MIXED HODGE STRUCTURES

ON THE HOMOTOPY OF LINKS

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MPI/SFB 85-37

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The general purpose of this paper is to use mixed Hodge theory to study the homotopy of a complex algebraic variety in the neighborhood of a singular point, or more generally, in the neighborhood of a subvariety. An important application will be to find topological restrictions on the links of isolated singular points.

First let us describe the applications of this theory. Recall that the link of an isolated singularity in an n-dimensional variety is a real (2n-1)-manifold.

Theorem 6.1. Let L be the link of an isolated singularity of an n-dimensional variety. If $s,t < n$ and $s+t \geq n$, then the cup product

$$H^s(L;Q)\otimes H^t(L;Q) \to H^{s+t}(L;Q)$$

vanishes.

This result shows that there are restrictions on the topology of such links and is, we believe, the first result of this type in dimensions $n > 2$. For example, any manifold L of the form $K\times M\times N$ where $\dim K < n$, $\dim M < n$, $\dim K + \dim M \geq n$ and $\dim K + \dim M + \dim M = 2n-1$ cannot occur as such a link. This generalizes a result from [Sullivan Topology] for n=2. The theorem is proved by showing that $H^\cdot(L)$ has a mixed Hodge structure which is preserved by the cup product, and by finding non-trivial restrictions on the weight filtration of $H^\cdot(L)$.

The next result is an example of a manifold M whose cohomology satisfies the restriction of Theorem 6.1 and hence could have a weight filtration, but whose rational homotopy type could not have a weight filtration compatible with that on its homology.

Theorem 6.3. There is a smooth simply-connected closed 11-manifold M with the rational cohomology ring of

$$2(S^2 \times S^9) \# (S^5 \times S^6) \# 2(S^4 \times S^7)$$

that does not have the homotopy type, or even the rational homotopy type, of the link of an isolated singularity of a six-dimensional variety.

Finally, a result on three-dimensional link complements:

Theorem 6.2. If K is the link in the three-sphere of an isolated singular point of a plane algebraic curve and $L = S^3 - K$, then L is a formal space (in the sense of Sullivan).

This result raises the question of whether the complement of any compound torus link in the three-sphere is a formal space.

Next let us make the notion of link more precise: Let X be a complex projective variety, and let Z be a closed subvariety with the singular locus of X contained in Z. Let T be a "sufficiently nice" neighborhood of Z in X. (For the precise definition, see Section 1.) The link of Z in X is by definition

$$L = L(X, Z) = T - Z.$$

This is the ordinary notion of link as in PL topology, for example. If X has complex dimension n, then L is homotopy equivalent to a real compact $(2n-1)$-manifold with boundary.

More generally, the applications and proof of the main result require that we consider not just links but the complement of one link in another: Let X, Z, and T be as above, let Z' be another closed subvariety, let $Y = Z \cup Z'$ and suppose that the singular locus of X is contained in Y. The link of Z in X (with Y removed) is by definition

$$L = L(X, Y, Z) = T - Y.$$

Thus $L(X, Y, Z)$ is the complement of $Z' \cap L(X, Z)$ in $L(X,Z)$.

The main technical result, Theorem 5.1.1, may be loosely stated as follows: If $L = L(X, Y, Z)$ is as above, then:

(i). For all k, $H^k(L)$ has a real mixed Hodge structure.

(ii). The cup product of $H^{\cdot}(L)$ is a morphism of mixed Hodge structures.

(iii). The real homotopy type of L has a mixed Hodge structure.

Part (i) of this theorem was already proved by a number of authors; it was proved in [Durfee, Duke], for instance, by using a Mayer-Vietoris construction. Unfortunately the algebra structure is lost in this construction, so the other two parts of the theorem cannot be proved this way. Although Deligne's original construction of a complex for computing the mixed Hodge structure on the cohomology of an algebraic variety yields a differential graded algebra, the algebra is not commutative, so that one cannot conclude that the homotopy of a variety has a mixed Hodge structure. In [Morgan], it was shown in the case of smooth varieties that the mixed Hodge structure passes to rational homotopy. In [Hain DHT] this result was extended to all varieties. In this paper, we use simplicial techniques to get a mixed Hodge structure on the real homotopy type of the link L. In doing so, we reprove the first result above. For technical simplicity, we work only with real mixed Hodge structures in this paper, even though there is no obstruction to working over the rational numbers.

In Section 1, the notion of link is defined more carefully, and its basic properties are described. Sections 2 and 3 contain the machinery for the rest of the paper. This machinery is necessary for dealing efficiently with the formalisms of mixed Hodge complexes, particularly the quasi-isomorphisms. Section 2 describes simplicial objects in various categories, principally the categories of smooth manifolds and algebraic varieties. The two basic examples are the simplicial objects associated to a finite open cover of a smooth manifold, and the simplicial object associated to a divisor with normal crossings expressed as a union of its irreducible components. Next, cosimplicial objects are described; the basic example is a cosimplicial differential graded algebra. The de Rham functor is defined; this takes a cosimplicial chain complex to a cosimplicial chain complex by taking collections of "compatible forms" in the sense of Thom or Sullivan. For example, the de Rham functor of the complex of smooth forms on a simplicial manifold $X.$ is an algebra whose cohomology is is isomorphic to the cohomology of the geometric realization $|X.|$, which is naturally isomorphic to the cohomology of $X.$

Section 3 describes mixed Hodge complexes. According to [Deligne IHES], a mixed Hodge complex consists of three objects in various derived categories. We do not adopt this approach here, since we need concrete objects and morphisms in order to apply the de Rham functor. Instead we use global mixed Hodge complexes, that is, a chain complex over the real numbers with a weight filtration and a chain complex over the complex numbers with Hodge and weight filtrations. With this definition, mixed Hodge complexes form a category in an obvious way. A multiplicative mixed Hodge complex is defined to be a mixed Hodge complex which is a differential graded algebra as well, and a de Rham mixed Hodge complex for a topological space X is defined to be one whose real part is quasi-isomorphic with the de Rham complex of the singular simplices of X. These also form categories. Hence we may consider cosimplicial versions of the three types of mixed Hodge complexes mentioned above. The first main result , which comes from [Hain DHT], suitably rephrased, is that the de Rham complex of a cosimplicial multiplicative mixed Hodge complex is a multiplicative mixed Hodge complex. The second main result, from [Hain DHT] or [Morgan], is that if a space has a de Rham mixed Hodge complex, then the real homotopy type of the space has a mixed Hodge structure.

A de Rham mixed Hodge complex for a link L as above is then constructed in Section 4 in the case where X is smooth and projective, $Y = D$ is a divisor with normal crossings in X and $Z = E$ consists of components of D. When E has one component, the obvious way to construct a de Rham mixed Hodge complex for L is to take the log complex $E^{\cdot}(X \log D+E)$ and localize it along E. This is done in section 4.2, Proposition 4.2.3 being the main result. The case of arbitrary E is done in section 4.3, the main technical result being Proposition 4.3.1. Here is an informal description of the construction: Let $E = E_1 \cup \ldots \cup E_s$ be the irreducible components of E. Choose neighborhoods T_i of E_i and let $T_i^* = T_i - D$. By 4.2, each T_i^* and all the intersections of the T_i^* have de Rham mixed Hodge complexes. The link L is homotopy equivalent to $T^* = \cup T_i^*$, and the space T^* can be replaced by the simplicial object $T^*_{\cdot}$ associated to this cover. Each of the components of this simplicial manifold then has a de Rham mixed Hodge complex, so that the whole forms a cosimplicial de

Rham mixed Hodge complex. Applying the de Rham functor from Section 2 then gives a de Rham mixed Hodge complex for the geometric realization $|T^*.|$, which in this case is homotopy equivalent to T^*.

In Section 5 the main results on the existence of a mixed Hodge structure on the real homotopy type of the link L are proved by using resolution of singularities to reduce to the case of Section 4. This section also contains the following results: Let $W = Z \cap Z'$. Then the obviously defined map $L \to X - Y$, and the correspondence defined by $L \to Z - W$ (the composite of the map $L \to T - Z'$ with an inverse to the homotopy equivalence $T - Z' \leftarrow Z - W$) induce mixed Hodge structure morphisms on cohomology and real homotopy. As a corollary, it is shown that real mixed Hodge structure on the cohomology of L obtained here agrees with that of [Durfee Duke]. Finally, we prove that Alexander and Lefschetz duality preserve mixed Hodge structures.

The first author spent a very pleasant year working on this material at the Max Planck Institute for Mathematics, Bonn, BRD, and he thanks them for their support. He also paid shorter visits to the University of Utrecht, the Netherlands, and the Institute des Hautes Etudes Scientifique, France, and he thanks them for their support as well. The second author would like to thank Jim Carlson for helpful discussions on asymptotic Hodge theory that led to some of the ideas in this paper.

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0. NOTATION.

0.1 If A and B are subsets of a set X, we let

$$A - B = \{x : x \text{ is an element of } A \text{ but not of } B\}.$$

0.2 Two chain complexes A and B are quasi-isomorphic if there is a sequence of chain complexes and maps

$$A \to C_1 \leftarrow C_2 \to \dots \to C_n \leftarrow B$$

such that the composite is a cohomology isomorphism. We write

$$A \leftrightarrow B.$$

0.3. Suppose we are given morphisms and quasi-isomorphisms as in the following diagram:

$$\begin{array}{ccc} A & \longleftrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longleftrightarrow & B' \end{array}$$

This square is an abbreviated form of

$$\begin{array}{ccccccccc} A & \to & C_1 & \leftarrow \ldots \to & C_n & \leftarrow & B \\ \downarrow & & & & & & \downarrow \\ A' & \to & C'_1 & \leftarrow \ldots \to & C'_n & \leftarrow & B \end{array}$$

(Without loss of generality, the chains are of the same length.) A <u>congruence</u> of such a square is a collection of maps

$$h_i \colon C_i \to C'_i$$

giving commmutative diagrams, for $i = 1,..,n$.

0.4. A <u>differential graded algebra</u> is defined, for example, in [Halperin]. In this paper, all differential graded algebras will be assumed to be (graded) commutative. A <u>filtered differential graded algebra</u> is a differential graded algebra with a filtration which is preserved by the differential and the product. The notion of quasi-isomorphism and congruence is extended to filtered chain complexes [Deligne 1.3.6], differential graded algebras and filtered differential graded algebras in the obvious way.

0.5. Here is some of the notation used in this paper:

$L(X, Z)$ is the link of Z in X (Section 1)

$L(X, Y, Z)$ is the link of Z in X with Y removed (Section 1)

$D(C)$ is the de Rham complex of the cosimplicial differential graded algebra C (2.3)

$F_R(X)$ (respectively $F_C(X)$) denotes real (respectively complex) valued functions on a smooth manifold X. The symbol $F(X)$ means $F_R(X)$.

$E_R(X)$ (respectively $E_C(X)$) is the algebra of real (respectively complex) valued forms on a smooth manifold X. The symbol $E(X)$ is shorthand for $E_R(X)$.

$A(X)$ is the Thom-deRham complex of a topological space X (2.5)

$E_R(X \log D)$ (respectively $E_C(X \log D)$) is the smooth real (respectively complex) log complex for $X - D$ (4.1)

$K(X)$ is the mixed Hodge complex for the smooth projective variety X (3.2.1)

$K(X, D)$ is the mixed Hodge complex for $X - D$, where D is a divisor with normal crossings in X (4.1.2)

$K(X, D, D_I)$ is the mixed Hodge complex for the link $L(X, D, D_I)$ (4.2.2)

1. LINKS.

Let X be a complex projective variety, and let Z and Z' be closed (and hence compact) subvarieties. Let

$$Y = Z \cup Z' \quad \text{and} \quad W = Z \cap Z'.$$

(The emphasis will be on the triple $X \supset Y \supset Z$.) Assume that the singular locus of X is contained in Y. Let T be a neighborhood of Z in X with the property that there is a sequence of neighborhoods

$$T = T_0 \supset T_1 \supset \ldots \supset Z$$

with $\cap T_i = Z$, and such that $Z - W$ is a strong deformation retraction of $T_i - Z'$, for all i. The _link of Z in X with Y removed_ is by definition

$$L = L(X,Y,Z) = T - Y.$$

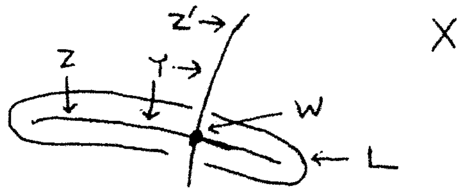
When Z' is the empty set (so that $Y = Z$), we let

$$L(X,Z) = L(X,Z,Z).$$

We sometimes write

$$T(X,Y,Z)$$

for a neighborhood T as above. Here is a schematic picture:



Neighborhods T as above can be constructed by triangulation, or by using level sets of real polynomials as in [Durfee, Neighborhoods].

The homotopy type of L is independent of the choice of T. Thus L(X,Y,Z) as an isomorphim class of objects in the homotopy category depends only on X, Y and Z. (This follows from the usual nesting argument.)

If X has (complex) dimension n, then L(X, Y, Z) has the homotopy type of a real (2n-1)-manifold. This manifold is compact if Y = Z. In fact, L can always be chosen to be a manifold.

Let X, Y, Z and X_1, Y_1, Z_1 be as above. An algebraic map

$$f: X_1 \to X$$

with the properties $Y_1 = f^{-1}(Y)$ and $Z_1 = f^{-1}(Z)$ induces a map (well-defined in the homotopy category)

$$L(X_1,Y_1,Z_1) \to L(X,Y,Z).$$

If the map f gives an algebraic isomorphism of X_1-Y_1 to X-Y, then $f^{-1}(L(X,Y,Z)) \to L(X_1,Y_1,Z_1)$ is a homotopy equivalence.

If the map f is an inclusion, we say that $L(X_1,Y_1,Z_1)$ is a <u>sublink</u> of L(X,Y,Z). If $L(X_1,Z_1)$ is a sublink of L(X,Y), note that $Z_1 = X_1 \cap Z$, and hence that

$$L(X,Z) - L(X_1,Z_1) = L(X,X_1 \cup Z,Z)$$

(equality of topological spaces, with proper choice of neighborhood T). Thus our notion of "link" includes the complement of one link in another. Conversely, given X, Y, Z, Z' and W as in the beginning of this section, then

$$L(X,Y,Z) = L(X,Z) - L(Z',W).$$

Thus L(X,Y,Z) is the complement of one (absolute) link in another.

2. SIMPLICIAL AND COSIMPLICIAL OBJECTS

2.1. Simplicial objects.

Some of the basic objects in this paper are simplicial objects in various categories. The categories we will use most often are the categories of topological spaces, smooth manifolds and complex algebraic varieties. Let $\underline{\Delta}$ be the category whose objects are the finite ordinals $[n] = \{0,1,\ldots,n\}$ and whose maps are order-preserving functions. As usual, a simplicial object $X.$ in a category C is defined to be a contravariant functor from the category $\underline{\Delta}$ to C. Simplicial objects in a category C, with morphisms defined to be natural transformations, themselves form a category.

2.1.1 Example Let X be a topological space, and let $X = \cup_{i \in I} X_i$ be a cover of X by a finite collection of subspaces. Let $X.$ be the simplicial space defined as follows: For an n-simplex $\sigma = (i_0,\ldots,i_n) \in I^{n+1}$, let

$$X_\sigma = X_{i_0} \cap \ldots \cap X_{i_n}$$

and let

$$X_n = \coprod_{\sigma \in I^{n+1}} X_\sigma.$$

The face maps

$$d_i \colon X_n \to X_{n-1}$$

are induced by the natural inclusions

$$X_{i_0} \cap \ldots \cap X_{i_n} \to X_{i_0} \cap \ldots \cap X_{i_{i-1}} \cap X_{i_{i+1}} \cap \ldots \cap X_{i_n}$$

and the degeneracy maps

$$s_j \colon X_n \to X_{n+1}$$

are induced by the natural isomorphism

$$X_{i_0} \cap \ldots \cap X_{i_n} \to X_{i_0} \cap \ldots \cap X_{i_j} \cap X_{i_j} \cap \ldots \cap X_{i_n}.$$

For example, X could be a variety with normal crossings, covered by the union of its irreducible components.

2.2. Cosimplicial objects.

Similarly, a cosimplicial object in a category A (in our case, a category of chain complexes, differential graded algebras, mixed Hodge complexes, and so forth) is a covariant functor $\underline{\Delta} \to A$. One way of obtaining a cosimplicial object is to take a simplicial object X. in a category C, ie, a contravariant functor $\underline{\Delta} \to C$, together with another contravariant functor T: C → A , and compose them; this gives a cosimplicial object T(X.) in A. For example, let X. be a simplicial manifold and let E be the functor from manifolds to differential graded algebras which associates to a manifold its algebra of smooth real-valued forms. Then E(X.) is a cosimplicial differential graded algebra.

Two cosimplicial chain complexes A and B are quasi-isomorphic if there is a sequence of objects and morphisms

$$A \to C_1 \leftarrow C_2 \to \ldots \to C_n \leftarrow B$$

such that in each simplicial degree n the induced map $H^{\cdot}(A^n) \to H^{\cdot}(B^n)$ is an isomorphism. (This is a strong definition. A weaker definition would require only that the cohomology of the single complexes associated to the double complexes be isomorphic.) A filtered quasi-isomorphism is defined similarly.

2.3. The de Rham complex of a cosimplicial chain complex.

Let C be a cosimplicial real or complex chain complex. The value of C on the object [q] of $\underline{\Delta}$ is then a chain complex, and will be denoted C[q]. The p-th part of this chain complex will be denoted $C^p[q]$. The de Rham complex D(C) is by definition the chain complex defined as follows: $D^m = \oplus_{s+t=n} D^{st}$, where an element c of D^{st} is a sequence $(c_n)_{n=0,1,\ldots}$ with c_n an element of $C^s[n]\otimes E^t(\Delta^n)$ which satisfies the compatibility condition

$$(f_*\otimes id)c_n = (id\otimes |f|^*)c_m$$

as elements of $C^s[m]\otimes E^t(\Delta^n)$, for all $f: [n] \to [m]$ in $\underline{\Delta}$. (Here Δ^n is the standard n-simplex, $E^t(\Delta^n)$ is the algebra of real or complex smooth forms on Δ^n and $|f|: \Delta^n \to \Delta^m$ and $f_*: C^s[n] \to C^s[m]$ denote the induced maps.) The differentials in $C^{\cdot}[n]$ and $E^{\cdot}(\Delta^n)$

make $D^{\cdot\cdot}$ into a double complex, and the differential in D is the associated single differential. If C is a differential graded algebra, the products in $C^{\cdot}[n]$ and $E^{\cdot}(\Delta^n)$ define a product in $D^{\cdot\cdot}$ and hence in D. Thus D is a functor from the category of real or complex cosimplicial differential graded algebras to the category of differential graded algebras. In particular, it takes quasi-isomorphisms to quasi-isomorphisms. The same is true in the filtered categories [Hain DHT]. We will apply this construction most often when the cosimplicial differential graded algebra C is $E(X.)$, the algebra of forms on a simplicial manifold $X.$

Another way of obtaining a chain complex from the cosimplicial chain complex C is by taking the associated simple complex sC, defined by

$$(sC)^n = \oplus_{p+q=n} C^p[q]$$

with differential defined as usual [Deligne 5.1.9.1]. Integration then defines a quasi-isomorphism

$$D(C) \to sC.$$

(For the proof, see [Hain DHT 5.10].)

For example, let K be a simplicial set and let F be the contravarient functor from sets to k-vector spaces (k=R or C) which associates to a set X the vector space $\mathrm{Hom}(k, X)$. Then FK is a cosimplicial chain complex, sFK is the usual complex of simplicial cochains on K, and $D(FK)$ is Thom's algebra of "compatible forms" on K [Halperin 13.5], [Sullivan IHES].

2.4. Geometric realization. The geometric realization $|X.|$ of a simplicial object $X.$ in the category of topological spaces is defined as usual [Deligne 5.2.1.1]. This defines a functor from the category of simplicial topological spaces to the category of topological spaces. Now suppose that X is a topological space, and let $X = \cup X_i$ be a finite cover of X. Let $X.$ be the simplicial space associated to this cover as in (2.1.1). There is a natural map $|X.| \to X$ defined by collapsing all the simplexes. The following result is well known [Segal]:

2.4.1. Proposition. If $X.$ is the simplicial space defined above, then the map $|X.| \to X$

is a homotopy equivalence.

2.5. Real homotopy type.

Next, we need to make precise what is meant by the real homotopy type of a topological space X. (Similar remarks apply to the rational homotopy type; we only use the real homotopy type in this paper.) If X is a smooth manifold, its real homotopy type is given by its de Rham complex of real smooth forms. However, we need a concept which will work for more general spaces such as singular varieties or geometric realizations of simplicial varieties.

Let X be a topological space. The singular simplices $\Delta(X)$ on X forms a simplicial set. The usual singular cochain complex $S(X)$ of X is defined by

$$S(X) = sF\Delta(X)$$

where s and F are defined in (2.3). The Thom-deRham complex $A(X)$ of X is defined by

$$A(X) = DF\Delta(X).$$

This has the following properties [Halperin Section 15]:

1. A continuous map $f: X \to Y$ of topological spaces induces a differential graded algebra map $f^*: A(Y) \to A(X)$.
2. There is a quasi-isomorphism $A(X) \to S(X)$, and in particular, an isomorphism $H^{\cdot}(A(X)) \to H^{\cdot}(X;R)$ which preserves algebra structure.
3. If X is a smooth manifold, there is a natural (differential graded algebra) quasi-isomorphism of $A(X)$ with $E_R(X)$, the algebra of real forms on X.

2.5.1. Proposition. If $X.$ is a split simplicial space, then there is a natural quasi-isomorphism

$$D(A(X.)) \to A(|X.|).$$

This is proved in [Hain DHT].

3. MIXED HODGE COMPLEXES.

3.1. Mixed Hodge complexes

Recall that a mixed Hodge complex K consists of

(i) A filtered real chain complex $(K_R, W.)$, where $W.$ is an increasing filtration

(ii) A bifiltered complex chain complex $(K_C, W., F^\cdot)$, where $W.$ is increasing and $F^\cdot$ is decreasing, and a filtered quasi-isomorphism of $(K_C, W.)$ to $(K_R, W.)\otimes C$ such that

(a) the differential in the zero term of the spectral sequence of the filtration $W.$, i.e.

$$d_0: {}_WE_0{}^{pq} \to {}_WE_0{}^{p,q+1}$$

is strictly compatible with the filtration induced by F, and

(b) $F^\cdot$ induces a Hodge structure of pure weight q on ${}_WE_1{}^{pq}$.

Although a mixed Hodge complex can be defined over any ring between the integers and the real numbers [Deligne III.0.4], ours will always be over the real numbers R. Note that our mixed Hodge complexes are not objects in a derived category, as in [Deligne III], but are actual chain complexes. This makes it possible to define a morphism of mixed Hodge complexes in an obvious way: Let

$$K = ((K_R, W), (K_C, W, F)) \text{ and } K' = ((K'_R, W'), (K'_C, W', F'))$$

be mixed Hodge complexes. A morphism α: $K \to K'$ consists of chain maps α_R: $K_R \to K'_R$ and α_C: $K_C \to K'_C$, together with a congruence (0.3) of the square

$$\begin{array}{ccc} K_R\otimes C & \longleftrightarrow & K_C \\ \downarrow & & \downarrow \\ K'_R\otimes C & \longleftrightarrow & K'_C \end{array}$$

Thus mixed Hodge complexes form a category.

3.2 De Rham mixed Hodge complexes

A mixed Hodge complex K as above is multiplicative if K_R and K_C are (commutative, as always) differential graded algebras which are filtered (i.e., the products preserve the filtrations), and if the filtered quasi-isomorphism is a map of differential graded algebras. These also form a category in the obvious way.

Let X be a topological space, for example, a variety or the geometric realization of a simplicial variety. A de Rham mixed Hodge complex for X is a multiplicative mixed Hodge complex K together with a quasi-isomorphism of K_R to $A(X)$, the Thom-deRham algebra of singular forms on X (Section 2.5).

Let K_i be a de Rham mixed Hodge complex for X_i, for $i=1,2$. A morphism of de Rham mixed Hodge complexes consists of a morphism $\alpha\colon K_1 \to K_2$ of multiplicative mixed Hodge complexes, a continuous map $f\colon X_2 \to X_1$, and a congruence of the square

$$\begin{array}{ccc} K_{1R} & \longleftrightarrow & A(X_1) \\ \downarrow & & \downarrow \\ K_{2R} & \longleftrightarrow & A(X_2) \end{array}$$

Thus de Rham mixed Hodge complexes form a category whose objects are pairs (X,K) and whose morphisms are pairs (f,α).

3.2.1. Example. The standard mixed Hodge complex $K(X)$ for a smooth projective variety X which is defined by $K_R = E_R(X)$ with $0 = W_{-1} \subset W_0 = E_R(X)$, and $K_C = E_C(X)$ with W as before and F as usual, is a de Rham mixed Hodge complex for X. In fact, the correspondence $X \mapsto (X, K(X))$ defines a functor from the category of smooth projective varieties to the category of de Rham mixed Hodge complexes.

If K is a mixed Hodge complex, then the cohomology groups $H^n(K_R)$ have a mixed Hodge structure, for all n [Deligne 8.1.9]. Clearly if K is also multiplicative, then the induced product in the cohomology ring $H^\cdot(K_R)$ preserves both filtrations, ie, the cup product $H^\cdot(K_R)\otimes H^\cdot(K_R) \to H^\cdot(K_R)$ is a morphism of mixed Hodge structures. This proves the following result:

3.2.2. Proposition. Let X be a topological space. If K is a de Rham mixed Hodge complex for X, then $H^n(X)$ has a mixed Hodge structure, for all n, and cup products in $H^{\cdot}(X)$ preserve this mixed Hodge structure.

More important, and much more difficult, are results of Morgan and Hain which show that the mixed Hodge structure actually passes to real homotopy:

3.2.3. Theorem. Let X be a topological space and let K be a de Rham mixed Hodge complex for X.

(a) [Morgan 8.6,9.1]. If

$$\rho: M \to E(X)$$

is a minimal model (or a 1-minimal model) for the algebra of real-valued forms on X, then M has a family of mixed Hodge structures such that the product, differential and the cohomology isomorphism (respectively isomorphism on H^1, injection on H^2)

$$\rho^*: H(M) \to H(X)$$

are morphisms of mixed Hodge structures.

(b) [Hain DHT]. The real Lie algebra model L_X of X has a (not necessarily unique) mixed Hodge structure such that the bracket, differential and the isomorphism (to reduced homology)

$$L_X/[L_X, L_X] \cong \tilde{H}.(X)$$

are morphisms of mixed Hodge structures.

(c). In particular, the Malcev Lie algebra of the fundamental group of X has a mixed Hodge structure, and, when X is simply connected, the higher homotopy groups of X have a unique functorial mixed Hodge structure. In addition, Massey products preserve mixed Hodge structure.

Actually, Morgan works with a "mixed Hodge diagram", which is a more restricted notion than a de Rham mixed Hodge complex, but it is not hard to generalize his proof to the above situation. In addition, both Morgan and Hain find mixed Hodge structures over the rational numbers rather than the real numbers; we use the real numbers here for simplicity. The statement about Massey products is proved in [Durfee Notes]. Functoriality for the homotopy groups of non-simply connected spaces is more subtle, and will not be discussed in this paper. The following result is well known ([Deligne Weil 5.3.7] [Durfee Notes], [Hain DHT]):

3.2.4. Corollary. Let X be a topological space, and let K be a de Rham mixed Hodge complex for X. If $H^m(X)$ has a pure Hodge structure of weight m for all m (or km, for some fixed natural number k), then X is a formal space.

3.2.5. Proposition. Let K be a de Rham mixed Hodge complex for X and let L be a de Rham mixed Hodge complex for Y. Then $K\otimes L$ is a de Rham mixed Hodge complex for $X\times Y$.

The proof is straightforward.

3.2.6. Corollary. Under the above hypotheses, the cross product

$$H^\cdot(X)\otimes H^\cdot(Y) \to H^\cdot(X\times Y)$$

is a morphism of mixed Hodge structures.

Proof. The cross product can be defined by $a\times b = (p_1{}^*a)\cup(p_2{}^*b)$, where the p_i are the projections. ■

3.3. Cosimplicial mixed Hodge complexes.

A cosimiplical (multiplicative) mixed Hodge complex K is simply a cosimplicial object in the category of (multiplicative) mixed Hodge complexes It thus consists of a cosimplicial real filtered chain complex K_R, a cosimplicial complex bifiltered chain complex K_C, and a filtered quasi-isomorphism (2.2.1) $K_R\otimes C \leftrightarrow K_C$.

Deligne in [Deligne IHES 8.1.15] defines a functor from cosimplicial mixed Hodge complexes to mixed Hodge complexes by taking the associated single complex: Let K be a cosimplicial mixed Hodge complex. Its complex part K_C has a bigrading $K_C{}^p[n]$, where p is the differential degree and n is the simplicial degree. To this cosimplicial object K_C a bigraded cochain complex is associated in the standard way, and sK_C is defined to be its associated single complex. (See 2.3.) The weight filtration on sK_C is defined to be the "diagonal filtration"

$$W_n(sK_C) = \oplus_{p,q} W_{n+q}(K_C{}^p[q])$$

and the Hodge filtration is defined to be

$$F^r(sK_C) = \oplus_{p,q} F^r(K_C{}^p[q]).$$

The same is done for K_R. The object sK associated to K is then a mixed Hodge complex.

We will now describe how the de Rham functor of (2.3) takes a cosimplicial multiplicative mixed Hodge complex to a multiplicative mixed Hodge complex, thus providing an alternative to the preceeding construction when the complexes in question are multiplicative. Let K be a cosimplicial multiplicative mixed Hodge complex. Its real part K_R is then a cosimplicial differential graded algebra, so its de Rham complex $D(K_R)$ is a real differential graded algebra. Similarly, $D(K_C)$ is a complex differential graded algebra. Furthermore, the de Rham functor takes the filtered quasi-isomorphism $K_R \otimes C \leftrightarrow K_C$ into a filtered quasi-isomorphism $D(K_R) \otimes C \cong D(K_R \otimes C) \leftrightarrow D(K_C)$. Recall from (2.3) that the de Rham complex D has a bigrading D^{st}. Define filtrations on $D(K_R)$ and $D(K_C)$ by

$$W_m D(K_R) = \oplus_{t>0,\ s} D^{st}(W_{m+t} K_R)$$

and similarly for K_C, and

$$F^p D(K_C) = D(F^p K_C).$$

These filtrations are preserved by products. Hence $D(K_R)$ is a filtered differential graded algebra, $D(K_C)$ is a bifiltered differential graded algebra, and the above quasi-isomorphism is a filtered one.

3.3.1. Lemma [Hain DHT 5.6]. If K is a cosimplicial multiplicative mixed Hodge complex, then $D(K)$ is a multiplicative mixed Hodge complex. In fact, D is a functor between these categories.

3.3.2. Lemma [Hain DHT 5.10]. If K is a cosimplicial multiplicative mixed Hodge complex, then there is a natural quasi-isomorphism of mixed Hodge complexes $D(K) \to sK$.

3.4. Cosimplicial de Rham mixed Hodge complexes.

A cosimplicial de Rham mixed Hodge complex is of course just a cosimplicial object in the category of de Rham mixed Hodge complexes. It consists thus of a cosimplicial multiplicative mixed Hodge complex K, a simplicial space $X.$, and a quasi-isomorphism $K_R \leftrightarrow A(X.)$. The functor K of (3.2.1) from the category of smooth projective varieties to de Rham mixed Hodge complexes extends naturally to a functor from the category of simplicial smooth projective varieties to the category of cosimplicial de Rham mixed Hodge complexes.

3.4.1. Lemma. If K is a cosimplicial de Rham mixed Hodge complex for a simplicial manifold $X.$, then $D(K)$ is a de Rham mixed Hodge complex for the geometric realization $|X.|$ of $X.$.

Proof. The differential graded algebra $D(K)$ is a multiplicative mixed Hodge complex by (3.3.1). Furthermore, the de Rham functor takes the quasi-isomorphism $K_R \leftrightarrow A(X.)$ to a quasi-isomorphism $D(K_R) \leftrightarrow D(A(X.))$, and the latter is quasi-isomorphic to $A(|X.|)$ by (2.5.1). ■

3.4.2. Example. Let $E = \cup_{i\in I} E_i$ be a variety with normal crossings, where as usual the E_i

are smooth and projective. The simplicial space E. associated to the union $E = \cup E_i$ as in (2.1.1) is a simplicial object in the category of smooth projective varieties. Thus K(E.) as in (3.2.1) is a cosimplicial de Rham mixed Hodge complex for the simplicial manifold E. , so that D(K(E.)) is a de Rham mixed Hodge complex for the geometric realization |E.| by Lemma 3.4.1, and hence for E, by Proposition 2.4.1. (Similarly, sK(E.) is a mixed Hodge complex for E.)

3.4.3. Example. Let $f: (X,K) \to (Y,L)$ be a morphism of de Rham mixed Hodge complexes. Then the mapping cone of f is a cosimplicial de Rham mixed Hodge complex, so applying the de Rham functor gives a de Rham mixed Hodge complex for it. In particular, if f is an inclusion, the relative cohomology groups $H^{\cdot}(Y,X)$ have a mixed Hodge structure, which is preserved by cup products. (See [Deligne 8.3.8], [Durfee Duke Section 2].) Similarly, the relative cross product is a morphism of mixed Hodge structures.

4. THE LINK OF A DIVISOR WITH NORMAL CROSSINGS

4.1. Open varieties

Let X be a smooth projective variety, let $D \subset X$ be a divisor with normal crossings and let $U = X - D$. First we describe the standard real mixed Hodge complex for U, as done in [Deligne] and modified in [Hain, DHT]. The real part of this complex, called $E_R(X \log D)$, consists of global sections of a sheaf L. The sheaf, a subsheaf of the sheaf of all smooth real-valued forms on U, is defined as follows: Let N be an open set in X with holomorphic coordinates $(z_1,\dots,z_n)$ such that $N \cap D$ is the subvariety $z_1 \dots z_k = 0$. Let L(N) be generated as ring by the algebra E(N) of smooth real-valued forms on N together with the real forms $\theta_1,\dots\theta_k$, where

$$\theta_j = (1/4\pi i)(dz_j/z_j - d\bar{z}_j/\bar{z}_j).$$

There is an algebra isomorphism

$$L(N) \cong E(N)\otimes\Lambda(\theta_1,\ldots,\theta_k).$$

It is easy to see that this description is independent of the choice of coordinates. Define a weight filtration on $L(N)$ by letting $W_mL(N)$ be the $E(N)$-submodule generated by forms of the type $\theta_{i_1}\wedge\ldots\wedge\theta_{i_k}$ for $k \leq m$. This is preserved by wedge product, and is independent of the choice of coordinates. Let $E_R(X \log D)$ be global sections of the sheaf L ; this is a filtered differential graded algebra. The next result follows by standard sheaf theory techniques.

4.1.1 Proposition. The inclusion of differential graded algebras

$$E_R(X \log D) \to E_R(X - D)$$

is a quasi-isomorphism. ∎

Similarly, let $\tilde{K}_C$ and $E_C(X \log D)$ be defined as global sections of the sheaves whose sections over N are defined by

$$E_C(N)\otimes\Lambda(dz_1/z_1,\ldots,dz_k/z_k,\ d\bar{z}_1/\bar{z}_1,\ldots,d\bar{z}_k/\bar{z}_k)\otimes C[\log |z_1|,\ldots\log |z_k|]$$

and

$$E_C(N)\otimes\Lambda(dz_1/z_1,\ldots,dz_k/z_k)$$

respectively. Define weight filtrations by letting the weights of dz_j/z_j, $d\bar{z}_j/\bar{z}_j$ and $\log |z_j|$ be 1. The filtrations-preserving differential graded algebra homomorphisms

$$E_R(X \log D)\otimes C \to \tilde{K}_C \leftarrow E_C(X \log D)$$

are filtered quasi-isomorphisms by standard arguments. Define the Hodge filtration on $E_C(X \log D)$ as usual.

4.1.2. Theorem. [Deligne II, Hain DHT 5.2] $((E_R(X \log D), W), (E_C(X \log D), W, F))$ is a (real) de Rham mixed Hodge complex for $X - D$. This complex is functorial in maps of the pair (X,D).

We denote this mixed Hodge complex by $K(X,D)$.

4.1.3. Example. Let us generalize Example 3.4.2. (This generalization will be needed in (4.3.3). Let E be as in that example and let E' be another variety with normal crossings with no components in common with E. Let $E'' = E \cap E'$, let $E''_\sigma = E_\sigma \cap E'$ for any n-simplex $\sigma \in I^{n+1}$, let $E''.$ be the simplicial space associated to the union $E'' = \cup_{i \in I} E''_i$, and let $(E - E'').$ be the simplicial space associated to the union $E - E'' = \cup(E_i - E''_i)$. Then $(E - E'').$ is a simplicial variety with cosimplicial de Rham mixed Hodge complex $K(E., E''.)$, so $D(K(E., E''.))$ is a de Rham mixed Hodge complex for $|(E - E'').|$ and hence for $E - E''$. (Similarly, $sK(E.,E''.)$ is a mixed Hodge complex for $E - E''$ with a non-commutative product.)

4.2. The link of a smooth complete intersection.

Let X and D be as in (4.1). Suppose that

$$D = D_1 \cup \ldots \cup D_r$$

is the decomposition of D into irreducible components. We may assume that each component is smooth. Fix some subset σ of $\{1,\ldots,r\}$, and let

$$D_\sigma = \cap_{i \in \sigma} D_i.$$

In this section we will find a de Rham mixed Hodge complex for the link $L(X, D, D_\sigma)$ of D_σ in X with D removed, as defined in Section 1. Throughout this section, X, D and D_σ will be fixed. We let

$$L_\sigma = L(X, D, D_\sigma).$$

Recall that we may always choose L_σ to be a smooth manifold. We also let $T_\sigma = T(X, D, D_\sigma)$ be a neighborhood of D_σ as in Section 1. Let $F(V)$ (respectively $F_C(V)$) denote real (respectively complex) valued smooth functions on a manifold V. Consider the differential graded algebra $E_R(X \log D) \otimes F(D_\sigma)$, where the tensor product is over $F(X)$; since

$$F(D_\sigma) = F(X)/(\text{ideal of functions which vanish on } D_\sigma)$$

both $F(D_\sigma)$ and $E_R(X \log D)$ are modules over $F(X)$.

4.2.1. Lemma. There is a natural (differential graded algebra) quasi-isomorphism

$$\alpha: E_R(X \log D) \otimes F(D_\sigma) \longleftrightarrow E_R(L_\sigma).$$

Proof. The quasi-isomorphism is the composition

$$E_R(X \log D) \otimes F(D_\sigma) \to E_R(T_\sigma \log D) \otimes F(D_\sigma) \leftarrow E_R(T_\sigma \log D) \to E_R(T_\sigma - D) \to E_R(L_\sigma).$$

This completes the proof. ■

We now make this into a multiplicative mixed Hodge complex. Consider the three algebras

$$E_R(X \log D) \otimes F(D_\sigma)$$

$$\tilde{K}_C \otimes F_C(D_\sigma)$$

$$E_C(X \log D) \otimes F_C(D_\sigma).$$

Define filtrations by localizing the filtrations from (4.1) along D_σ, that is, by setting

$$W_m(E_R(X \log D) \otimes F(D_\sigma)) = (W_m E_R(X \log D)) \otimes F(D_\sigma)$$

and so forth. These filtrations are multiplicative. Standard sheaf theory arguments show that the filtration-preserving differential graded algebra homomorphisms

$$E_R(X \log D) \otimes F(D_\sigma) \otimes C \to \tilde{K}_C \otimes F_C(D_\sigma) \leftarrow E_C(X \log D) \otimes F_C(D_\sigma)$$

are filtered quasi-isomorphims.

4.2.2. Lemma. If X, D and σ are as above, then

$$((E_R(X \log D) \otimes F(D_\sigma), W), (E_C(X \log D) \otimes F_C(D_\sigma), W, F))$$

is a (real) multiplicative mixed Hodge complex.

We denote the complex of this lemma by $K(X, D, D_\sigma)$.

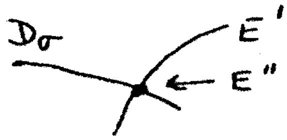
Proof of Lemma 4.2.2. We will check that $K(X, D, D_\sigma)$ satisfies the conditions of (3.1). First some notation. Let

$$\sigma' = \{1,\dots,r\} \cdot \sigma$$

$$E' = \cup_{i\in\sigma'} D_i$$

$$E'' = D_\sigma \cap E'$$

Here is a schematic picture:



Choose coordinates $(z_1,\dots,z_n)$ in a neighborhood N such that

$$D \cap N = \{z_1 \dots z_k = 0\}$$

$$D_\sigma \cap N = \{z_1 = \dots = z_m = 0\}$$

for some $1 \le m \le k \le n$. Thus

$$E' = \{z_{m+1} \dots z_k = 0\}$$

$$E'' = \{z_1 = \dots = z_m = 0,\ z_{m+1} \dots z_n = 0\}.$$

The weight filtration on $E_C(X \log D) \otimes F_C(D_\sigma)$ splits into two peices: In local coordinates on N, a form in this complex is a wedge product of a form on D_σ with forms of the type dz_i/z_i, where i is chosen from $1,\dots,k$. The external weight filtration W^E is defined by letting $W^E{}_q$ be all forms with at most q terms of type dz_i/z_i, with $1 \le i \le m$. The internal weight filtration W^I is defined similarly by letting $W^I{}_q$ be all forms with at most q terms of type dz_i/z_i with $m+1 \le i \le k$. It is easy to check that

$$W_q = \Sigma_{r+s=q}\, W^I{}_r \otimes W^E{}_s\ .$$

The external residue operator is defined on $W^E{}_q$ and vanishes on $W^E{}_{q-1}$. It removes these dz/z's and induces an isomorphism

$$W^E{}_q E_C{}^m(X \log D) \otimes F_C(D_\sigma)\ /\ W^E{}_{q-1} \cong E_C{}^{m-q}(D_\sigma \log E'').$$

Let $E^{\cdot\cdot}$ denote the weight spectral sequence of $K(X, D, D_\sigma)$ and let $'E^{\cdot\cdot}$ denote the weight spectral sequence of $K(D_\sigma, E'')$. Now, applying this and [Zucker A.2], we have

$$E_0{}^{-q,m} = (W_q/W_{q-1})\ E^{m-q}(X \log D) \otimes F_C(D_\sigma)$$

$$= \oplus_{r+s=q}\ (W^I{}_s/W^I{}_{s-1})\ (W^E{}_r/W^E{}_{r-1})\ E_C{}^{m-q}(X \log D) \otimes F_C(D_\sigma)$$

$$= \oplus_{r+s=q} (W_s/W_{s-1})\ E_C^{m-q-r}(D_\sigma \log E'')$$
$$= \oplus_{r+s=q}\ {}'E_1^{-s,m-2r}\ .$$

The differential d_0 respects this decomposition. Since d_0 is strict on $'E_0^{\cdot\cdot}$, it is strict on $E_0^{\cdot\cdot}$. Also,

$$E_1^{-q,m} = \oplus_{r+s=q}\ {}'E_1^{-s,m-2r}(-r)$$

where $(-r)$ denotes tensor product with the Hodge structure of Tate, which increases weights by $2r$. Thus $E_1^{-q,m}$ has a pure Hodge structure of weight m. ■

Assembling (4.2.1) and (4.2.2), we get the next result, the goal of this section:

4.2.3. Proposition. The complex $K(X, D, D_\sigma)$ is a de Rham mixed Hodge complex for $L(X, D, D_\sigma)$.

For fixed X and D, the correspondence

$$D_\sigma \mapsto (\ (LX, D, D_\sigma), K(X, D, D_\sigma)\ )$$

is clearly a contravarient functor from the category of such D_σ to the category of de Rham mixed Hodge complexes.

The next two results are needed to show that there are morphisms from the mixed Hodge complexes of the complement and divisor, respectively, to the link. They are needed to prove (4.3.2) and (4.3.3).

4.2.4. Proposition. There is a morphism of de Rham mixed Hodge complexes

$$K(X, D) \to K(X, D, D_\sigma)$$

corresponding to the inclusion

$$L_\sigma \to X - D.$$

Proof. The morphism of de Rham mixed Hodge complexes is defined by the obvious maps, and to check that it is a morphism is straight-forward diagram chasing. ■

Let E' and E'' be as in the proof of (4.2.2). Let $K(D_\sigma, E'')$ denote the standard de Rham mixed Hodge complex for $D_\sigma - E''$ from (4.1) and let $T_\sigma = T(X, D, D_\sigma)$. Note that the inclusion $D_\sigma - E'' \to T_\sigma - E'$ is a homotopy equivalence.

4.2.5. Proposition. There is a morphism of de Rham mixed Hodge complexes

$$K(D_\sigma, E'') \to K(X, D, D_\sigma)$$

corresponding to the map

$$L_\sigma \to T_\sigma - E' \leftarrow D_\sigma - E''.$$

Proof. The morphism of multiplicative mixed Hodge complexes is simply the inclusion; in fact, the external weight zero part of $K(X, D, D_\sigma)$ is isomorphic to $K(D_\sigma, E'')$. To check that it is a morphism of de Rham mixed Hodge complexes is once again a straight-forward diagram chase. ■

4.3. The link of a divisor with normal crossings.

Now we assemble the pieces above. Let X be a smooth projective variety, let D be a divisor with normal crossings in X, and let E be a subdivisor, i.e., a union of irreducible components of D. We will construct a multiplicative mixed Hodge complex for the link of E in X with D removed, as defined in Section 1. Suppose

$$E = E_1 \cup \ldots \cup E_s$$

is the decomposition of E into its irreducible components. Let $E.$ be the simplicial variety associated to $E = \cup E_i$ as in (2.1.1). Thus we may assemble the mixed Hodge complexes of (4.2.2) into a cosimplicial mixed Hodge complex $DK(X, D, E.)$. The main technical result of Section 4 is the following proposition.

4.3.1. Proposition. $DK(X, D, E.)$ is a de Rham mixed Hodge complex for $L(X, D, E)$.

Proof. Just as in Examples (3.4.2) and (4.1.3), we may compose the functor defining the simplicial variety $E.$ with the functor $E_\sigma \mapsto ((L(X, D, E_\sigma) , K(X, D, E_\sigma))$ of (4.2). Proposition 4.2.3 gives that $K(X, D, E.)$ is a cosimplicial de Rham mixed Hodge complex for $L(X, D, E.)$. Lemma 3.4.1 gives that $DK(X, D, E.)$ is a de Rham mixed Hodge complex for $|L(X, D, E.)|$. Finally, with proper choice of neighborhoods, we claim that

$$L(X, D, E) = L(X, D, E_1) \cup \ldots \cup L(X, D, E_s) .$$

In fact, choose neighborhoods $T_i = T(X, D, E_i)$ of E_i as in Section 1 such that $\cup T_i$ is a neighborhood of the form $T(X, D, E)$. This can be done using [Durfee Neighborhoods], for example. Then $T_i - D$ is a link of the form $L(X, D, E_i)$ and $\cup(T_i - D) = \cup T_i - D$ is a link of the form $L(X, D, E)$. The theorem now follows from (2.4.1). ■

This de Rham mixed Hodge complex is clearly functorial for maps as in Section 1. The next results are needed to show that the mixed Hodge complexes for the complement and divisor, respectively, map to the mixed Hodge complex for the link.

4.3.2. Proposition. There is a morphism of de Rham mixed Hodge complexes

$$K(X, D) \to D(K(X, D, E.))$$

corresponding to the inclusion

$$L \hookrightarrow X - D.$$

Proof. By (4.2.4), for each σ there is a morphism of de Rham mixed Hodge complexes

$$K(X, D) \to K(X, D, E_\sigma)$$

corresponding to the inclusion

$$L_\sigma \hookrightarrow X - D .$$

Hence there is a morphism of cosimplicial de Rham mixed Hodge complexes

$$K(X, D) \to K(X, D, E.)$$

corresponding to the inclusion

$$L. \to X - D .$$

(Here $X - D$ is a "constant" simplicial object, with $X - D$ in each degree and all face and degeneracy maps the identity. Similarly, $K(X,D)$ is a constant cosimplicial object.) Thus there is a morphism of de Rham mixed Hodge complexes

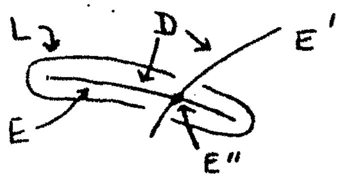
$$DK(X,D) \to DK(X, D, E.)$$

corresponding to the map

$$|L.| \to |X - D|.$$

However, $DK(X, D)$ is quasi-isomorphic to $K(X, D)$, $|X - D|$ is just $X - D$, and the map $|L.| \to X - D$ can be replaced by the inclusion $L \hookrightarrow X - D$. ∎

Let E' be the divisor with no irreducible components in common with E with the property that $D = E \cup E'$. Let $E'' = E \cap E'$ and let $DK(E., E''.)$ denote the de Rham mixed Hodge complex for $E - E''$ from 4.1.3. Note that the inclusion $E - E'' \to T - E'$ is a homotopy equivalence. Here is a schematic picture:



4.3.3. Proposition. There is a morphism of de Rham mixed Hodge complexes

$$DK(E., E''.) \to DK(X, D, E.)$$

corresponding to the map

$$L \to T - E' \leftarrow E - E'' .$$

Proof. We will use the notation of (4.1.3). By (4.2.5), for each σ there is a morphism of de Rham mixed Hodge complexes

$$K(E_\sigma, E''_\sigma) \to K(X, D, E_\sigma)$$

corresponding to the map

$$L_\sigma \to T_\sigma - E' \leftarrow E_\sigma - E''_\sigma .$$

Hence there is a morphism of cosimplicial de Rham mixed Hodge complexes

$$K(E_., E''_.) \to K(X, D, E_.)$$

corresponding to the map

$$L_. \to T_. - E' \leftarrow E_. - E''_.$$

Thus there is a morphism of de Rham mixed Hodge complexes

$$DK(E_., E''_.) \to DK(X, D, E_.)$$

corresponding to the map

$$|L_.| \to |T_. - E'| \leftarrow |E_. - E''_.|$$

which can be replaced by

$$L \to T - E' \leftarrow E - E''$$

by (2.4.1). ∎

4.4. Comparison with the Mayer-Vietoris complex.

The Mayer-Vietoris construction also gives a (non-multiplicative) mixed Hodge complex for a link. The next result is a slight generalization of [Durfee Duke 3.4].

4.4.1. Theorem. Let X, D and E be defined as in the beginning of (4.3) and let E' and E'' be defined as in (4.3.3). Then

$$K(X, D) \oplus sK(E_., E''_.) \,/\, K(X, E')$$

is a mixed Hodge complex for $L(X, D, E)$.

Proof. Since $L = (X - D) \cap (T - E')$ with $(X - D) \cup (T - E') = X - E'$, the construction of [Durfee Duke 2.13] applies. ∎

4.4.2. Proposition. The mixed Hodge complex $DK(X, D, E_.)$ of (4.3.1) and the mixed Hodge complex of (4.4.1) are quasi-isomorphic as mixed Hodge complexes.

Proof. Since $sK(E_., E''_.)$ is quasi-isomorphic to the complex $DK(E_., E''_.)$ by (3.3.2), the complex of (4.4.1) can be replaced by $K(X, D) \oplus DK(E., E''.) / K(X, E')$. By the universal mapping property of this complex and (4.3.2) and (4.3.3), there is a map of this complex into $DK(X, D, E.)$. This map induces an isomorphism of mixed Hodge structures on cohomology, and hence is a quasi-isomorphism. ■

5. THE MAIN RESULTS

5.1. Mixed Hodge structures on the homotopy of links.

The main results of this paper are contained in the following theorem. The basic idea is to use resolution of singularities to reduce to the case of a divisor with normal crossings, which has already been treated in Section 4.

5.1.1. Theorem. If $L = L(X, Y, Z)$ is the link of Z in X with Y removed, as in Section 1, then:

1. The cohomology groups of L have a unique functorial (real) mixed Hodge structure.
2. The cup product in cohomology preserves this mixed Hodge structure.
3. The mixed Hodge structure on cohomology is the same as the mixed Hodge structure obtained by the Mayer-Vietoris construction [Durfee Duke].
4. The minimal model and the Lie algebra model for L have mixed Hodge structures, as described in Theorem 3.2.3. In particular, the Malcev Lie algebra of the fundamental group of X has a mixed Hodge structure, and, when, X is simply connected, the higher homotopy groups of X have a unique functorial mixed Hodge structure. Furthermore, Massey products preserve mixed Hodge structure.
5. Let Z' and W be defined as in Section 1. Then the inclusion map $L \to X - Y$ and the correspondence $L \to Z - W$ defined by $L \to T - Z' \leftarrow Z - W$ induce maps of mixed Hodge structures on cohomology, and, when the spaces are simply-connected, on higher homotopy groups.

Proof. Let $\pi: \tilde{X} \to X$ be a resolution of singularities such that $D = \pi^{-1}(Y)$ is a divisor with normal crossings and $E = \pi^{-1}(Z)$ consists of components of D. Then $\pi(L(\tilde{X}, D, E)) = L(X, Y, Z)$, and π is a diffeomorphism of these spaces. Now $L(\tilde{X}, D, E)$ has a de Rham mixed Hodge complex by (4.3.1). Furthermore, given any two resolutions, a third one can be found which dominates them, and the corresponding maps will induce maps of de Rham mixed Hodge complexes. Thus statements (1) and (2) follow from (3.2.2), statement (3) follows from (4.4.2), statement (4) follows from (3.2.3) and statement (5) follows from (4.3.2) and 4.3.3). ∎

Functorality of the homotopy groups of non-simply connected spaces is more subtle, and will not be discussed in this paper.

5.2. Duality.

In this section we prove that various dualities in the cohomology of a link preserve mixed Hodge structure. The proofs are similar to those in the global case [Durfee Euler]. Recall that a link L(X, Z), where X has dimension n, has the homotopy type of a compact real (2n-1)-manifold.

5.2.1. Lemma. If $L = L(X, Z)$ where X has dimension n, then $H^{2n-1}(L)$ has pure type (n,n).

Proof. By resolution of singularities, we may assume that X is smooth. Choose an open neighborhood T of Z in X such that the boundary of T is smooth, and let L be the boundary of T. This can be done by [Durfee Neighborhoods], for example. Then

$$H^{2n-1}(L) \cong H^{2n}(X - T, L) \cong H^{2n}(X, Z) \cong H^{2n}(X) .$$

These maps preserve mixed Hodge structures by [Durfee Duke 3.2]. Furthermore, $H^{2n}(X)$ has pure type (n,n). ■

5.2.2. Theorem. If $L = L(X, Z)$ where X has dimension n, then Poincaré duality

$$H^q(L) \to H_{2n-q}(L)$$

is an isomorphism of mixed Hodge structures of type $(-n,-n)$.

Proof. Since the cup product in $H^{\cdot}(L)$ preserves mixed Hodge structures, so does the cap product. The duality map is cap product with the fundamental homology class, which has type $(-n,-n)$. ■

Recall that a mixed Hodge structure on relative cohomology groups is defined in (3.4.3), and that the notion of sublink is defined in Section 1. Next we will prove that a general duality [Spanier 6.2.17] preserves mixed Hodge structures.

5.2.3. Theorem. Let $L_2 \subset L_1 \subset L$ be links, where $L_2 = L(X_2, Z_2)$, $L_1 = L(X_1, Z_1)$, $L = L(X, Z)$ and X has dimension n. Then

$$H_q(L - L_2, L - L_1) \cong H^{2n-1-q}(L_1, L_2)$$

is an isomorphism of mixed Hodge structures of type (n,n).

Proof. Recall that products and mapping cones have de Rham mixed Hodge complexes by (3.2.5) and (3.4.3). The normal bundle to the diagonal Δ in $L \times L$ has a Thom class, which by excision gives a class $u \in H^{2n-1}(L\times L, L\times L - \Delta)$. The class u is the image of 1 under the Thom isomorphism, and hence has type (n,n). Let

$$i\colon (L_1, L_2) \times (L-L_1, L-L_2) \hookrightarrow (L\times L, L\times L - \Delta)\ .$$

Duality is provided by the morphism

$$H^{2n-1}((L_1 \times L_2) \times (L-L_2, L-L_1)) \otimes H_q(L-L_2, L-L_1) \longrightarrow H^{2n-1-q}(L_1, L_2)$$

defined by $(i^*u, c) \mapsto i^*u/c$. The slant product is dual to the cross product, and hence preserves mixed Hodge structures (3.2.6). ■

The *cohomology of a link* with compact supports has a mixed Hodge structure, and the duality theorem with compact supports [Spanier 6.9.10] can be easily reduced to the above duality theorem, exactly as in the global case.

5.2.4. Corollary. If $L_1 = L(X_1, Z_1)$ is a sublink of $L = L(X, Z)$, where X has dimension n, then Alexander duality

$$H^q(L, L_1) \to H_{2n-1-q}(L - L_1)$$

is an isomorphism of mixed Hodge structures of type $(-n,-n)$.

6. APPLICATIONS

First we prove the theorem mentioned in the introduction on the vanishing of cup products.

6.1. Theorem. Let L be the link of an isolated singularity of an n-dimensional variety. If $s,t < n$ and $s+t \geq n$, then the cup product

$$H^s(L;Q)\otimes H^t(L;Q) \to H^{s+t}(L;Q)$$

vanishes.

Proof. The groups $H^k(L)$ have a mixed Hodge structure which is preserved by cup products, by Theorem 5.1.1. In particular, the weight filtration satisfies $W_k \cup W_m \subset W_{k+m}$, for all k,m. Now for isolated singularities, it is proved in [Durfee Duke 3.8], or see [Durfee Japan] for an elementary proof, that

$$W_k H^k(L) = 0 \quad \text{for} \quad k \geq n, \text{ and}$$
$$W_k H^k(L) = H^k(T) \quad \text{for } k < n.$$

It follows that the cup product is zero. ■

6.2. Theorem. If Y is an algebraic curve in C^2 with an isolated singularity at p and $L = L(C^2, Y, p)$ is the link of p in C^2 with Y removed, then $H^k(L)$ has pure type (k,k), for all k. In particular, L is a formal space.

Proof. First some notation. Let

$$S = L(C^2, p)$$
$$K = L(Y, p).$$

Then S is the three-sphere, K is an "algebraic link" in S, and (see Section 1) $L = S - K$ is the link complement. The group $H^1(K)$ has type (1,1), and hence (by the long exact sequence of a pair [Durfee Duke 2.7]) so does $H^2(S,K)$. By Alexander duality (5.2.4), this group is isomorphic to $H_1(L)$, which hence has type (-1,-1). Thus $H^1(L)$ has type (1,1). A similar argument applies to $H^2(L)$. Finally, $H^0(L)$ has type (0,0), and all other groups are zero. Formality follows from (3.2.4). ∎

6.3. Theorem. There is a simply-connected closed 11-manifold M with the rational cohomology of

$$N = 2(S^2 \times S^9) \# (S^5 \times S^6) \# 2(S^4 \times S^7)$$

that does not have the rational homotopy type of a the link of an isolated singularity of a (six-dimensional) variety.

Proof. It suffices to construct the rational homotopy type of M, for [Sullivan Inf, Theorem 13.2] then implies that there is a smooth closed 11-manifold realizing this rational homotopy type. We obtain the rational homotopy type of M by deforming that of the space N above.

It follows from [Hain Memoir 2.4, 4.16] that the Lie algebra model of N is

$$L_N = (L(X_1, X_2, Y_1, Y_2, X_1^*, X_2^*, Y_1^*, Y_2^*, U, V, Z), \delta)$$

where $\deg X_j = 1$, $\deg Y_j = 3$, $\deg X_j^* = 8$, $\deg Y_j^* = 6$, $\deg U = 4$, $\deg V = 5$, $\deg Z = 10$, and $\delta X_j = \delta Y_j = \delta X_j^* = \delta Y_j^* = \delta U = \delta V = 0$ and $\delta Z = [X_1, X_1^*] + [X_2, X_2^*] + [Y_1, Y_1^*] + [Y_2, Y_2^*] + [U, V]$. We now construct a perturbation $\partial = \delta + p$ of the differential ∂ of L_N. In order that ∂ be a differential, it is necessary and sufficient for p to be a derivation of L_N of degree -1, that $p^2 = 0$ and that $\delta p + p\delta = 0$. In order that the perturbed rational homotopy type (L_N, ∂) have the same rational cohomology ring as that of L_N, it is necessary and sufficient that the image of p be in $[[L_N, L_N], L_N]$ (cf. [Hain Mem 2.5]). Define p by

$$pX_j = pY_j = pV = pZ = 0$$

and

$$pX_1^* = [V, [X_2, X_2]] + [X_1, [Y_1, Y_1]]$$
$$pX_2^* = 2[X_2, [X_1, V]] + [X_2, [Y_2, Y_2]]$$
$$pU = [[X_2, X_2], X_1]$$
$$pY_j^* = [Y_j, [X_j, X_j]].$$

One can easily check that $p^2 = 0$ and $p\delta + \delta p = 0$.

As remarked before, there is a simply-connected closed 11-manifold M with the rational homotopy type of $L_M = (L_N, \partial)$. Denote by $x_j, x_j^*, y_j, y_j^*, u, v, z$ the basis of $\widetilde{H}^{\cdot}(M; Q)$ dual to the basis $X_j, X_j^*, Y_j, Y_j^*, U, V, Z$. According to [Tanre V.7(4)], the Massey products $\langle x_1, x_2, x_2 \rangle$ and $\langle x_j, y_j, y_j \rangle$ for $j = 1,2$ are defined and non-zero.

Suppose the L_M has a mixed Hodge structure and that the mixed Hodge structure on $H^{\cdot}(M;Q)$ satisfies the weight restrictions given in the proof of Theorem 6.1. Recall from Section 5 that Massey products preserve weights. Since $\langle x_1, x_2, x_2 \rangle$ is a non-zero element of $H^5(M) = W_5H^5(M)$, it follows that either x_1 or x_2 is in $W_1H^1(M)$. Suppose $x_1 \in W_1H^1(M)$. Since $y_1 \in H^4(M) = W_4H^4(M)$, it follows that $\langle x_1, y_1, y_1 \rangle \in W_9H^9(M)$, which is zero. This contradicts the fact that $\langle x_1\ y_1, y_1 \rangle$ is not zero. Thus L_M cannot have the rational homotopy type of the link of an isolated singularity. ∎

Actually, [Sullivan Inf. Theorem 13.2] allows one to specify the rational Pontrajagin classes of this manifold as well.

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June 1985

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