HIGHER ORDER LAPLACIANS I.
HARMONIC AND
TWO–POINTS–HOMOGENEOUS SPACES

by

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Introduction

A Riemannian manifold (M^n,g) is defined to be harmonic if the Riemann–Lebesques density function $\omega_p = \sqrt{|\det g^{ij}|}$ (resp. if the polar density function $\theta_p = r_p^{n-1}\omega_p$) is depending only on the geodesics distance r_p from p in any normal coordinate neighbourhood around any point p . These spaces can be characterized also by the Mean Value Property of the harmonic functions [10].

The harmonic manifolds are Einstein manifolds and therefore all these spaces are real analytic w.r.t. the normal coordinate neighbourhoods by the Kazdan–de Turck Theorem [2]. So the distinguation of the infinitesimal, local and global harmonicity is not necessary as all these properties are equivalent.

Any two point homogeneous space is obviously harmonic. The classical Lichnerowicz conjecture [5] asserts the converse statment: Any harmonic manifold is two–point homogeneous.

This conjecture was proved by the present author in the simply connected compact case (resp. in more general, for compact manifolds with finite fundamental groups) [8]. The methods of this proof are of global character which do not work in the non–compact case. The main tricks if this treatment can be summarized as follows:

The compact harmonic manifolds considered are Blaschke manifolds, which have simple closed geodesics with the same length, say 2π. It turned out also, that the squared density function $\theta^2(r)$ is a trigonometric polynomial of the form $\theta^2(r) = T(\cos r)$ furthermore the symmetric (globally defined) eigenfunctions $\varphi_\lambda(r) = \varphi_\lambda(-r)$ of the radial Laplacian

$$\frac{d^2}{dr^2} + \frac{\theta'}{\theta}\frac{d}{dr}$$

are also trigonometric polynomial of the form $\varphi_\lambda(r) = P_\lambda(\cos r)$.

One can prove from these properties, that the density function $\theta(r)$ is of the form: $\theta(r) = (\sin r)^p(1 - \cos r)^q$ furthermore the first non–trivial eigenfunction φ_λ is simple linear, i.e. $\varphi_{\lambda_1}(r) = A \cos r + B$. Using an imbedding procedure, the symmetricity of the space follows. This proves the conjecture in the case considered.

In the present paper we consider the harmonic manifolds without any topological conditions. On the other hand we prove the Lichnerowicz conjecture under a stronger assumption namely for sharply harmonic manifolds.

It is a well known fact that the invariant differential operators of a two–point–homogeneous space are the polynomials of the Laplacian. In Chapter 2 we converse this statement, asserting that a Riemannian space is two–point–homogeneous if and only if suitable differential operators (the so called higher order Laplacians) are the polynomial of the Laplacian. Also characterizations of harmonic as well as of Einstein and super Einstein manifolds are given there.

In chapter 3 we characterize the symmetric spaces.

At last we mention that in an earlier paper [9] of the author a simple topological proof is given for the symmetricity of the two–point–homogeneous spaces. The basic statement of this proof is the following.

The $R'_{\underline{m}_p}(\,.\,) := (\nabla_{\underline{m}_p} R)(\,.\,,\underline{m}_p)\underline{m}_p$ denotes the indicated covariant derivative of the curvature tensor with respect to a unit vector $\underline{m}_p$ at a point p. So $R'_{\underline{m}_p}(\,.\,)$ is a self adjoint endomorphism.

If for any p the eigenvalues (or the invariants) of $R'_{\underline{m}_p}$ are constant along the euclidean unit sphere $S_p^{n-1} \subset T_p(M^n)$ then $R'_{\underline{m}_p} = 0$ $\nabla R = 0$ follows, i.e. the space is symmetric.

We use this statement also in this paper.

§ 1. The Lichnerowicz conjecture on sharply harmonic manifolds

Let $A_{p;r}$ be the Jacobian endomorphism field along an arc–wise parametrized geodesics $\gamma(r)$, $\gamma(0) = p$, defined as usual by

$$A''_{p;r} + R_{\dot\gamma(r)} \circ A_{p;r} = 0\,;\quad A_{p;0} = 0\,;\quad A'_{p;0} = \mathrm{Id}\,, \tag{1.1}$$

where $R_{\dot\gamma}(X) = R(X) = R(X,\dot\gamma)\dot\gamma$ is the Jacobian curvature–operator field. The endomorphism $A_{p;r}$ acts in the $(n-1)$–dimensional subspace of the tangent space $T_{\gamma(r)}(M^n)$ standing orthogonal to $\dot\gamma(r)$. The density function $\theta_p(\gamma(r))$ along $\gamma(r)$ is one of the invariants of $A_{p;r}$ namely it is the determinant: $\theta_p = \det A_p$. In the following we consider all the invariants $\sigma_p^{(1)}$; $\sigma_p^{(2)}, \dots ,\sigma_p^{(n-1)}$ of A_p defined by the characteristic equation:

$$\det(A_p + \lambda \mathrm{Id}) = \lambda^{n-1} + \sigma_p^{(1)}\lambda^{n-2} + \dots + \sigma_p^{(n-1)} . \tag{1.2}$$

Therefore $\sigma_p^{(1)} = \mathrm{Tr}\, A_p$; $\sigma_p^{(n-1)} = \theta_p$ satisfy. For a technical simplification we introduce also the invariants $\alpha_p^{(\rho)}$ defined by

$$\alpha_p^{(1)} := \mathrm{Tr}\, A_p\, ;\ \alpha_p^{(2)} := \mathrm{Tr}\, A_p \circ A_p = \mathrm{Tr}\, A_p^2\, ;\ \alpha_p^{(\rho)} := \mathrm{Tr}\, A_p^\rho . \tag{1.3}$$

The connections between these invariants are described by the well known formulas ([15], p. 92)

$$\begin{aligned} &\alpha^{(\rho)} - \alpha^{(\rho-1)}\sigma^{(1)} + \dots + (-1)^\rho \rho\, \sigma^{(\rho)} = 0 \ \text{ for } \rho < n ; \\ &\alpha^{(\rho)} - \alpha^{(\rho-1)}\sigma^{(1)} + \dots + (-1)^n \alpha^{(\rho-n+1)}\sigma^{(n-1)} = 0 \ \text{ for } \rho \geq n . \end{aligned} \tag{1.4}$$

Using induction, we get that any invariant $\alpha^{(i)}$ is a polynomial of the invariants $\sigma^{(1)}, \dots, \sigma^{(n-1)}$, and conversely, any invariant $\sigma^{(i)}$ is a well defined polynomial of the invariants $\alpha^{(1)}, \dots, \alpha^{(n-1)}$. It is also well known, that any invariant system (σ_p or α_p) uniquely determines the eigenvalues of the endomorphism A_p, including the multiplicities as well.

A Riemannian manifold is called <u>sharply harmonic</u> if all the invariants $\alpha_p^{(1)}; \dots ;\alpha_p^{(n-1)}$ (or equivalently all the invariants $\sigma_p^{(1)}; \dots ;\sigma_p^{(n-1)}$) are radial functions around any point p .

Any two–point homogeneous space is obviously sharply harmonic. The following theorem says the converse statement in the odd dimensional case.

Theorem 1.1. Any odd dimensional sharply harmonic manifold is a symmetric space, rather more it is a space of constant sectional curvature.

Proof Fix a point $p \in M^n$ and pick up a unit vector $\underline{m} \in T_p(M^n)$. The Taylor series of A_p w.r.t. $\underline{m}$ is

$$(1.5) \qquad A_p = r\mathrm{Id} - r^3 \tfrac{1}{6} R_{\underline{m}} - r^4 \tfrac{1}{12} R'_{\underline{m}} + \tfrac{r^5}{5!} \left[R^2_{\underline{m}} - 3R''_{\underline{m}} \right] + 0(r^6)$$

where the self adjoint operators $R_{\underline{m}}(X) := R(X,\underline{m})\underline{m}$; $R'_{\underline{m}}(X) := (\nabla_{\underline{m}}R)(X,\underline{m})\underline{m}$ are considered acting in the tangent space of the unit sphere $S^{n-1}_p \subset T_p(M^n)$ at the point $\underline{m} \in S^{n-1}_p$. In the case considered also the invariants

$$(1.6) \qquad \beta^{(\rho)}_p = \mathrm{Tr}\left[\tfrac{1}{r^3} (A_p - r\mathrm{Id}) \right]^\rho = \mathrm{Tr}\, \tfrac{1}{6^\rho} R^\rho_{\underline{m}} + 0(r)$$

are radial functions, which means that also the invariants of $R_{\underline{m}}$ are constant along the sphere S^{n-1}_p . Therefore also the eigenvalues of $R_{\underline{m}}$ are constant (with constant multiplicities) along S^{n-1}_p . These eigenvalues must be equal because in the opposite case the eigensubspaces of $R_{\underline{m}}$ split the tangent space $T(S^{n-1}_p)$ into non–trivial orientable (the first Stiefeld–Whitney class of S^{n-1}_p is 0 !) and continuous distributions: $T(S^{n-1}_p) = \xi_1 \oplus \xi_2 \oplus \dots \oplus \xi_k$. This is impossible because the Euler classes $\chi(\xi_i)$ are zeros, furthermore the Euler class $\chi(T(S^{n-1})) = \chi(\xi_1) \vee \chi(\xi_2) \vee \dots \vee \chi(\xi_k)$ of the even dimensional sphere S^{n-1}_p is non–zero.

Therefore $R_{\underline{m}} = \lambda_p \mathrm{Id}$ holds for some constant λ_p and the space is of constant curvature by the Schure Theorem.

Q.e.d.

The situation is more complicated in the even dimensional case because the Euler class of an odd dimensional sphere is zero rather more the tangent space $T(S_p^{n-1})$ can be splitted into non–trivial continuous distributions. Also in this case the invariants as well as the eigenvalues of $R_{\underline{m}}$ are constants on S_p^{n-1} rather more these constants are independent also from the point p . By our guess these properties implies that $R_{\underline{m}}$ can be identified with one of the curvature operators belonging to the two–point–homogeneous spaces but the details seem to be complicated.

Because of these technical difficulties we introduce a new endomorphism field, namely the field

$$B_p := A_p - \tfrac{1}{3} r_p A'_p = \tfrac{2}{3} r_p \,\mathrm{Id} + \tfrac{1}{36} r_p^4 R'_{\underline{m}} + 0(r_p^5) \tag{1.7}$$

along the geodesics $\gamma(s)$ with $p = \gamma(0)$. All the invariants of B_p define radial functions around p in the two point homogeneous spaces. This statement can be conversed for any dimension as follows.

Theorem 1.2 A Riemann space (M^n,g) is two point homogeneous if and only if the invariants of B_p define radial functions around any point p .

Proof If the invariants of B_p define radial functions around p then also the invariants

$$\delta_p^{(\rho)} := \mathrm{Tr}\left[\frac{36}{r_p^4}(B_p - \tfrac{2}{3} r_p \,\mathrm{Id})\right]^{\rho} = \mathrm{Tr}(R'_{\underline{m}})^{\rho} + 0(r) \tag{1.8}$$

define radial functions around p . This means that the invariants as well as the eigenvalues of $R'_{\underline{m}}$ are constant along S_p^{n-1} . By the methods of the paper [9] we get $R'_{\underline{m}} = 0$; $\nabla R = 0$, i.e. the space is locally symmetric. Therefore for B_p we get

$$B_p = \frac{2}{3} r_p \mathrm{Id} - \frac{r_p^5}{180} R_{\underline{m}}^2 + 0(r^6) ,$$

(1.9)

$$\mathrm{Tr}\left[-\frac{180}{r_p^5} (B_p - \frac{2}{3} r_p \,\mathrm{Id}) \right]^{\rho} = \mathrm{Tr} R_{\underline{m}}^{2\rho} + 0(r) ,$$

which means that the invariants of $R_{\underline{m}}^2$ as well as of $R_{\underline{m}}$ are constant along S_p^{n-1} . So also the invariants of the Jacobian field A_p define radial functions around p . Specially, the space is a symmetric harmonic space which is a two–point homogeneous space by a Lichnerowicz theorem [1].

Q.e.d.

§ 2. The higher order Laplacians

It is a well known fact that the invariant differential operators of a two–point–homogeneous space are the polynomial of the Laplacian. Following we converse this statement, showing, that if some differential operators of a Riemannian space are the polynomial of the Laplacian than the space is two–point–homogeneous.

For preparing of these operators let us consider a C^{∞}–kernel function (or double function)

(2.1) $$H(p,q) : M^n \times M^n \longrightarrow \mathbb{R}$$

on a C^∞–Riemannian manifold M^n such that the function $H(p,p) : M^n \longrightarrow \mathbb{R}$ never vanishes. For a point p and for a unit vector $\dot{e}_p \in T_p(M^n)$ the $e_p(r)$ denotes the arc–wise parametrized geodesics through p with the tangent vector $\dot{e}_p$ furthermore $d\dot{e}_p$ means the normalized euclidean measure of the unit vectors $\dot{e}_p$ in the euclidean tangent space $T_p(M^n)$. Using the function $H(p,q)$ as weight function we introduce the averaging operator $E_{H;p;r}$ on a geodesics sphere $S_{p;r}$ with the centre p and radius r by

$$(2.2) \qquad E_{H;p;r}(\varphi) = \frac{1}{\int H(p,e_p(r))d\dot{e}_p} \int \varphi(e_p(r))H(p,e_p(r))d\dot{e}_p ,$$

where φ is an arbitrary C^∞–function . The odd order derivatives of $E_{H;p;r}$ w.r.t. the r at $r = 0$ vanish. The higher order Laplacian $\Delta_H^{(k)}$ generated by H are defined by the even order derivatives as follows:

$$(2.3) \qquad \Delta_H^{(k)}(\varphi)/p := \frac{\partial^{2k} E_{H;p;r}(\varphi)}{\partial r^{2k}}/r = 0$$

The operators $\Delta_H^{(k)} := \Delta^{(k)}$ generating by the constant function $H(p,q) = 1$ were introduced by Willmore [13] and these were studied by many other authors.. In the present paper we study more such operators namely generating by the several invariants of the Jacobian field. From the point of view of harmonic spaces the operators $\Delta^{(k)}$ and $\Delta_\theta^{(k)}$ play an important role, where $\theta_p = \det A_p = \sigma_p^{(n-1)}$ is one of the invariants. These operators are obviously equal on harmonic spaces (i.e. $\Delta^{(k)} = \Delta_\theta^{(k)}$) .

A function u is called of a common harmonic function w.r.t. the operators $\Delta_H^{(k)}$ if

(2.4) $$\Delta_H^{(k)} u = 0 \; ; \; k = 1,2, \ldots$$

satisfy.

Let us assume that the space (M^n,g) is analytic w.r.t. the normal coordinate neighbourhood (it is so called normal anylytic) and also the function H is analytic. In this case the Taylor serie

(2.5) $$E_{H;p;r}(\varphi) = \Sigma \frac{\Delta_H^{(k)} \varphi / p}{(2k)!}$$

gives the following theorem immediatly.

Theorem 2.1 A function u is common harmonic w.r.t. the operators $\Delta_H^{(k)}$ if and only if for it the mean value property:

(2.6) $$u(p) = E_{H;p;r}(\varphi)$$

holds for any small radius value $r > 0$.

In the next step we consider the Willmore's operators $\Delta^{(k)}$ and the operators $\Delta_\theta^{(k)}$.

Theorem 2.2 On any harmonic space the operators $\Delta^{(k)} = \Delta_\theta^{(k)}$ are the polynomials of the Laplacian.

Proof For a Laplacian eigenfunction with the eigenvalue λ we get

$$\int_{S_{p;r}} (\theta\varphi')' d\dot{e}_p = \lambda \int_{S_{p;r}} \theta\, \varphi\, d\dot{e}_p \,, \tag{2.7}$$

which follows from the classical decomposition

$$\Delta = \Delta_S + \frac{\partial^2}{\partial r^2} + \frac{\theta'}{\theta}\frac{\partial}{\partial r} \tag{2.8}$$

and from the Stokes theorem easily.

By the $(n-1+2k)$–th derivatives w.r.t. r at $r = 0$ we have

(2.9)

$$\int \sum_{a=1}^{2k+2} \begin{bmatrix} n+2k \\ a-1 \end{bmatrix} \varphi^{(a)}_{\dot{e}_p}\, \theta^{(n+k-a+1)}_{\dot{e}_p} d\dot{e}_p = \lambda \int \sum_{a=0}^{2k} \begin{bmatrix} n-1+2k \\ a \end{bmatrix} \varphi^{(a)}_{\dot{e}_p}\, \theta^{(n-1+2k-a)}_{\dot{e}_p} d\dot{e}_p \,,$$

where $\varphi^{(a)}_{\dot{e}_p}$ resp. $\theta^{(a)}_{\dot{e}_p}$ means the a–th covariant derivative: $\nabla^{(a)}_{\dot{e}_p}\varphi$ resp. $\nabla^{(a)}_{\dot{e}_p}\theta$ of the functions w.r.t. $\dot{e}_p$.

The above formula gives the recursion:

$$(n-1)! \begin{bmatrix} n+2k \\ 2k+1 \end{bmatrix} \Delta^{(k+1)} = \left[\sum_{a=0}^{2k} \begin{bmatrix} n-1+2k \\ a \end{bmatrix} \int \theta^{(n-1+2k-a)}_{\dot{e}_p} \nabla^{(a)}_{\dot{e}_p} d\dot{e}_p \right] \Delta$$

(2.10)

$$- \sum_{a=0}^{2k} \begin{bmatrix} n+2k \\ a-1 \end{bmatrix} \int \theta^{(n+2k-a+1)}_{\dot{e}_p} \nabla^{(a)}_{\dot{e}_p} d\dot{e}_p$$

for the Willmore's operators:

$$\Delta^{(k)}_{/p} = \int \nabla^{(2k)}_{\dot{e}_p} \, d\dot{e}_p \, ; \; \Delta^{(1)} = n\Delta \, . \tag{2.11}$$

On harmonic manifolds the derivatives $\theta^{(a)}_{\dot{e}_p}$ are constant therefore from (2.10) a recursion formula of the form

$$\Delta^{(k)} = \Delta^{(k-1)}\Delta + P_k(\Delta^{(1)},\Delta^{(2)}, \dots ,\Delta^{(k-1)}) \tag{2.12}$$

follows, where $P_k(\Delta^{(1)}, \dots ,\Delta^{(k-1)})$ is a polynomial of the arguments. The proof can be completed by induction.

The following is the converse statement.

<u>Theorem 2.3</u> A space is harmonic if and only if the operators $\Delta^{(k)}_\theta$ or the Willmore's operators $\Delta^{(k)}$ are the polynomial of the Laplacian.

<u>Proof</u> First notice that in an arbitrary Riemannian manifold the integral formula

$$\int \phi(e_p(r))u'(e_p(r))d\dot{e}_p = 0 \, , \text{ where } \phi(e_p(r)) := \frac{\theta_p(e_p(r))}{\int \theta_p(e_p(r))d\dot{e}_p} \tag{2.13}$$

holds for any harmonic function u , which can be proved by the Stokes theorem easily. Notice too, that from (2.10)

(2.14)
$$n(n+2)\Delta^{(2)} = 3\Delta^2 + 2\rho^{ij}\nabla_i\nabla_j + 2(\nabla_j\rho^{ij})\nabla_i\,,$$
$$n(n+2)\Delta_\theta^{(2)} = n(n+2)\Delta^{(2)} - 2R\Delta - 4\rho^{ij}\nabla_i\nabla_j - (\nabla^i R)\nabla_i$$

follow, where ρ^{ij} is the Ricci curvature und R is the curvature scalar. Therefore each of the operators $\Delta^{(2)}$, $\Delta_\theta^{(2)}$ is the polynomial of the Laplacian Δ if and only if the space is an Einstein space. So the spaces considered are normal analytic by the Kazdan–de Turck theorem furthermore the harmonic functions are analytic by the Bernstein theorem.

First let us assume that the operators $\Delta_\theta^{(k)}$ are polynomials of the Laplacian. In this case any harmonic function (w.r.t. the Laplacian Δ) is a common harmonic function w.r.t. the operators $\Delta_\theta^{(k)}$, therefore by Theorem 2.1 the mean value property $u(p) = E_{\theta;p;r}(u)$ follows for any harmonic function u. By derivation of $E_{\theta;p;r}(u)$ w.r.t. r and by (2.13) we get

(2.15)
$$\int_{S_{p;r}} \phi_p' u\, d\underline{e}_p = 0$$

for any harmonic function u. As any continuous function u of $S_{p;r}$ can be extended into a harmonic function of the inside of $S_{p;r}$ (Dirichlet–problem), so the above integral equation says: $\phi_p' = 0$, i.e. ϕ_p is constant on the geodesics $e_p(r)$. As $\phi_p(p) = 1$, so $\phi_p = 1$ follows and therefore θ_p is a rational function. This proves the harmonicity of the manifold in the first case.

If the operators $\Delta^{(k)}$ are the polynomial of Δ then from the mean value property $u(p) = \int u(e_p(r))d\dot{e}_p$ of the harmonic functions u we have

(2.16)
$$\int_{S_{p;r}} u'(e_p(r))\,d\,\underline{e}_p = 0\,.$$

So by (2.13) we get

$$\int_{S_{p;r}} \frac{\phi_p - 1}{\theta_p} u' d\,\underline{e}_p = 0 , \tag{2.17}$$

where for the function $v = (\phi_p - 1)/\theta_p$ on $S_{p;r}$ the integral formula

$$\int_{S_{p;r}} v\theta_p d\,\underline{e}_p = 0 \tag{2.18}$$

satisfies. Solving the Neumann problem, a harmonic function u in the inside of $S_{p;r}$ exists, such that $u' = v$ on the boundary $S_{p;r}$. Using this function u , from (2.17) we get: $\phi_p = 1$. This proves the harmonicity in the second case completely.

Q.e.d.

In the last step we are interested in the question, that in which case is a Riemannian manifold of a two–point homogeneous space.

<u>Theorem 2.4</u> A compact Riemannian manifold with finite fundamental group is a two–point–homogeneous space if and only if the higher order Laplacians $\Delta_\theta^{(k)}$ or the Willmore operators $\Delta^{(k)}$ are the polynomial of the Laplacian.

This statement follows easily from the proof of the Lichnerowicz conjecture in the compact case [8] and from Theorem 2.3.

<u>Theorem 2.5</u> An odd dimensional Riemannian manifold is a two–point–homogeneous space (rather more a space of constant sectional curvature) if and only if all the higher

order Laplacians $\Delta^{(k)}_{\sigma^{(1)}}; \Delta^{(k)}_{\sigma^{(2)}}; \dots ; \Delta^{(k)}_{\sigma^{(n-1)}} = \Delta^{(k)}_{\theta}$, generated by the invariants $\sigma_p^{(i)}(q) = \sigma^{(i)}(p,q)$ of the Jacobian field $A_{p;q}$ are the polynomials of the Laplacian.

Proof If all these operators are the polynomials of Δ, then the space is harmonic (i.e. $\sigma_p^{n-1} = \theta_p$ is radial) by Theorem 2.3. Furthermore for any harmonic function u we have

$$u(p) = E_{\sigma^{(1)};p;r}(u) = E_{\sigma^{(2)};p;r}(u) = E_{\theta;p;r}(u) ,$$

therefore

$$\frac{\sigma_p^{(1)}(e_p(r))}{\int \sigma_p^{(1)}(e_p(r))d\dot{e}_p} = \frac{\sigma_p^{(2)}(e_p(r))}{\int \sigma_p^{(2)}(e_p(r))d\dot{e}_p} = \dots = \frac{\theta_p(e_p(r))}{\int \theta_p(e_p(r))d\dot{e}_p}$$

follows from the Dirichlet problem, i.e. all the invariants $\sigma_p^{(i)}$ define radial functions around p. The proof can be completed by using of Theorem 1.1.

Q.e.d.

In the general case let $\omega_p^{(1)}, \omega_p^{(2)}, \dots, \omega_p^{(n-1)}$ be the invariants of the endomorphism field $B_p = A_p - \frac{1}{3} r_p A'_p$ defined in (1.7) (i.e. $\det(B_p + \lambda Id) = \lambda^{n-1} + \omega_p^{(1)}\lambda^{n-2} + \dots + \omega_p^{(n-1)})$. Then by the same argument and from Theorem 1.2 we have

Theorem 2.6 A Riemannian space is two–point homogeneous if and only if the operators $\Delta_\theta^{(k)}$ and the operators $\Delta^{(k)}_{\omega^{(1)}}, \dots, \Delta^{(k)}_{\omega^{(n-1)}}$ are the polynomial of the Laplacian Δ.

At last we characterize the Einstein and also the super Einstein manifolds (An Einstein manifold is called to be super Einstein if also $R_i^{abc} R_{abcj} = \lambda g_{ij}$ satisfies for a constant value λ) .

Theorem 2.7 A space is Einstein if and only if for an arbitrary fixed index i the operator $\Delta^{(2)}_{\sigma(i)}$ is the polynomial of the Laplacian.

A space is super Einstein if and only if for an arbitrary fixed index i the operators $\Delta^{(2)}_{\sigma(i)}$, $\Delta^{(2)}_{\sigma(i)}$ are the polynomial of the Laplacian.

The theorem is only a reformulation of a Gray–Willmore theorem [3] asserting, that a manifold is Einstein (resp. super Einstein) iff the mean value property of harmonic functions holds up to the order 6 (resp. up to the order 8). The proof is based upon the remark that the operator $\Delta^{(2)}_{\sigma(i)}$ is a linear combination of the operators: $\Delta^2 = \Delta\Delta$; $\rho^{ij}\nabla_i\nabla_j$, $R\Delta$, $(\nabla_j \rho^{ij})\nabla_i$, $(\nabla^i R)\nabla_i$ and the operator $\Delta^{(3)}_{\sigma(i)}$ can be built as a polynomial of the operators Δ , $\Delta^{(2)}$, $R_{abci}R_j^{abc}\nabla^i\nabla^j$; $R_{abcd}R^{abcd}\nabla_i\nabla^i$;
$\int \nabla_{\dot{e}_p}\left[R_{\dot{e}_p a \dot{e}_p b} R_{\dot{e}_p}{}^{a}{}_{\dot{e}_p}{}^{b} \right]\nabla_{\dot{e}_p} d\dot{e}_p$. For super Einstein manifold the last operator vanishes and the others are polynomials of the Laplacian. These technical details are left to the reader.

§ 3. Symmetricity of manifolds

In this chapter we are interested in the question that which symmetricity properties of the invariants $\sigma^{(1)}, \dots ,\sigma^{(n-1)},\omega^{(1)}, \dots ,\omega^{(n-1)}$ guarantee the symmetry of the space.

In this consideratons we initiate the holonomy group $\mathscr{H}_p$ acting in the tangent space $T_p(M^n)$. The local action $\tilde{h} : M^n \longrightarrow M^n$ of the elements $h \in \mathscr{H}_p$ is defined by the exponential map:

$$\tilde{h} = \exp \circ h \circ \exp^{-1} .$$

For symmetric spaces these actions are isometrics.

We draw into the consideration also the Simons theorem [7], asserting that the holonomy group $\mathscr{H}_p$ of an irreducible Riemannian space is transitive on the unit sphere exept the space is a symmetric space with rank ≥ 2.

In the proof of the following theorems we use de Rahm decomposition and the proof follows immediately from the Simons theorem, from the proof of the Lichnerowicz conjecture in the compact case [8] resp. for sharply harmonic manifolds.

Theorem 3.1 Let M^n be a compact Riemannian space with finite fundamental groups. The M^n is a symmetric space if and only if the elements $h \in \mathscr{H}_p$ induce volume preserving maps $\tilde{h}$ on M^n.

Indeed, if one component at the de Rahm decomposition is not a symmetric space of rank ≥ 2, then it is a two–point–homogeneous space by [8].

Theorem 3.2 Let M^n be an odd dimensional irreducible Riemannian space. The M^n is symmetric if and only if the functions $\sigma_p^{(1)}, \ldots, \sigma_p^{(n-1)} = \theta_p$ are invariant under the actions $\tilde{h}$ where $h \in \mathscr{H}_p$.

Theorem 3.3 A Riemannian space is symmetric if and only if the functions $\theta_p, \omega_p^{(1)}, \ldots, \omega_p^{(n-1)}$ are invariant under the actions $\tilde{h}$, $h \in \mathscr{H}_p$, for any point $p \in M^n$.

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