b(ab^{n-1} - b^{n-1}a) \\
&= (n-1)[L_a, L_b](b) \cdot b^{n-2} + (n-2)[L_a, L_b](b) \cdot b^{n-2} + \ldots + [L_a, L_b](b) \cdot b^{n-2} \\
&= \frac{n(n-1)}{2}[L_a, L_b](b) \cdot b^{n-2}.
\end{aligned}$$

The last expression coincides with the right-hand side of (13). □

Let $x, y, z$ be a set of generators for the Lie algebra $so(3)$ with $[x, y] = z$, $[y, z] = x$ and $[z, x] = y$. We shall consider $so(3)$ as a simple L.t.s. by setting $[a, b, c] = [[a, b], c]$. Let $\widetilde{S}_2$ be the 2-dimensional subsystem spanned by $x$ and $y$. Over the complex numbers $\widetilde{S}_2$ is isomorphic to the L.t.s. $S_2$ mentioned in Section 2; the isomorphism is given by $e = -x + y\sqrt{-1}$, $f = x + y\sqrt{-1}$.

**Proposition 25.** *Both $so(3)$ and $\widetilde{S}_2$ satisfy Conjecture 3.*

*Proof.* The products of the form $z^n(x^p y^q)$ with $n, p, q$ non-negative integers, form a basis for the universal enveloping algebra of $so(3)$ considered as a Lie triple system. In our case, (13) reads as

$$\begin{aligned}
&z^n(x^p y^q) \cdot z - z \cdot z^n(x^p y^q) \\
&\qquad = -\frac{n(p+q)}{2} z^{n-1}(x^p y^q) + \frac{p(p-1)}{2} z^{n+1}(x^{p-2} y^q) + \frac{q(q-1)}{2} z^{n+1}(x^p y^{q-2}) + \ldots
\end{aligned}$$

where the omitted terms are of degree $n + p + q - 2$ and smaller. Similarly,

$$\begin{aligned}
&z^n(x^p y^q) \cdot x - x \cdot z^n(x^p y^q) \\
&\qquad = \frac{n(n-1)}{2} z^{n-2}(x^{p+1} y^q) - \frac{p(n+q)}{2} z^n(x^{p-1} y^q) + \frac{q(q-1)}{2} z^n(x^{p+1} y^{q-2}) + \ldots
\end{aligned}$$

and

$$\begin{aligned}
&z^n(x^p y^q) \cdot y - y \cdot z^n(x^p y^q) \\
&\qquad = \frac{n(n-1)}{2} z^{n-2}(x^p y^{q+1}) + \frac{p(p-1)}{2} z^n(x^{p-2} y^{q+1}) - \frac{q(n+p)}{2} z^n(x^p y^{q-1}) + \ldots
\end{aligned}$$

Now, suppose that there exists an element $r$ of the universal enveloping algebra of $so(3)$ considered as a Lie triple system, of degree $N > 1$, which commutes with $x, y$ and $z$. This element has the form

$$r = \sum_{n+p+q=N} \alpha_{n,p,q} z^n(x^p y^q) + \text{lower degree terms.}$$

The requirement that $rz - zr$ has no terms of degree $N-1$ imposes linear conditions on the coefficients $\alpha_{n,p,q}$, similar conditions come from $rx - xr$ and $ry - yr$. Explicitly, these conditions are as follows:

$$\begin{aligned}
-(p+q)(n+2)\alpha_{n+2,p,q} + (p+1)(p+2)\alpha_{n,p+2,q} + (q+1)(q+2)\alpha_{n,p,q+2} &= 0,\\
(n+1)(n+2)\alpha_{n+2,p,q} - (n+q)(p+2)\alpha_{n,p+2,q} + (q+1)(q+2)\alpha_{n,p,q+2} &= 0,\\
(n+1)(n+2)\alpha_{n+2,p,q} + (p+1)(p+2)\alpha_{n,p+2,q} - (n+p)(q+2)\alpha_{n,p,q+2} &= 0.
\end{aligned}$$

The determinant of the corresponding $3\times 3$-matrix is equal to $2(n+2)(p+2)(q+2)(n+p+q+1)^2$ and it follows that all the $\alpha_{n,p,q}$ are zero and, hence, $\deg r < N$, which gives a contradiction.

The argument for $\widetilde{S}_2$ is entirely similar. □

Let $(\,,)$ be a non-degenerate symmetric bilinear form on a vector space $V$ of dimension greater than 1. Define a ternary bracket on $V$ by

$$[a,b,c] = (a,c)b - (b,c)a.$$

A straightforward verification shows that $V$ with this bracket satisfies all the axioms of a Lie triple system.

If $I$ is an ideal in $V$, $[I,V,V] \subseteq I$, that is, $(v,x)u - (v,u)x \in I$ for any $x \in I$ and $u,v \in V$. Hence, $(v,x)u \in I$ for any $x \in I$ and this means that $I$ is either trivial, or coincides with $V$. Therefore, $V$ is simple.

**Proposition 26.** *The L.t.s. $V$ satisfies Conjecture 3.*

*Proof.* Fix a basis $\{x_k\}$ for $V$, $n \geq k \geq 1$, and let $r \in U(V)$ be homogeneous of degree greater than 1. The condition $x_k r - r x_k = 0$ implies, by (13), that

$$\sum_{i,j}[x_k, x_i, x_j]\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}r = 0,$$

that is,

$$\sum_{i,j}((x_k,x_j)x_i - (x_i,x_j)x_k)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}r = 0.$$

Assuming that the basis $\{x_k\}$ is orthonormal, we get

$$x_k\sum_i \frac{\partial^2}{\partial x_i^2}r = \sum_i x_i\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}r = (m-1)\frac{\partial}{\partial x_k}r,$$

where $m = \deg r$. If $\frac{\partial}{\partial x_k}r = 0$ for some $k$ it follows that $\sum_i \frac{\partial^2}{\partial x_i^2}r = 0$ and, hence, that $\frac{\partial}{\partial x_k}r = 0$ for all $k$. In this case $r$ is a constant, so we can assume that $\frac{\partial}{\partial x_k}r \neq 0$ for all $k$ and that $\sum_i \frac{\partial^2}{\partial x_i^2}r \neq 0$.

Let us write $\psi$ for $\sum_i \frac{\partial^2}{\partial x_i^2}r$. We have

$$(m-1)\frac{\partial}{\partial x_k}r = x_k\psi \tag{14}$$

and, hence,

$$(m-1)x_k\frac{\partial}{\partial x_k}r = x_k^2\psi$$

and

$$m(m-1)r = q\psi$$

with $q = \sum_{i=1}^n x_i^2$. It follows that

$$m(m-1)\frac{\partial}{\partial x_k}r = 2x_k\psi + q\frac{\partial}{\partial x_k}\psi$$

which implies

$$(m-2)x_k\psi = q\frac{\partial}{\partial x_k}\psi.$$

If $m = 2$ this means that $r$ is a scalar multiple of $q$. If $m \neq 2$ we have that $\psi = q\psi_0$ with $\psi_0 \neq 0$, and $m(m-1)r = q^2\psi_0$. It is readily seen that $\psi_0$ satisfies

$$(m-4)x_k\psi_0 = q\frac{\partial}{\partial x_k}\psi_0.$$

If $m = 4$ this implies that $r$ is a scalar multiple of $q^2$; otherwise the above manipulations can be repeated. Eventually, this process has to stop and in the end we get that $m = 2l$ and that, up to a multiplication by a scalar, $r = q^l$.

Now, (14) can be re-written as

$$(2l-1)\cdot 2x_k l q^{l-1} = x_k(2nlq^{l-1} + 4l(l-1)q^{l-1}).$$

This gives $n = 1$ and it follows that $x_k r - r x_k = 0$ cannot be satisfied for all $k$. □

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