Profinite topologies in just infinite branch groups

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Abstract

A just infinite branch group is an automorphism group of some tree. As such, it has two natural topologies, one induced on it as a subgroup of the full automorphism group and the other its own profinite topology. A natural question is when these two topologies coincide. We answer this question positively for periodic Grigorchuk's groups and for so-called GGS-groups and construct a class of groups for which this question is answered negatively.

Contents

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Introduction

A profinite group is *just infinite* if its every proper (continuous) quotient is finite; equivalently, if every closed normal subgroup is open. It is hereditarily just infinite if every open subgroup is just infinite.

Notice that any finitely generated profinite group which is virtually a pro-p group can be mapped onto a virtually pro-p just infinite group [6].

Another result from [6], based partly on the results of [13], states that any profinite just infinite group either contains an open normal subgroup which is isomorphic to the direct product of a finite number of copies of some hereditarily just infinite profinite group, or is a profinite branch group. The latter groups can be defined as profinite groups with a tree-like structure lattice of subnormal subgroups [14].

It is such groups that are the main subject of the present paper. More specifically, we consider abstract just infinite *branch groups*. As explained in some greater detail in Section 1, every such group is in fact some automorphism group G of a rooted spherically homogenous tree T (see also [1]).

As a subset of $\operatorname{Aut} T$, group G is equipped with the induced topology τ_1. It is proven in [6] that the closure in $\operatorname{Aut} T$ of any branch group is a profinite branch group. However, G has its own profinite topology τ_2 (and as G is residually finite, its completion with respect to τ_2 is a profinite group). The main question of the current paper will be, *when do the two topologies coincide?*

This question admits an equivalent form. It is easy to see that the level stabilizers of vertices of T form a base of profinite topology in $\operatorname{Aut} T$. Define *principal congruence subgroups* in G to be the intersections of those level stabilizers with G. Let us say that G has *congruence property* if every finite index subgroup of G is a congruence subgroup, i.e. if it contains some principal congruence subgroup. The two topologies coincide iff G has congruence property [2].

In what follows we first establish that all groups from two rather well-known classes, periodic Grigorchuk's groups G_ω [5] and GGS-groups [7, 4], do possess this property. Then we introduce a class of periodic, and even p-, groups that do not have it(!). The existence of such groups is somewhat unexpected and answers negatively Question 3 of [1].

These main results are stated in Theorems 3.1, 3.2, 4.1, and 5.1.

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1 Main definitions

Let us start with a precise definition of a branch group.

Definition 1.1. A group G is said to be a *branch group*, if there exist two decreasing sequences of subgroups $(L_i)_{i\in N}$, $(M_i)_{i\in N}$ and a sequence of integers $(k_i)_{i\in N}$ such that

$$L_0 = M_0 = G, \quad k_0 = 1,$$

$$\cap_{i\in N} M_i = 1,$$

and for each i the following four conditions hold,

(1) M_i is a normal finite-index subgroup of G;

(2) M_i is a direct product of k_i copies of the subgroup L_i, i.e. there are subgroups $L_i^{(1)}$, …, $L_i^{(k_i)}$ of G such that

$$M_i = L_i^{(1)} \times \ldots \times L_i^{(k_i)}, \tag{1.1}$$

and each of the factors is isomorphic to L_i;

(3) k_i properly divides k_{i+1}, i.e. $m_{i+1} = \frac{k_{i+1}}{k_i} \geq 2$, and the product decomposition (1.1) of M_{i+1} refines the product decomposition (1.1) of M_i, meaning that each factor $L_i^{(j)}$ of M_i contains m_{i+1} factors of M_{i+1}, namely the factors $L_{i+1}^{(l)}$ for $l = (j-1)m_{i+1}+1$, …, jm_{i+1};

(4) conjugations by elements of G transitively permute the factors in the product decomposition (1.1).

Notice that the set of subgroups $L_i^{(j)}$ ordered by inclusion forms a locally finite rooted spherically homogeneous tree T. The group G acts by automorphisms on T and is transitive on each layer of T (comprised by all subgroups $L_i^{(j)}$ for a fixed i).

In general, this action may not be faithful. However, in case of *just-infinite* branch groups, which are what we are primarily interested in, it is faithful. Otherwise, as is easily seen, the kernel of the action would be a non-trivial normal subgroup of infinite index. Thus, a just-infinite branch group is a group of automorphisms of a rooted spherically homogeneous tree.

Let us specify what conditions a group of tree automorphisms must meet in order to be branch. Notice that if $A = (A_1, A_2, \ldots)$ is a sequence of finite sets of cardinality 2 or greater, then the set of all finite sequences

$$a_1 a_2 \ldots a_n, \ a_i \in A_i, \ n \in N,$$

forms a rooted spherically homogeneous tree, where the sequences correspond to vertices, the root is the empty sequence, and two vertices are joined by an edge if they have the form $a_1a_2\ldots a_n$ and $a_1a_2\ldots a_na_{n+1}$ for some n and some $a_i \in A_i$. For example, the tree corresponding to a branch group G can be constructed over a sequence of arbitrary sets A_i such that the cardinality of A_i is equal to m_i.

The number n is called the *length* of a vertex $u = a_1 a_2 \dots a_n$ and is denoted by $|u|$. The set of all vertices of length n is called the nth level of T.

Of special interest is the case when all sets A_i are equal to one set X. In this case the tree is called *regular*, and the set of its vertices is precisely the set of all words in the alphabet X.

Just as we constructed a tree T over an infinite sequence of sets $A = (A_1, A_2, \dots)$, we can construct a tree over a finite sequence $(A_1, A_2, \dots, A_n)$. This tree will be denoted by $T_{[n]}$; geometrically, it is obtained by removing everything that lies below the nth level.

Suppose that $u = \hat{a}_1 \hat{a}_2 \dots \hat{a}_n$ is a vertex. The set of all vertices $\hat{a}_1 \hat{a}_2 \dots \hat{a}_n a_{n+1} a_{n+2} \dots a_{n+m}$, where $m \in N$ and a_{n+i} range over the set A_{n+i}, forms a subtree of T. We will denote that subtree by T_u. It is easy to see that T_u is naturally isomorphic to the tree $T_{(n)}$ constructed over the sequence $(A_{n+1}, A_{n+2}, \dots)$, via the map

$$\hat{a}_1 \hat{a}_2 \dots \hat{a}_n a_{n+1} a_{n+2} \dots a_{n+m} \mapsto a_{n+1} a_{n+2} \dots a_{n+m},$$

This map also allows to identify subtrees T_v for all vertices v of length n, with one fixed tree $T_{(n)}$.

Of course, if T is regular then for any vertex u T_u is isomorphic to T.

Consider now an arbitrary subgroup X in $\operatorname{Aut} T$ and a vertex v of T. The *rigid stabilizer* of v in X is the subgroup

$$\mathrm{rist}_X(v) = \{x \in X | \text{ for any } u \in T \setminus T_v \ u^x = u\}.$$

We also denote by $\mathrm{rist}_X(n)$ the subgroup $\prod_{|v|=n} \mathrm{rist}_X(n)$. This is, of course, a normal subgroup in X.

We are now ready to give an equivalent definition of a just-infinite branch group.

Definition 1.2. Let G be a group of automorphisms of a spherically homogenous tree T. G is called a *branch group*, if it is transitive on each level, and for every integer $n \geq 1$ $\mathrm{rist}_G(n)$ has finite index in G.

There are several notions that are particularly useful in studying branch groups. Let G be a branch group and let v be a vertex of T. The *stabilizer* of v in G is the subgroup

$$\mathrm{st}_G(v) = \{x \in G | v^x = v\}.$$

Denote also by $\mathrm{st}_G(n)$ the subgroup $\cap_{|v|=n} \mathrm{st}_G(v)$, which keeps all vertices of level n fixed.

Subgroups $\mathrm{st}_G(n)$ are called *principal congruence subgroups* in G. A subgroup of G which contains a principal congruence subgroup is called a *congruence subgroup*.

Definition 1.3. Group G is said to possess the *congruence property* if any its finite index subgroup is a congruence subgroup.

Notice that to every $x \in \mathrm{st}_{\mathrm{Aut}\,T}(v)$ we can assign a unique automorphism $x_v \in \mathrm{Aut}\,T_v$ which is obtained by taking the restriction of x onto the subtree T_v. Moreover, for each vertex v of length n x_v can be considered as an element of $\mathrm{Aut}\,T_{(n)}$. Thus, for each vertex v there is a fixed homomorphism $\varphi_v : \mathrm{st}_{\mathrm{Aut}\,T}(v) \to \mathrm{Aut}\,T_{(|v|)}$. We will denote $\varphi_v(\mathrm{st}_G(v))$ by G_v.

In general, for different vertices u, v of the same length G_u and G_v are conjugate in $\mathrm{Aut}\,T_{(|u|)}$ but may not coincide. Let us fix for each pair of vertices u, v of the same length a conjugating element $\psi_{u,v}$ such that $G_u^{\psi_{u,v}} = G_v$, and $\psi_{u,u} = 1$ for any u. Nonetheles, in the examples we will be considering G_u and G_v do coincide for any two vertices of the same length, and all elements $\psi_{u,v}$ will be equal to the trivial element.

Definition 1.4. A branch group G acting on a regular tree T is *fractal*, if for each vertex u of T $G_u = G$.

As will be evidenced many times throughout this paper, the property of a group to be fractal is very helpful in inductive proofs.

It is clear that the direct product of φ_v taken over all vertices v of length n defines a monomorphism of $\mathrm{st}_{\mathrm{Aut}\,T}(n)$ into the direct product of $|A_1||A_2|\ldots|A_n|$ copies of $\mathrm{Aut}\,T_{(n)}$. This product map will be denoted by Φ_n.

Of course, collections of maps φ_u and Φ_n vary with and are determined by the tree T. Since at each given moment a tree in question will be fixed, we will keep the same notation for maps on different trees.

The above maps give a way to describe some, but not all, automorphisms of T. A more general way is given by an observation that for any vertex u of length n x induces a permutation $x(u)$ of A_{n+1}, which acts according to the rule $a \mapsto ((ua)^x)_{n+1}$. Such permutations are called *accompanying* permutations of x. It is easy to see that they fully determine x.

Let us now describe our major examples.

1.1 Grigorchuk's groups

We deal here with two types of examples, *periodic Grigorchuk's groups* G_ω and the second Grigorchuk's group G_4. The former are known as the first examples of groups of intermediate growth [5]. They are just-infinite branch groups ([5]) acting on the infinite binary tree. That tree can be constructed as a tree over a sequence of sets $(X, X, \ldots)$ for $X = \{0, 1\}$.

Groups G_ω are parameterized by infinite sequences $\omega = (\omega_1, \omega_2, \ldots)$ such that for each i $\omega_i \in \{0, 1, 2\}$, and each of the numbers 0, 1, 2 occurs in ω infinitely many times. The set of all such ω is denoted by Ω_0. The expression $\sigma\omega$ denotes the sequence $(\omega_2, \omega_3, \ldots)$. Clearly, $\sigma\omega$ is also in Ω_0.

Each Grigorchuk's group G_ω, $\omega \in \Omega_0$, is generated by four automorphisms a, b_ω, c_ω, d_ω. (Although just three of these automorphisms are actually enough). The only nontrivial accompanying permutation of a is at the root vertex and is the cyclic permutation, denoted by π, of two elements. The other three automorphisms have infinitely many nontrivial accompanying permutations, but all of them are located in vertices $u_n = \underbrace{11\ldots1}_{n}0$, $n = 0, 1, 2, \ldots$. Permutation $b_\omega(u_n)$ is trivial if $\omega_{n+1} = 2$ and is equal to π otherwise. Likewise, permutation $c_\omega(u_n)$ is trivial if $\omega_{n+1} = 1$ and is equal to π otherwise, and permutation $d_\omega(u_n)$ is trivial if $\omega_{n+1} = 0$ and is equal to π otherwise.

Thus, for any vertex u of the form $u = \underbrace{11\ldots1}_{n}$ or $u = \underbrace{11\ldots1}_{n}0 = u_n$ generators b_ω, c_ω, d_ω are in $\mathrm{st}_{G_\omega}(u)$, and we have

$$\varphi_u(b_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 2 \\ b_{\sigma^n\omega}, & \text{if } u = \underbrace{11\ldots1}_{n} \end{cases}$$

$$\varphi_u(c_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 1 \\ c_{\sigma^n\omega}, & \text{if } u = \underbrace{11\ldots1}_{n} \end{cases}$$

$$\varphi_u(d_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 0 \\ d_{\sigma^n\omega}, & \text{if } u = \underbrace{11\ldots1}_{n} \end{cases}$$

In particular, for any vertex u $\varphi_u(G_\omega) = G_{\sigma^n\omega}$ for $n = |u|$.

It also follows from the above definitions that exactly one of generators b_ω, c_ω, d_ω has trivial accompanying permutation at $u_0 = (0)$. That generator will be called the *d-generator* of G_ω. Now, let $m(\omega)$ be the maximal number n such that the finite sequence $\omega_1\omega_2\ldots\omega_n$ is constant. Suppose that y_ω is the generator of G_ω such that $y_{\sigma^{m(\omega)}\omega}$ is the d-generator of $G_{\sigma^{m(\omega)}\omega}$. Then y_ω is called the *c-generator* of G_ω. That of generators b_ω, c_ω, d_ω which is neither the d-generator nor the c-generator is called *b-generator*.

Directly applying Proposition 1 from [8], we get the following fact.

Lemma 1.1. Let G_ω be a periodic Grigorchuk's group. Then

$$G_\omega/[G_\omega, G_\omega] \cong < a > \times < b_\omega, c_\omega > \cong Z_2 \times Z_2 \times Z_2.$$

Corollary 1.1. *Let x_ω be the d-generator of G_ω and let y_ω be its c-generator. Then the factorgroup of G_ω by the normal closure N of x_ω is isomorphic to the group generated by a and y_ω.*

Proof. It is sufficient to show that the intersection $< a, y_\omega > \cap N$ is trivial. Notice that N is contained in $\mathrm{st}_{G_\omega}(m(\omega)+1)$. On the other hand, the intersection $< a, y_\omega > \cap\, \mathrm{st}_{G_\omega}(m(\omega)+1)$ is generated by element $(ay_\omega)^{2^{m(\omega)+1}}$. Since the order of this element is equal to 2, it is sufficient to prove that the element does not belong to N.

The image of $(ay_\omega)^{2^{m(\omega)+1}}$ under $\Phi_{m(\omega)+1}$ equals $(\underbrace{y_{\sigma^{m(\omega)+1}\omega}, y_{\sigma^{m(\omega)+1}\omega}, \ldots, y_{\sigma^{m(\omega)+1}\omega}}_{2^{m(\omega)+1}})$. However, since x_ω is the d-generator of G_ω, for any vertex u of length $m(\omega)$ $\varphi_u(N)$ is in the normal closure of $x_{\sigma^{m(\omega)}\omega}$. And since $x_{\sigma^{m(\omega)+1}\omega}$ is not the d-generator of $G_{\sigma^{m(\omega)+1}\omega}$, for any vertex u of length $m(\omega)+1$ $\varphi_u(N)$ is in the normal closure of the set $\{a, x_{\sigma^{m(\omega)+1}\omega}\}$. Hence the image of any $\varphi_u(N)$ in the factorgroup of $G_{\sigma^{m(\omega)+1}\omega}$ by its derived subgroup is in the subgroup

$$(< a, x_{\sigma^{m(\omega)+1}\omega} > [G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}])/[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}].$$

By Lemma 1.1 we get that the image of $y_{\sigma^{m(\omega)+1}\omega}$ is not in this subgroup, hence $(ay_\omega)^{2^{m(\omega)+1}}$ is not in N. ■

Let us introduce a length function ∂_ω on each G_ω. Each element of G_ω can be presented by a word of the form,

$$a^\epsilon x_1 a \dots a x_m a^\mu,$$

where $x_1, \dots, x_m \in \{b_\omega, c_\omega, d_\omega\}$, and ϵ, μ are in GF(2). The length of such a word, by definition, is equal to m. The length $\partial_\omega(x)$ is defined as the minimum of lengths of words representing x.

The **second Grigorchuk's group** G_4 is also a 2-group, but it acts on the regular 4-tree, constructed over the sequence $(X, X, \dots)$ for $X = \{0, 1, 2, 3\}$. This group is generated by two automorphisms, a and b. The former acts "at the root", i.e. its only nontrivial accompanying permutation is located at the root vertex and equal to the cyclic permutation (0123). The second generator, b, is defined by the equality,

$$\Phi_1(b) = (a, 1, a, b).$$

Directly applying Proposition 1 from [8], we get the following fact.

Lemma 1.2. Let G_4 be the second Grigorchuk's group. Then

$$G_4/[G_4, G_4] \cong < a > \times < b > \cong Z_4 \times Z_4.$$

Group G_4 is periodic [12]; using Proposition 2.2 from Section 2.1, we can prove that

Proposition 1.1. *Group G_4 is just infinite branch.*

Proof. Any $\mathrm{rist}_{G_4}(n)$ has finite index in G_4; indeed, since G_4 is finitely generated periodic, $\gamma_3(G_4)$ has finite index in G_4. By Proposition 2.2 $\Phi_n(\mathrm{rist}_{G_4}(n))$ is equal to the direct product of 4^n copies of $\gamma_3(G_4)$. Thus $\Phi_n(\mathrm{rist}_{G_4}(n))$ has finite index in $\Phi_n(\mathrm{st}_{G_4}(n))$, because the latter is contained in the direct product of 4^n copies of G_4. Thus, $\mathrm{rist}_{G_4}(n)$ has finite index in $\mathrm{st}_{G_4}(n)$ and hence in G_4. This proves that G_4 is branch.

Again, since G_4 is finitely generated periodic, then so is $\gamma_3(G_4)$. Hence the derived subgroup $[\gamma_3(G_4), \gamma_3(G_4)]$ has finite index. Then by Theorem 4 of [6] G_4 is just infinite. ■

Let us introduce a length function on G_4, which we will need later on. Each element $x \in G_4$ can be represented (not uniquely) by a word of the form,

$$a^{\delta_1} b^{\beta_1} \dots a^{\delta_m} b^{\beta_m} a^{\delta_{m+1}},$$

where $\delta_2, \dots, \delta_m, \beta_1, \dots, \beta_m \in Z/4Z \setminus \{0\}$ and $\delta_1, \delta_{m+1} \in Z/4Z$. The number m will be called the length of such a word. The length $\partial_{G_4}(x)$ is defined as the minimal length of words representing x.

Also, denote by $\partial_a(x)$ the sum of all δ_i and by $\partial_b(x)$ the sum of all β_i in some representation of x. By Lemma 1.2, the sums are independent of the choice of a particular representation, and $\partial_a(x)$ and $\partial_b(x)$ are well-defined.

1.2 Periodic GGS-groups

The term GGS-group was introduced by Gilbert Baumslag [4] and refers to Rostislav ***G***rigorchuk, Narain ***G***upta and Said ***S***idki. Those groups act on regular p-trees, where $p \geq 3$. We will always assume that p is prime.

Each GGS-group is fully defined by a vector

$$\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{p-1}) \in \underbrace{\mathrm{GF}(p) \oplus \mathrm{GF}(p) \oplus \ldots \oplus \mathrm{GF}(p)}_{p-1} \tag{1.2}$$

in the following way.

Construct the regular p-tree T_p as the tree over the sequence $(X, X, \ldots)$, $X = \{0, 1, 2, \ldots, p-1\}$. Each GGS-group $G_{\bar{\alpha}}$ acting on T_p is generated by two automorphisms a and b. The only accompanying permutation of a is at the root and is the cyclic permutation $(01 \ldots p-1)$, i.e. a cyclically permutes edges coming out of the root. The second generator is in the stabilizer of level 1, and

$$\Phi_1(b) = (a^{\alpha_1}, a^{\alpha^2}, \ldots, a^{\alpha_{p-1}}, b).$$

The vector (1.2) is called the *accompanying vector* of G.

By [12], all GGS-groups are just infinite branch; they are periodic if and only if $\sum_{i=1}^{p-1} \alpha_i = 0$. Notice that for any vertex u $\varphi_u(G_{\bar{\alpha}}) = G_{\bar{\alpha}}$. Hence all $G_{\bar{\alpha}}$ are fractal as well.

All GGS-groups are just infinite as well; they are periodic if and only if $\sum_{i=1}^{p-1} \alpha_i = 0$ [12]. By Proposition 1 from [8], we have

Lemma 1.3. Let $G_{\bar{\alpha}}$ be a GGS-group. Then

$$G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \cong < a > \times < b > \cong Z_p \times Z_p.$$

Later on we will need the following length function on $G_{\bar{\alpha}}$. Each element $x \in G_{\bar{\alpha}}$ can be represented (not uniquely) by a word of the form,

$$a^{\delta_1} b^{\beta_1} \ldots a^{\delta_m} b^{\beta_m} a^{\delta_{m+1}},$$

where $\delta_2, \ldots, \delta_m, \beta_1, \ldots, \beta_m \in \mathrm{GF}(p) \setminus \{0\}$ and $\delta_1, \delta_{m+1} \in \mathrm{GF}(p)$. The number m will be called the length of such a word. The length $\partial_{G_{\bar{\alpha}}}(x)$ is defined as the minimal length of words representing x.

Also, denote by $\partial_a(x)$ the sum of all δ_i and by $\partial_b(x)$ the sum of all β_i in some representation of x. By Lemma 1.3, the sums are independent of the choice of a particular representation, and $\partial_a(x)$ and $\partial_b(x)$ are well-defined.

1.3 EGS-groups

The term EGS-group stands for "extended Gupta-Sidki group", but the word "extended" should not be understood in the usual algebraic sense. While each EGS-group contains some specific GGS-group as a subgroup, it is not an extension of it. However, in a certain sense a periodic GGS-group, or, more precisely, its accompanying vector, does define a unique EGS-group.

Let $G_{\bar{\alpha}}$ be a periodic GGS-group with generators a and b and accompanying vector $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1})$. The corresponding EGS-group $\Gamma_{\bar{\alpha}}$ is generated by automorphisms a, b, and the automorphism c such that

$$\Phi_1(c) = (c, a^{\alpha_1}, a^{\alpha^2}, \ldots, a^{\alpha_{p-1}}).$$

The group $G_{\bar{\alpha}}$ is then called the *associated* GGS*-group* of $\Gamma_{\bar{\alpha}}$.

Let us establish a few needed properties of EGS-groups. First of all, by Theorem 2 of [9], all EGS-groups are *periodic*. Just as with GGS-groups, for any vertex u $\varphi_u(\Gamma_{\bar{\alpha}}) = \Gamma_{\bar{\alpha}}$.

Lemma 1.4. Let $\Gamma_{\bar{\alpha}}$ be a EGS-group. Then

$$\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong < a > \times < b > \times < c > \cong Z/pZ \times Z/pZ \times Z/pZ.$$

Proof. Consider a group $H =< a, b, c | a^p, b^p, c^p >$, the free product of three copies of Z/pZ. Each element x of H is uniquely represented in the form

$$a^{s_1} b^{\beta_1} c^{\delta_1} \ldots a^{s_m} b^{\beta_m} c^{\delta_m}, \tag{1.3}$$

where s_i, β_i, $\delta_i \in \mathrm{GF}(p)$, and for each $i = 1, 2, \ldots, m-1$ either β_i or δ_i is not equal to 0, for each $i = 1, 2, \ldots, m-1$ either δ_i or s_{i+1} is not equal to 0, and for each $i = 2, 3, \ldots, m$ either s_i or β_i is not equal to 0. The amount of non-zero numbers among β_1, β_2, ..., β_m, δ_1, δ_2, ..., δ_m is called the *length* of x and is denoted by $\partial(x)$. Denote the sum of all β_i by $\partial_b(x)$ and the sum of all δ_i by $\partial_c(x)$.

Consider the subgroup $L(H)$ that consists of all elements x such that $s_1+s_2+\ldots+s_m = 0$. This is clearly a normal subgroup generated by elements b, b^a, ..., $b^{a^{p-1}}$, c, c^a, ..., $c^{a^{p-1}}$. Moreover, each element of $L(H)$ is uniquely represented in the form

$$b^{\beta_1 a^{s_1}} c^{\delta_1 a^{t_1}} \ldots b^{\beta_m a^{s_m}} c^{\delta_m a^{t_m}}, \tag{1.4}$$

where s_i, t_i β_1, $\delta_m \in \mathrm{GF}(p)$, and β_2, ..., β_m, δ_1, ..., δ_{m-1} are in $\mathrm{GF}(p) \setminus \{0\}$. Let us denote by $\partial_{b,i}(x)$ the sum of the β_j such that $s_j = i$, and by $\partial_{c,i}(x)$ the sum of the δ_j such that $t_j = i$.

Consider a map $\tilde{\Phi} : L(H) \to \underbrace{H \times \ldots \times H}_{p}$ acting via the rule:

$$\tilde{\Phi}(b) = (a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}, b),$$

$$\tilde{\Phi}(c) = (c, a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}),$$

and $\tilde{\Phi}(b^{a^i})$ and $\tilde{\Phi}(c^{a^i})$ are obtained from, respectively, $\tilde{\Phi}(b^{a^{i-1}})$ and $\tilde{\Phi}(c^{a^{i-1}})$ by one cyclic shift to the right.

Let us show first that

$$\tilde{\Phi}^{-1}(\underbrace{[H,H] \times \ldots \times [H,H]}_{p}) \leq [H,H].$$

Consider an arbitrary element $x \in L(H)$. Suppose that $\tilde{\Phi}(x) = (x_1, x_2, \ldots, x_{p-1}, x_0)$. Then $\partial_b(x_i) = \partial_{b,i}(x)$ and $\partial_c(x_i) = \partial_{c,i}(x)$. Hence

$$\sum_{i \in \mathrm{GF}(p)} \partial_b(x_i) = \partial_b(x)$$

and

$$\sum_{i \in \mathrm{GF}(p)} \partial_c(x_i) = \partial_c(x).$$

Suppose that x is not in $[H,H]$. Then either $\partial_b(x)$ or $\partial_c(x)$ is not equal to zero. By the above equalities, not all of numbers $\partial_b(x_i)$, $\partial_c(x_i)$ are equal to zero. Hence $\tilde{\Phi}(x)$ is not in $([H,H] \times \ldots \times [H,H])$, which is equivalent to saying that $\tilde{\Phi}^{-1}([H,H] \times \ldots \times [H,H]) \leq [H,H]$.

Consider now the following sequence of subgroups. $K_0 = 1$; $K_n = \tilde{\Phi}^{-1}(K_{n-1} \times \ldots \times K_{n-1})$, $n \geq 1$. Take $K = \cup_{n=0}^{\infty} K_n$. It is easy to show that K is a normal subgroup of H. It follows, for instance, from Proposition 1 of [9] that $\Gamma_{\bar{\alpha}}$ is isomorphic to the factor-group H/K. Hence $\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong H/[H,H]$, and we have the desired statement. ∎

Relying on a fact to be established in Section 2.3, we obtain

Proposition 1.2. *Let $\Gamma_{\bar{\alpha}}$ be a EGS-group with nonsymmetric accompanying vector. Then $\Gamma_{\bar{\alpha}}$ is just infinite branch.*

Proof. Any $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(n)$ has finite index in $\Gamma_{\bar{\alpha}}$; indeed, since $\Gamma_{\bar{\alpha}}$ is finitely generated periodic, its derived subgroup is of finite index. By Proposition 2.4 $\Phi_n(\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(n))$ is equal to the direct product of p^n copies of $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. Thus $\Phi_n(\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(n))$ has finite index in $\Phi_n(\mathrm{st}_{\Gamma_{\bar{\alpha}}}(n))$, because the latter is contained in the direct product of p^n copies of $\Gamma_{\bar{\alpha}}$. Thus, $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(n)$ has finite index in $\mathrm{st}_{\Gamma_{\bar{\alpha}}}(n)$ and hence in $\Gamma_{\bar{\alpha}}$. This proves that $\Gamma_{\bar{\alpha}}$ is branch.

Again, since $\Gamma_{\bar{\alpha}}$ is finitely generated periodic, then so is its derived subgroup. Hence the second derived subgroup has finite index. Then by Theorem 4 of [6] $\Gamma_{\bar{\alpha}}$ is just infinite. ∎

2 Rigid stabilizers

Knowing rigid stabilizers of a given branch group is particularly important, since by Theorem 4 of [6] their derived subgroups form the base of the profinite topology on the group in question. Moreover, it often happens that the base is formed by the rigid stabilizers themselves. As we will see in later sections, such is the case with all examples we consider.

2.1 Grigorchuk's groups

We first compute the rigid stabilizers of the series of groups G_ω, $\omega \in \Omega_0$. Previously, they were only known for the first Grigorchuk's group [10].

Proposition 2.1. *Suppose $n \geq 1$ and $x_{\sigma^n\omega} \in \{b_{\sigma^n\omega}, c_{\sigma^n\omega}, d_{\sigma^n\omega}\}$ is such that $x_{\sigma^{n-1}\omega}$ is the d-generator of $G_{\sigma^{n-1}\omega}$. Let $N_{\sigma^n\omega}$ denote the normal closure of $(ax_{\sigma^n\omega})^2$ in $G_{\sigma^n\omega}$, and let y_ω be the d-generator of G_ω. Then for any vertex u of length n, if $1 \leq n \leq m(\omega)$, we have*

$$\mathrm{rist}_{G_\omega}(u)_u =< y_{\sigma^n\omega}, N_{\sigma^n\omega} >, \tag{2.1}$$

and if $n > m(\omega)$, we have

$$\mathrm{rist}_{G_\omega}(u)_u = N_{\sigma^n\omega}. \tag{2.2}$$

Proof. First, we prove by induction on n that for any n and any vertex u of length n

$$\mathrm{rist}_{G_\omega}(u)_u \geq N_{\sigma^n\omega}. \tag{2.3}$$

The base of induction is $n = 1$. Let x_ω be the d-generator of G_ω. Consider element $[y_\omega^a, x_\omega]$, where $y_\omega \in \{b_\omega, c_\omega, d_\omega\} \setminus x_\omega$. We have

$$\Phi_1([y_\omega^a, x_\omega]) = (1, (ax_{\sigma\omega})^2).$$

Since for any vertex u of T $\mathrm{st}_{G_\omega}(u)_u = G_{\sigma^{|u|}\omega}$, the base of induction is proven.

Assume that $n > 1$, and for all smaller numbers the statement is true; let us prove it for n. Since $\mathrm{rist}_{G_\omega}(u)$ is normal in $\mathrm{st}_{G_\omega}(u)$ and, as mentioned above, $\mathrm{st}_{G_\omega}(u)_u = G_{\sigma^{|u|}\omega}$, it suffices to prove that $N_{\sigma^{n-1}\omega}$ contains subgroup $\Phi_1^{-1}(1 \times N_{\sigma^n\omega})$. Let $y_{\sigma^{n-2}\omega}$ be the d-generator of $G_{\sigma^{n-2}\omega}$, and let $z_{\sigma^{n-1}\omega}$ be the d-generator of $G_{\sigma^{n-1}\omega}$. Then $N_{\sigma^{n-1}\omega}$ is the normal closure of $(ay_{\sigma^{n-1}\omega})^2$, and $N_{\sigma^n\omega}$ is the normal closure of $(az_{\sigma^n\omega})^2$.

There are two possibilities.

1) Suppose that $y_{\sigma^{n-1}\omega} = z_{\sigma^{n-1}\omega}$. Then $\Phi_1((ay_{\sigma^{n-1}\omega})^2) = (y_{\sigma^n\omega}, y_{\sigma^n\omega})$. Consider some $x_{\sigma^{n-1}\omega} \in \{b_{\sigma^{n-1}\omega}, c_{\sigma^{n-1}\omega}, d_{\sigma^{n-1}\omega}\}$ that is not the d-generator.

Notice, then, that

$$\Phi_1([x^a_{\sigma^{n-1}\omega}, (ay_{\sigma^{n-1}\omega})^2]) = (1, (ay_{\sigma^n\omega})^2).$$

Since commutator $(ay_{\sigma^{n-1}\omega})^2$ does belong to $N_{\sigma^{n-1}\omega}$ and the latter subgroup is normal in $G_{\sigma^{n-1}\omega}$, we get the desired inclusion (2.3).

2) Suppose that $y_{\sigma^{n-1}\omega} \neq z_{\sigma^{n-1}\omega}$. Then $y_{\sigma^n\omega}$ is not the d-generator, in particular, $\Phi_1((ay_{\sigma^{n-1}\omega})^2) = (ay_{\sigma^n\omega}, y_{\sigma^n\omega}a)$. Then

$$\Phi_1([(ay_{\sigma^{n-1}\omega})^2, z_{\sigma^{n-1}\omega}]) = (1, (az_{\sigma^n\omega})^2).$$

Just as before, this gives us inclusion (2.3).

Let us now prove equalities (2.1) and (2.2). We start by showing that for vertex $u = 1$ of length 1 there is equality

$$\mathrm{rist}_{G_\omega}(u)_u =< y_{\sigma\omega}, N_{\sigma\omega} >, \tag{2.4}$$

where y_ω is the d-generator of G_ω.

Inclusion $\mathrm{rist}_{G_\omega}(u)_u \geq < y_{\sigma\omega}, N_{\sigma\omega} >$ follows from (2.3) and the definition of d-generator. To prove its reverse, notice that subgroup $\Phi_1^{-1}(< y_{\sigma\omega}, N_{\sigma\omega} \times < y_{\sigma\omega}, N_{\sigma\omega} >)$ of G_ω coincides with the normal closure N of y_ω in G_ω. By Corollary 1.1, the cosets of N are the elements of $< a, z_\omega >$, where z_ω is the c-generator. A direct calculation shows that neither of those elements is in $\mathrm{rist}_{G_\omega}(1)$. Thus, we have equality (2.4).

Notice that for any two vertices u and v, $v = uv'$,

$$\mathrm{rist}_{G_\omega}(v)_v = \mathrm{rist}_{\mathrm{rist}_{G_\omega}(u)_u}(v')_{v'} = (\mathrm{rist}_{G_\omega}(u)_u \cap \mathrm{rist}_{G_{\sigma^{|u|}\omega}}(v'))_{v'}. \tag{2.5}$$

Suppose $1 \leq n \leq m(\omega)$. Then it is obvious that $< y_{\sigma^{n-1}\omega}, N_{\sigma^{n-1}\omega} >$ contains $\Phi_1^{-1}(< y_{\sigma^n\omega}, N_{\sigma^n\omega} > \times < y_{\sigma^n\omega}, N_{\sigma^n\omega} >)$, which implies equality (2.1).

Suppose now that $n = m(\omega)+1$ and u is a vertex of length n. Let us show (2.2). It follows from (2.5) that $\mathrm{rist}_{G_\omega}(u)_u$ is contained in $< x_{\sigma^n\omega}, N_{\sigma^n\omega} >$, where $x_{\sigma^{n-1}\omega}$ is the d-generator of $G_{\sigma^{n-1}\omega}$. Yet, by definition of $m(\omega)$ $x_{\sigma^{n-1}\omega}$ does not equal $y_{\sigma^{n-1}\omega}$. Hence (by Lemma 1.1) $x_{\sigma^{n-1}\omega}$ does not belong to $< y_{\sigma^{n-1}\omega}, N_{\sigma^{n-1}\omega} >$. Hence $y_{\sigma^n\omega}$ is not in $\mathrm{rist}_{G_\omega}(u)_u$, which implies (2.2).

Finally, notice that by Lemma 1.1 $N_{\sigma^n\omega}$ contains neither of generators $b_{\sigma^n\omega}$, $c_{\sigma^n\omega}$, $d_{\sigma^n\omega}$, for any n. Hence we get (2.2) for all $n > m(\omega)$. ■

Let us now proceed to the second Grigorchuk's group.

Lemma 2.1. The index in $[G_4, G_4]$ of the third term $\gamma_3(G_4)$ of the lower central series of G_4 is equal to 4.

Proof. Since the derived subgroup of G_4 is the normal closure of $[b, a]$, the factor group $[G_4, G_4]/\gamma_3(G_4)$ is cyclic, and we only need to know its order.

Let us show first that $\Phi_1(\gamma_3(G_4)) \geq \gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4)$. Indeed, since $\Phi_1([b^a, b]) = ([a, b], 1, 1, [b, a])$, we have

$$\Phi_1([[b, b^a], b^{a^{-1}}]) = (1, 1, 1, [[b, a], a]), \tag{2.6}$$

$$\Phi_1([[b, b^a], b][[b, b^a], b^{a^2}]^{-1}) = (1, 1, 1, [[b, a], b]). \tag{2.7}$$

As $\gamma_3(G_4)$ is in fact the normal closure in G_4 of the set $\{[[b, a], b], [[b, a], a]\}$, equalities (2.6) and (2.7) give us the desired inclusion.

Consider now the element $[b, a]^4$. Since $b^4 = 1$, we get that

$$1 = [b^4, a] = [b, a]^4 (\mathrm{mod}\, \gamma_3(G_4)),$$

i.e. that $[b, a]^4$ is in $\gamma_3(G_4)$. Hence the index of $\gamma_3(G_4)$ in $[G_4, G_4]$ can be equal to 1, 2, or 4.

Let us show that $[b, a]^2$ is not in $\gamma_3(G_4)$. Denote by ε_1 the natural projection $G_4 \to G_4/ < a, [G_4, G_4] >$ and by ε_2 the isomorphism between $G_4 \to G_4/ < a, [G_4, G_4] >$ and the ring $Z/4Z = \{0, 1, 2, 3\}$ such that $\varepsilon_2(b) = 1$. Now, let ε denote the composition $(\varepsilon_2 \circ \varepsilon_1 \times \varepsilon_2 \circ \varepsilon_1 \times \varepsilon_2 \circ \varepsilon_1 \times \varepsilon_2 \circ \varepsilon_1) \circ \Phi_1$. Then

$$\varepsilon([b, a]^2) = (2, 0, 0, 2),$$

$$\varepsilon([[b,a],a]) = (2,1,0,1),$$
$$\varepsilon([[b,a],b]) = (0,0,0,0).$$

Obviously, $\varepsilon(\mathrm{st}_{G_4}(1))$ is a submodule of module $Z/4Z \oplus Z/4Z \oplus Z/4Z \oplus Z/4Z$. Since $\gamma_3(G_4)$ is the normal closure of elements $[[b,a],a]$ and $[[b,a],b]$, the image $\varepsilon(\gamma_3(G_4))$ is the submodule generated by the elements

$$(2,1,0,1),$$
$$(1,2,1,0),$$
$$(0,1,2,1),$$
$$(1,0,1,2).$$

Thus, $[b,a]^2$ is in $\gamma_3(G_4)$ if and only if the following system of equations has a solution in $Z/4Z$:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Let us show that this system, in fact, does not have a solution. Indeed, applying elementary transformations to its extended matrix, we get

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The resulting system, while not equivalent to the initial one, is still its consequence, and does not have a solution. Thus, the initial system does not have a solution either.

Thus, $[b,a]^2$ is not in $\gamma_3(G_4)$. Hence, the index of $\gamma_3(G_4)$ in $[G_4,G_4]$ is no less than 4. Earlier we proved that the index is not greater than 4 either. Thus, we have the statement of the lemma. ■

Remark 2.1. It can be easily inferred from the concluding arguments of the proof that for every element x of factorgroup $[G_4,G_4]/\gamma_3(G_4)$ $\varepsilon(x)$ is nontrivial.

Remark 2.2. By (2.6) and (2.7) we have, in fact, that

$$\Phi_1(\gamma_4(G_4)) \geq \gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4). \tag{2.8}$$

Indeed, the left parts of (2.6) and (2.7) are in $\gamma_4(G_4)$.

Proposition 2.2. *For any vertex u*

$$\mathrm{rist}_{G_4}(u)_u = \gamma_3(G_4).$$

Proof. The inclusion $\mathrm{rist}_{G_4}(u)_u \geq \gamma_3(G_4)$ follows from (2.8). Let us prove the equality.

By (2.5), it is sufficient to prove the equality for a vertex u of length 1. To be definite, let us assume that u is the rightmost vertex. Denote by ε the natural projection of G_4 to $G_4/[G_4, G_4]$. Then for any $x \in \mathrm{st}_{G_4}(1)$ the image $(\varepsilon \times \varepsilon \times \varepsilon \times \varepsilon) \circ \Phi_1(x)$ has the form,

$$(b^{\alpha_1} a^{\alpha_2+\alpha_4}, b^{\alpha_2} a^{\alpha_1+\alpha_3}, b^{\alpha_3} a^{\alpha_2+\alpha_4}, b^{\alpha_4} a^{\alpha_1+\alpha_3}),$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in Z/4Z$. Thus, if $x \in \mathrm{rist}_{G_4}(u)$, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_2 + \alpha_4 = 0$. Hence $\alpha_4 = 0$ as well, and $\mathrm{rist}_{G_4}(u)_u \leq [G_4, G_4]$.

Consider an arbitrary x in $\Phi_1^{-1}([G_4, G_4] \times [G_4, G_4] \times [G_4, G_4] \times [G_4, G_4]) \cap G_4$. As any element of $\mathrm{st}_{G_4}(1)$, x can be represented by a word in a form,

$$b^{\beta_1 a^{i_1}} b^{\beta_2 a^{i_2}} \dots b^{\beta_m a^{i_m}}, \tag{2.9}$$

where $\beta_j \in Z/4Z \setminus \{0\}$, $i_j \in Z/4Z$ for $j = 1, 2, \dots, m$, and $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, m-1$. Denote by $\partial_i(x)$, $i \in Z/4Z$, the sum of all β_j such that $i_j = i$. Since $\partial_i(x) = \partial_b(\varphi_{i-1}(x))$, the numbers $\partial_i(x)$ do not depend on the choice of a particular representation (2.9).

For the same reason, all numbers $\partial_i(x)$ are actually equal to 0 (recall that x is in $\Phi_1^{-1}([G_4, G_4] \times [G_4, G_4] \times [G_4, G_4] \times [G_4, G_4]) \cap G_4$). Hence, x can be represented as a product of commutators $[b^{a^i}, b^{a^j}]$ and their conjugates. This means, in particular, that $\Phi_1^{-1}([G_4, G_4] \times [G_4, G_4] \times [G_4, G_4] \times [G_4, G_4]) \cap G_4$ is the normal closure of the set of commutators $[b^{a^i}, b^{a^j}]$.

By Lemma 2.1, the group $([G_4, G_4]/\gamma_3(G_4) \times [G_4, G_4]/\gamma_3(G_4) \times [G_4, G_4]/\gamma_3(G_4) \times [G_4, G_4]/\gamma_3(G_4))$ can be regarded as module $Z/4Z \oplus Z/4Z \oplus Z/4Z \oplus Z/4Z$. There is a natural map, which we'll denote by ϵ, from $\Phi_1^{-1}([G_4, G_4] \times [G_4, G_4] \times [G_4, G_4] \times [G_4, G_4]) \cap G_4$ to this module. The map is obtained as the composition of Φ_1, the fourth direct power of the natural projection $[G_4, G_4] \to [G_4, G_4]/\gamma_3(G_4)$, and the fourth direct power of the isomorphism $[G_4, G_4]/\gamma_3(G_4) \cong Z/4Z = \{0, 1, 2, 3\}$ taking $[b, a]\gamma_3(G_4)$ to 1. The image of ϵ is denoted by L.

We have

$$\epsilon([b, b^a]) = (3, 0, 0, 1),$$

$$\epsilon([b, b^{a^2}]) = (0, 0, 0, 0),$$

$$\epsilon([b, b^{a^{-1}}]) = (0, 0, 3, 1).$$

Thus, L is generated by all cyclic shifts of vectors $(-1, 0, 0, 1)$, $(0, 0, -1, 1)$. And since the sum of coordinates of either of these vectors is equal to 0, L does not contain vectors $(0, 0, 0, 1)$ or $(0, 0, 0, 2)$. This is equivalent to saying that G_4 does not contain elements $\Phi_1^{-1}((1, 1, 1, [b, a]))$ or $\Phi_1^{-1}((1, 1, 1, [b, a]^2))$. Hence by Lemma 2.1, $\mathrm{rist}_{G_4}(u) = \gamma_3(G_4)$. ∎

2.2 GGS-groups

The upcoming description of rigid stabilizers of GGS-groups is obtained as a partial case of Theorem 2.7.1 from [11].

Proposition 2.3. *Let $G_{\bar{\alpha}}$ be a periodic GGS-group with accompanying vector $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{p-1})$. Then if $\bar{\alpha}$ is symmetric, i.e. if $\alpha_i = \alpha_{p-i}$ for all $i = 1, 2, \ldots, \frac{p-1}{2}$, we have inclusion*

$$\Phi_1(\gamma_3(G_{\bar{\alpha}})) \geq \underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p}.$$

In all other cases we have inclusion

$$\Phi_1(\gamma_3(G_{\bar{\alpha}})) \geq \underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p}.$$

Proof. First, we need to check that $\bar{\alpha}$ is not ε-diagonal, i.e. is not proportional to a vector of the form $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$, where ε is a unity root of pth degree in the field $\mathrm{GF}(p^n)$ for a natural n.

Since $G_{\bar{\alpha}}$ is periodic, $\alpha_1 + \ldots + \alpha_{p-1} = 0$. But the sum of coordinates of a vector $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$ equals to

$$\varepsilon + \varepsilon^2 + \ldots + \varepsilon^{p-1} = \begin{cases} \varepsilon\frac{1-\varepsilon^{p-1}}{1-\varepsilon}, & \text{if } \varepsilon \neq 1 \\ p-1 & \text{if } \varepsilon = 1 \end{cases}$$

Thus, if $\varepsilon = 1$, this sum is not equal to 0. If $\varepsilon \neq 1$, then clearly $1 - \varepsilon^{p-1} \neq 0$, and the sum is not equal to 0, either.

We obtain that $\bar{\alpha}$ is not ε-diagonal. Then by Theorem 2.7.1 of [11], if $\bar{\alpha}$ is ε-symmetric (i.e. has form $(\alpha_1, \alpha_2, \ldots, \alpha_{s-1}, \alpha_s, \varepsilon^s\alpha_s, \varepsilon^{s-1}\alpha_{s-1}, \ldots, \varepsilon^2\alpha_2, \varepsilon\alpha_1)$, $s = \frac{p-1}{2}$), then $\Phi(\gamma_3(G_{\bar{\alpha}}))$ contains the direct product $\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p}$. Since the only unity root of pth degree in $\mathrm{GF}(p)$ is 1, all ε-symmetric vectors are the vectors satisfying the condition $\alpha_i = \alpha_{p-i}$, $i = 1, 2, \ldots, \frac{p-1}{2}$.

By the same theorem, in all other cases there is inclusion $\Phi_1(\gamma_3(G_{\bar{\alpha}})) \geq \underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p}$. ∎

Corollary 2.1. *If the accompanying vector of a periodic GGS-group $G_{\bar{\alpha}}$ is symmetric, then for any vertex u $\mathrm{rist}_{G_{\bar{\alpha}}}(u)_u$ contains $\gamma_3(G_{\bar{\alpha}})$. Otherwise for any vertex u $\mathrm{rist}_{G_{\bar{\alpha}}}(u)_u$ contains $[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$.*

Proof. Follows immediately from Proposition 2.3. ∎

2.3 EGS-groups with nonsymmetric accompanying vector

Proposition 2.4. *Let $\Gamma_{\bar{\alpha}}$ be a EGS-group with nonsymmetric accompanying vector. Then for any vertex u $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(u)_u$ contains $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$.*

Proof. Let $G_{\bar{\alpha}}$ be the associated GGS-group of $\Gamma_{\bar{\alpha}}$. Then by Proposition 2.3 $\Phi_1(\gamma_3(G_{\bar{\alpha}})) \geq \underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p}$. Now, we can prove in the same fashion that if $F_{\bar{\alpha}}$ is the subgroup of $\Gamma_{\bar{\alpha}}$ generated by elements a and c, then $\Phi(\gamma_3(F_{\bar{\alpha}}))$ contains the direct product $\underbrace{\gamma_3(F_{\bar{\alpha}}) \times \ldots \times \gamma_3(F_{\bar{\alpha}})}_{p}$.

Notice that $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ is the normal closure in $\Gamma_{\bar{\alpha}}$ of the following set: $[a, b]$, $[a, c]$, $[b, c]$. It follows from the above that for any vertex u of length 1 $\Phi_1(\gamma_3(\Gamma_{\bar{\alpha}}))$ contains elements $([a, b], 1, \ldots, 1)$ and $([a, c], 1, \ldots, 1)$. Further, $\Phi_1([b^a, c]) = ([b, c], 1, \ldots, 1)$. Hence

$$\Phi_1([\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]) \geq [\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \times \underbrace{\{1\} \times \ldots \times \{1\}}_{p-1}, \tag{2.10}$$

i.e. $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(u)_u \geq [\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ for all vertices of length 1. Taking into account (2.5) and inclusion (2.10), we get the desired inclusion for an arbitrary vertex u. ∎

3 Grigorchuk's groups

In the next two sections we establish the congruence property for three of our major examples, groups G_ω, the second Grigorchuk's group, and periodic GGS-groups. In all three cases we follow a similar pattern, by proving first that rigid stabilizers are congruence subgroups (a necessary condition always) and then showing that in fact every normal subgroup of groups in question contains some rigid stabilizer (a necessary and by the first part sufficient condition).

3.1 Groups G_ω

Lemma 3.1. For each $\omega \in \Omega_0$ the derived subgroup $[G_\omega, G_\omega]$ is a congruence subgroup.

Proof. Since the derived subgroup of any G_ω has finite index and the intersection of a finite number of principal congruence subgroups is again a principal congruence subgroup, it is sufficient to show that any nontrivial coset of $[G_\omega, G_\omega]$ has empty intersection with some principal congruence subgroup. Those cosets are $a[G_\omega, G_\omega]$, $ab_\omega[G_\omega, G_\omega]$, $ac_\omega[G_\omega, G_\omega]$, $ad_\omega[G_\omega, G_\omega]$, $b_\omega[G_\omega, G_\omega]$, $c_\omega[G_\omega, G_\omega]$, $d_\omega[G_\omega, G_\omega]$. Since $[G_\omega, G_\omega]$ is contained in $\mathrm{st}_{G_\omega}(1)$, the first four cosets clearly have empty intersection with $\mathrm{st}_{G_\omega}(1)$.

For definiteness' sake, let us assume that ω begins with 0. Then by definition $\Phi_1(b_\omega) = (a, b_{\sigma\omega})$, $\Phi_1(c_\omega) = (a, c_{\sigma\omega})$, $\Phi_1(d_\omega) = (1, d_{\sigma\omega})$.

It is easy to see that $[G_\omega, G_\omega]$ is generated by elements $(ab_\omega)^2$, $(ac_\omega)^2$, $(ad_\omega)^2$. By our assumption we have

$$\Phi_1((ab_\omega)^2) = (b_{\sigma\omega}a, ab_{\sigma\omega}),\ \Phi_1((ac_\omega)^2) = (c_{\sigma\omega}a, ac_{\sigma\omega}),\ \Phi_1((ad_\omega)^2) = (d_{\sigma\omega}, d_{\sigma\omega}).$$

Thus, for any $y \in [G_\omega, G_\omega]$ and for any vertex u of length 1 $\varphi_u(y)$ belongs to one of the following cosets, $[G_{\sigma\omega}, G_{\sigma\omega}]$, $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$, $ab_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$, $ac_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$. Moreover, if u and v are the two vertices of length 1, $\varphi_u(y)$ and $\varphi_v(y)$ belong to same coset. Thus, an element $x_\omega y$, where $x_\omega \in \{b_\omega, c_\omega, d_\omega\}$, is in $\mathrm{st}_{G_\omega}(2)$ only in case when $x_\omega = d_\omega$ and $\varphi_u(y)$ is either in $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$ or in $[G_{\sigma\omega}, G_{\sigma\omega}]$. In the latter case one of $\varphi_u(x_\omega y)$, $\varphi_v(x_\omega y)$ is in $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$ and the other is in $[G_{\sigma\omega}, G_{\sigma\omega}]$.

Let us consider coset $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$. Suppose first that $m(\omega) = 1$ (and thus $\omega_2 \neq \omega_1$). Then $\Phi_1(d_{\sigma\omega}) = (a, d_{\sigma^2\omega})$, and, as above, for any $z \in [G_{\sigma\omega}, G_{\sigma\omega}]$ there is a vertex u of length 1 such that $\varphi_u(d_{\sigma\omega} z)$ is not in $\mathrm{st}_{G_{\sigma^2\omega}}(1)$. This would mean that our initial element $x_\omega y = d_\omega y$ is not in $\mathrm{st}_{G_\omega}(3)$.

Now we use an obvious induction on $m(\omega)$ to prove that there always is a vertex u (of length no greater than $m(\omega)+1$) such that $\varphi_u(x_\omega y)$ is not in $\mathrm{st}_{G_{\sigma^{|u|}\omega}}(1)$, which implies that $x_\omega y$ is not in $\mathrm{st}_{G_\omega}(|u|+1)$. ■

Remark 3.1. It follows from the proof of the lemma that $[G_\omega, G_\omega]$ contains $\mathrm{st}_{G_\omega}(m(\omega)+2)$.

Proposition 3.1. *For each $\omega \in \Omega_0$ and each generator $x_\omega \in \{b_\omega, c_\omega, d_\omega\}$ the normal closure of $(x_\omega a)^2$ is a congruence subgroup.*

Proof. As in Lemma 3.1, consider cosets of the normal closure N of some $(x_\omega a)^2$. Denote by y_ω the element of $\{b_\omega, c_\omega, d_\omega\}$ chosen in the following way. If x_ω is not the d-generator, then y_ω is the d-generator. If x_ω is the d-generator, then y_ω is the c-generator.

It is obvious that the factorgroup of G_ω by N is isomorphic to a factorgroup of finite group $< x_\omega > \times < a, y_\omega >$. Fix the following system of representatives of cosets of N: $a^i x_\omega^j s$, where $i, j \in \mathrm{GF}(2)$ and $s \in < y_\omega, y_\omega^a >$.

Notice that since $N \leq \mathrm{st}_{G_\omega}(1)$, the desired fact has to be verified only for the cosets whose representatives are in $\mathrm{st}_{G_\omega}(1)$. Thus, we can assume that $i = 0$.

There are two possibilities.

1) Let x_ω be the d-generator of G_ω, i.e. $\Phi_1(x_\omega) = (1, x_{\sigma\omega})$. Consider $t \in \mathrm{st}_{G_\omega}(1) \setminus N$ and write it as $t = x_\omega^j s r$, where j, s are as above and r is an element of N. Let u, v be the two vertices of length 1, and let us consider the cosets containing $\varphi_u(t)$ and $\varphi_v(t)$.

Since $\Phi_1^{-1}([G_{\sigma\omega}, G_{\sigma\omega}] \times [G_{\sigma\omega}, G_{\sigma\omega}]) \cap G_\omega \leq [\mathrm{st}_{G_\omega}(1), \mathrm{st}_{G_\omega}(1)]$, for any two elements s and s' in the same coset of $[\mathrm{st}_{G_\omega}(1), \mathrm{st}_{G_\omega}(1)]$ $\varphi_u(x_\omega^j s r)$ and $\varphi_u(x_\omega^j s' r)$ are in the same coset of $[G_{\sigma\omega}, G_{\sigma\omega}]$. The same can be said about $\varphi_u(x_\omega^j s r)$ and $\varphi_u(x_\omega^j s r')$ for any r, r' in one coset of $[\mathrm{st}_{G_\omega}(1), \mathrm{st}_{G_\omega}(1)]$. Also, the same holds for the second vertex v.

In $< y_\omega, y_\omega^a >$, there are exactly four cosets of $< y_\omega, y_\omega^a > \cap [\mathrm{st}_{G_\omega}(1), \mathrm{st}_{G_\omega}(1)]$. Their representatives are, for instance, elements 1, y_ω, y_ω^a, $y_\omega y_\omega^a$. Similarly, there are exactly two cosets of $N \cap [\mathrm{st}_{G_\omega}(1), \mathrm{st}_{G_\omega}(1)]$ in N. Their representatives are 1 and $(x_\omega a)^2$.

Thus, we have 16 possibilities for different choices of s and r. They are shown in the following table, where on the vertical are representatives $x_\omega^j s$, on the horizontal are r, and the cells contain the representatives of the cosets containing $\varphi_u(x_\omega^j s r)$ and $\varphi_v(x_\omega^j s r)$.

	1	$(x_\omega a)^2$
1	(1,1)	$(x_{\sigma\omega}, x_{\sigma\omega})$
y_ω	$(a, y_{\sigma\omega})$	$(ax_{\sigma\omega}, z_{\sigma\omega})$
y_ω^a	$(y_{\sigma\omega}, a)$	$(z_{\sigma\omega}, ax_{\sigma\omega})$
$y_\omega y_\omega^a$	$(ay_{\sigma\omega}, ay_{\sigma\omega})$	$(az_{\sigma\omega}, az_{\sigma\omega})$
x_ω	$(1, x_{\sigma\omega})$	$(x_{\sigma\omega}, 1)$
z_ω	$(a, z_{\sigma\omega})$	$(ax_{\sigma\omega}, y_{\sigma\omega})$
$x_\omega y_\omega^a$	$(y_{\sigma\omega}, ax_{\sigma\omega})$	$(z_{\sigma\omega}, a)$
$z_\omega y_\omega^a$	$(ay_{\sigma\omega}, az_{\sigma\omega})$	$(az_{\sigma\omega}, ay_{\sigma\omega})$

It follows from the table that, for a t in any coset other than a one of the form $(y_\omega y_\omega^a)^{2^k}N$, either $\varphi_u(t)$ or $\varphi_v(t)$ is in a nontrivial coset of $[G_{\sigma\omega}, G_{\sigma\omega}]$ and hence, by the remark after Lemma 3.1, is not in $\mathrm{st}_{G_{\sigma\omega}}(m(\sigma\omega)+2)$. Thus, all cosets but the ones of the form $(y_\omega y_\omega^a)^{2^k}N$, have empty intersection with $\mathrm{st}_{G_\omega}(m(\sigma\omega)+3)$.

Let us now consider cosets $(y_\omega y_\omega^a)^{2^k}N$. Notice that $m(\omega)$ is the smallest number such that $y_{\sigma^{m(\omega)}\omega}$ is the d-generator of the relevant group. In particular, this implies that $y_\omega y_\omega^a$ has order $2^{m(\omega)+1}$, and that for every $k = 1, 2, \ldots, m(\omega)-1$ and for any vertex u of length $k+1$ $\varphi_u((y_\omega y_\omega^a)^{2^k})$ is either $ay_{\sigma^{k+1}\omega}$ or $y_{\sigma^{k+1}\omega}a$, hence, is in the coset $ay_{\sigma^{k+1}\omega}[G_{\sigma^{k+1}\omega}, G_{\sigma^{k+1}\omega}]$. Further, for any vertex u of length $m(\omega)+1$ $\varphi_u((y_\omega y_\omega^a)^{2^k}) = y_{\sigma^{m(\omega)+1}\omega}$ and therefore is in the coset $y_{\sigma^{m(\omega)+1}\omega}[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$.

Consider a vertex u of length $k+1$, $k = 1, 2, \ldots, m(\omega)$ and an element r of $\mathrm{st}_N(k+1)$. Let us see which cosets of the derived subgroup $[G_{\sigma^{k+1}\omega}, G_{\sigma^{k+1}\omega}]$ $\varphi_u(r)$ can belong to. Since $x_{\sigma^k\omega}$ is the d-generator of $G_{\sigma^k\omega}$ for $k = 1, 2, \ldots, m(\omega)-1$, for any element x of the normal closure of $x_{\sigma^k\omega}$ in $G_{\sigma^k\omega}$ $\varphi_0(x)$ and $\varphi_1(x)$ can be only in the cosets $x_{\sigma^{k+1}\omega}[G_{\sigma^{k+1}\omega}, G_{\sigma^{k+1}\omega}]$ and $[G_{\sigma^{k+1}\omega}, G_{\sigma^{k+1}\omega}]$. However, for $k = 1, 2, \ldots, m(\omega)-1$ $\varphi_v(r)$, $|v| = k$, actually are in the normal closure of $x_{\sigma^k\omega}$.

Thus, for any $k = 1, 2, \ldots, m(\omega)-1$, any vertex u of length $k+1$, and for any $r \in \mathrm{st}_N(k+1)$ $\varphi_u(r)$ is in a nontrivial coset of the relevant derived subgroup. By Remark 3.1 no such element is in the principal congruence subgroup $\mathrm{st}_{G_{\sigma^{k+1}\omega}}(m(\sigma^{k+1}\omega)+2)$. And since the set $(y_\omega y_\omega^a)^{2^k}(N \setminus \mathrm{st}_N(k+1))$ has empty intersection with $\mathrm{st}_{G_\omega}(k+1)$, we have that for $k = 1, 2, \ldots, m(\omega)-1$ coset $(y_\omega y_\omega^a)^{2^k}N$ has empty intersection with $\mathrm{st}_{G_\omega}(k+1+m(\sigma^{k+1}\omega))$.

Now let us consider coset $(y_\omega y_\omega^a)^{2^{m(\omega)}}N$. Since $x_{\sigma^{m(\omega)}\omega}$ is not the d-generator of $G_{\sigma^{m(\omega)}\omega}$, for an x in the normal closure of $x_{\sigma^{m(\omega)}\omega}$ in $G_{\sigma^{m(\omega)}\omega}$, $\varphi_0(x)$ and $\varphi_1(x)$ are in one of the following cosets, $x_{\sigma^{m(\omega)+1}\omega}[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$, $a[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$, $ax_{\sigma^{m(\omega)+1}\omega}[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$, and $[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$. However, even in this case for any vertex u of length $m(\omega)+1$ and for any t in $(y_\omega y_\omega^a)^{2^{m(\omega)}}\mathrm{st}_N(1)$ $\varphi_u(t)$ is in a nontrivial coset of the derived subgroup. Just as above, we obtain that coset $(y_\omega y_\omega^a)^{2^{m(\omega)}}N$ has empty intersection with $\mathrm{st}_{G_\omega}(m(\omega)+3+m(\sigma^{m(\omega)+1}\omega))$.

2) Suppose now that x_ω is not the d-generator. Then, by our choice, y_ω is the d-generator. This means that subgroup $< y_\omega, y_\omega^a >$ consists of four elements, 1, y_ω, y_ω^a, $y_\omega y_\omega^a$. Let us write down a table similar to the previous one (here z_ω is the b-generator).

	1	$(x_\omega a)^2$
1	(1,1)	$(ax_{\sigma\omega}, ax_{\sigma\omega})$
y_ω	$(1, y_{\sigma\omega})$	$(ax_{\sigma\omega}, az_{\sigma\omega})$
y_ω^a	$(y_{\sigma\omega}, 1)$	$(az_{\sigma\omega}, ax_{\sigma\omega})$
$y_\omega y_\omega^a$	$(y_{\sigma\omega}, y_{\sigma\omega})$	$(az_{\sigma\omega}, az_{\sigma\omega})$
x_ω	$(a, x_{\sigma\omega})$	$(x_{\sigma\omega}, a)$
z_ω	$(a, z_{\sigma\omega})$	$(x_{\sigma\omega}, ay_{\sigma\omega})$
$x_\omega y_\omega^a$	$(ay_{\sigma\omega}, x_{\sigma\omega})$	$(z_{\sigma\omega}, a)$
$z_\omega y_\omega^a$	$(ay_{\sigma\omega}, z_{\sigma\omega})$	$(z_{\sigma\omega}, ay_{\sigma\omega})$

As we see, for any element t of a nontrivial coset of N either $\varphi_0(t)$ or $\varphi_1(t)$ is in a nontrivial coset of $[G_{\sigma\omega}, G_{\sigma\omega}]$. Thus, by Remark 3.1 N contains principal congruence subgroup $\mathrm{st}_{G_\omega}(m(\sigma\omega)+2)$. ■

Theorem 3.1. *For each $\omega \in \Omega_0$ G_ω has congruence property.*

Proof. Since all G_ω are just infinite, we can just prove that the normal closure $N(g)$ of any $g \in G_\omega$ is a congruence subgroup. We will prove this fact by induction on the length $\partial_\omega(g)$ of g, see Section 1.1 for definition of the length function.

Suppose g has length 1, i.e. has form $a^\epsilon x_\omega a^\mu$, where $x_\omega \in \{b_\omega, c_\omega, d_\omega\}$, ϵ, μ are as above. Then its normal closure does contain the normal closure of $(ax_\omega)^2$, which is a congruence subgroup by Proposition 3.1. Hence $N(g)$ is a congruence subgroup.

If g has length greater than 1, then by Lemma 8.2 of [5], there exist a natural number n and an element f of the normal closure of g such that $f \in \mathrm{st}_{G_\omega}(u)$, $u = \underbrace{11\ldots1}_{n}$, and $\varphi_u(f) \in \{b_{\sigma^n\omega}, c_{\sigma^n\omega}, d_{\sigma^n\omega}\}$. Fix some such f and choose $m \geq n$ such that $\omega_m \neq \omega_{m+1}$ and $x_{\sigma^m\omega} = \varphi_v(f)$, $v = \underbrace{11\ldots1}_{m}$, is the d-generator.

Let $y_{\sigma^{m-1}\omega}$ be the d-generator of $G_{\sigma^{m-1}\omega}$. Consider $h = [\Phi_m^{-1}(\underbrace{1,\ldots,1}_{2^m-1}, (ay_{\sigma^m\omega})^2), f]$. Since $f \in \mathrm{st}_{G_\omega}(v)$, h belongs to $\mathrm{rist}_{G_\omega}(v)$, and $\varphi_v(h) = [(ay_{\sigma^m\omega})^2, x_{\sigma^m\omega}]$. It follows from ω_m being not equal to ω_{m+1} and Proposition 2.1 that $y_{\sigma^m\omega}$ is not the d-generator of $G_{\sigma^m\omega}$, and $\Phi_1(\varphi_v(h)) = (1, [y_{\sigma^{m+1}\omega}a, x_{\sigma^{m+1}\omega}]) = (1, (ax_{\sigma^{m+1}\omega})^2)$.

Thus, the normal closure $N(h)$ of h in G_ω contains $\Phi_{m+1}^{-1}(\underbrace{N_{\sigma^{m+1}\omega} \times \ldots \times N_{\sigma^{m+1}\omega}}_{2^{m+1}})$. By Proposition 3.1, $N_{\sigma^{m+1}\omega}$ contains principal congruence subgroup $\mathrm{st}_{G_{\sigma^{m+1}\omega}}(k)$ for some k. Hence $N(h)$ contains $\Phi_{m+1}^{-1}(\underbrace{\mathrm{st}_{G_{\sigma^{m+1}\omega}}(k) \times \ldots \times \mathrm{st}_{G_{\sigma^{m+1}\omega}}(k)}_{2^{m+1}})$, i.e. contains $\mathrm{st}_{G_\omega}(m+1+k)$.

Since $N(g) \geq N(h)$, the theorem is proven. ■

3.2 The second Grigorchuk's group.

Lemma 3.2. The third term of the lower central series of G_4 contains $\mathrm{st}_{G_4}(3)$.

Proof. Let us prove first that $[G_4, G_4]$ contains $\mathrm{st}_{G_4}(2)$. Indeed, take $x \in \mathrm{st}_{G_4}(2)$. Then

$$\Phi_1(x) = (b^{\alpha_1} a^{\alpha_2+\alpha_4} x_1, b^{\alpha_2} a^{\alpha_1+\alpha_3} x_2, b^{\alpha_3} a^{\alpha_2+\alpha_4} x_3, b^{\alpha_4} a^{\alpha_1+\alpha_3} x_4),$$

where $x_1, x_2, x_3, x_4 \in [G_4, G_4]$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in Z/4Z$. We must have $\alpha_1 + \alpha_3 = 0$ and $\alpha_2 + \alpha_4 = 0$. Hence $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0$. However, since $\partial_b(x)$ equals exactly $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, we have that $x \in [G_4, G_4]$. Hence $[G_4, G_4]$ contains $\mathrm{st}_{G_4}(2)$.

By Lemma 2.1 $\gamma_3(G_4)$ has finite index in G_4. Thus, it is sufficient to prove now that every nontrivial coset of $\gamma_3(G_4)$ has empty intersection with $\mathrm{st}_{G_4}(3)$.

Consider a coset of $\gamma_3(G_4)$. Unless it is contained in $[G_4, G_4]$, it has empty intersection with $\mathrm{st}_{G_4}(2)$ by the above. Suppose that it is contained in $[G_4, G_4]$. Then by Remark 2.1 for any element x of that coset there is a vertex u of length 1 such that $\varphi_u(x)$ is not in $[G_4, G_4]$, hence not in $\mathrm{st}_{G_4}(2)$. Thus, the coset has empty intersection with $\mathrm{st}_{G_4}(3)$, which is what we need. ∎

Proposition 3.2. *For any $x \in G_4$ there is a vertex u such that $\mathrm{st}_N(u)_u$ contains G_4, where N is the normal closure of x in G.*

Proof. It is sufficient to prove that there is a vertex u such that $\mathrm{st}_N(u)_u \ni b$. Indeed, in that case $\mathrm{st}_N(u)_u \ni b^a$ as well, and therefore $\mathrm{st}_N(u3)_{u3} \ni a, b$.

Let us prove the desired fact by induction on the length $\partial_{G_4}(x)$. The base of induction is $\partial_{G_4}(x) \le 1$. First suppose that $\partial_{G_4}(x) = 0$, i.e. $x = a^i$. It suffices to consider case $x = a^2$. But we have

$$\Phi_1([a^2, b]) = (1, b^{-1}, 1, b),$$

and we can take the rightmost vertex as u.

Suppose that $\partial_{G_4}(x) = 1$, i.e. $x = a^i b^j a^k$. First assume that $x \in \mathrm{st}_{G_4}(1)$, i.e. $i + k = 0$. Then we can assume that $x = b^i$, and as before, we only need to consider $x = b^2$. We have

$$\Phi_1([b^2, b^a]) = ([a^2, b], 1, 1, [b^2, a]),$$

and

$$\Phi_1([a^2, b]) = (1, b^{-1}, 1, b),$$

just as we wanted.

Suppose that x is an arbitrary (of any length) element of $G_4 \setminus \mathrm{st}_{G_4}(1)$. It has form $a^i z$ for some $i \in Z/4Z \setminus \{0\}$ and $z \in \mathrm{st}_{G_4}(1)$. Then

$$\varphi_2([x, b^{a^{1-i}}]) = \begin{cases} b, & \text{if } i = 2 \\ a, & \text{otherwise.} \end{cases}$$

In the first case we have what we want right away; otherwise, as above, there exists a vertex $u = 2v$ such that $\mathrm{st}_N(u)_u \ni b$.

We have completed the base of induction and even a part of the inductive step. Let $n > 1$ and $\partial_{G_4}(x) = n$. By the above, it is sufficient to consider the case when $x \in \mathrm{st}_{G_4}(1)$. However, for any vertex u of length 1 the length of $\varphi_u(x)$ is less or equal to $\frac{\partial_{G_4}(x)+1}{2}$, and since $\partial_{G_4}(x) > 1$, it is strictly less than $\partial_{G_4}(x)$. Thus, we can apply the inductive assumption. This completes the proof of the proposition. ∎

Theorem 3.2. *The second Grigorchuk's group has congruence property.*

Proof. As any subgroup of finite index contains a normal subgroup, it is sufficient to prove that the normal closure of any element of G_4 is a congruence subgroup. Consider the normal closure N of some $x \in G_4$. By Proposition 3.2, there is a vertex u such that $\mathrm{st}_N(u)_u \geq G_4$. Since by Proposition 2.2 $\mathrm{rist}_{G_4}(u)_u \geq \gamma_3(G_4)$, then $\mathrm{rist}_N(u)_u \geq \gamma_4(G_4)$. Since by Remark 2.2 $\Phi_1(\gamma_4(G_4)) \geq (\gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4) \times \gamma_3(G_4))$, we have

$$N \geq \Phi^{-1}_{|u|+1}(\underbrace{\gamma_3(G_4) \times \ldots \times \gamma_3(G_4)}_{4^{|u|+1}}).$$

Then by Lemma 3.2 $N \geq \mathrm{st}_{G_4}(|u|+4)$. ∎

4 GGS-groups

4.1 Normal subgroups of GGS-groups

As we mentioned in Section 1.2, all GGS-groups are fractal, i.e. for any vertex u $\mathrm{st}_{G_{\bar{\alpha}}}(u)_u$ coincides with $G_{\bar{\alpha}}$. Here we prove at first that, with the exception of a finite number of vertices, this holds for any normal subgroup of $G_{\bar{\alpha}}$.

Lemma 4.1. Let $G_{\bar{\alpha}}$ be a periodic GGS-group. Then for any $x \in G_{\bar{\alpha}}$ there is a vertex u such that $\mathrm{st}_X(u)_u \ni b$, where X is the normal closure of x in $G_{\bar{\alpha}}$.

Proof. Let us prove the statement by induction on length $\partial_{G_{\bar{\alpha}}}(x)$. The base of induction is $\partial_{G_{\bar{\alpha}}}(x) \leq 1$.

The elements of length 0 are a^i, $i \in \mathrm{GF}(p)$. We have

$$\Phi_1([a^i,b]) = (a^{-\alpha_{p-(i-1)}}, a^{-\alpha_{p-(i-2)}}, \ldots, a^{-\alpha_{p-1}}, b^{-1}, a^{-\alpha_1}, \ldots, a^{-\alpha_{p-i}})(a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}}, b) =$$

$$(a^{\alpha_1-\alpha_{p-(i-1)}}, a^{\alpha_2-\alpha_{p-(i-2)}}, \ldots, b^{-1}a^{\alpha_i}, a^{\alpha_{i+1}-\alpha_1}, \ldots, a^{-\alpha_{p-i}}b).$$

If $\alpha_i = 0$ or $\alpha_{p-i} = 0$, then for either $u = i-1$ or $u = p-i-1$, respectively, we have $\varphi_u(X) \ni b$. If $\alpha_i\alpha_{p-i} \neq 0$, we proceed to case $\partial_{G_{\bar{\alpha}}}(x) = 1$.

If $\partial_{G_{\bar{\alpha}}}(x) = 1$, then $x = a^i b^\varepsilon a^j$. Conjugating by a^i if needed, we can assume that $x = b^\varepsilon a^j$.

If $j = 0$, we have the desired fact right away. Suppose $j \neq 0$. But in this case $\varphi_{p-1}(x^p) = b^\varepsilon$. Since p is prime, this completes the base of induction.

Suppose now that x is an element of length $m > 1$, and for all elements of smaller length the statement is already proven. If x is in $\mathrm{st}_{G_{\bar{\alpha}}}(1)$, for any vertex u of length 1 $\partial_{G_{\bar{\alpha}}}(\varphi_u(x)) \leq \frac{\partial_{G_{\bar{\alpha}}}(x)+1}{2} < \partial_{G_{\bar{\alpha}}}(x)$. Since $G_{\bar{\alpha}}$ is fractal, we can apply the inductive assumption to any $\varphi_u(x)$.

Suppose that x is not in $\mathrm{st}_{G_{\bar{\alpha}}}(1)$, i.e. $x = ya^i$ for some $y \in \mathrm{st}_{G_{\bar{\alpha}}}(1)$ and some $i \in \mathrm{GF}(p)$. Denote $\Phi_1(y) = (y_0, y_1, \ldots, y_{p-1})$. The we have

$$\Phi_1(x^p) = (y_0 y_{\pi^i(0)} \ldots y_{\pi^{(p-1)i}(0)}, y_1 y_{\pi^i(1)} \ldots y_{\pi^{(p-1)i}(1)}, \ldots, y_{p-1} y_{\pi^i(p-1)} \ldots y_{\pi^{(p-1)i}(p-1)}),$$

where π denotes the (nontrivial) accompanying permutation of a. As for any element of $\mathrm{st}_{G_{\bar\alpha}}(1)$, there is equality

$$\partial_a(y_0)+\ldots+\partial_a(y_{p-1})=(\alpha_1+\ldots+\alpha_{p-1})(\partial_b(y_0)+\ldots+\partial_b(y_{p-1}))=0.$$

Therefore, since for any $i\in\mathrm{GF}(p)$ a^i is transitive on vertices of length 1, x^p is actually in $\mathrm{st}_{G_{\bar\alpha}}(2)$. On the other hand, for any vertex u of length 1 $\partial_{G_{\bar\alpha}}(\varphi_u(x))\le\partial_{G_{\bar\alpha}}(x)$. Thus, for any vertex v of length 2 $\partial_{G_{\bar\alpha}}(\varphi_v(x))\le\frac{\partial_{G_{\bar\alpha}}(x)+1}{2}<\partial_{G_{\bar\alpha}}(x)$. Again, since $G_{\bar\alpha}$ is fractal, we can apply the inductive assumption to any $\varphi_v(x)$. ∎

Proposition 4.1. *Let $G_{\bar\alpha}$ be a periodic GGS-group. Then for any $x\in G_{\bar\alpha}$ there is a vertex u such that $\mathrm{st}_X(u)_u=G_{\bar\alpha}$.*

Proof. By Lemma 4.1, there is a vertex u such that $\mathrm{st}_X(u)_u\ni b$. Since $G_{\bar\alpha}$ is fractal and $\mathrm{st}_X(u)$ is normal in $G_{\bar\alpha}$, we have $\mathrm{st}_X(u)_u\ni b^{a^i}$ for any $i\in\mathrm{GF}(p)$ as well. Thus, if $v=u(p-1)$ is a vertex adjacent to u, then $\mathrm{st}_X(v)_v\ni b,\ a^{\alpha_1},\ a^{\alpha_2},\ \ldots,\ a^{\alpha_{p-1}}$. Since among the numbers $\alpha_1,\alpha_2,\ldots,\alpha_{p-1}$ there is a nontrivial one, $\mathrm{st}_X(v)_v=G_{\bar\alpha}$. ∎

4.2 Congruence subgroups of GGS-groups with nonsymmetric accompanying vector

Despite the title of the section, the lemma below does apply to arbitrary GGS-groups.

Lemma 4.2. The derived subgroup of any periodic GGS-group $G_{\bar\alpha}$ contains principal congruence subgroup $\mathrm{st}_{G_{\bar\alpha}}(2)$.

Proof. Consider an element $x\in\mathrm{st}_{G_{\bar\alpha}}(1)$; denote $\Phi_1(x)=(x_0,x_1,\ldots,x_{p-1})$. Clearly, $\partial_b(x)=\sum_{i=0}^{p-1}\partial_b(x_i)$, and vectors $(\partial_b(x_0),\ldots,\partial_b(x_{p-1}))$ and $(\partial_a(x_0),\ldots,\partial_a(x_{p-1}))$ are related by the following expression,

$$\begin{pmatrix} 0 & \alpha_{p-1} & \ldots & \alpha_1\\ \alpha_1 & 0 & \ldots & \alpha_2\\ \alpha_2 & \alpha_1 & \ldots & \alpha_3\\ \ldots & \ldots & \ldots & \ldots\\ \alpha_{p-2} & \alpha_{p-3} & \ldots & \alpha_{p-1}\\ \alpha_{p-1} & \alpha_{p-2} & \ldots & 0 \end{pmatrix}\begin{pmatrix}\partial_b(x_0)\\ \partial_b(x_1)\\ \partial_b(x_2)\\ \ldots\\ \partial_b(x_{p-2})\\ \partial_b(x_{p-1})\end{pmatrix}=\begin{pmatrix}\partial_a(x_0)\\ \partial_a(x_1)\\ \partial_a(x_2)\\ \ldots\\ \partial_a(x_{p-2})\\ \partial_a(x_{p-1})\end{pmatrix}.$$

For x to be in $\mathrm{st}_{G_{\bar\alpha}}(2)$, it is necessary and sufficient that $\partial_a(x_i)=0$ for all $i=0,1,\ldots,p-1$. Thus, $x\in\mathrm{st}_{G_{\bar\alpha}}(2)$ if and only if vector $(\partial_b(x_0),\ldots,\partial_b(x_{p-1}))$ is a solution to the homogenous system of linear equations with matrix

$$A=\begin{pmatrix} 0 & \alpha_{p-1} & \ldots & \alpha_1\\ \alpha_1 & 0 & \ldots & \alpha_2\\ \alpha_2 & \alpha_1 & \ldots & \alpha_3\\ \ldots & \ldots & \ldots & \ldots\\ \alpha_{p-2} & \alpha_{p-3} & \ldots & \alpha_{p-1}\\ \alpha_{p-1} & \alpha_{p-2} & \ldots & 0 \end{pmatrix}$$

over GF(p).

Assume that $x \in \mathrm{st}_{G_{\bar{\alpha}}}(2)$; let us show that $x \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. In order to do that, it is sufficient to show that if a vector $(\beta_0, \ldots, \beta_{p-1})$ is a solution to the aforementioned system, then $\sum_{i=0}^{p-1} \beta_i = 0$.

Notice that the rank of A is at least 1, i.e. the dimension of the solution space is no greater than $p-1$. Consider a vector $(\beta_0, \ldots, \beta_{p-1})$ in the solution space. Since each row of A is obtained by a cyclic shift from any other row, the cyclic space generated by $(\beta_0, \ldots, \beta_{p-1})$ is in the solution space as well. On the other hand, the dimension of this cyclic space is equal to $p - \mathrm{GCD}(x^p - 1, f(x))$, where $f(x) = \beta_0 + \beta_1 x + \ldots + \beta_{p-1} x^{p-1}$.

Since p is the characteristic of GF(p), $x^p - 1 = (x-1)^p$. Suppose that $\sum_{i=0}^{p-1} \beta_i \neq 0$. Then 1 is not a root of $f(x)$, hence $\mathrm{GCD}(x^p - 1, f(x)) = 1$. Thus, the dimension of the cyclic space generated by $(\beta_0, \ldots, \beta_{p-1})$ is p. But the dimension of the solution space, which contains the cyclic space, does not exceed $p-1$. This contradiction proves that for any vector from the solution space the sum of its coordinates is equal to zero.

In particular, the sum of coordinates of $(\partial_b(x_0), \ldots, \partial_b(x_{p-1}))$ is equal to zero. Hence, $\partial_b(x) = 0$, which, given that x is in $\mathrm{st}_{G_{\bar{\alpha}}}(1)$, implies $x \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. ∎

Proposition 4.2. *Any periodic GGS-group with a nonsymmetric accompanying vector has congruence property.*

Proof. We need to prove that every finite index subgroup of each $G_{\bar{\alpha}}$ is a congruence subgroup. This is equivalent to proving that every normal finite index subgroup is a congruence subgroup, and since all $G_{\bar{\alpha}}$ are just infinite, we can prove that the normal closure X of any element $x \in G_{\bar{\alpha}}$ is a congruence subgroup.

By Proposition 4.1, there is a vertex u such that $\mathrm{st}_X(u)_u = G_{\bar{\alpha}}$. Notice that $[\mathrm{st}_X(u), \mathrm{rist}_{G_{\bar{\alpha}}}(u)] \leq \mathrm{rist}_X(u)$. By Corollary 2.1, $\mathrm{rist}_{G_{\bar{\alpha}}}(u)_u \geq [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. Hence, $\mathrm{rist}_X(u)_u \geq \varphi_u([\mathrm{st}_X(u), \mathrm{rist}_{G_{\bar{\alpha}}}(u)]) \geq \gamma_3(G_{\bar{\alpha}})$. By Proposition 2.3, $\gamma_3(G_{\bar{\alpha}}) \geq \Phi_1^{-1}(\underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p})$. Hence, there is inclusion

$$X \geq \Phi_{|u|+1}^{-1}(\underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p^{|u|+1}}).$$

However, by Lemma 4.2 $[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$ contains $\mathrm{st}_{G_{\bar{\alpha}}}(2)$. Therefore, X contains $\Phi_{|u|+1}^{-1}(\underbrace{\mathrm{st}_{G_{\bar{\alpha}}}(2) \times \ldots \times \mathrm{st}_{G_{\bar{\alpha}}}(2)}_{p^{|u|+1}}) = \mathrm{st}_{G_{\bar{\alpha}}}(|u|+3)$. Theorem is proven. ∎

4.3 Congruence subgroups of GGS-groups with symmetric accompanying vector

Lemma 4.3. Let $G_{\bar{\alpha}}$ be a periodic GGS-group with a symmetric accompanying vector. Then $\gamma_3(G_{\bar{\alpha}})$ contains principal congruence subgroup $\mathrm{st}_{G_{\bar{\alpha}}}(3)$.

Proof. Let us show first that $|[G_{\bar\alpha}, G_{\bar\alpha}] : \gamma_3(G_{\bar\alpha})| = p$. Since $G_{\bar\alpha}$ is generated by two elements a and b of finite order, it is clear that the factorgroup of $[G_{\bar\alpha}, G_{\bar\alpha}]$ by $\gamma_3(G_{\bar\alpha})$ is a finite cyclic group. The coset generating it is $[a, b][G_{\bar\alpha}, G_{\bar\alpha}]$.

Let us prove that $[a, b]^p$ is in $\gamma_3(G_{\bar\alpha})$. Indeed,

$$1 = [a^p, b] = [a, b]^p (\operatorname{mod} \gamma_3(G_{\bar\alpha}).$$

Thus, $|[G_{\bar\alpha}, G_{\bar\alpha}] : \gamma_3(G_{\bar\alpha})|$ equals either p or 1.

Let us show now that $[a, b]^p$ is not in $\gamma_3(G_{\bar\alpha})$. Consider map $\varepsilon : \mathrm{st}_{G_{\bar\alpha}}(1) \to \underbrace{\mathrm{GF}(p) \oplus \ldots \oplus \mathrm{GF}(p)}_{p}$ acting via the rule, $\varepsilon(x) = (\partial_b(\varphi_0(x)), \partial_b(\varphi_1(x)), \ldots, \partial_b(\varphi_{p-1}(x)))$.

Let L_1 denote $\varepsilon([G_{\bar\alpha}, G_{\bar\alpha}])$ and L_2 denote $\varepsilon(\gamma_3(G_{\bar\alpha}))$. Since $[G_{\bar\alpha}, G_{\bar\alpha}]$ is the normal closure of $[a, b]$ in $G_{\bar\alpha}$ and

$$\Phi_1([a, b]) = (b^{-1}a^{\alpha_1}, a^{\alpha_2-\alpha_1}, \ldots, a^{\alpha_{p-1}-\alpha_{p-2}}, a^{-\alpha_{p-1}}b),$$

L_1 is the cyclic space generated by vector $(1, -1, 0, \ldots, 0)$. Since $\gamma_3(G_{\bar\alpha})$ is the normal closure of the set $\{[[a, b], a], [[a, b], b]\}$ and

$$\Phi_1([[a, b], a]) = (a^{-\alpha_1}ba^{-\alpha_{p-1}}b, a^{\alpha_1-\alpha_2}b^{-1}a^{\alpha_1}, a^{\alpha_1-\alpha_3}, \ldots, a^{\alpha_{p-3}-\alpha_{p-1}}, b^{-1}a^{\alpha_2}),$$

$$\Phi_1([[a, b], b]) = ([b^{-1}a^{\alpha_1}, a^{\alpha_1}], 1, \ldots, 1, [a^{-\alpha_1}b, b]),$$

L_2 is the cyclic space generated by vector $(-1, 2, -1, 0, \ldots, 0)$.

Thus, the dimension of L_1 equals $p - \deg f(x)$, where $f(x) = \mathrm{GCD}(x^p - 1, x - 1)$. Since p is equal to the charasteric of $\mathrm{GF}(p)$, $x^p - 1 = (x-1)^p$. Thus, $\deg f(x) = 1$ and $\dim L_1 = p-1$.

Similarly, the dimension of L_2 equals $p - \deg g(x)$, where $g(x) = \mathrm{GCD}(x^p - 1, x^2 - 2x + 1)$. Since $x^2 - 2x + 1 = (x-1)^2$, then $\deg g(x) = 2$ and $\dim L_2 = p - 2$. Hence $|[G_{\bar\alpha}, G_{\bar\alpha}] : \gamma_3(G_{\bar\alpha})|$ is at least p. By the above, it has to be equal to p.

What we in fact have just proven is that every element of $[G_{\bar\alpha}, G_{\bar\alpha}] \setminus \gamma_3(G_{\bar\alpha})$ has a nontrivial image under ε. In other words, for every x in $[G_{\bar\alpha}, G_{\bar\alpha}] \setminus \gamma_3(G_{\bar\alpha})$ there is a vertex u of length 1 such that $\varphi_u(x)$ is not in $[G_{\bar\alpha}, G_{\bar\alpha}]$.

Since by Lemma 4.2 $[G_{\bar\alpha}, G_{\bar\alpha}] \geq \mathrm{st}_{G_{\bar\alpha}}(2)$, any nontrivial coset of $[G_{\bar\alpha}, G_{\bar\alpha}]$ has empty intersection with $\mathrm{st}_{G_{\bar\alpha}}(2)$. Thus, any coset of $\gamma_3(G_{\bar\alpha})$ which is not in $[G_{\bar\alpha}, G_{\bar\alpha}]$ has empty intersection with $\mathrm{st}_{G_{\bar\alpha}}(2)$, and nontrivial coset of $\gamma_3(G_{\bar\alpha})$ which is in $[G_{\bar\alpha}, G_{\bar\alpha}]$ has empty intersection with $\mathrm{st}_{G_{\bar\alpha}}(3)$. Indeed, as has been noted above, for any element x of such coset there is a vertex u of length 1 such that $\varphi_u(x)$ is not in $[G_{\bar\alpha}, G_{\bar\alpha}]$ and hence is not in $\mathrm{st}_{G_{\bar\alpha}}(2)$. Then x is not in $\mathrm{st}_{G_{\bar\alpha}}(3)$. This proves the lemma. ∎

Lemma 4.4. Let $G_{\bar\alpha}$ be a periodic GGS-group with a symmetric accompanying vector. Then

$$\Phi_1(\gamma_4(G_{\bar\alpha})) \geq \underbrace{\gamma_3(G_{\bar\alpha}) \times \ldots \times \gamma_3(G_{\bar\alpha})}_{p}.$$

Proof. 1. By definition of a symmetric vector, $\alpha_i = \alpha_{p-i}$. Fix an i such that $\alpha_{p-i} \neq 0$ and consider $[[b, b^{a^{p-i}}], bb^{-a^{p-2i}}]$. We have

$$\Phi_1(b^{a^{p-i}}) = (a^{\alpha_{i+1}}, \dots, a^{\alpha_{p-1}}, \underbrace{b}_{p-i}, a^{\alpha_1}, \dots, a^{\alpha_i}),$$

$$\Phi_1([b, b^{a^{p-i}}]) = (\underbrace{1, \dots, 1}_{p-i-1}, [a^{\alpha_{p-i}}, b], \underbrace{1, \dots, 1}_{i-1}, [b, a^{\alpha_i}]),$$

$$\Phi_1(b^{a^{p-2i}}) = (a^{\alpha_{2i+1}}, \dots, a^{\alpha_{p-1}}, \underbrace{b}_{p-2i}, a^{\alpha_1}, \dots, \underbrace{a^{\alpha_i}}_{p-i}, a^{\alpha_i+1}, \dots, a^{\alpha_{2i}}).$$

Since $\bar{\alpha}$ is symmetric, we have

$$\Phi_1(bb^{-a^{p-2i}}) = (\underbrace{*, \dots, *}_{p-i-1}, 1, \underbrace{*, \dots, *}_{i-1}, ba^{-\alpha_{2i}}),$$

where stars stand for some elements of $G_{\bar{\alpha}}$ (we don't care to know which). Thus,

$$\Phi_1([[b, b^{a^{p-i}}], bb^{-a^{p-2i}}]) = (1, \dots, 1, [[b, a^{\alpha_i}], ba^{-\alpha_{2i}}]).$$

Put $x = [[b, b^{a^{p-i}}], bb^{-a^{p-2i}}]$.

2. Suppose that there is j such that $\alpha_j = 0$. Suppose also that there is i such that $\alpha_i = \alpha_{p-i} \neq 0$ and $\alpha_{p-2i} = \alpha_{2i} = 0$. Then

$$\Phi_1([[b, b^{a^{p-i}}], b^{a^{p-2i}}]) = (\underbrace{1, \dots, 1}_{p-i-1}, [[a^{\alpha_i}, b], a^{\alpha_i}], \underbrace{1, \dots, 1}_{i-1}, [[b, a^{\alpha_i}], a^{\alpha_{2i}}]) =$$

$$= (\underbrace{1, \dots, 1}_{p-i-1}, [[a^{\alpha_i}, b], a^{\alpha_i}], \underbrace{1, \dots, 1}_{i}).$$

We now show that such i can always be found (under our assumption that one of numbers α_j is equal to 0). It follows from a Fermat's theorem that the remainders modulo p of numbers 2, 2^2, …, 2^{p-1} are all the numbers 1, 2, …, $p-1$. Therefore, if we start with an arbitrary l such that $\alpha_l = \alpha_{p-l} \neq 0$, the numbers α_{p-2l}, $\alpha_{p-2^2 l}$, …, $\alpha_{p-2^{p-1}l}$ are all the numbers α_1, α_2, …, α_{p-1} in a different order. Hence there is a $k \in \{1, 2, \dots, p-1\}$ such that $\alpha_{p-2^{k-1}l} \neq 0$ and $\alpha_{p-2^k l} = 0 = \alpha_{2^k l}$. Then the desired i is $2^{k-1}l$, and we have

$$\Phi_1([[b, b^{a^{p-i}}], b^{a^{p-2i}}]) = (\underbrace{1, \dots, 1}_{p-i}, [[a^{\alpha_{p-i}}, b], a^{\alpha_{p-i}}], \underbrace{1, \dots, 1}_{i-1}, [[b, a^{\alpha_i}], a^{\alpha_{2i}}]) =$$

$$((\underbrace{1, \dots, 1}_{p-i}, [[a^{\alpha_{p-i}}, b], a^{\alpha_{p-i}}], \underbrace{1, \dots, 1}_{i}).$$

Put $y = [[b, b^{a^{p-i}}], b^{a^{p-2i}}]$ for this i. Notice that since $\gamma_3(G_{\bar{\alpha}})$ coincides with the normal closure of $[[b, a^{\alpha_i}], b]$ and $[[a^{\alpha_i}, b], a^{\alpha_i}]$, the normal closure of x, y coincides with $\Phi_1^{-1}(\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p})$.

3. Suppose now that for any i $\alpha_i \neq 0$. Consider an arbitrary element of the form,

$$\Phi_1(bb^{-a^{p-2j}}) = (\underbrace{*,\ldots,*}_{p-j-1}, 1, \underbrace{*,\ldots,*}_{j-1}, ba^{-\alpha_{2j}}),$$

where stars stand for some elements of $G_{\bar{\alpha}}$. Take an $i < j$. Then

$$\Phi_1((bb^{-a^{p-2j}})^{a^{j-i}}) = (\underbrace{*,\ldots,*}_{p-i-1}, 1, \underbrace{*,\ldots,*}_{i-1}, a^{\alpha_{p-j+i}-\alpha_{p-j-i}}).$$

We show now that there are i and j such that $\alpha_{p-j+i} \neq \alpha_{p-j-i}$. Assume the reverse and consider $i = n$, $j = n+1$, $1 \leq n < \frac{p-1}{2}$. Then $\alpha_{p-1} = \alpha_{p-1-2n}$, i.e. $\alpha_{p-1} = \alpha_{p-3} = \ldots = \alpha_2$. Now consider $i = n$, $j = n+2$, $1 \leq n < \frac{p-3}{2}$. Then $\alpha_{p-2} = \alpha_{p-4} = \ldots = \alpha_1$. On the other hand, $\bar{\alpha}$ is symmetric, so we have $\alpha_{\frac{p-1}{2}} = \alpha_{\frac{p+1}{2}}$. As numbers $\frac{p+1}{2}$ and $\frac{p+1}{2}$ are of different parity, this means that all α_i are equal to each other, i.e. $\bar{\alpha}$ is diagonal. However, it was shown in the proof of Proposition 2.3 that in such a case $G_{\bar{\alpha}}$ cannot be periodic.

Thus, we can take i and j such that $\alpha_{p-j+i} \neq \alpha_{p-j-i}$. Then

$$\Phi_1([[b, b^{a^{p-i}}], (bb^{-a^{p-2j}})^{a^{j-i}}]) = (1, \ldots, 1, [[b, a^{\alpha_i}], a^{\alpha_{p-j+i}-\alpha_{p-j-i}}]),$$

and we put $y = [[b, b^{a^{p-i}}], (bb^{-a^{p-2j}})^{a^{j-i}}]$ for these i and j.

4. Thus, in all possible cases we have found two elements x, y such that their normal closure coincides with $\Phi_1^{-1}(\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p})$. It remains to notice that x, y are actually in $\gamma_4(G_{\bar{\alpha}})$. ∎

We are ready to establish the main result of the section.

Proposition 4.3. *Any periodic GGS-group with a symmetric accompanying vector has congruence property.*

Proof. As in the nonsymmetric case, we need to prove that every normal finite index subgroup of a $G_{\bar{\alpha}}$ is a congruence subgroup. This is equivalent to proving that the normal closure X of any element $x \in G_{\bar{\alpha}}$ is a congruence subgroup.

By Proposition 4.1, there is a vertex u such that $\mathrm{st}_X(u)_u = G_{\bar{\alpha}}$. Again, $[\mathrm{st}_X(u), \mathrm{rist}_{G_{\bar{\alpha}}}(u)] \leq \mathrm{rist}_X(u)$. By Corollary 2.1, $\mathrm{rist}_{G_{\bar{\alpha}}}(u)_u \geq \gamma_3(G_{\bar{\alpha}})$. Hence, $\mathrm{rist}_X(u)_u \geq \varphi_u([\mathrm{st}_X(u), \mathrm{rist}_{G_{\bar{\alpha}}}(u)]) \geq \gamma_4(G_{\bar{\alpha}})$. By Lemma 4.4, $\gamma_4(G_{\bar{\alpha}}) \geq \Phi_1^{-1}(\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p})$. Hence, there is inclusion

$$X \geq \Phi_{|u|+1}^{-1}(\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p^{|u|+1}}).$$

However, by Lemma 4.3 $\gamma_3(G_{\bar{\alpha}})$ contains $\mathrm{st}_{G_{\bar{\alpha}}}(3)$. Therefore, X contains $\Phi_{|u|+1}^{-1}(\underbrace{\mathrm{st}_{G_{\bar{\alpha}}}(3) \times \ldots \times \mathrm{st}_{G_{\bar{\alpha}}}(3)}_{p^{|u|+1}}) = \mathrm{st}_{G_{\bar{\alpha}}}(|u|+4)$. The theorem is proven. ∎

Combining the statements of Theorems 4.2 and 4.3, we get

Theorem 4.1. *Any periodic GGS-group has congruence property.*

5 Groups without congruence property: EGS-groups

In this section we establish the important result on existence of branch p-groups without the congruence property.

Lemma 5.1. Let $\Gamma_{\bar{\alpha}}$ be a EGS-group with nonsymmetric accompanying vector. Then for any natural number n there is an element t_n of coset $b[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ such that $t_n \equiv c(\operatorname{mod} \operatorname{st}_{\Gamma_{\bar{\alpha}}}(n))$.

Proof. Put $t_1 = b$ and $t_2 = b^a$. Suppose $n > 1$ and an element t_n with the required property has already been defined. Notice that $b^{-1}t_n$ is in $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. Then by Proposition 2.4 automorphism $x_{n+1} = \Phi_1^{-1}((b^{-1}t_n, \underbrace{1, \ldots, 1}_{p-1}))$ is in $\Gamma_{\bar{\alpha}}$. Put $t_{n+1} = b^a x_{n+1}$.

We have

$$\Phi_1(t_{n+1}) = (t_n, a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}).$$

Thus,

$$\Phi_1(c^{-1}t_{n+1}) = (c^{-1}t_n, 1, \ldots, 1).$$

Since by assumption $c^{-1}t_n$ is in $\operatorname{st}_{\Gamma_{\bar{\alpha}}}(n)$, $c^{-1}t_{n+1}$ is in $\operatorname{st}_{\Gamma_{\bar{\alpha}}}(n+1)$.

Finally, it follows from the proof of Proposition 2.4 that x_{n+1} is in $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. Therefore t_{n+1} is in $b[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. ■

Corollary 5.1. *The derived subgroup $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ is not a congruence subgroup.*

Proof. It follows from Lemma 5.1 that for every n $c^{-1}t_n$ is in $\operatorname{st}_{\Gamma_{\bar{\alpha}}}(n)$. However, by Lemma 1.4 it is not in $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. Therefore, $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ does not contain $\operatorname{st}_{\Gamma_{\bar{\alpha}}}(n)$ for any n. ■

Thus, we obtain the following important, and surprising, result.

Theorem 5.1. *No periodic EGS-group with nonsymmetric accompanying vector has congruence property.*

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