COMPRESSION FUNCTIONS OF UNIFORM EMBEDDINGS OF GROUPS INTO HILBERT AND BANACH SPACES

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Abstract. We construct finitely generated groups with arbitrary prescribed Hilbert space compression $\alpha \in [0,1]$. For a large class of Banach spaces $\mathcal{E}$ (including all uniformly convex Banach spaces), the $\mathcal{E}$–compression of these groups coincides with their Hilbert space compression. Moreover, the groups that we construct have asymptotic dimension at most 2, hence they are exact. In particular, the first examples of groups that are uniformly embeddable into a Hilbert space (moreover, of finite asymptotic dimension and exact) with Hilbert space compression 0 are given. These groups are also the first examples of groups with uniformly convex Banach space compression 0.

1. Introduction

1.1. **Uniform embeddings.** The property of uniform embeddability of groups into Hilbert spaces (and, more generally, into Banach spaces) became popular after Gromov [Gr1] suggested that this property might imply the Novikov conjecture. Indeed, following this suggestion, Yu [Yu] and later Kasparov and Yu [KY] proved that a finitely generated group uniformly embeddable into a Hilbert space, respectively into a uniformly convex Banach space, satisfies the Novikov conjecture.

This raised the question whether every finitely generated group can be embedded uniformly into a Hilbert space, or more generally, into a uniformly convex Banach space. Gromov constructed [Gr2] finitely generated *random* groups whose Cayley graphs (quasi)-contain some infinite families of expanders and thus cannot be embedded uniformly into a Hilbert space (or into any ℓ^p with $1 \leq p < \infty$, e.g. [Roe, Ch.11.3]). The recent results of V. Lafforgue [Laf] yield a family of expanders that is not uniformly embeddable into any uniformly convex Banach space. Nevertheless, one cannot apply Gromov's argument to deduce that random groups corresponding to Lafforgue's family of graphs do not embed uniformly into any uniformly convex Banach space. Indeed, Lafforgue's expanders are Cayley graphs of finite quotients of a non-free group, therefore there are loops of bounded size in all of them, and the graphs have bounded girth. On the other hand, Gromov's argument succeeds only if the girth of a graph in the family of expanders is of the same order as the diameter of the graph.

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1.2. **Compression and compression gap.**

Definition 1.1 (cf. [GuK]). Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\phi\colon X \to Y$ be an 1-Lipschitz map. The *compression of* ϕ is the supremum over all $\alpha \geq 0$ such that

$$\mathrm{d}_Y(\phi(u), \phi(v)) \geq \mathrm{d}_X(u, v)^\alpha$$

for all u, v with large enough $\mathrm{d}_X(u, v)$.

If $\mathcal{E}$ is a class of metric spaces closed under rescaling of the metric, then the $\mathcal{E}$–*compression of* X is the supremum over all compressions of 1-Lipschitz maps $X \to Y$, $Y \in \mathcal{E}$. In particular, if $\mathcal{E}$ is the class of Hilbert spaces, we get the *Hilbert space compression* of X.

The $\mathcal{E}$–compression measures the least possible distortion of distances when one tries to draw a copy of X inside a space from $\mathcal{E}$. It is a quasi-isometry invariant of X and it takes values in the interval $[0, 1]$. Similar concepts of distortion have been extensively studied (mostly for finite metric spaces mapped into finite dimensional Hilbert spaces) by combinatorists for many years (see [Bou, DL], for example).

Since any finitely generated group G can be endowed with a word length metric and all such metrics are quasi-isometric, one can speak about the $\mathcal{E}$–*compression of a group* G. Guentner and Kaminker proved in [GuK] that if the Hilbert space compression of a finitely generated group G is larger than $\frac{1}{2}$ then the reduced C^*-algebra of G is exact (in other words, G is *exact* or G satisfies Guoliang Yu's *property A* [Yu]).

One of the goals of this paper is to describe all possible values of $\mathcal{E}$–compression for finitely generated groups, when $\mathcal{E}$ is either the class of Hilbert spaces or, more generally, the class of uniformly convex Banach spaces.

A very limited information was known about the possible values of Hilbert space compression of finitely generated groups. For example, word hyperbolic groups have Hilbert space compression 1 [BS], and so do groups acting properly and co-compactly on a cubing [CN]; co-compact lattices in arbitrary Lie groups, and all lattices in semi-simple Lie groups have Hilbert space and, moreover, L^p–compression 1 [Te]; any group that is not uniformly embeddable into a Hilbert space (such groups exist by [Gr2]) has Hilbert space compression 0, etc. (see the surveys in [AGS], [Te]). The first groups with Hilbert space compressions strictly between 0 and 1 were found in [AGS]: R. Thompson's group F has Hilbert space compression $\frac{1}{2}$, the Hilbert space compression of the wreath product $Z \wr Z$ is between $\frac{1}{2}$ and $\frac{3}{4}$ (later Tessera proved in [Te] that it is actually at least $\frac{2}{3}$), the Hilbert space compression of $Z \wr (Z \wr Z)$ is between 0 and $\frac{1}{2}$.

The notion of $\mathcal{E}$–compression can be generalized to the notion of $\mathcal{E}$–*compression gap* of a space X, where $\mathcal{E}$ is any class of metric spaces closed under rescaling of the metric (see Definition 2.6). It measures even more accurately than the compression the best possible (least distorted) way of embedding X in a space from $\mathcal{E}$. For example, it is proved in [AGS] that R. Thompson's group F has Hilbert space compression gap $(\sqrt{x}, \sqrt{x}\log x)$. This means there exists an 1-Lipschitz embedding of F into a Hilbert space with compression function $\sqrt{x}$, and every 1-Lipschitz embedding of F into a Hilbert space has compression function at most $\sqrt{x}\log x$. This is much more precise than simply stating that the Hilbert space compression of F is $\frac{1}{2}$. Another example: it follows from [Te] that every lattice in a semi-simple Lie group has a Hilbert space compression gap $\left(\frac{x}{\sqrt{\log x}\log\log x}, x\right)$ and the upper bound of the gap cannot be improved. This is a much more precise statement than the statement that the Hilbert space compression of the lattice is 1.

In this paper, we show that a large class of functions appear as Hilbert space compression functions of graphs of bounded degree, and as upper bounds of Hilbert space compression gaps of logarithmic size of finitely generated groups. This class of functions is defined as follows.

Definition 1.2. Let $\mathcal{C}$ be the collection of continuous functions $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $a > 0$:

(1) ρ is increasing on $[a, \infty)$, and $\lim_{x\to\infty} \rho(x) = \infty$;
(2) ρ is subadditive;
(3) the function $\tau(x) = \frac{x}{\rho(x)}$ is increasing, and the function $\frac{\tau(x)}{\log x}$ is non-decreasing on $[a, \infty)$.

Remark 1.3. The collection $\mathcal{C}$ contains all functions x^α, for $\alpha \in (0, 1)$, as well as functions $\log x$, $\log\log x$, $\frac{x}{\log^\beta(x+1)}$, $\frac{x^\alpha}{\log^\gamma(x+1)}$ for $\alpha \in (0, 1), \beta > 1, \gamma > 0$, etc.

1.3. **Results of the paper.** Let $\mathcal{E}$ be the class of all uniformly convex Banach spaces.

Proposition 1.4 (see Proposition 4.2)**.** *Let ρ be a function in $\mathcal{C}$. There exists a graph Π of bounded degree such that ρ is the Hilbert space compression function of Π, and also the $\mathcal{E}$–compression function of Π.*

In particular, for any $\alpha \in [0, 1]$ there exists a graph of bounded degree whose Hilbert space compression equals the $\mathcal{E}$–compression, and both are equal to α.

Using the construction of the graph in Proposition 1.4, we realize every function of $\mathcal{C}$ as the upper bound of a Hilbert space compression gap of logarithmic size of a finitely generated group.

Theorem 1.5 (see Theorem 5.5)**.** *For every function $\rho \in \mathcal{C}$ there exists a finitely generated group of asymptotic dimension at most 2 such that for every $\epsilon > 0$, $\left(\frac{\rho}{\log^{1+\epsilon}(x+1)}, \rho\right)$ is a Hilbert space compression gap and an $\mathcal{E}$–compression gap of the group.*

In particular, for every $\alpha \in [0, 1]$ there exists a finitely generated group G_α of asymptotic dimension at most 2 and with the Hilbert space compression equal to the $\mathcal{E}$–compression and equal to α.

Since the groups G_α have finite asymptotic dimension, they are all exact and uniformly embeddable[1] into Hilbert spaces even when $\alpha = 0$. Thus we construct the first examples of groups uniformly embeddable into Hilbert spaces, moreover exact and even of finite asymptotic dimension, that have Hilbert space compression 0, and even {uniformly convex Banach space}–compression zero. Note that since the construction in [Gr2] does not immediately extend to uniformly convex Banach spaces, our groups seem to be the only existing examples of groups with {uniformly convex Banach space}–compression 0.

[1]A finitely generated group G of finite asymptotic dimension has Guoliang Yu's property A [HR, Lemma 4.2]. That property is equivalent to the exactness of the reduced C^*-algebra of G [Oz, HR] and guarantees uniform embeddability into a Hilbert space [Yu, Th.2.2].

1.4. **The plan of the proofs.** The plan for proving Proposition 1.4 and Theorem 1.5 is the following. We use a family of V. Lafforgue's expanders Π_k, $k \geq 1$, which are Cayley graphs of finite factor-groups M_k of a lattice Γ of $\mathrm{SL}_3(F)$ for a local field F. Lafforgue proved [Laf] that this family of expanders does not embed uniformly into a uniformly convex Banach space. Now taking any function ρ in $\mathcal{C}$, we choose appropriate scaling constants $\lambda_k, k \geq 1$, such that the family of rescaled metric spaces $(\lambda_k \Pi_k)_{k\geq 1}$ has {uniformly convex Banach space}-compression function ρ and Hilbert space compression function ρ as well. This gives Proposition 1.4.

The group satisfying the conditions of Theorem 1.5 is constructed as a graph of groups. We use the fact that each M_k is generated by some number m involutions (that can be achieved by choosing a lattice Γ generated by involutions). One of the vertex groups of the graph of groups is the free product F of the groups M_k, other vertex groups are m copies of the free product $H = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$. Edges connect F with each of the m copies of H. The edge groups are free products of countably many copies of $\mathbb{Z}/2\mathbb{Z}$. We identify such a subgroup in H with a subgroup of F generated by involutions, one involution from the generating set of each factor M_k. As a result, the group G is finitely generated, and each finite subgroup M_k in G is distorted by a scaling constant close to λ_k. Hence the Cayley graph of G contains a quasi-isometric copy of the family of metric spaces $(\lambda_k \Pi_k)_{k\geq 1}$. This allows us to apply Proposition 1.4 and get an upper bound for a compression gap. A lower bound is achieved by a careful analysis of the word metric on G.

In order to show that G has asymptotic dimension at most 2, we use a result by Dranishnikov and Smith [DS] on the asymptotic dimension of countable groups, and results by Bell and Dranishnikov [BD], as well as by Bell, Dranishnikov and Keesling [BDK] on the asymptotic dimension of groups acting on trees.

1.5. **Other Banach spaces.** The class of Banach spaces to which our arguments apply cannot be extended much beyond the class of uniformly convex Banach spaces; for instance it cannot be extended to reflexive strictly convex Banach spaces. Indeed, our proof is based on the fact that a family of V. Lafforgue's expanders [Laf] does not embed uniformly into a uniformly convex Banach space. But by a result of Brown and Guentner [BG], any countable graph of bounded degree can be uniformly embedded into a Hilbertian sum $\oplus l^{p_n}(\mathbb{N})$ for some sequence of numbers $p_n \in (1, +\infty)$, $p_n \to \infty$. The Banach space $\oplus l^{p_n}(\mathbb{N})$ is reflexive and strictly convex, but it is not uniformly convex.

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2. Embeddings of metric spaces

Given two metric spaces (X, d_X) and (Y, d_Y) and an 1-Lipschitz map $\phi\colon X \to Y$ we define the *distortion* of ϕ [HLW] as follows:

$$\mathrm{dtn}(\phi) = \max_{x\neq y} \frac{\mathrm{d}_X(x,y)}{\mathrm{d}_Y(\phi(x),\phi(y))}. \tag{1}$$

For a metric space (X, d) and a collection of metric spaces $\mathcal{E}$ we define the *$\mathcal{E}$–distortion of* (X, d), which we denote by $\mathrm{dtn}_{\mathcal{E}}(X, \mathrm{d})$, as the infimum over the distortions of all

1-Lipschitz maps from X to a metric space from $\mathcal{E}$. Note that given λ a positive real number $\mathrm{dtn}_{\mathcal{E}}(X, \mathrm{d}) = \mathrm{dtn}_{\mathcal{E}}(X, \lambda\mathrm{d})$ provided that the class $\mathcal{E}$ is closed under rescaling of the metrics by λ.

Remark 2.1. If $\mathcal{E}$ contains a space with n points at pairwise distance at least 1 from each other then for every graph X with n vertices and edge-length metric, $\mathrm{dtn}_{\mathcal{E}} X \leq \mathrm{diam}\, X$.

The notion of distortion originated in combinatorics is related to the following notion of uniform embedding with origin in functional analysis.

Definition 2.2. Given two metric spaces (X, d_X) and (Y, d_Y), and two non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$, with $\lim_{x\to\infty} \rho_\pm = \infty$, a map $\phi\colon X \to Y$ is called a (ρ_-, ρ_+)–*embedding* (also called a *uniform embedding* or a *coarse embedding*) if

$$\rho_-(\mathrm{d}_X(x_1, x_2)) \leq \mathrm{d}_Y(\phi(x_1), \phi(x_2)) \leq \rho_+(\mathrm{d}_X(x_1, x_2)), \tag{2}$$

for all x_1, x_2 with large enough $\mathrm{d}(x_1, x_2)$.

If $\rho_+(x) = Cx$, i.e. if ϕ is C-Lipschitz for some constant $C > 0$, then the embedding is called a ρ_-*–embedding*.

Definition 2.3. For a family of metric spaces $X_i, i \in I$, by a (ρ_-, ρ_+)*–embedding* (resp. ρ_-*–embedding*) of the family we shall mean the (ρ_-, ρ_+)–embedding (resp. ρ_-–embedding) of the wedge union of X_i.

Let (X, d_X) be a metric space such that for every $x \neq x'$ the distance $\mathrm{d}_X(x, x')$ is at most δ, for some uniform $\delta > 0$ (e.g., the set of vertices of a graph). Then it is easy to see that any (ρ_-, ρ_+)–embedding of X is also a ρ_-–embedding. The same holds for a family of metric spaces.

Convention 2.4. Since in this paper we discuss mainly embeddings of graphs, in what follows we restrict ourselves to ρ_-–embeddings, and denote the function ρ_- simply by ρ.

Notation 2.5. For two functions $f, g : X \to \mathbb{R}$ we write $f \ll g$ if there exist $a, b, c > 0$ such that $f(x) \leq a g(bx) + c$ for every $x \in X$. If $f \ll g$ and $g \ll f$ then we write $f \asymp g$.

Definition 2.6. Let (X, d) be a metric space, and let $\mathcal{E}$ be a collection of metric spaces. Let $f, g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be two increasing functions such that $f \ll g$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

We say that (f, g) is an $\mathcal{E}$*–compression gap of* (X, d) if

1. there exists an f–embedding of X into a space from $\mathcal{E}$;
2. for every ρ–embedding of X into a space from $\mathcal{E}$, we have $\rho \ll g$.

If $f = g$ then we say that f is the $\mathcal{E}$*–compression function* of X.

The quotient $\frac{g}{f}$ is called the *size of the gap*. The functions f and g are called, the *lower* and *upper bound of the gap* respectively.

Remark 2.7.

(i) If α is the supremum of all numbers such that $x^\alpha \ll \rho$ and ρ is the $\mathcal{E}$–compression function of (X, d) then α is the $\mathcal{E}$–compression of (X, d).

(ii) If $\mathcal{E}$ is closed under rescaling of the metrics, then any compression gap of a metric space X is a quasi-isometry invariant.

3. Embeddings of expanders

For every finite m-regular graph, the largest eigenvalue of its incidence matrix is m. We denote by λ_2 the second largest eigenvalue.

We start by a well known metric property of expanders.

Lemma 3.1 ([Lub], Ch.1). *Let $\epsilon > 0$ and let $\mathcal{G}_{m,\epsilon}$ be the family of all m-regular graphs with $\lambda_2 \leq m - \epsilon$. Then there exist two constants κ, κ' such that for any graph Π in $\mathcal{G}_{m,\epsilon}$ with set of vertices V, and any vertex $x \in V$, the set $\{y \in V \mid \mathrm{d}(x, y) \geq \kappa \operatorname{diam} \Pi\}$ has cardinality at least $\kappa' |V|$.*

We now recall properties of expanders related to embeddings into Hilbert spaces. We denote by $\mathcal{H}$ the class of all separable Hilbert spaces.

Theorem 3.2 ([LLR], Theorem 3.2(6), [HLW], Theorem 13.8 and its proof). *Let $\epsilon > 0$ and let $\mathcal{G}_{m,\epsilon}$ be the family of all m-regular graphs with $\lambda_2 \leq m - \epsilon$.*

(i) *There exist constants $c > 0$ and $d > 1$ such that for any graph Π in $\mathcal{G}_{m,\epsilon}$ with set of vertices V*

$$c \log |V| \leq \mathrm{dtn}_{\mathcal{H}} \Pi \leq \operatorname{diam} \Pi \leq d \log |V| \,. \tag{3}$$

(ii) *Moreover, there exists $r > 0$ such that for every 1-Lipschitz embedding ϕ of Π into a Hilbert space Y there exist two vertices v_1 and v_2 in V with $\mathrm{d}(v_1, v_2) \geq \kappa \operatorname{diam} \Pi$ (where κ is the constant in Lemma 3.1) and*

$$\frac{\mathrm{d}(v_1, v_2)}{\|\phi(v_1) - \phi(v_2)\|} \geq r \operatorname{diam} \Pi \,.$$

Remark 3.3. Thus for all expanders in $\mathcal{G}_{m,\epsilon}$ the canonical embedding $\Pi \hookrightarrow \frac{1}{\sqrt{2}} \ell^2(V)$ has optimal distortion, see Remark 2.1.

V. Lafforgue [Laf] proved that for some smaller family of expanders one can replace $\mathcal{H}$ by a large class of Banach spaces. Namely, for every prime number p and natural number $r > 0$, he defined a class of Banach spaces $\mathcal{E}^{p,r} = \bigcup_{\alpha > 0} \mathcal{E}^{p,r,\alpha}$ satisfying the following:

(1) For any p, r, the class $\mathcal{E}^{p,r}$ contains all uniformly convex Banach spaces[2] (including all Hilbert spaces, and even all spaces l^q, $q > 1$);

(2) $\bigcup_r \mathcal{E}^{2,r}$ is the set of all B-convex Banach spaces.

Lafforgue's family of expanders that does not embed uniformly into any Banach space from $\mathcal{E}^{p,r}$ is constructed as follows. Given numbers p and r as above, let F be a local field such that the cardinality of its residual field is p^r. Let Γ be a lattice in $\mathrm{SL}(3, F)$.

The group Γ is residually finite with Kazhdan's property (T). In fact, Γ satisfies the *Banach version* of property (T) with respect to the family $\mathcal{E}^{p,r}$ [Laf, §3 and Proposition 4.5].

[2]A Banach space is *uniformly convex* if for every $R > 0$ and every $\delta > 0$ there exists $\varepsilon = \varepsilon(R, \delta) > 0$ such that if x, y are two points in the ball around the origin of radius R at distance at least δ then their middle point $\frac{1}{2}(x + y)$ is in the ball around the origin of radius $R - \varepsilon$.

Let $(\Gamma_k)_{k\geq 1}$ be a decreasing sequence of finite index normal subgroups of Γ such that $\bigcap_{k\geq 1}\Gamma_k = \{1\}$, and let $M_k = \Gamma/\Gamma_k$, $k \geq 1$, be the sequence of quotient groups. Given a finite symmetric set of generators of Γ of cardinality $m \geq 2$, we consider each M_k endowed with the induced set of m generators; we denote by d_k the corresponding word metric on M_k and by Π_k the corresponding m-regular Cayley graph. Since Γ has property (T), the family $(\Pi_k, \mathrm{d}_k)_{k\geq 1}$ is a family of expanders [Lub, Proposition 3.3.1]. Moreover, the Banach version of property (T) for Γ yields Proposition 3.4 below, which implies that the family $(\Pi_k, \mathrm{d}_k)_{k\geq 1}$ cannot be uniformly embedded into a Banach space from $\mathcal{E}^{p,r}$ (see Remark 3.6).

Proposition 3.4 ([Laf], Proposition 5.2)**.** *For every $\alpha > 0$ there exists a constant $C = C(\alpha)$ such that for any $k \geq 1$ and any space Y from $\mathcal{E}^{p,r,\alpha}$ an 1-Lipschitz map $\phi : \Pi_k \to Y$ satisfies:*

$$\frac{1}{(\operatorname{card}\Pi_k)^2}\sum_{x,y\in\Pi_k}\|\phi(x)-\phi(y)\|^2 \leq C\,\frac{1}{\operatorname{card}\Pi_k}\sum_{x,y\ \ neighbors}\|\phi(x)-\phi(y)\|^2\,. \tag{4}$$

Corollary 3.5.

(i) *For every $\alpha > 0$ there exists a constant $D = D(\alpha)$ such that for any $k \geq 1$,*

$$D\operatorname{diam}\Pi_k \leq \operatorname{dtn}_{\mathcal{E}^{p,r,\alpha}}\Pi_k \leq \operatorname{diam}\Pi_k\,. \tag{5}$$

(ii) *Moreover, there exists $R = R(\alpha)$ such that for every 1-Lipschitz embedding ϕ of Π_k into a space $Y \in \mathcal{E}^{p,r,\alpha}$ there exist two vertices v_1 and v_2 in Π_k with $\mathrm{d}(v_1, v_2) \geq \kappa\operatorname{diam}\Pi_k$, where κ is the constant from Lemma 3.1, and*

$$\frac{\mathrm{d}(v_1,v_2)}{\|\phi(v_1)-\phi(v_2)\|} \geq R\operatorname{diam}\Pi_k\,. \tag{6}$$

Proof. The second inequality in (5) follows from Remark 2.1 and from the fact that no Banach space can be covered by finitely many balls of radius 1. We prove the first inequality in (5). Let Y be an arbitrary space from $\mathcal{E}^{p,r,\alpha}$ and let $\phi : \Pi_k \to Y$ be an 1-Lipschitz map. Inequality (4) implies that

$$\frac{1}{[\operatorname{dtn}(\phi)\operatorname{card}\Pi_k]^2}\sum_{x,y\in\Pi_k}\mathrm{d}_k(x,y)^2 \leq Cm\,. \tag{7}$$

Recall that each graph Π_k is m-regular. Lemma 3.1 and (7) imply that $Cm\,[\operatorname{dtn}(\phi)]^2 \geq \kappa^2\kappa'\,(\operatorname{diam}\Pi_k)^2$.

Now take $0 < R < \kappa\sqrt{\frac{\kappa'}{Cm}}$, where κ, κ' are the constants from Lemma 3.1 and C is the constant from (4). Assume that there exists an 1-Lipschitz embedding ϕ of Π_k into a space $Y \in \mathcal{E}^{p,r,\alpha}$ such that for any two vertices v_1 and v_2 in Π_k with $\mathrm{d}(v_1,v_2) \geq \kappa\operatorname{diam}\Pi_k$ the following inequality holds:

$$\frac{\mathrm{d}(v_1,v_2)}{\|\phi(v_1)-\phi(v_2)\|} < R\operatorname{diam}\Pi_k\,.$$

Inequality (4) implies that

$$Cm > \frac{1}{(\mathrm{card}\,\Pi_k)^2} \sum_{\mathrm{d}(v_1,v_2)\geq \kappa \mathrm{diam}\,\Pi_k} \frac{\mathrm{d}(v_1,v_2)^2}{R^2[\mathrm{diam}\,\Pi_k]^2} \geq \frac{\kappa^2}{R^2}\kappa' .$$

This contradicts the choice of R. □

Since $(\Pi_k, \mathrm{d}_k)_{k\geq 1}$ is a family of expanders, it is contained in some family $\mathcal{G}_{m,\epsilon}$. Then according to Theorem 3.2, we have $\mathrm{diam}\,\Pi_k \asymp \log |V_k|$.

Remark 3.6. Theorem 3.2(ii) implies that no sub-family of the family $\mathcal{G}_{m,\epsilon}$ can be uniformly embedded into a Hilbert space, in the sense of Definition 2.3.

Similarly, Corollary 3.5(ii) implies that no sub-family of the family $(\Pi_k)_{k\geq 1}$ can be uniformly embedded into a space from $\mathcal{E}^{p,r}$.

The construction in the following lemma was provided to us by Dave Witte-Morris.

Lemma 3.7. *Let F be a nonarchimedean local field of characteristic 0. There exists a lattice in the group $\mathrm{SL}(3, F)$ containing infinitely many noncentral involutions.*

Proof. Choose an algebraic number field K, such that

- F is one of the (nonarchimedean) completions of K, and
- F is totally real.

Let p be the characteristic of the residue field of F, and let $c = 1 - p^3$. Thus $c < 0$, but c is a square in F. Consider $L = K[\sqrt{c}]$, and denote by τ the Galois involution of L over K. We likewise denote by τ the involution defined on the 3×3 matrices by applying τ to each entry. Let $G = SU(3; L, \tau) = \left\{g \in SL(3, L)\ ;\ \tau\left(g^T\right) \cdot g = \mathrm{Id}_3\right\}$. Then

(a) G is a K-form of $\mathrm{SL}(3)$;
(b) $G_F = \mathrm{SL}(3, F)$;
(c) G is compact at each real place.

Properties (a), (b), (c) can be proved using arguments similar to the ones in [Wi, Chapter 10].

Let S be the collection of all the Archimedean places of K and the place corresponding to F. According to the theorem due to Borel, Harish-Chandra, Behr and Harder [Ma, §I.3.2], the S-integer points of G form a lattice Γ in $\mathrm{SL}(3, F)$. By [Ta], any lattice in a p-adic algebraic group is co-compact, in particular it is the case for Γ. The lattice Γ obviously contains noncentral diagonal matrices that are involutions.

Let σ be one of these involutions and let $Z(\sigma)$ be its centralizer in $\mathrm{SL}(3, F)$. The subgroup $\Gamma \cap Z(\sigma)$ has infinite index in Γ. Otherwise, for some positive integer n one would have that $\gamma^n \in Z(\sigma)$ for any $\gamma \in \Gamma$. Since $Z(\sigma)$ is an algebraic subgroup in $\mathrm{SL}(3, F)$ and since Γ is Zariski dense in $\mathrm{SL}(3, F)$, it would follow that $g^n \in Z(\sigma)$ for any $g \in \mathrm{SL}(3, F)$. This is impossible.

For a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of representatives of distinct left cosets in $\Gamma/(\Gamma \cap Z(\sigma))$ the involutions in the sequence $(\gamma_n \sigma \gamma_n^{-1})_{n\in\mathbb{N}}$ are pairwise distinct. □

Lemma 3.8. *For every prime number p and natural number $r > 0$, there exists $m = m(p, r) > 0$ and a lattice Γ in $\mathrm{SL}(3, F)$ such that F is a local field with residue field of cardinality p^r, and Γ is generated by finitely many involutions $\sigma_1, ..., \sigma_m$.*

Proof. Let Γ be a lattice of $\mathrm{SL}(3, F)$ containing infinitely many non-central involutions. According to Lemma 3.7 such a lattice exists. Consider the (infinite) subgroup Γ' of Γ generated by involutions. Since Γ' is a normal subgroup in Γ, by Margulis' Theorem Γ' has finite index in Γ, thus it is a lattice itself. $\square$

Notation 3.9. Let Γ be one of the lattices from Lemma 3.8. We keep the notation given before Proposition 3.4, referring, in addition, to the generating set consisting of involutions. More precisely, consider a maximal ideal I in the ring of S-integers of the global field defining Γ (S is the corresponding set of valuations containing all Archimedian ones) and the congruence subgroup Γ_k of Γ corresponding to the ideal I^k. Let $M_k = \Gamma/\Gamma_k$, $k \geq 1$. Let $U_k = \{\sigma_1(k), \ldots, \sigma_m(k)\}$ be the image of the generating set of Γ in M_k (each $\sigma_i(k)$ is an involution). We shall denote the corresponding word metric on M_k again by d_k. Since each $\sigma_i(k)$ is an involution, the Cayley graph Π_k of M_k is m-regular. Let $v(k)$ denote the cardinality of the group M_k.

Remark 3.10. It is easy to see that $v(k) = |M_k|$ satisfies
$$ck - c_1 < \log v(k) < ck + c_1$$
for some constants c, c_1.

4. Metric spaces with arbitrary compression functions

Let (X, d) be a metric space and $\lambda > 0$. We denote by λX the metric space $(X, \lambda \mathrm{d})$.

Let $(\Pi_n)_{n\geq 1}$ be the sequence of Cayley graphs of finite factor groups M_n, $n \geq 1$, of the lattice Γ from Lemma 3.8, see Notation 3.9.

Let ρ be a function in $\mathcal{C}$ (see Definition 1.2). For every $n \geq 1$, let x_n be a fixed vertex in Π_n. We are going to choose appropriate scaling constants λ_n, $n \geq 1$, so that the wedge union Π of the metric spaces $\lambda_n \Pi_n$, obtained by identifying all the vertices x_n to the same point x, has the required compression function ρ.

Notation 4.1. Throughout this section we denote $\log v(n)$ by y_n. According to Remark 3.10, $cn - c_1 < y_n < cn + c_1$ for some constants c, c_1.

We are looking for a sequence of rescaling constants λ_n satisfying $\lambda_n = \rho(\lambda_n y_n)$. This is equivalent to the fact that $\tau(\lambda_n y_n) = y_n$, where $\tau(x) = \frac{x}{\rho(x)}$. Since $\tau(x)$ is increasing by the definition of $\mathcal{C}$, continuous and $\lim_{x\to\infty} \tau(x) = +\infty$, $\tau^{-1}(x)$ exists for large enough x. Thus, we can take $\lambda_n := \frac{\tau^{-1}(y_n)}{y_n}$.

Proposition 4.2. *Let Π be the wedge union of $\lambda_n \Pi_n$, $n \geq 1$. Then ρ is both the Hilbert space compression function and the $\mathcal{E}^{p,r}$–compression function of Π (up to the equivalence relation $\asymp$).*

In particular, ρ is the $\{$uniformly convex Banach space$\}$–compression function of Π.

The proof of Proposition 4.2 will show that the same conclusions hold if we replace the sequence $(\lambda_n)_{n\geq 1}$ by any sequence of numbers $(\mu_n)_{n\geq 1}$ satisfying $\mu_n \asymp \lambda_n$.

Proof of Proposition 4.2. We are going to prove that ρ is the Hilbert space compression function of Π. The proof that ρ is the $\mathcal{E}^{p,r}$–compression function is essentially the same (with reference to Theorem 3.2 replaced by a reference to Corollary 3.5).

Let d_τ be the canonical distance on Π, and let V be the set of vertices of Π. Consider the map f from V to $\ell_2(V)$ defined as follows. For every $n \geq 1$ and every $v \in V_n \setminus \{x_n\}$ let $f(v)$ be equal to $\lambda_n \delta_v$, where δ_v is the Dirac function at v. Also let $f(x) = 0$.

We prove that the following inequalities are satisfied, with $d \geq 1$ the constant in Theorem 3.2(i).

$$\frac{1}{\sqrt{2}}\rho\left(\frac{\mathrm{d}_\tau(v,v')}{d}\right) \leq \|f(v)-f(v')\| \leq \sqrt{2}\mathrm{d}_\tau(v,v')\,. \tag{8}$$

Assume that $v, v' \in \Pi_n$ for some n. Then $\|f(v)-f(v')\| \leq \lambda_n\sqrt{2}$ (when one of the vertices is x_n the first term is λ_n). Since $\mathrm{d}_\tau(v,v') \geq \lambda_n$, we get the second inequality in (8).

To prove the first inequality, recall that by Theorem 3.2 the diameter of $\lambda_n\Pi_n$ is at most $d\lambda_n\, y_n$. Hence

$$\rho\left(\frac{\mathrm{d}_\tau(v,v')}{d}\right) \leq \rho\left(\lambda_n y_n\right) = \lambda_n \leq \|f(v)-f(v')\|\,.$$

Assume now that $v \in \Pi_m \setminus \{x_m\}$ and $v' \in \Pi_n \setminus \{x_n\}$. Then

$$\|f(v)-f(v')\| = \sqrt{\lambda_m^2+\lambda_n^2} \leq \lambda_m+\lambda_n \leq \mathrm{d}_\tau(v,x)+\mathrm{d}_\tau(x,v') = \mathrm{d}_\tau(v,v')\,.$$

On the other hand

$$\|f(v)-f(v')\| \geq \frac{1}{\sqrt{2}}(\lambda_m+\lambda_n) = \frac{1}{\sqrt{2}}\left(\|f(v)-f(x)\|+\|f(v')-f(x)\|\right)\,.$$

By the previous case, the last term is at least

$$\frac{1}{\sqrt{2}}\left[\rho\left(\frac{\mathrm{d}_\tau(v,x)}{d}\right)+\rho\left(\frac{\mathrm{d}_\tau(x,v')}{d}\right)\right] \geq \frac{1}{\sqrt{2}}\rho\left(\frac{\mathrm{d}_\tau(v,v')}{d}\right)\,.$$

The latter inequality is due to the sub-additivity and the monotonicity of ρ.

This proves that ρ has property (1) of the Hilbert space compression function of Π, see Definition 2.6(1).

To prove property (2) of the Hilbert space compression function, let us consider an increasing function $\bar\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty}\bar\rho(x) = \infty$, and an 1-Lipschitz $\bar\rho$–embedding g of Π into a Hilbert space. For any pair of vertices v, v' in Π we have

$$\bar\rho(\mathrm{d}_\tau(v,v')) \leq \|g(v)-g(v')\| \leq \mathrm{d}_\tau(v,v')\,. \tag{9}$$

We shall prove that $\bar\rho \ll \rho$.

For every $n \geq 1$, for any two vertices v, v' in the same graph $\lambda_n\Pi_n$, the inequality (9) can be re-written as:

$$\bar\rho(\lambda_n \mathrm{d}(v,v')) \leq \|g(v)-g(v')\| \leq \lambda_n\mathrm{d}(v,v')\,. \tag{10}$$

We denote by h the restriction of g to $\lambda_n\Pi_n$, rescaled by the factor $\frac{1}{\lambda_n}$. The sequence of inequalities (10) divided by λ_n yields

$$\frac{1}{\lambda_n}\bar\rho(\lambda_n\mathrm{d}(v,v')) \leq \|h(v)-h(v')\| \leq \mathrm{d}(v,v')\,. \tag{11}$$

Theorem 3.2 implies that for some constants r, d and κ there exist vertices v_1 and v_2 such that

$$ry_n \leq \frac{\mathrm{d}(v_1, v_2)}{\|h(v_1) - h(v_2)\|}$$

and $\kappa y_n \leq \mathrm{d}(v_1, v_2) \leq dy_n$. From (11) and the monotonicity of $\bar{\rho}$, we get

$$\bar{\rho}(\kappa\lambda_n y_n) \leq \bar{\rho}(\lambda_n \mathrm{d}(v_1, v_2)) \leq \frac{\lambda_n \mathrm{d}(v_1, v_2)}{ry_n} \leq \frac{d}{r}\lambda_n = \frac{d}{r}\rho(\lambda_n y_n).$$

Denote by z_n the product $\lambda_n y_n$, also equal to $\tau^{-1}(y_n)$, by the definition of λ_n. Then we get

$$\bar{\rho}(\kappa z_n) \leq \frac{d}{r}\rho(z_n). \tag{12}$$

We have that $y_n \leq cn + c_1 \leq y_{n-1} + C_1$ where $C_1 = c + 2c_1$.

By property (3) of Definition 1.2, $\tau(x) = \frac{x}{\rho(x)}$ is an increasing map defining a bijection $[a, \infty) \to [b, \infty)$. Moreover, the condition that $\frac{\tau(x)}{\log x}$ is non-decreasing easily implies that for some $\theta > 1$, $\tau(x) + 1 \leq \tau(\theta x)$ for every $x \geq a$. It follows that $\tau(x) + C_1 \leq \tau(Cx)$ for every $x \geq a$, where C is a power of θ depending on C_1. This implies that for every $y \geq b$, $\tau^{-1}(y+C_1) \leq C\tau^{-1}(y)$. Consequently, for n large enough, $z_n = \tau^{-1}(y_n) \leq \tau^{-1}(y_n+C_1) \leq C\tau^{-1}(y_{n-1}) = z_{n-1}$. We thus get

$$\frac{z_n}{z_{n-1}} \leq C\,.$$

Now take any sufficiently large $x > 0$. Then for some n, x is between z_{n-1} and z_n. Hence (by the monotonicity of $\bar{\rho}$ and ρ), we get:

$$\bar{\rho}(\kappa x) < \bar{\rho}(\kappa z_n) < \frac{d}{r}\rho(z_n) < \frac{d}{r}\rho(Cz_{n-1}) \ll \rho(Cx).$$

Therefore, $\bar{\rho} \ll \rho$ as required. □

One can obviously replace Π in Proposition 4.2 by a uniformly proper space (graph of bounded degree) with the same property.

The following corollary immediately follows from Proposition 4.2.

Corollary 4.3. *For every number α in $[0, 1]$ there exists a proper metric space (graph of bounded degree) whose the Hilbert space compression and the $\mathcal{E}^{p,r}$–compression are equal to α.*

Note that $\alpha = 0$ can be obtained by taking, say, $\rho(x) = \log x$ in Proposition 4.2. To achieve $\alpha = 1$ take the one-vertex graph.

5. Discrete groups with arbitrary Hilbert space compressions

For every prime number p and every natural number $r > 0$, we denote $\mathcal{E}^{p,r}$ simply by $\mathcal{E}$ in what follows. Recall that $\mathcal{E}$ contains all the uniformly convex Banach spaces.

Pick a function $\rho \in \mathcal{C}$. For simplicity we assume that $\rho(1) > 0$. As in Definition 1.2, we denote by τ the function $\frac{x}{\rho(x)}$.

Let $(M_k, \mathrm{d}_k)_{k\geq 1}$ be the sequence of finite groups defined as above, for fixed p and r, see Lemma 3.8 and Notation 3.9. Recall that d_k is the word metric associated to the generating set $U_k = \{\sigma_1(k), \dots, \sigma_m(k)\}$ consisting of $m = m(p,r)$ involutions, and that $v(k)$ is the cardinality of M_k.

Notation 5.1. We denote by $|\cdot|_k$ the length function on M_k associated to U_k.

Notation 5.2. Let us fix three sequences of numbers: $(\lambda_k)_{k\geq 1}$, $(m_k)_{k\geq 1}$, and $(\mu_k)_{k\geq 1}$.

For $k \geq 1$, we set λ_k such that $\rho(\lambda_k \log v(k)) = \lambda_k$; equivalently $\lambda_k = \frac{\tau^{-1}(\log v(k))}{\log v(k)}$ with $v(k)$ as above. Without loss of generality we assume (by taking a suitable subsequence of M_k) that for every $k \geq 1$, $\lambda_{k+1} - \lambda_k \geq 4$.

For every $k \geq 2$, let m_k be the integer part of $\frac{\lambda_k - 1}{2}$, and let $m_1 = 0$. Then we put $\mu_k = 2m_k + 1$ for every $k \geq 1$.

According to our choice, for every $k \geq 2$

$$\mu_k \leq \lambda_k < \mu_k + 2\,. \tag{13}$$

We are now ready to construct our group.

Let F be the free product of the groups M_k, $k \geq 1$.

For every $i \in \{1, \dots, m\}$, consider a copy H_i of the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$, where the generator of the $\mathbb{Z}/2\mathbb{Z}$-factor of H_i is denoted by σ_i, while the generator of the $\mathbb{Z}$-factor of H_i is denoted by t_i.

Notation 5.3. We denote by $|\cdot|_{H_i}$ the length function on H_i relative to the generating set $\{\sigma_i, t_i\}$. For every $k \in \mathbb{Z}$, we denote the element $t_i^k \sigma_i t_i^{-k}$ in H_i by $\sigma_i^{(k)}$. Note that $|\sigma_i^{(k)}|_{H_i} = 2k+1$, and that H_i is the semidirect product of $\langle \sigma_i^{(k)}, k \in \mathbb{Z}\rangle$ and $\langle t_i \rangle$.

Let G be the fundamental group of the following graph of groups (see [Ser, Ch.5.1] for the definition). The vertex groups are F and $H_1, ..., H_m$, the only edges of the graph are (F, H_i), $i = 1, ..., m$. The edge groups $F \cap H_i$ are free products $*_{k\geq 1} (\mathbb{Z}/2\mathbb{Z})_k$ of countably many groups of order 2, where the k-th factor $(\mathbb{Z}/2\mathbb{Z})_k$ is identified with $\langle \sigma_i(k)\rangle < M_k$ in F and with $\langle \sigma_i^{(m_k)} \rangle$ in H_i.

Notation 5.4. We denote by d_G and by $|\cdot|_G$ the word metric and respectively the length function on G associated to the generating set $U = \{\sigma_1(1), ..., \sigma_m(1), t_1, ..., t_m\}$.

Theorem 5.5. *With the notation above and the terminology in Definition 2.6, the following hold:*

- **(I)** *The group G has as a Hilbert space compression gap, and an $\mathcal{E}$–compression gap $(\sqrt{\rho}, \rho)$.*
- **(II)** *The group G has as a Hilbert space compression gap, and an $\mathcal{E}$–compression gap $\left(\frac{\rho}{\log^{1+\epsilon}(x)}, \rho\right)$, for every $\epsilon > 0$. Hence the Hilbert space compression and the $\mathcal{E}$–compression of G are equal to the supremum over all the non-negative numbers α such that $x^\alpha \ll \rho$.*
- **(III)** *The asymptotic dimension of G is at most 2.*

We start our proof by describing the length function $|\cdot|_G$.

By the standard properties of amalgamated products [LS77, Ch.IV, Th.2.6], the free product $F = \ast_{k\geq 1} M_k$ and the groups H_i, $i \in \{1, 2, \ldots, m\}$, are naturally embedded into G. Moreover, the subgroup $T = \langle t_i, 1 \leq i \leq m\rangle$ is free of rank m and a retract of G, so its length function coincides with $|\cdot|_G$ restricted to T.

Definition 5.6. With every element $s \in M_k$ we assign the *weight* $\mathrm{wt}(s) = \mu_k\, |s|_k$. With every element $b \in H_i$ we assign the *weight* $\mathrm{wt}(b) = |b|_{H_i}$. The *weight* $\mathrm{wt}(W)$ of any word $W = s_1 b_1 ... s_n b_n$ with s_1 and b_n possibly trivial and $n \in \mathbb{N}$ is the sum $\sum_{i=1}^{n}[\mathrm{wt}(s_i)+\mathrm{wt}(b_i)]$.

Let F' be the free product of $F = \ast_{k\geq 1} M_k$ and all the 2-element groups $\left\langle \sigma_i^{(k)} \right\rangle$ with $k \in \mathbb{Z} \setminus \{m_k\ ,\ k \geq 1\}$ and $i \in \{1, \ldots, m\}$. Then G is the multiple HNN extension of F' with free letters $t_1, \ldots, t_m$ shifting $\sigma_i^{(k)}$. This is the presentation of G used in the following lemma.

A subword $t_i^{\pm 1}\sigma_i^{(k)} t_i^{\mp 1}$ of a word in generators $\sigma_i^{(k)}, t_i$ of G is called a *pinch*. It can be removed and replaced by $\sigma_i^{(k\pm 1)}$; we call this operation *removal of a pinch*.

Lemma 5.7. *For every $k \geq 1$ and $g \in M_k$, $|g|_G = \mathrm{wt}(g) = \mu_k\, |g|_k$.*

Proof. Let $s = |g|_k$ and let $g = \sigma_{i_1}(k)\sigma_{i_2}(k)...\sigma_{i_s}(k)$ be a shortest representation of g as a product of generators of M_k. Then we can represent g as

$$t_{i_1}^{m_k}\sigma_{i_1}(1)t_{i_1}^{-m_k} t_{i_2}^{m_k}\sigma_{i_2}(1)t_{i_2}^{-m_k} ... t_{i_s}^{m_k}\sigma_{i_s}(1)t_{i_s}^{-m_k}.$$

Hence $|g|_G \leq (2m_k + 1)\, s = \mu_k\, |g|_k$.

Let W be any shortest word in the alphabet U representing g in G. Since $g \in F$, there exists a sequence of removals of pinches and subwords of the form aa^{-1} that transfers W into a word W' in generators of F (i.e. letters of the form $\sigma_j(l)$) representing g. Since the weight of every $\sigma_i(1)$ and $t_i^{\pm 1}$ is 1, the total weight of the word W is the length of W. The removal of a pinch does not increase the total weight, and the removal of a word aa^{-1} decreases it. Hence the total weight of the letters in W' is at most $|W|$. On the other hand, since W' is a word in generators of the free product F representing an element of one of the factors M_k, all letters from W' must belong to M_k, i.e. they have the form $\sigma_i(k)$ with $i \in \{1, \ldots, m\}$. Therefore the total weight of W' is $\mu_k\, |W'|$. Since W' represents g in M_k, we have that the total weight of W' is at least $\mu_k\, |g|_k$. Hence $|g|_G = |W| \geq \mu_k\, |g|_k$ as required. □

The choice of μ_k given by (13), Proposition 4.2 and the construction in its proof, and Lemma 5.7 imply the following.

Lemma 5.8.

(1) *The function ρ is the $\mathcal{E}$–compression function and the Hilbert space compression function of $S = \bigcup_{k\geq 1} M_k$ endowed with the restriction of the metric d_G.*

(2) *The map $f : S \to l_2(S)$ defined by $f(1) = 0$ and $f(s) = \mu_k \delta_s$ for every $s \in M_k \setminus \{1\}$ is a ρ–embedding of (S, d_G).*

Lemma 5.8 immediately implies

Lemma 5.9. *The function ρ is an upper bound for an $\mathcal{E}$–compression gap of G (in particular, also for a Hilbert space compression gap of G).*

Definition 5.10. We are going to use the standard normal forms of elements in the amalgamated product G (see [LS77, Ch.IV.2]):

$$s_1b_1...s_nb_n \tag{14}$$

where

(N_0) possibly $s_1 = 1$ or $b_n = 1$;
(N_1) $s_i \in M_{k_i}$ and $b_i \in H_{l_i}$ (s_i, b_i are called *syllables*);
(N_2) if $b_i = 1$, $i < n$, then s_i and s_{i+1} are not in the same M_k; if $s_i = 1$, $i > 1$, then b_{i-1}, b_i are not in the same H_k.

One can get from one normal form to another by sequences of operations as follows:

(r_1) replacing s_ib_i with $(s_i\sigma_k(p))(\sigma_k(p)b_i)$ (*left insertion*);
(r_2) replacing b_is_{i+1} with $(b_i\sigma_k(p))(\sigma_k(p)s_{i+1})$ (*right insertion*);
(r_3) replacing two-syllable words with syllables from the same factor by the one-syllable word, their product.

One can easily check that this collection of moves is confluent, so one does not need the inverse of rule (r_3).

Choose one representative for each left coset of $M_k/\langle\sigma_i(k)\rangle$ and one representative in each left coset of $H_i/\langle\sigma_i(k)\rangle$, the representative of $\langle\sigma_i(k)\rangle$ is 1. Let $\mathcal{M}$ be the set of all these representatives. We consider only the normal forms (14) in which all the s_i and b_i (except possibly for the last non-identity syllable) are in $\mathcal{M}$. We shall call these normal forms *good*. Every element of G has a unique good normal form.

Lemma 5.11. *Let $V = s_1b_1s_2...s_nb_n$ be the good normal form of an element g in the amalgamated product G. Then the length $|g|_G$ is between $\mathrm{wt}(V)/3$ and $\mathrm{wt}(V)$, i.e.*

$$\sum_{i=1}^{n} \mu_{k_i}|s_i|_{k_i} + \sum_{i=1}^{n} |b_i|_{H_{l_i}} \geq |g|_G \geq \frac{1}{3}\left(\sum_{i=1}^{n} \mu_{k_i}|s_i|_{k_i} + \sum_{i=1}^{n} |b_i|_{H_{l_i}}\right). \tag{15}$$

For every normal form V' of g, the length of g is between $\mathrm{wt}(V')/9$ and $\mathrm{wt}(V')$.

Proof. The first inequality in (15) follows from Lemma 5.7 and it holds for every normal form.

Consider a shortest word W in the generators of G representing g. Making moves of type (r_3), we rewrite W in a normal form W' without increasing its weight (note that a removal of pinches can be seen as a succession of two (r_3)-type moves). We have that $|g|_G = \mathrm{wt}(W) \geq \mathrm{wt}(W') \geq |g|_G$; the last inequality holds because any word of the form (14) and of weight ℓ is equal in G to a word of length ℓ in the generators in U and their inverses. It follows that $\mathrm{wt}(W') = |g|_G$. By doing insertion moves (r_1) and (r_2), we can rewrite the normal form W' into the good normal form V of g. Note that each syllable is multiplied by at most two involutions during the process. The weight of each of these involutions does not exceed the weight of the syllables s_i, s_{i+1} involved in the moves. Hence the total weight of the word cannot more than triple during the process. Hence $\mathrm{wt}(V) \leq 3\mathrm{wt}(W') = 3|g|_G$ which proves the second inequality in (15).

The second statement is proved in a similar fashion: one needs to analyze the procedure of getting the good normal form from any normal form, and then use the first part of the lemma. □

Definition 5.12. Let $s_1b_1...s_nb_n$ be the good normal form of g. Represent each b_i as the shortest word $w(b_i)$ in the alphabet of generators $\{\sigma_{l_i}, t_{l_i}\}$ of the corresponding subgroup H_{l_i}. Then the word $s_1w(b_1)...s_nw(b_n)$ in the alphabet $\bigcup_{k\geq 1} M_k \cup \{\sigma_i, t_i, i \in \{1, 2, ..., m\}\}$ is called the *extended normal form* of g.

The unicity of the good normal form and of each word $w(b_i)$ implies that every element in G has a unique extended normal form.

Definition 5.13. For every pair of elements g and h in G with extended normal forms $g = s_1w(b_1)...s_lw(b_l)$ and $h = s'_1w(b'_1)...s'_nw(b'_n)$ let $p(g, h) = p(h, g)$ be the longest common prefix of these words, of length $i(g, h)$. Thus

$$g \equiv p(g, h)f(g, h)q(g, h) \quad \text{and} \quad h \equiv p(g, h)f(h, g)q(h, g)$$

where the words $f(g, h)q(g, h)$, $f(h, g)q(h, g)$ have different first syllables $f(g, h)$ and $f(h, g)$ respectively. Here the *syllables* are either in some M_k , $k \geq 1$, or they are in $\{t_1^{\pm 1}, t_2^{\pm 1}, ..., t_m^{\pm 1}\}$.

Note that $p(g, h), q(g, h), q(h, g)$ are extended normal forms; $f(g, h)$ (resp. $f(h, g)$) is either an element in M_k for some $k \geq 1$ or it is in $\{\sigma_i, t_i^{\pm 1}\}$ for some $i \in \{1, 2, ..., m\}$.

Definition 5.14. Let $s_1w(b_1)...s_nw(b_n)$ be the extended normal form of g. Then for every i let $g[i]$ be the i-th letter of the extended normal form, let $\hat{g}_i$ be the prefix ending in $g[i]$, and let $\bar{g}_i$ be the suffix starting with $g[i]$ of the extended normal form.

Lemma 5.15. *For every $g, h \in G$, the distance $\mathrm{d}_G(g, h)$ is in the interval $[A/9, A]$ where A is equal to:*

(**S**) $\mu_k\, \mathrm{d}_k(f(g, h), f(h, g)) + \mathrm{wt}(q(g, h)) + \mathrm{wt}(q(h, g))$ *if* $f(g, h), f(h, g) \in M_k$, *or*
(**B**) $\mathrm{wt}(f(g, h)q(g, h)) + \mathrm{wt}(f(h, g)q(h, g))$ *otherwise.*

Proof. In Case (**S**), $q(g, h)^{-1}(f(g, h)^{-1}f(h, g))q(h, g)$ becomes a normal form for $g^{-1}h$ if we combine all neighbor letters from the same H_k into one syllable, and $f(g, h)^{-1}f(h, g)$ into one syllable. In Case (**B**), $q(g, h)^{-1}f(g, h)^{-1}f(h, g)q(h, g)$ becomes a normal form for $g^{-1}h$ after a similar procedure. Then one can use Lemma 5.11. □

For every element g in G whose extended normal form has the last syllable in Y, where $Y = M_k$ for some $k \geq 1$ or $Y = \{\sigma_i, t_i^{\pm 1}\}$ for some $i \in \{1, \ldots, m\}$, we consider a copy ϕ_g of an 1-Lipschitz embedding of Y into a Hilbert space $\mathcal{H}_g$ with optimal distortion: either the embedding from Remark 3.3 rescaled by the factor μ_k if $Y = M_k$ (i.e. the embedding sending all non-trivial elements in M_k in pairwise orthogonal vectors of length μ_k) or the embedding defined by any choice of an orthonormal basis in a copy of $\mathbb{R}^2$ otherwise. Note that since $\mu_1 = 1$ this is coherent with the identification of $\sigma_i \in H_i$ with $\sigma_i(1) \in M_1$.

Let $\mathcal{H}$ be the Hilbertian sum of all $\mathcal{H}_g$. Consider the following map from G into $\mathcal{H}$: for every $g \in G$ let

$$\psi(g) := \sum_i \phi_{\hat{g}_i}(g[i]). \tag{16}$$

We use the map ψ to prove the following.

Lemma 5.16. *The group G has as a Hilbert space and an $\mathcal{E}$–compression gap $(\sqrt{\rho}, \rho)$.*

Proof. In view of Lemma 5.9, it suffices to show that ψ is a $\sqrt{\rho}$–embedding to finish the proof.

Let $g, h \in G$. Then, by (16), since any two elements of $\mathcal{H}$ of the form $\phi_{\hat{g}_i}(g[i])$, $\phi_{\hat{h}_j}(h[j])$ either coincide or are othogonal to each other, we have:

$$\begin{array}{rcl} \|\psi(g) - \psi(h)\|^2 & = & \|\phi_{p(g,h)f(g,h)}(f(g,h))\|^2 \quad + \quad \|\phi_{p(g,h)f(h,g)}(f(h,g))\|^2 \\ & + & \sum_{j>i(g,h)+1} \|\phi_{\hat{g}_j}(g[j])\|^2 \quad + \quad \sum_{j>i(g,h)+1} \|\phi_{\hat{h}_j}(h[j])\|^2 \quad \leq C\mathrm{d}_G(g,h)^2 \end{array}$$

for some constant C (by Lemma 5.15 and the fact that all embeddings ϕ are 1-Lipschitz). Thus ψ is a Lipschitz embedding.

To prove the lower bound, note that for $g[i]$ in H_k,

$$|g[i]|_G = \|\phi_{\hat{g}_i}(g[i])\| = 1 \gg \rho(|g[i]|_G).$$

Also if $g[i] \in M_k$ then by Lemma 5.8 (2), and by the definition of ϕ_g we may write

$$|g[i]|_G = \mu_k \, |g[i]|_k = \|\phi_{\hat{g}_i}(g[i])\| \gg \rho(|g[i]|_G)\,.$$

Thus, in all cases,

$$\|\phi_{\hat{g}_i}(g[i])\| \gg \rho(|g[i]|_G).$$

Now, in case (**B**),

$$\|\psi(g) - \psi(h)\|^2 \gg \rho(\mathrm{d}_G(f(g,h), f(h,g)))^2 + \sum_{j>i(g,h)+1} \rho(|g[j]|_G)^2 + \sum_{j>i(g,h)+1} \rho(|h[j]|_G)^2.$$

The fact that $\rho(g[i]) \geq \rho(1)$ for non-identity $g[i]$, and the subadditivity of ρ (plus Lemma 5.15) imply

$$\|\psi(g) - \psi(h)\|^2 \gg \rho(\mathrm{d}_G(g,h))$$

as desired. In case (**S**), the proof is similar, only one needs to take into account that

$$\begin{array}{rl} \|\phi_{p(g,h)f(g,h)}(f(g,h))\|^2 & + \ \|\phi_{p(g,h)f(h,g)}(f(h,g))\|^2 \\ & = \|\phi_{p(g,h)f(g,h)}(f(g,h)) - \phi_{p(g,h)f(h,g)}(f(h,g))\|^2 \\ & \geq \rho(\mathrm{d}_G(f(g,h), f(h,g)))^2. \end{array}$$

□

Lemma 5.16 gives part (**I**) of Theorem 5.5.

Let us prove part (**II**). For simplicity, we take $\epsilon = 1$. The reader can easily modify the proof to make it work for every $\epsilon > 0$. Thus, we are going to prove that $\frac{\rho(x)}{\log^2(x+1)}$ is a lower bound for a Hilbert space compression gap and an $\mathcal{E}$–compression gap of G. That is, we are going to prove that there exists a $\frac{\rho(x)}{\log^2(x+1)}$–uniform embedding of G in a Hilbert space.

Consider the same Hilbert space $\mathcal{H}$ and the same embeddings ϕ_g for $g \in G$ as before. Let us define an embedding π of G into $\mathcal{H}$. Let g be given in an extended normal form. Then we set

$$\pi(g) := \sum_j \kappa_j \phi_{\hat{g}_j}(g[j])$$

where the coefficients κ_j are defined as follows:

$$\kappa_j = \frac{\sqrt{\mathrm{wt}(\bar{g}_j)}}{\log(\mathrm{wt}(\bar{g}_j)+1)\sqrt{\mathrm{wt}(g[j])}}.$$

We shall need the following two elementary inequalities, the first of which is obvious.

Lemma 5.17. *For every sequence of positive numbers $a_1, ..., a_n$, we have*

$$\sum_{i=1}^{n}\left(\sum_{r=i}^{n} a_r\right) a_i \le \left(\sum_{r=1}^{n} a_i\right)^2.$$

Lemma 5.18. *Let $a_1, ..., a_n$ be positive real numbers with $a_n \ge 1$. Then*

$$\frac{a_1}{(a_1+...+a_n)\log^2(a_1+...+a_n+1)} + \frac{a_2}{(a_2+...+a_n)\log^2(a_2+...+a_n+1)} + ... + \frac{a_n}{a_n\log^2(a_n+1)} \le C. \tag{17}$$

for some constant C.

Proof. For every $k = 1, ..., n$ denote $s_k = a_k + ... + a_n$. Then (17) can be rewritten as follows:

$$\frac{s_1 - s_2}{s_1\log^2(s_1+1)} + ... + \frac{s_{n-1} - s_n}{s_{n-1}\log^2(s_{n-1}+1)} + \frac{1}{\log^2(s_n+1)}. \tag{18}$$

Since the function $\frac{1}{x\log^2(x+1)}$ is decreasing, one can estimate (provided $b \ge a \ge 1$)

$$\frac{b-a}{b\log^2(b+1)} \le \int_a^b \frac{dx}{x\log^2(x+1)} \le 2\int_a^b \frac{dx}{(x+1)\log^2(x+1)} = \frac{2}{\log(a+1)} - \frac{2}{\log(b+1)}.$$

Applying this inequality to (18) and using $a_n \ge 1$, we get (17). □

Let g, h be two elements in G given in their extended normal forms. Let $i = i(g,h)$, $f = f(g,h)$, $f' = f(h,g)$, $p = p(g,h)$, $q = q(g,h)$, $q' = q(h,g)$.

Then $\pi(g) - \pi(h) = \Sigma_1 + \Sigma_2 + \Sigma_3$ where

$$\Sigma_1 = \sum_{r=1}^{i}\left(\frac{\sqrt{\mathrm{wt}(\bar{g}_r)}}{\log(\mathrm{wt}(\bar{g}_r)+1)} - \frac{\sqrt{\mathrm{wt}(\bar{h}_r)}}{\log(\mathrm{wt}(\bar{h}_r)+1)}\right)\frac{1}{\sqrt{\mathrm{wt}(g[r])}}\phi_{\hat{g}_r}(g[r]),$$

$$\Sigma_2 = \frac{\sqrt{\mathrm{wt}(\bar{g}_{i+1})}}{\log(\mathrm{wt}(\bar{g}_{i+1})+1)\sqrt{\mathrm{wt}(g[i+1])}}\phi_{pf}(f) - \frac{\sqrt{\mathrm{wt}(\bar{h}_{i+1})}}{\log(\mathrm{wt}(\bar{h}_{i+1})+1)\sqrt{\mathrm{wt}(h[i+1])}}\phi_{pf'}(f'),$$

$$\Sigma_3 = \sum_{r\ge i+2}\frac{\sqrt{\mathrm{wt}(\bar{g}_r)}}{\log(\mathrm{wt}(\bar{g}_r)+1)\sqrt{\mathrm{wt}(g[r])}}\phi_{\hat{g}_r}(g[r]) - \sum_{r\ge i+2}\frac{\sqrt{\mathrm{wt}(\bar{h}_r)}}{\log(\mathrm{wt}(\bar{h}_r)+1)\sqrt{\mathrm{wt}(h[r])}}\phi_{\hat{h}_r}(h[r]).$$

Lemma 5.19 (Upper bound). $\|\pi(g) - \pi(h)\| \ll \mathrm{d}_G(g,h)$, *that is π is Lipschitz.*

Proof. Since G is finitely generated, it suffices to prove that the norm $\|\pi(g) - \pi(gs)\|$ is bounded uniformly in $s \in U \cup U^{-1}$ and $g \in G$. In this case the sum Σ_3 does not appear, and $\Sigma_2 = \frac{1}{\log 2}\phi_{gs}(s)$ has norm $\frac{1}{\log 2}$. By eventually replacing g with gs and s with s^{-1}, we can always assume that $h = gs$ satisfies $\mathrm{wt}(\bar{h}_r) = \mathrm{wt}(\bar{g}_r) + 1$ for all $r \leq i$. Then

$$\|\Sigma_1\|^2 = \sum_{r=1}^{i} \left(\frac{\sqrt{\mathrm{wt}(\bar{g}_r) + 1}}{\log(\mathrm{wt}(\bar{g}_r) + 2)} - \frac{\sqrt{\mathrm{wt}(\bar{g}_r)}}{\log(\mathrm{wt}(\bar{g}_r) + 1)} \right)^2 \mathrm{wt}(g[r])\,.$$

Since the function $x \mapsto \frac{\sqrt{x}}{\log(x+1)}$ is increasing for $x \geq e^2 - 1$ it follows that

$$\|\Sigma_1\|^2 \ll \sum_{r=1}^{i} \frac{\left[\sqrt{\mathrm{wt}(\bar{g}_r) + 1} - \sqrt{\mathrm{wt}(\bar{g}_r)}\right]^2}{\log^2(\mathrm{wt}(\bar{g}_r) + 1)} \mathrm{wt}(g[r]) \leq \sum_{r=1}^{i} \frac{\mathrm{wt}(g[r])}{\mathrm{wt}(\bar{g}_r) \log^2(\mathrm{wt}(\bar{g}_r) + 1)}.$$

Lemma 5.18 implies now that

$$\|\Sigma_1\|^2 = O(1). \tag{19}$$

□

Lemma 5.20 (Lower bound). $\|\pi(g) - \pi(h)\| \gg \frac{\rho(\mathrm{d}_G(g,h))}{\log^2(\mathrm{d}_G(g,h)+1)}$ *if* $g \neq h$.

Proof. We can write

$$\begin{aligned} \|\Sigma_3\|^2 \;&\gg \textstyle\sum_{r \geq i+2} \frac{\mathrm{wt}(\bar{g}_r)}{\log^2(\mathrm{wt}(\bar{g}_r)+1)\mathrm{wt}(g[r])} \rho\left(\mathrm{wt}(g[r])\right)^2 \\ &\quad + \textstyle\sum_{r \geq i+2} \frac{\mathrm{wt}(\bar{h}_r)}{\log^2(\mathrm{wt}(\bar{h}_r)+1)\mathrm{wt}(h[r])} \rho\left(\mathrm{wt}(h[r])\right)^2\,. \end{aligned} \tag{20}$$

If g, h are in Case (**B**) then

$$\|\Sigma_2\|^2 \gg \frac{\mathrm{wt}(\bar{g}_{i+1})}{\log^2(\mathrm{wt}(\bar{g}_{i+1}) + 1)\mathrm{wt}(f)} \rho\left(\mathrm{wt}(f)\right)^2 + \frac{\mathrm{wt}(\bar{h}_{i+1})}{\log^2(\mathrm{wt}(\bar{h}_{i+1}) + 1)\mathrm{wt}(f')} \rho\left(\mathrm{wt}(f')\right)^2\,.$$

Thus in case (**B**) we have

$$\begin{aligned} \|\Sigma_2\|^2 + \|\Sigma_3\|^2 \;&\gg \textstyle\sum_{r > i} \frac{\mathrm{wt}(\bar{g}_r)}{\log^2(\mathrm{wt}(\bar{g}_r)+1)\mathrm{wt}(g[r])} \rho\left(\mathrm{wt}(g[r])\right)^2 \\ &\quad + \textstyle\sum_{r > i} \frac{\mathrm{wt}(\bar{h}_r)}{\log^2(\mathrm{wt}(\bar{h}_r)+1)\mathrm{wt}(h[r])} \rho\left(\mathrm{wt}(h[r])\right)^2\,. \end{aligned}$$

By the subadditivity of ρ, we have

$$\mathbf{F} = \left[\sum_{r>i} \rho(\mathrm{wt}(g[r])) + \sum_{r>i} \rho(\mathrm{wt}(h[r]))\right]^2 \geq \rho(\mathrm{d}_G(g,h))^2.$$

By the Cauchy-Schwartz inequality, the squared sum $\mathbf{F}$ does not exceed the product $F_1 F_2$ where

$$F_1 = \sum_{r>i} \frac{\mathrm{wt}(g[r]) \log^2(\mathrm{wt}(\bar{g}_r) + 1)}{\mathrm{wt}(\bar{g}_r)} + \sum_{r>i} \frac{\mathrm{wt}(h[r]) \log^2(\mathrm{wt}(\bar{h}_r) + 1)}{\mathrm{wt}(\bar{h}_r)}$$

$$F_2 = \sum_{r>i} \frac{\mathrm{wt}(\bar{g}_r)}{\log^2(\mathrm{wt}(\bar{g}_r)+1)\mathrm{wt}(g[r])}\rho(\mathrm{wt}(g[r]))^2 + \sum_{r>i} \frac{\mathrm{wt}(\bar{h}_r)}{\log^2(\mathrm{wt}(\bar{h}_r)+1)\mathrm{wt}(h[r])}\rho(\mathrm{wt}(h[r]))^2 .$$

Then $\frac{\rho(\mathrm{d}_G(g,h))^2}{\log^4(\mathrm{d}_G(g,h)+1)}$ does not exceed $\frac{1}{\log^4(\mathrm{d}_G(g,h)+1)}\mathbf{F}$ that, in turn, does not exceed

$$\frac{1}{\log^4(\mathrm{d}_G(g,h)+1)} F_1F_2 \leq \left(\sum_{r>i} \frac{\mathrm{wt}(g[r])}{\log^2(\mathrm{wt}(\bar{g}_r)+1)\mathrm{wt}(\bar{g}_r)} + \sum_{r>i} \frac{\mathrm{wt}(h[r])}{\log^2(\mathrm{wt}(\bar{h}_r)+1)\mathrm{wt}(\bar{h}_r)} \right) F_2.$$

By Lemma 5.18, the latter sum does not exceed a constant times F_2 that, in turn does not exceed a constant times $\|\Sigma_2\|^2 + \|\Sigma_3\|^2$.

Combining all these inequalities, we get

$$\|\pi(g)-\pi(h)\|^2 \geq \|\Sigma_2\|^2 + \|\Sigma_3\|^2 \geq \frac{\rho(\mathrm{d}_G(g,h))^2}{\log^4(\mathrm{d}_G(g,h)+1)}.$$

Assume that g,h are in case (**S**), so $f,f' \in M_k$ for some $k \geq 1$. Theorem 3.2 implies that $\mathrm{diam}\,(M_k, \mathrm{d}_k) \leq d \log v(k)$. Therefore by inequalities (13)

$$\rho\left(\mathrm{d}_G\left(f,f'\right)\right)^2 \leq \rho\left(\mu_k \,\mathrm{diam}\, M_k\right)^2 \leq \rho(d\lambda_k \log v(k))^2 \leq (d+1)^2(\lambda_k)^2 \ll (\mu_k)^2 .$$

The last but one inequality above follows by monotonicity and sub-additivity of ρ, as well as by the equality $\lambda_k = \rho(\lambda_k \log v(k))$.

Therefore

$$\|\Sigma_2\|^2 \gg \frac{\mathrm{wt}(\bar{g}_{i+1})}{\log^2(\mathrm{wt}(\bar{g}_{i+1})+1)\mathrm{wt}(f)}\rho\left(\mathrm{d}_G\left(f,f'\right)\right)^2 + \frac{\mathrm{wt}(\bar{h}_{i+1})}{\log^2(\mathrm{wt}(\bar{h}_{i+1})+1)\mathrm{wt}(f')}\rho\left(\mathrm{d}_G\left(f,f'\right)\right)^2 . \tag{21}$$

An argument similar to the one in case (**B**) allows to obtain the lower bound in this case also. □

Lemma 5.20 completes the proof of Part (**II**) of Theorem 5.5.

Now let us prove Part (**III**).

Lemma 5.21. *Consider the group F (the free product of finite groups M_k, $k \geq 1$) with the metric induced by the word metric on G. Then F has asymptotic dimension one.*

Proof. It is a straightforward consequence of [DS, Theorem 2.1], stating that the asymptotic dimension of a countable group endowed with a proper left invariant metric is the supremum over the asymptotic dimensions of its finitely generated subgroups. Any finitely generated subgroup of F is inside a free product of finitely many M_k (hence of asymptotic dimension ≤ 1, according to [BDK]). □

Note that the asymptotic dimension of each H_i is 1 by [BDK]. It remains to use [BD, Corollary 24] and conclude that the asymptotic dimension of G is at most 2.

Theorem 5.5 is proved.

We conclude with some open questions.

Question 5.22. Does every finitely generated group have a Hilbert space compression gap of the form $(\frac{f}{\log x}, f)$ for some function $f : \mathbb{R}_+ \to \mathbb{R}_+$?

Question 5.23. Is there an amenable group with Hilbert space compression $< \frac{1}{2}$?

Question 5.24. What is a {uniformly convex Banach space}–compression of R. Thompson's group F or of the wreath product $\mathbb{Z} \wr \mathbb{Z}$?

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