Weierstrass Formulae for Surfaces of Constant Mean curvature 1 in $\mathbb{H}^3$

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0. Introduction

$I\!PSL(2,\mathbf{C})$ is naturally endowed with a conformal structure by left translation of its Cartan-Killing form. Viewing $I\!PSL(2,\mathbf{C})$ as the oriented orthonormal frame bundle of $I\!H^3$, the 3-dimensional hyperbolic space of curvature -1, Bryant [4] showed that a holomorphic curve in $I\!PSL(2,\mathbf{C})$ that is null with respect to this conformal structure projects to $I\!H^3$ to give a surface of constant mean curvature 1, and furthermore that every such surface in $I\!H^3$ arises in this way. This is analogous to the fact that holomorphic curves in $\mathbf{C}^3$ that are null with respect to the complexification of the Euclidean structure on $I\!R^3$ project to $I\!R^3$ to give surfaces of constant mean curvature 0, i.e. minimal surfaces.

In this paper we give a simple characterization of such null curves in $I\!PSL(2,\mathbf{C})$ in terms of the geometry of its compactification, $I\!P_3$. This facilitates the study of a natural correspondence which exists between these null curves and *free* holomorphic curves on a non-singular quadric surface in the dual $I\!P_3^*$, see 2.7. This correspondence is the analogue of the correspondence between null curves in $\mathbf{C}^3$ and curves on the *singular* quadric surface in $I\!P_3^*$ given by the cone over a quadric curve. That correspondence was first discovered by Lie and underlies the classical Weierstrass representation formulae for minimal surfaces in $I\!R^3$, see [12]. Both correspondences are particular instances of the classical duality between curves in $I\!P_3$ and $I\!P_3^*$ determined by osculation. In another direction they may be generalized to a correspondence for null curves in an Einstein-Weyl space, see [14].

We describe analogues of the Weierstrass formulae that generate null meromorphic curves in $I\!PSL(2,\mathbf{C})$ from pairs of meromorphic functions on a Riemann surface of arbitrary genus and thus determine explicit formulae for surfaces of constant mean curvature 1 in $I\!H^3$. We describe how various features of the geometry of the null curve are determined by the meromorphic functions. In addition we describe moduli for null meromorphic curves in $I\!PSL(2,\mathbf{C})$ and calculate rational and elliptic examples. Finally we outline a possible application of this work to the study of monopoles on hyperbolic space.

There is renewed interest in surfaces in $\mathbb{H}^n$ and surfaces of constant mean curvature in particular, see [10] and the references cited therein. Recently, Bobenko [3], using methods from soliton theory, has given explicit constructions for all constant mean curvature tori in $\mathbb{R}^3$, S^3 and $\mathbb{H}^3$ in terms of theta functions.

The differential geometric significance of our work rests on Bryant's results [4]. Our approach however, which is similiar to that in [12], is derived from Hitchin's work in [7]. It has been drawn to our attention that Kerbaugh [9] has derived Weierstrass formulae similiar to those described in section 3.

1. Duality and the Einstein-Weyl Structure

(1.1) $\mathbb{P}SL(2,\mathbf{C})$ may be viewed as the complement in $\mathbb{P}_3$ of the non-singular quadric surface, $Q_2 = (ad - bc = 0)$. Let $Q_2^* \subset \mathbb{P}_3^*$ parameterize the collection of hyperplanes in $\mathbb{P}_3$ that lie tangent to Q_2; it is clear from duality that Q_2 parameterizes the hyperplanes in $\mathbb{P}_3^*$ that are tangent to Q_2^*, and $Q_2^* \cong Q_2$. Thus points of $\mathbb{P}SL(2,\mathbf{C})$ are characterized by the fact that they are dual to hyperplanes in $\mathbb{P}_3^*$ that are not tangent to Q_2^*.

Q_2^* is isomorphic to $\mathbb{P}_A \times \mathbb{P}_B$, where $\mathbb{P}_A$ and $\mathbb{P}_B$ parameterize the families of A-lines and B-lines on Q_2^* respectively, see [5] . Each factor is isomorphic to $\mathbb{P}_1$ and non-tangential hyperplane intersections comprise the (1,1)-homology class. Consequently, identification of the factors by a choice of a non-tangential hyperplane intersection together with an ordering of the factors identifies $\mathbb{P}SL(2,\mathbf{C})$ with $\mathrm{Aut}(\mathbb{P}_1)$, since $\mathbb{P}_q$, the (1,1)-curve on Q_2^* dual to $q \in \mathbb{P}SL(2,\mathbf{C})$, then gives the graph of an automorphism of $\mathbb{P}_1$. We identify the factors using the curve that corresponds with, e, the usual identity element of $\mathbb{P}SL(2,\mathbf{C})$.

(1.2) Hitchin [7] has shown that the moduli space of a complete family of rational curves on a complex surface which have self-intersection number 2 is naturally an Einstein-Weyl space. So, in particular, $\mathbb{P}SL(2,\mathbf{C})$ has such a structure. He uses a theorem of Kodaira which describes the tangent space at a point of the moduli space. Here this says that there is a canonical isomorphism:

$$\kappa : T_q\mathbb{P}SL(2,\mathbf{C}) \cong H^0(\mathbb{P}_q, \mathcal{O}(N_q)),$$

where $N_q \longrightarrow \mathbb{P}_q$ is the normal bundle. Since N_q has degree 2, the set of global holomorphic sections which possess a double root on $\mathbb{P}_q$ gives, via this isomorphism, a null cone in $T_q\mathbb{P}SL(2,\mathbf{C})$: thus one obtains a conformal structure on $\mathbb{P}SL(2,\mathbf{C})$. A direction at $q \in \mathbb{P}SL(2,\mathbf{C})$ is determined by a pair of points $\{\eta,\mu\} \subset \mathbb{P}_q$ and the set of (1,1)-curves whose intersection with $\mathbb{P}_q$ is $\{\eta,\mu\}$ gives a curve on $\mathbb{P}SL(2,\mathbf{C})$ with this direction at q. This determines a distinguished class of curves on $\mathbb{P}SL(2,\mathbf{C})$: Hitchin shows that they are the geodesics of a projective connection and furthermore that if $\eta = \mu$ then the corresponding geodesic is null.

(1.3) **Proposition** The totally geodesic null hypersurfaces of the Einstein-Weyl structure on $\mathbb{P}SL(2,\mathbf{C})$ are cut out by the hyperplanes in $\mathbb{P}_3$ that lie tangent to Q_2.

Proof Fix $\mu \in \mathbf{Q}_2^*$ and consider the dual hyperplane μ^* and the hypersurface $\mathcal{S}_\mu = \mu^* \cap \mathbb{P}SL(2,\mathbf{C})$. The tangent directions $T_q\mathcal{S}_\mu \subset H^0(\mathbb{P}_q, \mathcal{O}(N_q))$ give the infintesimal deformations of $\mathbb{P}_q$ in directions on $\mathcal{S}_\mu$: these are of the form $\{\sigma; \sigma(\mu) = 0\}$. It follows immediately from 1.2 that the geodesics thus determined lie on $\mathcal{S}_\mu$ and furthermore that there is a unique null geodesic through q that lies on $\mathcal{S}_\mu$. Hence $\mathcal{S}_\mu$ is a totally geodesic null hypersurface of $\mathbb{P}SL(2,\mathbf{C})$.

Conversely, suppose that $q \in \mathbb{P}SL(2,\mathbf{C})$ lies on a totally geodesic null hypersurface $\mathcal{S}$. $T_q\mathcal{S}$ is a null plane in $H^0(\mathbb{P}_q, \mathcal{O}(N_q))$ and is therefore of the form $\{\sigma; \sigma(\mu) = 0\}$ for some fixed $\mu \in \mathbb{P}_q$. For any other $q' \in \mathcal{S}$, $\mathbb{P}_{q'}$ intersects $\mathbb{P}_q$ at 2 points, counted with multiplicity, giving the geodesic in $\mathbb{P}SL(2,\mathbf{C})$ that passes through q and q'. Since $\mathcal{S}$ is totally geodesic this geodesic lies on $\mathcal{S}$ and consequently must give a tangent direction at q, which implies that $\mu \in \mathbb{P}_{q'} \cap \mathbb{P}_q$. So for any $q' \in \mathbb{P}SL(2,\mathbf{C})$, $\mathbb{P}_{q'}$ passes through μ and hence $\mathcal{S} \subset \mathcal{S}_\mu$. We suppose that every totally geodesic surface is extended to its maximal domain of definition and hence $\mathcal{S} = \mathcal{S}_\mu$.

Remark Observe that $\mathbb{P}_q$ parameterizes the set of totally geodesic null hypersurfaces which pass through $q \in \mathbb{P}SL(2,\mathbf{C})$.

(1.4) A deformation of a (1,1)-curve $\mathbb{P}_q$ on $\mathbf{Q}_2^*$ amounts to the same thing as a curve of automorphisms of $\mathbb{P}_1$ which passes through q, viewed as an element of $\mathrm{Aut}(\mathbb{P}_1)$ following 1.1. Let $\pi_1, \pi_2 : \mathbf{Q}_2^* \longrightarrow \mathbb{P}_1$ denote the projection maps into the factors of $\mathbf{Q}_2^* \simeq \mathbb{P}_1 \times \mathbb{P}_1$. The following gives an isomorphism, $N_q \cong (q \circ \pi_1|_{\mathbb{P}_q})^{-1} T\mathbb{P}_1$:

$$T_{(\zeta,q(\zeta))}(\mathbb{P}_1 \times \mathbb{P}_1) \longrightarrow T_{q(\zeta)}\mathbb{P}_1,$$

where $(u,v) \longrightarrow v - dq(u)$, since the kernel is $T_{(\zeta,q(\zeta))}\mathbb{P}_q$. Hence the bijectivity of $\pi_1|_{\mathbb{P}_q}$ gives the isomorphism:

$$\iota : H^0(\mathbb{P}_1, \mathcal{O}(q^{-1}T\mathbb{P}_1)) \cong H^0(\mathbb{P}_q, \mathcal{O}(N_q)),$$

where $\iota(\sigma) = \sigma \circ \pi_1|_{\mathbb{P}_q}$.

In these terms Kodaira's isomorphism may be described as follows: suppose that $w : U \longrightarrow \mathbb{P}SL(2,\mathbf{C}), U \subset \mathbf{C}$ open, is such that $w(u_0) = q$, then writing $w : U \times \mathbb{P}_1 \longrightarrow \mathbb{P}_1$ we have

$$\iota^{-1} \circ \kappa(\partial w(\frac{\partial}{\partial u}|_{u_0}))(\zeta) = \frac{\partial w}{\partial u}(u_0,\zeta)\frac{d}{d\zeta}|_{q(\zeta)}$$

(1.5) **Proposition** The conformal structure on $\mathbb{P}SL(2,\mathbf{C})$ determined by Kodaira's isomorphism coincides with that induced by left translation of the Cartan-Killing form.

Proof At the identity element, $\iota^{-1} \circ \kappa : T_e\mathbb{P}SL(2,\mathbf{C}) \simeq H^0(\mathbb{P}_1, \mathcal{O}(T\mathbb{P}_1))$ gives the usual isomorphism of Lie algebras determined by the identification of $\mathbb{P}SL(2,\mathbf{C})$ with $\mathrm{Aut}(\mathbb{P}_1)$ described in 1.1. It is easily checked that the Cartan-Killing form on $H^0(\mathbb{P}_1, \mathcal{O}(T\mathbb{P}_1))$ determines the same null cone as described in 1.2 and hence it only remains to observe that the conformal structure induced by $\iota^{-1} \circ \kappa$ is left invariant, which is clear.

2. The Gauss Transform, Osculation and the Correspondence

(2.1) Let M be a Riemann surface and recall that a non-constant holomorphic curve $w : M \longrightarrow \mathbb{P}SL(2,\mathbf{C})$ is said to be *null* if $\partial w(\frac{\partial}{\partial u})$ is a null vector for all $u \in M$.

Away from the zeros of ∂w, $\kappa(\partial w(\frac{\partial}{\partial u}))$ has a double root on $\mathbb{P}_{w(u)}$, at $\Gamma_w(u)$ say. Γ_w extends over the zeros of ∂w in the usual way and thus one obtains a holomorphic map $\Gamma_w : M \longrightarrow Q_2^*$ which, because of the close analogy with the Euclidean case [12], we call the *Gauss transform* of w. Projection of Γ_w to the $\mathbb{P}_1$-factors yields a pair of *Gauss maps* to $\mathbb{P}_1$: $\Gamma_w = (\gamma_1, \gamma_2)$.

Remark $\Gamma_w(u)$ gives the totally geodesic null hypersurface of $I\!PSL(2,\mathbf{C})$ that is determined at $w(u)$ by $\partial w(\frac{\partial}{\partial u})$.

From 1.4 it follows that if $\partial w(u) \neq 0$ then $\frac{\partial w}{\partial u}(u,\_)$ has a double zero at $\gamma_1(u) \in I\!P_1$. This gives:

Proposition A non-constant map $w : M \longrightarrow I\!PSL(2,\mathbf{C})$ is null iff

$$\frac{\partial w}{\partial u}(u,\zeta) = \mathcal{O}[(\zeta - \gamma_1(u))^2].$$

Note that $\gamma_2(u) = w(u, \gamma_1(u))$.

(2.2) Recall that a holomorphic curve $\mu : M \longrightarrow I\!P_3$ is said to be *full* if $\mathcal{A} = \mu(M)$ does not lie on any hyperplane (and that if $\mathcal{A}$ is algebraic then its degree is at least 3). The map $\mu^* : M \longrightarrow I\!P_3^*$, given on a dense open set by

$$\mu^*(u) = span\{\mu_U(u), \partial\mu_U(\frac{\partial}{\partial u}), \partial^2\mu_U(\frac{\partial}{\partial u})\},$$

where $\mu_U : U \longrightarrow \mathbf{C}^4$ is a lift of μ over U, the domain of a coordinate chart in M, is well-defined and gives the *dual curve* of μ. The same construction applied to μ^* yields μ.

At a point $u \in M$ where $\mathcal{A}$ is smooth, $\mu^*(u)$ gives the hyperplane of $I\!P_3$ that intersects $\mathcal{A}$ at $\mu(u)$ with multiplicity (at least) 3. If $\mathcal{A} \subset I\!P_3$ is an algebraic curve this determines a birational map between $\mathcal{A}$ and a dual curve $\mathcal{A}^* \subset I\!P_3^*$. See [5] or [6] for further details.

(2.3) It is useful for our purposes to observe that when $\mathcal{A} = \mu(M) \subset I\!P_3^*$ lies on $\mathbf{Q}_2^* \simeq I\!P_1 \times I\!P_1$, osculation may be described as follows.

An automorphism of $I\!P_1$ is determined by its 2-jet at any point and hence there is a (canonical) holomorphic map

$$\nu : \mathcal{E}_{\boldsymbol{P}_1} \longrightarrow Aut(I\!P_1),$$

where $\mathcal{E}_{\boldsymbol{P}_1}$ is the étalé space of the sheaf whose stalk at $\zeta \in I\!P_1$ comprises the germs of holomorphic functions with non-zero derivative at ζ. ν is given on the stalk at ζ by sending a germ at ζ to the uniquely determined automorphism that has the same 2-jet there.

Provided that μ is full, it lifts into $\mathcal{E}_{\mathbf{P}_1}$ over a dense open subset $M_* \subset M$, and composition of this lift with ν determines a holomorphic map

$$\mu_* : M_* \longrightarrow Aut(\mathbb{P}_1).$$

It is clear that the graph of the *osculating automorphism* thus determined at $u \in M_*$ is cut out by the hyperplane of $\mathbb{P}_3$ that osculates there in the classical sense, and hence $\mu_* = \mu^*|_{M_*}$.

(2.4) **Proposition** The holomorphic curve $\nu : \mathcal{E}_{\mathbf{P}_1} \longrightarrow \mathbb{P}SL(2,\mathbf{C})$ is null and its Gauss transform is given by:

$$\Gamma_\nu([f]_\zeta) = (\zeta, f(\zeta)).$$

(Cf. Theorem 3.6 of [12].)

Proof By definition of ν, for any $[f]_{\zeta_0} \in \mathcal{E}_{\mathbf{P}_1}$ there is some neighbourhood of ζ_0 on which the following equation holds:

$$f(\zeta) - \nu([f]_{\zeta_0}, \zeta) = \mathcal{O}[(\zeta - \zeta_0)^3].$$

In the local chart $[f]_{\zeta_0} \longrightarrow \zeta_0$ on $\mathcal{E}_{\mathbf{P}_1}$, differentiation of this equation gives

$$\frac{\partial \nu}{\partial \zeta_0}([f]_{\zeta_0}, \zeta) = \mathcal{O}[(\zeta - \zeta_0)^2],$$

so it follows from 2.1 that ν is a null curve and $\gamma_1([f]_{\zeta_0}) = \zeta_0$.

$$\begin{aligned}
\Gamma_\nu([f]_{\zeta_0}) &= (\gamma_1([f]_{\zeta_0}), \nu([f]_{\zeta_0}, \gamma_1([f]_{\zeta_0})) \\
&= (\zeta_0, \nu([f]_{\zeta_0}, \zeta_0)) \\
&= (\zeta_0, f(\zeta_0)),
\end{aligned}$$

from the first equation above.

(2.5) **Corollary** Suppose that the image of a full curve $\mu : M \longrightarrow \mathbb{P}_3^*$ lies on Q_2^*. Then $\mu^*(M) \cap \mathbb{P}SL(2,\mathbf{C})$ is a null curve in $\mathbb{P}SL(2,\mathbf{C})$.

(2.6) We now show that all full null curves in $\mathbb{P}SL(2,\mathbf{C})$ arise in this way and that they are dual, as curves in $\mathbb{P}_3$, to their Gauss transforms.

Proposition Let M be a Riemann surface and suppose that $w : M \longrightarrow \mathbb{P}SL(2,\mathbf{C})$ is a null holomorphic curve such that γ_1 is non-constant. Then $w = \Gamma_w^*$.

Proof Suppose $\partial\gamma_1(u_0) \neq 0$ and that γ_1^{-1} is an inverse for γ_1 on a neighbourhood of $\zeta_0 = \gamma_1(u_0)$ such that $\gamma_1^{-1}(\zeta_0) = u_0$.

The Gauss transform is given by $\Gamma_w(u) =$ [redacted] $w(u, \gamma_1(u))$ and hence $f(\zeta) := w(\gamma_1^{-1}(\zeta), \zeta)$ gives a local implicit description over $\mathbb{P}_1$ of part of it. Now,

$$f'(\zeta_0) = \frac{\partial w}{\partial u}(u_0, \zeta_0)\frac{d\gamma_1}{d\zeta}^{-1}(u_0, \zeta_0) + \frac{\partial w}{\partial \zeta}(u_0, \zeta_0).$$

But, from the nullity criterion of 2.1, $\frac{\partial w}{\partial u}(u_0, \zeta_0) \equiv 0$ and hence

$$f''(\zeta_0) = \frac{\partial^2 w}{\partial u \partial \zeta}(u_0, \zeta_0)\frac{d\gamma_1}{d\zeta}^{-1}(u_0, \zeta_0) + \frac{\partial^2 w}{\partial \zeta^2}(u_0, \zeta_0).$$

Again from 2.1, $\frac{\partial^2 w}{\partial u \partial \zeta}(u_0, \zeta_0) \equiv 0$. Consequently,

$$\nu([f]_{\zeta_0}) = w(\gamma_1^{-1}(\zeta_0), \_).$$

So w coincides with Γ_w^* on an open subset and hence, by uniqueness of analytic continuation, they coincide where the latter is defined.

Remark From this, together with the fact that $\mu^{**} = \mu$, it follows that $\Gamma_w = w^*$, i.e. the Gauss transform of a null curve in $\mathbb{P}SL(2,\mathbf{C})$ is given by osculating the curve as a subset of $\mathbb{P}_3$ in the classical sense. Note that if $w : M \longrightarrow \mathbb{P}SL(2,\mathbf{C})$ is full then γ_1 is non-constant.

(2.7) **Corollary** *Null curves in $\mathbb{P}SL(2,\mathbf{C})$ are characterized by the fact that the hyperplanes of $\mathbb{P}_3$ that osculate them lie tangent to Q_2, the quadric at infinity of $\mathbb{P}SL(2,\mathbf{C})$.*

Accordingly, we call a curve $\mu : M \longrightarrow \mathbb{P}_3$ such that $\mu^*(M) \subset Q_2^*$, a null curve. We summarise these results in the following

Theorem If a full curve $\mathcal{A} \subset \mathbb{P}_3^*$ lies on Q_2^* then $\mathcal{A}^*$ is the extension to $\mathbb{P}_3$ of a full null curve in $\mathbb{P}SL(2,\mathbf{C})$. Every full null curve in $\mathbb{P}SL(2,\mathbf{C})$ arises in this way.

Remarks The above calculations could equally well have been phrased in terms of γ_2.

The degenerate, non-full, case is trivial and left as a simple exercise.

3. Weierstrass Formulae and the Geometry of Null Curves

(3.1) Suppose that (g, f) is a pair of meromorphic functions on a Riemann surface M and that g is not constant. It follows from inspection of $\nu : \mathcal{E}_{\mathbb{P}_1} \longrightarrow \mathbb{P}SL(2, \mathbf{C})$, as described in 2.3, that the following formulae give a null meromorphic curve $\omega : M^* \longrightarrow \mathbb{P}SL(2, \mathbf{C})$:

$$\omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where M^* is M punctured at a finite positive number of points and

$$\begin{aligned} \alpha &= (f')^{\frac{1}{2}} - \frac{1}{2} f (f')^{-\frac{3}{2}} f'' \\ \beta &= f\{(f')^{-\frac{1}{2}} + \frac{1}{2} g (f')^{-\frac{3}{2}} f''\} - g (f')^{\frac{1}{2}} \\ \gamma &= -\frac{1}{2} (f')^{-\frac{3}{2}} f'' \\ \delta &= (f')^{-\frac{1}{2}} + \frac{1}{2} g (f')^{-\frac{3}{2}} f'', \end{aligned}$$

where

$$f' = \frac{df}{dg}, f'' = \frac{d^2 f}{dg^2}.$$

If (g, f) are such that $f = \theta(g)$, for some $\theta \in \mathbb{P}SL(2, \mathbf{C})$ then the resulting curve is constant.

Observe that if both f and g are non-constant then the pair (f, g) generates w^{-1} from the above. This accords with the identification in 1.1 of $\mathbb{P}SL(2, \mathbf{C})$ with $\mathrm{Aut}(\mathbb{P}_1)$.

(3.2) Conversely, it follows from 2.7 that every full null meromorphic curve in $\mathbb{P}SL(2, \mathbf{C})$ has such a representation in terms of its Gauss maps. (We restrict attention here to meromorphic curves simply for the sake of simplicity. It is clear that similiar statements hold for general curves.)

(3.3) Explicit formulae for the surface of constant mean curvature 1 in $I\!H^3 \subset I\!R^{3,1}$ that is determined by projection of w are given by solving:

$$\omega\bar{\omega}^t = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$

(3.4) Given meromorphic functions (g, f) on a Riemann surface M, we characterize, in terms of (g, f), the *ends* of the null curve generated by 3.1, i.e. those points of M in the vicinity of which the null curve determined by (g, f) leaves every relatively compact subset of $I\!PSL(2, \mathbf{C})$. This gives the end structure of the corresponding constant mean curvature 1 surface in $I\!H^3$.

Recall from 1.1 that points on the quadric at infinity, Q_2, give the tangential hyperplane intersections with $Q_2^* \simeq I\!P_1 \times I\!P_1$. Hence it follows that we simply have to characterize those points where $(g, f)(M)$ osculates such an intersection:

Theorem Suppose that the meromorphic functions (g, f) are such that g is non-constant and $f \not\equiv \theta(g)$ for any $\theta \in I\!PSL(2, \mathbf{C})$. Let $\mathcal{D}_\infty(g)$, $\mathcal{D}_\infty(f)$ and $\mathcal{D}_\infty(f, g)$ denote the divisor of poles of g, f and their intersection respectively. Then the *ends* of the null meromorphic curve in $I\!PSL(2, \mathbf{C})$ generated by 3.1 are given as follows:

$$\{\xi \in M - (\mathcal{D}_\infty(g) \cup \mathcal{D}_\infty(f))\,;\, \frac{df}{dg}(\xi) = 0 \; or \; \frac{d^2 f}{dg^2}(\xi) = \infty\},$$

$$\{\xi \in \mathcal{D}_\infty(g) - \mathcal{D}_\infty(f, g)\,;\, \frac{df}{d(\frac{1}{g})}(\xi) = 0 \; or \; \frac{d^2 f}{d(\frac{1}{g})^2}(\xi) = \infty\},$$

$$\{\xi \in \mathcal{D}_\infty(f) - \mathcal{D}_\infty(f, g)\,;\, \frac{d(\frac{1}{f})}{dg}(\xi) = 0 \; or \; \frac{d^2(\frac{1}{f})}{dg^2}(\xi) = \infty\},$$

$$\{\xi \in \mathcal{D}_\infty(f, g)\,;\, \frac{d(\frac{1}{f})}{d(\frac{1}{g})}(\xi) = 0 \; or \; \frac{d^2(\frac{1}{f})}{d(\frac{1}{g})^2}(\xi) = \infty\}.$$

Remark Symmetry here in (g, f) follows from

$$\frac{d^2g}{df^2} = \frac{d^2f}{dg^2}\left(\frac{dg}{df}\right)^3.$$

(It is straightforward to recast the above for arbitrary holomorphic functions.)

(3.5) It follows from 3.1 that

$$\frac{d}{d\xi}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = -\frac{1}{2}\frac{dg}{d\xi}\left(\frac{df}{dg}\right)^{-\frac{1}{2}}\mathcal{S}_g(f)\begin{pmatrix} f & -gf \\ 1 & -g \end{pmatrix},$$

where $\mathcal{S}_g(f)$ is the *Schwarzian derivative* of f with respect to g. Away from ends, the vanishing of $\mathcal{S}_g(f)$ at a point means that the osculating section actually *hyperosculates* the curve in Q_2^*, i.e. agrees with the curve at that point to order 3.

Note that an automorphism hyperosculates iff the hyperplane of $\mathbb{P}_3^*$ that cuts out its graph hyperosculates the curve in the classical sense.

(3.6) Lifting ω locally over $U \subset M$ to a curve $\tilde{\omega}$ in $\mathbf{C}^4$ gives a null curve with respect to the complexified Euclidean structure on $\mathbb{R}^4$ and thus a minimal surface $\phi : U \longrightarrow \mathbb{R}^4$. Note that the branch points of the metric induced by the (branched) minimal immersion in $\mathbb{R}^4$ are given by the vanishing of $\mathcal{S}_g(f)$.

Remark For $f : M \longrightarrow \mathbb{P}_3$ let $f^* : M \longrightarrow \mathbb{P}_3^*$ describe the osculating hyperplanes and $f^\# : M \longrightarrow G(2,4)$ the osculating lines. $G(2,4) \simeq \mathbf{C}^4 \cup Q_3$ and such $f^\#$ give null curves in $\mathbf{C}^4$, and thus minimal surfaces in $\mathbb{R}^4$, see [11], [13]. Hence there is a globally defined minimal surface in $\mathbb{R}^4$ associated to a null curve in $\mathbb{P}SL(2,\mathbf{C})$. In the algebraic case the degrees of these curves are linked by the following Plücker formula:

$$deg(f) - 2deg(f^\#) + deg(f^*) = 2g - 2 - \beta^\#,$$

where g is the genus of M and $\beta^\#$ is the total ramification of $f^\#$, see [5].

(3.7) Suppose that $\mathcal{A} \subset Q_2^* \simeq \mathbb{P}_1 \times \mathbb{P}_1$ is an irreducible algebraic curve, which is full as a curve in $\mathbb{P}_3^*$. Let $\mathcal{E}$ denote the divisor $\mathbb{P}_1 \times \{0\}$ and $\mathcal{F}$ denote the divisor $\{0\} \times I\!P_1$. The corresponding null meromorphic curve $\psi_{\mathcal{A}}$ in $I\!PSL(2, \mathbf{C})$ has as domain of definition the desingularization $\delta_{\mathcal{A}} : \tilde{\mathcal{A}} \longrightarrow \mathcal{A}$.

$(\mathcal{A} \cdot \mathcal{F}, \mathcal{A} \cdot \mathcal{E}) = (k_1, k_2)$, is the bidegree of $\mathcal{A}$ and gives the bidegree of the Gauss transform $\Gamma_{\psi_{\mathcal{A}}} = (\gamma_1, \gamma_2)$.

$k_1 + k_2 = d$, the degree of $\mathcal{A}$ as a curve in $I\!P_3^*$.

$\mathcal{A}$ lies in the linear system $|k_1\mathcal{E} + k_2\mathcal{F}|$: fullness, together with irreducibility imply that $k_1, k_2 > 0$, see [6] section V.2. It follows from 2.7 that these linear systems give natural compactifications of the *moduli spaces* of null meromorphic curves in $I\!PSL(2, \mathbf{C})$ and thus the corresponding 'algebraic' surfaces of constant mean curvature 1 in $I\!H^3$.

The genus of a generic curve $\mathcal{A} \in |k_1\mathcal{E} + k_2\mathcal{F}|$, which is smooth, is given by the adjunction formula: $g = k_1k_2 - (k_1 + k_2) + 1$, see [4], [5].

Remark For such an $\mathcal{A}$, full in $I\!P_3^*$, there are $12k_1k_2 - 8(k_1 + k_2)$ points where $\mathcal{A}$ is hyperosculated.

(3.8) **Example** For $(g(\xi), f(\xi)) = (\xi^q, \xi^p)$, $p, q \geq 0$ coprime, 3.1 gives the null meromorphic curve $\psi_{\mathcal{A}}$:

$$\begin{aligned}
\alpha(\xi) &= \frac{p+q}{2\sqrt{pq}}\xi^{\frac{p-q}{2}} \\
\beta(\xi) &= \frac{q-p}{2\sqrt{pq}}\xi^{\frac{p+q}{2}} \\
\gamma(\xi) &= \frac{q-p}{2\sqrt{pq}}\xi^{-\frac{p+q}{2}} \\
\delta(\xi) &= \frac{p+q}{2\sqrt{pq}}\xi^{\frac{q-p}{2}}
\end{aligned}$$

$\mathcal{A} = (g, f)(I\!P_1)$ lies in $|q\mathcal{E} + p\mathcal{F}|$ and is smooth iff $q = 1$ or $p = 1$.

If $p = q = 1$ the $\mathcal{A}^*$ is simply a point, otherwise $\psi_{\mathcal{A}}$ has ends at 0 and ∞.

(3.9) **Example** Let $\Lambda \subset \mathbf{C}$ be a lattice with Eisenstein constants g_2, g_3 and $\wp$ be the associated Weierstrass function. The elliptic curve $\mathcal{A}$ in $I\!P_1 \times$

$I\!P_1 \simeq Q_2^*$, given by completion of $\eta^2 = 4\zeta^3 - g_2\zeta - g_3$, is parameterized by $(\wp, \wp') : \mathbf{C}/\Lambda \longrightarrow \mathcal{A}$. $\mathcal{A}$ lies in $|2\mathcal{E} + 3\mathcal{F}|$ and has degree 5 as a curve in $I\!P_3^*$. The virtual genus of $\mathcal{A}$ is 2, which is the genus of a smooth curve in $|2\mathcal{E}+3\mathcal{F}|$: the difference between this and the real genus of $\mathcal{A}$ is a contribution from a singularity.

Osculation of $\mathcal{A}$, gives the following genus 1 null curve in $I\!PSL(2, \mathbf{C})$:

$$
\begin{aligned}
\alpha &= \frac{3g_2{}^2 + 48g_3\wp - 24g_2\wp^2 + 240\wp^4}{2\sqrt{2}(-g_2 + 12\wp^2)^{\frac{3}{2}}(-g_3 - g_2\wp + 4\wp^3)^{\frac{1}{4}}} \\
\beta &= \frac{4g_2g_3 + g_2{}^2\wp - 96g_3\wp^2 - 40g_2\wp^3 - 48\wp^5}{2\sqrt{2}(-g_2 + 12\wp^2)^{\frac{3}{2}}(-g_3 - g_2\wp + 4\wp^3)^{\frac{1}{4}}} \\
\gamma &= \frac{g_2{}^2 + 48g_3\wp + 24g_2\wp^2 - 48\wp^4}{2\sqrt{2}(-g_2 + 12\wp^2)^{\frac{3}{2}}(-g_3 - g_2\wp + 4\wp^3)^{\frac{3}{4}}} \\
\delta &= \frac{4g_2g_3 + 3g_2{}^2\wp - 96g_3\wp^2 - 88g_2\wp^3 + 240\wp^5}{2\sqrt{2}(-g_2 + 12\wp^2)^{\frac{3}{2}}(-g_3 - g_2\wp + 4\wp^3)^{\frac{3}{4}}}
\end{aligned}
$$

This curve has 4 ends. Note that variation of g_2, g_3 gives a family of such curves.

4. Final Remarks

(4.1) Atiyah [1] has shown that finite energy solutions of the $SU(2)$-Bogomolny equations over $I\!H^3$ may be encoded into an auxiliary *spectral curve* $\mathcal{S}$, which is an algebraic curve on Q_2^* . $\mathcal{S}$ has bidegree (k, k) where k is the magnetic charge of the monopole. This is analogous to Hitchin's [8] enciphering of finite energy solutions of the $SU(2)$-Bogomolny equations over $I\!R^3$ into algebraic curves on a singular quadric surface in $I\!P_3$. Atiyah suggests that various aspects of the Euclidean case might be elucidated by the limiting behaviour of monopoles on $I\!H^3(-t) \longrightarrow I\!R^3$, as the curvature $t \longrightarrow 0$. This corresponds to the degeneration of a family of non-singular quadric surfaces in $I\!P_3$: $Q(-t) \longrightarrow$ singular quadric cone as $t \longrightarrow 0$.

(4.2) It follows from 2.7 that osculation of $\mathcal{S}$ determines a null meromorphic curve in $I\!PSL(2, \mathbf{C})$ and thus a surface Σ, of constant mean curvature 1 in $I\!H^3$. The monopole may be recovered from Σ. *How does the geometry of* Σ *reflect the structure of the monopole?* In particular, the following features of Σ might elucidate the monopole's structure:

- the ends of Σ
- the points of hyperosculation of $\mathcal{S}$
- the total Gaussian curvature of the metric induced on $\mathcal{S}$.

(4.3) Bryant asserts that the total Gaussian curvature induced by an algebraic mean curvature 1 immersion is some (negative) integer multiple of 4π. When the dual curve has bidegree (k_1, k_2) it seems likely that this is some expression symmetric in k_1, k_2. It would follow that the charge of a hyperbolic monopole may be written as the integral of the Gaussian curvature of the natural metric induced on the spectral curve.

This is analogous to the observation we make in the Euclidean case [15] where furthermore, we show that the null curve determined by osculation of the spectral curve generates the singularity set of the extended solution on $\mathbf{C}^3$ and the corresponding integral representation has a residue theoretic interpretation. (The latter is analogous to the fact that the charge of an instanton equals the degree of the corresponding hypersurface of jumping lines, see [2].)

(4.4) The analogue of Atiyah's limiting process for us realises minimal surfaces in $\mathbb{R}^3$ as limits of constant mean curvature t surfaces in $\mathbb{H}^3(-t)$ as $t \longrightarrow 0$. This follows from the degeneration of $\mathbf{Q}(-t)$ and the corresponding "$\mathbb{P}SL(2,\mathbf{C})$" $\longrightarrow \mathbf{C}^3$.

Does this process underlie the 'Lawson Correspondence' that is referred to in [4]?

(4.5) Polar decomposition gives a map $SL(2,\mathbf{C}) \longrightarrow SU(2)$. What surfaces in S^3 are generated by projection of null curves?

(4.6) Use of real structures should facilitate the construction of non-orientable examples of constant mean curvature 1 surfaces in $\mathbb{H}^3$.

(4.7) There should be an analogous construction for $\mathbb{H}^4$, see [7].

References

[1] M.F.Atiyah, *Magnetic monopoles on hyperbolic space*, in Vector Bundles on Algebraic Varieties, Oxford University Press 1987, 95-148.

[2] M.F.Atiyah and R.S.Ward, *Instantons and algebraic geometry*, Comm. Math. Phys. 55, (1977), 117-124.

[3] A.I.Bobenko, *All constant mean curvature tori in $\mathbb{R}^3$, S^3, $\mathbb{H}^3$ in terms of theta functions*, Math. Ann. 290 (1991), 209-245.

[4] R.L.Bryant, *Surfaces of Mean Curvature One in Hyperbolic Space*, Théorie des Variétés Minimal et Applications, Asterisque 154-155 (1987), 321–347.

[5] P.Griffiths and J.Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, 1978.

[6] R.Hartshorne, *Algebraic Geometry*, Springer-Verlag 1977.

[7] N.J.Hitchin, *Complex Manifolds and Einstein's Equations*, Twistor Geometry and Non-Linear Systems, Springer Lecture Notes 970, Springer-Verlag 1980.

[8] N.J.Hitchin, *Monopoles and geodesics* Comm. Math. Phys., 83 no. 4, (1982) 579-602.

[9] G.Kerbaugh, *Surfaces of constant mean curvature 1 in hyperbolic space*, Thesis, S.U.N.Y Stony Brook 1985.

[10] N.J.Korevaar, R.Kusner, W.H.Meeks III, B.Solomon, *Constant mean curvature surfaces in hyperbolic space*, Amer. J. Math. 114 no.1, (1992),1-43.

[11] W.T.Shaw, *Twistors, minimal surfaces and strings*, Class. Quantum Grav. 2 (1985), L113-L119.

[12] A.J.Small, *Minimal surfaces in $\mathbb{R}^3$ and algebraic curves*, to appear in Differential Geometry and Its Applications.

[13] A.J.Small, *Minimal surfaces in $\mathbb{R}^4$ and the Klein correspondence*, in preparation.

[14] A.J.Small, *Null curves in self-dual Einstein manifolds and Einstein-Weyl spaces*, in preparation.

[15] A.J.Small, *Monopole charge, null curves and jumping lines*, in preparation.

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