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BINARY RECURRENCES FOR WHICH POWERS OF 2 ARE DISCRIMINATING MODULI

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Abstract. Given a sequence of distinct positive integers $w_0, w_1, w_2, \ldots$ and any positive integer n, we define the discriminator function $\mathcal{D}_{\mathbf{w}}(n)$ to be the smallest positive integer m such that $w_0, \ldots, w_{n-1}$ are pairwise incongruent modulo m. In this paper, we classify all binary recurrent sequences $\{w_n\}_{n\geq 0}$ consisting of different integer terms such that $\mathcal{D}_{\mathbf{w}}(2^e) = 2^e$ for every $e \geq 1$. For all of these sequences it is expected that one can actually give a fairly simple description of $\mathcal{D}_{\mathbf{w}}(n)$ for every $n \geq 1$. For two infinite families of such sequences this has been done already in 2019 by Faye, Luca and Moree, respectively Ciolan and Moree.

1. Introduction

The *discriminator sequence* of a sequence $\mathbf{w} = \{w_n\}_{n\geq 0}$ of distinct integers is the sequence $\{\mathcal{D}_{\mathbf{w}}(n)\}_{n\geq 0}$ given by

$$\mathcal{D}_{\mathbf{w}}(n) = \min\{m \geq 1 : w_0, \ldots, w_{n-1} \text{ are pairwise distinct modulo } m\}.$$

In other words, $\mathcal{D}_{\mathbf{w}}(n)$ is the smallest integer m that allows one to discriminate (tell apart) the integers $w_0, \ldots, w_{n-1}$ on reducing them modulo m. If not all integers are distinct, but say $w_0, \ldots, w_k$, then we can define $\mathcal{D}_{\mathbf{w}}(j)$ for $j = 1, \ldots, k+1$. Obviously $\mathcal{D}_{\mathbf{w}}(n)$ is non-decreasing as a function of n. Note that since $w_0, \ldots, w_{n-1}$ are in n distinct residue classes modulo $\mathcal{D}_{\mathbf{w}}(n)$, we must have $\mathcal{D}_{\mathbf{w}}(n) \geq n$. On the other hand clearly

$$\mathcal{D}_{\mathbf{w}}(n) \leq \max\{w_0, \ldots, w_{n-1}\} - \min\{w_0, \ldots, w_{n-1}\} + 1.$$

The main problem is to give an easy description or characterization of $\mathcal{D}_{\mathbf{w}}(n)$ (in many cases such a characterization does not seem to exist).

In case w_j is a polynomial in j, the behavior of the discriminator is fairly well understood, see Moree [7] and Zieve [11] and references therein.

An intensively studied class of sequences is that of binary recurrent sequences, cf. the book by Everest et al. [4]. For a generic binary recurrent sequence there is currently no meaningful characterization of its discriminator. An example is provided by the discriminator for the Fibonacci sequence (see Table 1). However, if we have

$$\mathcal{D}_{\mathbf{w}}(2^e) = 2^e \quad \text{for every } e \geq 1, \tag{1}$$

the discriminator behavior tends to be much simpler. It is easy to see that then $\mathcal{D}_w(n) < 2n$. This allows one to exclude many potential discriminator values. Indeed, in general discriminator characterizations for a fixed n proceed by excluding all integers different from $\mathcal{D}_{\mathbf{w}}(n)$ as values. If (1) holds, then typically many powers of two occur as values (cf. Table 2). All known binary recurrent discriminators, described in Examples 1 and 2 below, satisfy (1). Thus, it is natural to ask for a classification of all binary recurrent sequences $\{w_n\}_{n\geq 0}$ such that (1) is satisfied. Note that for any such sequence the terms w_n must be distinct.

Our main result completely answers this question.

Theorem 1. *For integers w_0, w_1, p and q, let $\{w_n\}_{n\geq 0}$ be the sequence defined by*

$$w_{n+2} = pw_{n+1} + qw_n \quad \text{for all } n \geq 0. \tag{2}$$

If $(p, q) \equiv (2, 3) \pmod 4$ and $w_0 + w_1$ is odd, then $\mathcal{D}_{\mathbf{w}}(2^k) = 2^k$ for every $k \geq 1$.

If $(p, q) \not\equiv (2, 3) \pmod 4$ and $k \geq 3$, then $\#\{w_n \pmod{2^k} : 0 \leq n \leq 2^k - 1\} < 2^k$.

Representing the residue classes modulo m by $\overline{j}$, with $0 \leq j \leq m-1$, we can reformulate property (1) as saying that the map from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ given by $\overline{j} \mapsto \overline{u_j}$ is a *permutation* for every m that is a power of two.

We next describe the binary recurrent sequences for which the discriminator has been characterized. They fall into two families. Theorem 1 shows at a glance that for all of them (1) is satisfied.

Family 1. In Faye et al. [5] and its continuation by Ciolan et al. [2] the discriminator $\mathcal{D}_{\mathbf{U}(k)}(n)$ is studied, where the *Shallit sequence* $\mathbf{U}(k)$ is given by $\mathbf{U}(k) = \{U_n(k)\}_{n\geq 0}$ with $U_0(k) = 0,\ U_1(k) = 1$ and

$$U_{n+2}(k) = (4k+2)U_{n+1}(k) - U_n(k)$$

for all $n \geq 0$. By Theorem 1, we have $\mathcal{D}_{\mathbf{U}(k)}(2^e) = 2^e$ for every $e \geq 1$.

Family 2. Let $q \geq 5$ be a prime and put $q^* = (-1)^{(q-1)/2} \cdot q$. The sequence $u_q(1), u_q(2), \ldots$, with

$$u_q(j) = \frac{3^j - q^*(-1)^j}{4},$$

we call the *Browkin-Sălăjan sequence* for q. The sequence u_q satisfies the recursive relation $u_q(j) = 2u_q(j-1) + 3u_q(j-2)$ for $j \geq 3$, with initial values

$$u_q(1) = (3 + q^*)/4 \quad \text{and} \quad u_q(2) = (9 - q^*)/4.$$

We denote its discriminator by $\mathcal{D}_q$. In the context of the discriminator, the sequence u_5 $(2, 1, 8, 19, 62, 181, 548, 1639, 4922, \ldots)$ was first considered by Sabin Sălăjan during an internship carried out in 2012 under the guidance of Moree. The latter and Zumalacárregui in [11] determined $\mathcal{D}_5(n)$ (cf. Table 2).

Theorem 2. *Let $n \geq 1$ be an arbitrary integer. Let e be the smallest integer such that $2^e \geq n$ and f be the smallest integer such that $5^f \geq 5n/4$. Then $\mathcal{D}_5(n) = \min\{2^e, 5^f\}$.*

More recently Ciolan and Moree [3] completely characterized $\mathcal{D}_q$ for arbitrary primes $q > 5$. Noting that $u_q(1) + u_q(2) = 3$, one sees that Theorem 1 applies and hence $\mathcal{D}_q(2^e) = 2^e$ for every $e \geq 1$.

In order to prove Theorem 1, we will deal with the special case where $\mathbf{w}$ is a Lucas sequence first in Section 3. In the general case, we express $\mathbf{w}$ as a linear combination of a Lucas and a shifted Lucas sequence (Section 4). Our arguments require some consideration of the two divisibility of binomial coefficients (Section 2).

Beyond the polynomial and the recurrence sequence case there is very little known. Haque and Shallit [6] considered the discriminator for k-regular sequences. For these also property (1) is satisfied. Sun [10] made some conjectures regarding the discriminator for various sequences.

2. Preliminaries

2.1. **The exponent of 2 in binomial coefficients.** We recall a celebrated result of Kummer, cf. Ribenboim [9, pp. 30-33].

Theorem 3 (Kummer, 1852)**.** *Let p be a prime number. The exponent of p in $\binom{n}{m}$ is the number of base p carries when summing m with $n-m$ in base p.*

Here and in what follows we write $\nu_2(a)$ for the exponent of 2 in the factorization of the integer a.

Lemma 4. *We have*

$$\nu_2\left(\binom{\ell}{k}2^{3k}\right) > \nu_2(2\ell)$$

for all $k \geq 1$. Further,

$$\nu_2\left(\binom{2^k}{\ell}2^{\ell}\right) \geq k+3$$

for $\ell = 3$ and $\ell \geq 5$.

Proof. We use Theorem 3 with $p = 2$. For the first inequality, we note that it is clear for $k = 1$, so we may assume that $k \geq 2$. Write $\ell = 2^{\ell_0}\ell_1$ with integers $\ell_0 \geq 0$ and ℓ_1 odd. The inequality is clear for $\ell_0 \leq 1$. It is also clear if $k > (\ell_0+1)/3$. So, we may assume that $k \leq (\ell_0+1)/3$. Write $k = 2^{k_0}k_1$, where $k_0 \geq 0$ and k_1 is odd. Then $k_0 < k \leq (\ell_0+1)/3 < \ell_0$. It follows that by summing up k with $\ell - k$, we have at least $\ell_0 - k_0$ carries in base 2. Thus,

$$\nu_2\left(\binom{\ell}{k}2^{3k}\right) \geq (\ell_0 - k_0) + 3k > \ell_0 + 2k \geq \ell_0 + 2,$$

which is what we wanted to prove.

We will now prove the second inequality. Assume first that $\ell \in [3, 2^k - 1]$. Then the number of carries from summing up ℓ with $2^k - \ell$ is, by the previous argument, $k - \ell_0$, where again $\ell = 2^{\ell_0}\ell_1$ with ℓ_1 odd. Hence,

$$\nu_2\left(\binom{2^k}{\ell}2^{\ell}\right) = k - \ell_0 + \ell.$$

This is at least $k+3$ if $\ell \geq 3$ is odd (since then $\ell_0 = 0$). It is also at least $k-\ell_0+2^{\ell_0} > k+3$ if $\ell_0 \geq 3$. If $\ell_0 = 1$, then $\ell > 4$ so $k - \ell_0 + \ell \geq k+3$. Finally, if $\ell_0 = 2$, then since $\ell \neq 4$, we have $\ell \geq 8$ (since $4 \mid \ell$), so the above expression is at least $k-2+8 > k+3$. This was for $\ell < 2^k$. Finally, when $\ell = 2^k$, we have

$$\nu_2\left(\binom{2^k}{\ell}2^\ell\right) = 2^k > k+3$$

because $k \geq 3$. □

3. The Lucas sequence

A basic role in the theory of binary recurrent sequences is played by Lucas sequences.

Theorem 5. *Let $\{u_n\}_{n\geq 0}$ be a Lucas sequence with $u_0 = 0,\ u_1 = 1$ and*

$$u_{n+2} = pu_{n+1} + qu_n, \quad \text{for all } n \geq 0.$$

Then $\mathcal{D}_{\mathbf{u}}(2^k) = 2^k$ for all $k \geq 1$ if and only if $(p, q) \equiv (2, 3) \pmod 4$.

Proof. We look at $\{u_0, u_1, u_2, u_3\} = \{0, 1, p, p^2 + q\}$. Since these are all the residues modulo 4, it follows that either $(p, q) \equiv (2, 3) \pmod 4$ or $(p, q) \equiv (3, 1) \pmod 4$. The second possibility entails $(p, q) \in \{(3, 1),\ (7, 1),\ (3, 5),\ (7, 5) \pmod 8\}$ and one checks computationally that none of these 4 possibilities gives that $\{u_k \pmod 8 : 0 \leq k \leq 7\}$ covers all residue classes modulo 8. Thus, we must have $(p, q) \equiv (2, 3) \pmod 4$.

We consider the quadratic polynomial $x^2 - px - q$ having discriminant $\Delta = p^2 + 4q$. The equation $x^2 - px - q = 0$ is the characteristic equation for the Lucas sequence.

The degenerate case. In this case $\Delta = 0$ and $u_n = np_0^{n-1}$ with $p_0 = p/2$. We have $\{u_0, u_1, u_2, u_3\} = \{0, 1, 2p_0, 3p_0^2\}$ and since p_0 is odd, these are distinct modulo 4. We claim that $\nu_2(u_m - u_n) = \nu_2(m - n)$ for $m > n$. Notice that this claim implies (1).

We have $u_m - u_n \equiv m - n \pmod 2$. So $\nu_2(u_m - u_n) = 0$ if and only if $\nu_2(m - n) = 0$. Next assume that $m \equiv n \pmod 2$. Write $m = n + 2\ell$. Then

$$u_m - u_n = (n + 2\ell)p_0^{n+2\ell-1} - np_0^{n-1} = (n + 2\ell)p_0^{n-1}(p_0^{2\ell} - 1) + 2\ell p_0^{n-1}. \tag{3}$$

We can write $p_0^2 = 1 + 8p_1$ with p_1 an integer. Thus,

$$p_0^{2\ell} = (1 + 8p_1)^\ell = 1 + 8\ell p_1 + \binom{\ell}{2}(8p_1)^2 + \cdots .$$

From this and (3) we infer that

$$u_m - u_n = p_0^{n-1}\left(2\ell + \sum_{k\geq 1}(n + 2\ell)\binom{\ell}{k}(8p_1)^k\right).$$

Since by Lemma 4 for every $k \geq 1$ we have

$$\nu_2\left(\binom{\ell}{k}(8p_1)^k\right) > \nu_2(2\ell),$$

we conclude that

$$\nu_2(u_m - u_n) = \nu_2(2\ell) = \nu_2(m - n),$$

thus establishing the claim.

The non-degenerate case. Here $\Delta \neq 0$. Since $p = 2p_0$ and $q \equiv 3 \pmod 4$, it follows that $\Delta = 4(p_0^2 + q) = 16\Delta_0$, where Δ_0 is an integer. Let

$$\alpha = p_0 + 2\sqrt{\Delta_0} \qquad \text{and} \qquad \beta = p_0 - 2\sqrt{\Delta_0}$$

be the roots of $x^2 - px - q$. The Binet formula for u_n is

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{4}$$

While not necessary for this proof, we make a parenthesis and prove a property concerning the index of appearance of powers of 2. In the course of proving it, we will show that $2\|v_n$, with $\{v_n\}_{n\geq 0}$ the companion sequence of our Lucas sequence. This fact we actually do need in our proof.

For a positive integer m let $z(m)$ be the order of appearance of m in the sequence $\{u_n\}_{n\geq 0}$. It is the minimal positive integer k such that $m \mid u_k$. It is known that this exists for all m which are coprime q. Further, $m \mid u_n$ if and only if $z(m) \mid n$. For us, $z(2) = 2$ since $p \equiv 2 \pmod 4$ and $z(4) = 4$. It follows easily by induction on k that

$$z(2^k) = 2^k.$$

One way to see this is to introduce the companion sequence $\{v_n\}_{n\geq 0}$ given by $v_0 = 2,\ v_1 = p$ and $v_{n+2} = pv_{n+1} + qv_n$ for all $n \geq 0$. By induction, we get that $2\|v_n$ for all $n \geq 0$. The Binet formula for v_n is

$$v_n = \alpha^n + \beta^n \quad \text{for all } n \geq 0. \tag{5}$$

We have $u_{2n} = u_n v_n$ by the Binet formulas (4) and (5). We are now ready to show that $z(2^k) = 2^k$. Assume that $k \geq 3$ and that $2^k \mid u_n$. This implies that $n = 2^\ell n_1$ for some integers ℓ and n, with $\ell \geq 2$ and n_1 odd. Now we use repeatedly the formula $u_{2m} = u_m v_m$ for $m = n/2,\ n/4, \ldots$, resulting in

$$u_n = u_{2^\ell n_1} = v_{2^{\ell-1}n_1} v_{2^{\ell-2}n_1} \cdots v_{n_1} u_{n_1}.$$

Since $v_{2^i n_1} \equiv 2 \pmod 4$ for $i = 0, 1, \ldots, \ell - 1$ and u_{n_1} is odd, we infer that

$$\nu_2(u_{2^\ell n_1}) = \ell,$$

which shows that $k \geq \ell$. In particular, $z(2^k) = 2^k$.

Next we show that

$$u_{n+2^k} \equiv u_n + 2^k \pmod{2^{k+1}} \tag{6}$$

for all $k \geq 1$. One checks it easily by hand for $k = 1$ and $n = 0, 1$ as well as for $k = 2$ and $n = 0, 1, 2, 3$. Assume next $k \geq 3$. In what follows, for three algebraic integers a, b, c, we write $a \equiv b \pmod c$ if $(a - b)/c$ is an algebraic integer. We have

$$\begin{aligned} \alpha^{2^k} = (p_0 + 2\sqrt{\Delta_0})^{2^k} = & p_0^{2^k} + 2^k p_0^{2^k-1}(2\sqrt{\Delta_0}) + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2 \\ & + \binom{2^k}{4} p_0^{2^k-4}(2\sqrt{\Delta_0})^4 + \sum_{\substack{\ell\geq 3\\ \ell\neq 4}} \binom{2^k}{\ell} p_0^{2^k-\ell}(2\sqrt{\Delta_0})^\ell. \end{aligned}$$

Then, by Lemma 4,

$$\alpha^{2^k} \equiv p_0^{2^k} + 2^k p_0^{2^k-1}(2\sqrt{\Delta_0}) + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2 + \binom{2^k}{4} p_0^{2^k-1}(2\sqrt{\Delta_0})^4 \pmod{2^{k+3}\sqrt{\Delta_0}}.$$

Changing α to β, the same calculation yields

$$\beta^{2^k} \equiv p_0^{2^k} - 2^k p_0^{2^k-1}(2\sqrt{\Delta_0}) + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2 + \binom{2^k}{4} p_0^{2^k-4}(2\sqrt{\Delta_0})^4 \pmod{2^{k+3}\sqrt{\Delta_0}}.$$

Thus,

$$\begin{aligned}
\alpha^{n+2^k} - \beta^{n+2^k} &\equiv \alpha^n \left(p_0^{2^k} + 2^k p_0^{2^k-1}(2\sqrt{\Delta_0}) + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2 + \binom{2^k}{4} p_0^{2^k-4}(2\sqrt{\Delta_0})^4 \right) \\
&\quad - \beta^n \left(p_0^{2^k} - 2^k p_0^{2^k-1}(2\sqrt{\Delta_0}) + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2 + \binom{2^k}{4} p_0^{2^k-4}(2\sqrt{\Delta_0})^4 \right) \\
&\equiv p_0^{2k}(\alpha^n - \beta^n) + 2^k p_0^{2^k-1}(2\sqrt{\Delta_0})(\alpha^n + \beta^n) \\
&\quad + \binom{2^k}{2} p_0^{2^k-2}(2\sqrt{\Delta_0})^2(\alpha^n - \beta^n) \\
&\quad + \binom{2^k}{4} p_0^{2^k-4}(2\sqrt{\Delta_0})^4(\alpha^n - \beta^n) \pmod{2^{k+3}\sqrt{\Delta_0}}.
\end{aligned}$$

Dividing across by $\alpha - \beta$ (which is equal to $4\sqrt{\Delta_0}$), we obtain

$$\begin{aligned}
u_{n+2^k} &\equiv p_0^{2^k} u_n + 2^k p_0^{2^k-1}(v_n/2) + \binom{2^k}{2} p_0^{2^k-2}(4\Delta_0)u_n \\
&\quad + \binom{2^k}{4} p_0^{2^k-4}(16\Delta_0^2)u_n \pmod{2^{k+1}}.
\end{aligned} \tag{7}$$

We have $p_0^{2^k} \equiv 1 \pmod{2^{k+1}}$ and $v_n/2 \equiv 1 \pmod 2$. Finally,

$$\binom{2^k}{2} p_0^{2^k-2}(4\Delta_0) = 2^{k+1}(2^k - 1)p_0^{2^k-2}\Delta_0 \equiv 0 \pmod{2^{k+1}},$$

and also

$$\binom{2^k}{4} p_0^{2^k-4}(16\Delta_0^2) = \frac{2^{k-2}(2^k-1)(2^{k-1}-1)(2^k-3)}{3} 2^4 \Delta_0^2 \equiv 0 \pmod{2^{k+1}}.$$

We thus get from (7) that (6) holds for all $k \geq 1$. This implies by induction on k that $\mathcal{D}_{\mathbf{u}}(2^k) = 2^k$. □

4. The general case: the proof of Theorem 1

In the previous section we dealt with the Lucas sequence (Theorem 5). We will make crucial use of that result in order to deal with a more general recurrence $\{w_n\}_{n\geq 0}$ as in (2).

Proof of Theorem 1. If $\#\{w_n \pmod{2^k} : 0 \le n \le 2^k - 1\} = 2^k$ for all k, it is so for $k = 1$ in particular. Thus, $w_0,\ w_1$ have different parities which is equivalent to $w_0 + w_1$ being odd. Conversely, write

$$w_n = au_n + bu_{n+1}.$$

Note that $au_n + bu_{n+1}$ satisfies the same recurrence relation as w_n. On setting $n = 0$, respectively $n = 1$, we find $b = w_0$ and $a = w_1 - pw_0$. Thus, $a + b = (w_1 + w_0) - pw_0$ is odd. By (6), we obtain

$$\begin{aligned} w_{n+2^k} &= au_{n+2^k} + bu_{n+1+2^k} \equiv a(u_n + 2^k) + b(u_{n+1} + 2^k) \\ &\equiv (au_n + bu_{n+1}) + (a+b)2^k \equiv w_n + 2^k \pmod{2^{k+1}} \end{aligned}$$

for $k \ge 1$. This shows that $\mathcal{D}_{\mathbf{w}}(2^k) = 2^k$ for every $k \ge 1$.

It remains to prove the second assertion. Note that it is enough to prove it for $k = 3$. This can be done by doing a computer calculation modulo 8. We consider all integers a, b, p, q with $0 \le a, b, p, q \le 7$ and compute $\#\{w_n \pmod 8 : 0 \le n \le 7\}$. It turns out that if $(p, q) \not\equiv (2, 3) \pmod 4$, then this number is < 8. □

5. Tables

We tabulate the discriminator for a sequence that does not (Fibonacci sequence) and a sequence that does (Sălăjan sequence) satisfy the conditions of Theorem 1. We give the prime factorization of the values. Note the big difference in behavior.

n	$D_F(n)$	n	$D_F(n)$	n	$D_F(n)$
1	1	21 – 24	59	69 – 80	431
2	2	25 – 26	79	81 – 113	$3 \cdot 197$
3	3	27 – 32	83	114 – 115	$3 \cdot 283$
4	5	33 – 35	$2^3 \cdot 3 \cdot 5$	116 – 152	1039
5	2^3	36 – 39	157	153 – 158	$5 \cdot 13 \cdot 17$
6	3^2	40 – 44	173	159 – 162	1171
7 – 8	$2 \cdot 7$	45 – 55	193	163 – 166	1451
9 – 11	$3 \cdot 5$	56 – 59	311	167 – 184	$3 \cdot 487$
12 – 16	$2 \cdot 3 \cdot 5$	60 – 64	337	185 – 208	1609
17 – 20	$5 \cdot 7$	65 – 68	409	209 – 281	$3 \cdot 761$

TABLE 1. Discriminator for the Fibonacci sequence $1, 2, 3, 5, 8, 13, \ldots$

Table 2 demonstrates Theorem 2.

n	$D_S(n)$	n	$D_S(n)$
1	1	$129-256$	2^8
2	2	$257-512$	2^9
$3-4$	2^2	$513-1024$	2^{10}
$5-8$	2^3	$1025-2048$	2^{11}
$9-16$	2^4	$2049-2500$	5^5
$17-20$	5^2	$2501-4096$	2^{12}
$21-32$	2^5	$4097-8192$	2^{13}
$33-64$	2^6	$8193-12500$	5^6
$65-100$	5^3	$12501-16384$	2^{14}
$101-128$	2^7	$16385-32768$	2^{15}

TABLE 2. Discriminator for the Sălăjan sequence $2, 1, 8, 19, 62, 181, \ldots$

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