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TWISTED NOVIKOV HOMOLOGY OF COMPLEX HYPERSURFACE COMPLEMENTS

STEFAN FRIEDL AND LAURENTIU MAXIM

Abstract. We study the twisted Novikov homology of the complement of a complex hypersurface in general position at infinity. We give a self-contained topological proof of the vanishing (except possibly in the middle degree) of the twisted Novikov homology groups associated to positive cohomology classes of degree one defined on the complement.

1. Introduction

Novikov homology was originally introduced for the purpose of generalizing classical Morse theory to the context of arbitrary closed one-forms, e.g., see [5] for an overview. This theory and its variants have fascinating applications in dynamical systems, symplectic topology, geometry group theory, knot theory, etc. For example, twisted Novikov homology can detect fibering of knots. In relation to geometric group theory, the so-called BNSR-invariants of a group, which contain important information on the finiteness properties of certain subgroups, can be described in terms of vanishing results in Novikov homology (e.g., see [6, 18]). Other topological implications of vanishing of Novikov homology have been derived through the language of barcodes and Jordan cells (e.g., see [1]).

In this note, we study the twisted Novikov homology of complements to complex hypersurfaces in general position at infinity. We give a self-contained topological proof of the vanishing (except possibly in the middle degree) of the twisted Novikov homology groups associated to positive cohomology classes of degree one defined on the complement. Classical Novikov homology of (essential) hyperplane arrangement complements has been studied in [12] by Morse-theoretical methods.

Let M be a topological space. Throughout the paper we assume that all topological spaces are connected and that they admit universal coverings. Furthermore we make the canonical identifications $H^1(M;\mathbb{R}) = \operatorname{Hom}(H_1(M,\mathbb{Z}),\mathbb{R}) = \operatorname{Hom}(\pi_1(M),\mathbb{R})$. An *admissible pair* for M is an epimomorphism $\psi\colon \pi_1(M) \to \Gamma$ to a free abelian group together with some $\xi \in H^1(M;\mathbb{R}) = \operatorname{Hom}(\pi_1(M),\mathbb{R})$ such that $\xi\colon \pi_1(M) \to \mathbb{R}$ factors through ψ. By a slight abuse of notation we denote the unique induced homomorphism $\Gamma \to \mathbb{R}$ by ξ as well.

In Section 2.2, we will associate to (M,ψ,ξ), with (ψ,ξ) an admissible pair together with a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,S)$ over a domain S, the *twisted Novikov-Betti numbers* $b_i^\alpha(M,\psi,\xi)$ and the *twisted Novikov torsion numbers* $q_i^\alpha(M,\psi,\xi)$. The

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classical Novikov-Betti numbers $b_i(M,\xi)$ and Novikov torsion numbers $q_i(M,\xi)$ as defined in [5] can be recovered by taking α to be the trivial one-dimensional representation $\pi_1(M)\to \mathrm{GL}(1,\mathbb{Z})$ and by taking $\psi=\xi\colon \pi_1(M)\to\Gamma:=\mathsf{Im}(\xi)\subset\mathbb{R}$.

Before we state our main theorem we recall that for a complex hypersurface $X\subset\mathbb{C}^n$ the first homology of the complement M_X has a basis that is given by the choice of a positive meridian for each irreducible component of X. We say that a homomorphism $\xi\colon H_1(M_X;\mathbb{R})\to\mathbb{R}$ is *positive*, if ξ maps all meridians to positive real numbers.

Theorem 1.1. *Let $X\subset\mathbb{C}^n$ be a complex hypersurface in general position at infinity, with complement M_X. Then for any admissible pair $(\psi\colon \pi_1(M_X)\to\Gamma,\xi\in H^1(M_X;\mathbb{R}))$ such that ξ is positive, together with a representation $\alpha\colon\pi_1(M_X)\to \mathrm{GL}(k,S)$ over a domain S, we have*

$$b_i^\alpha(M_X,\psi,\xi)=\begin{cases} 0, & \textit{for } i\neq n,\\ (-1)^n k\chi(M_X), & \textit{for } i=n.\end{cases} \tag{1}$$

In particular,

$$(-1)^n\cdot\chi(M_X)\geq 0. \tag{2}$$

Moreover, all twisted Novikov torsion numbers of (M_X,ψ,ξ) vanish, that is:

$$q_i^\alpha(M_X,\psi,\xi)=0, \text{ for all } i\geq 0. \tag{3}$$

Conventions. All domains are understood to be commutative.

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2. Preliminaries

2.1. **Novikov rings.** Let Γ be a free abelian group and let S be a domain.

(1) We say that $p\in S[\Gamma]$ is a *monomial* if there exists $\gamma\in\Gamma$ and a unit $s\in S$ such that $p=s\gamma$.

(2) Given $\xi\in\mathrm{Hom}(\Gamma,\mathbb{R})$ and $p=\sum_{\gamma\in\Gamma}n_\gamma\gamma\in S[\Gamma]\setminus\{0\}$ we write

$$m_\xi(p):=\max\{\xi(\gamma)\,|\,n_\gamma\neq 0\}$$

and we write

$$t_\xi(p):=\sum_{\xi(\gamma)=m_\xi(p)}n_\gamma\gamma.$$

(3) Given $\xi\in\mathrm{Hom}(\Gamma,\mathbb{R})$ we write

$$T_\xi S[\Gamma]:=\{p\in S[\Gamma]\setminus\{0\}\,|\,t_\xi(\gamma)\text{ is a monomial}\}.$$

Furthermore we write

$$\mathcal{R}_\xi S[\Gamma]:=(T_\xi S[\Gamma])^{-1}S[\Gamma].$$

We refer to $\mathcal{R}_\xi S[\Gamma]$ as the *rational Novikov completion of $S[\Gamma]$ with respect to ξ*.

We recall the following well-known lemma.

Lemma 2.1. *Let Γ be a free abelian group and let S be a domain. Then $T_\xi S[\Gamma]$ consists precisely of the elements of $S[\Gamma]$ which are invertible in $\mathcal{R}_\xi S[\Gamma]$.*

Proof. Let $p \in S[\Gamma]$ be an element that is invertible in $\mathcal{R}_\xi S[\Gamma]$. This means that there there exists a $r \in T_\xi\Gamma$ and a $q \in S[\Gamma]$ such that $p \cdot r^{-1}q = 1$. In particular $p \cdot q = r$. Since S is a domain it follows that $t_\xi \colon S[\Gamma] \setminus \{0\} \to S[\Gamma] \setminus \{0\}$ is multiplicative. We now see that

$$t_\xi(p) \cdot t_\xi(q) \,=\, t_\xi(r) \,=\, 1.$$

It follows that $t_\xi(p)$ is a unit in $S[\Gamma]$. Since S is a domain we deduce that $t_\xi(p)$ is a monomial, i.e. $p \in T_\xi S[\Gamma]$. ☐

2.2. **Novikov-Betti and torsion numbers.** Let M be a topological space. We write $\pi = \pi_1(M)$. Let $(\psi \colon \pi \to \Gamma, \xi \in H^1(M;\mathbb{R}))$ be an admissible pair and let $\alpha \colon \pi_1(M) \to \mathrm{GL}(k,S)$ be a representation over a domain S. We denote by

$$\widetilde{M} \longrightarrow M$$

the universal covering of M. The canonical left π-action on $\widetilde{M}$ turns the cellular groups $C_i(\widetilde{M};\mathbb{Z})$ into left-modules over the group ring $\mathbb{Z}[\pi]$.

The representation α turns S^k into a right $\mathbb{Z}[\pi]$-module. We write

$$C_*^\alpha(M;S^k) := S^k \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})$$

and we write

$$H_i^\alpha(M;S^k) := H_i\big(C_*^\alpha(M;S^k)\big).$$

The homomorphism ψ turns $\mathbb{Z}[\Gamma]$, and thus also $\mathcal{R}_\xi\mathbb{Z}[\Gamma]$ into a right $\mathbb{Z}[\pi]$-module. Thus we can view $S[\Gamma]^k$ and $\mathcal{R}_\xi S[\Gamma]^k = \mathcal{R}_\xi\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} S^k$ as right $\mathbb{Z}[\pi]$-module. We write

$$C_*^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k) := \mathcal{R}_\xi S[\Gamma]^k \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})$$

and

$$H_i^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k) := H_i\big(C_*^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k)\big).$$

Definition. Let M be a topological space. We write $\pi = \pi_1(M)$. Let $(\psi \colon \pi \to \Gamma, \xi \in H^1(M;\mathbb{R}))$ be an admissible pair and let $\alpha \colon \pi_1(M) \to \mathrm{GL}(k,S)$ be a representation over a domain S. The *i-th twisted Novikov-Betti number* is defined as

$$b_i^\alpha(M,\psi,\xi) := \text{rank of the } \mathcal{R}_\xi S[\Gamma]\text{-module } H_i^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k).$$

The *i-th twisted Novikov torsion number* is defined as

$$q_i(M,\psi,\xi) := \begin{array}{c}\text{minimal number of generators of the torsion}\\ \text{submodule of the } \mathcal{R}_\xi S[\Gamma]\text{-module } H_i^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k).\end{array}$$

In the following proposition we list a few basic facts about twisted Novikov-Betti and torsion numbers. The proofs are verbatim the same as the proofs of the corresponding statements for untwisted Novikov-Betti and Novikov torsion numbers that are studied in [5, Chapter 1]:

Proposition 2.2. *The twisted Novikov-Betti and the twisted Novikov torsion numbers satisfy the following properties:*

(a) *The following equality holds:*

$$k\chi(M) = \sum_{i \geq 0} (-1)^i \cdot b_i^\alpha(M,\psi,\xi).$$

(b) *For any* $\lambda \in \mathbb{R}_{>0}$ *and any* i *we have*

$$H_i^\alpha(M; \mathcal{R}_{\lambda\xi} S[\Gamma]^k) \;=\; H_i^\alpha(M; \mathcal{R}_\xi S[\Gamma]^k).$$

In particular, we have

$$b_i^\alpha(M, \psi, \xi) = b_i^\alpha(M, \psi, \lambda\xi) \text{ and } q_i^\alpha(M, \psi, \xi) = q_i^\alpha(M, \psi, \lambda\xi).$$

2.3. **Topology of complex hypersurface complements.** Let X be a hypersurface in $\mathbb{C}^n$ ($n \geq 2$), with underlying reduced hypersurface X_{red} defined by the (square-free) equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of the polynomial f. Let $X_i = \{f_i = 0\}$ denote the irreducible components of X_{red}. Embed $\mathbb{C}^n$ in $\mathbb{CP}^n$ by adding the hyperplane at infinity, H, and let $\overline{X}$ be the projective completion of X in $\mathbb{CP}^n$. Let M_X denote the affine hypersurface complement

$$M_X := \mathbb{C}^n \setminus X = \mathbb{C}^n \setminus X_{red}.$$

Alternatively, M_X can be regarded as the complement in $\mathbb{CP}^n$ of the divisor $\overline{X} \cup H$. Then it is well-known that $H_1(M_X; \mathbb{Z})$ is a free abelian group, generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i, for $i = 1, \cdots, s$ (e.g., see [3], (4.1.3), (4.1.4)). Furthermore, since M_X is an n-dimensional affine variety, it has the homotopy type of a finite CW-complex of real dimension n (e.g., see [3], (1.6.7), (1.6.8)).

Let S^∞ be a $(2n-1)$-sphere in $\mathbb{C}^n$ of a sufficiently large radius (that is, the boundary of a small tubular neighborhood in $\mathbb{CP}^n$ of the hyperplane H at infinity). Denote by

$$X^\infty = S^\infty \cap X$$

the *link of* X *at infinity*, and by

$$M_X^\infty = S^\infty \setminus X^\infty$$

its complement in S^∞. Note that M_X^∞ is homotopy equivalent to $T(H) \setminus \overline{X} \cup H$, where $T(H)$ is the tubular neighborhood of H in $\mathbb{CP}^n$ for which S^∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism

$$\pi_i(M_X^\infty) \longrightarrow \pi_i(M_X)$$

induced by inclusion is an isomorphism for $i < n-1$ and it is surjective for $i = n-1$; see [4, Section 4.1] for more details. It follows that

$$\pi_i(M_X, M_X^\infty) = 0 \;, \text{ for all } \; i \leq n-1, \tag{4}$$

hence M_X has the homotopy type of a complex obtained from M_X^∞ by adding cells of dimension $\geq n$.

If, moreover, X is *in general position at infinity*, that is, the reduced underlying variety of $\overline{X}$ is transversal to H in the stratified sense, then M_X^∞ is a circle fibration over $H \setminus \overline{X} \cap H$, which is homotopy equivalent to the complement in $\mathbb{C}^n$ to the affine cone over the projective hypersurface $\overline{X} \cap H \subset H = \mathbb{CP}^{n-1}$ (for a similar argument see [4, Section 4.1]). Hence, by the Milnor fibration theorem (e.g., see [3, (3.1.9),(3.1.11)]), M_X^∞ fibers over $\mathbb{C}^* \simeq S^1$, with fiber F homotopy equivalent to a finite $(n-1)$-dimensional CW-complex.

3. Novikov homology of complex hypersurface complements

In this section we will give the proof of Theorem 1.1.

3.1. **Preliminary lemmas.** We start out with the following lemma.

Lemma 3.1. *Let $X \subset \mathbb{C}^n$ be a hypersurface with complement M_X. Let $(\psi\colon \pi_1(M_X) \to \Gamma, \xi \in H^1(M_X;\mathbb{R}))$ be an admissible pair and let $\alpha\colon \pi_1(M_X) \to \mathrm{GL}(k,S)$ be a representation over a domain S. Then the following hold:*

1. *We have $H_i(M_X;\mathcal{R}_\xi S[\Gamma]^k) = 0$ for $i > n$. In particular, we have the vanishing $b_i^\alpha(M_X,\psi,\xi) = 0$ and $q_i^\alpha(M_X,\psi,\xi) = 0$ for $i > n$.*
2. *We have $q_n^\alpha(M_X,\psi,\xi) = 0$.*

Proof. The first statement follows immediately from the fact that M_X has the homotopy type of a finite CW-complex M' of real dimension n. This fact also implies that $H_n^\alpha(M_X;\mathcal{R}_\xi S[\Gamma]^k) = H_n^\alpha(M';\mathcal{R}_\xi S[\Gamma]^k)$ is a submodule of the free $\mathcal{R}_\xi S[\Gamma]$-module $C_n^\alpha(M';\mathcal{R}_\xi S[\Gamma]^k)$. In particular $H_n^\alpha(M_X;\mathcal{R}_\xi S[\Gamma]^k)$ has no $\mathcal{R}_\xi S[\Gamma]$-torsion, which in turn implies that $q_n^\alpha(M_X,\psi,\xi) = 0$. □

In the following we adopt the convention that if $\varphi\colon \pi_1(M) \to G$ is a homomorphism and $N \subset M$ is a connected subspace, then, by a slight abuse of notation, we denote the induced homomorphism $\pi_1(N) \to \pi_1(M) \xrightarrow{\varphi} G$ by φ as well.

Proposition 3.2. *Let $X \subset \mathbb{C}^n$ be a hypersurface with complement M_X, and fix an admissible pair $(\psi\colon \pi_1(M_X) \to \Gamma, \xi \in H^1(M_X;\mathbb{R}))$. Let $\alpha\colon \pi_1(M_X) \to \mathrm{GL}(k,S)$ be a representation over a domain S. Then for any $i < n-1$, we have $\mathcal{R}_\xi S[\Gamma]$-isomorphisms*

$$H_i^\alpha(M_X^\infty;\mathcal{R}_\xi S[\Gamma]^k) \xrightarrow{\cong} H_i(M_X;\mathcal{R}_\xi S[\Gamma]^k)$$

and we have an epimorphism of $\mathcal{R}_\xi S[\Gamma]$-modules

$$H_{n-1}(M_X^\infty;\mathcal{R}_\xi S[\Gamma]^k) \to H_{n-1}(M_X;\mathcal{R}_\xi S[\Gamma]^k).$$

Proof. This is an immediate consequence of the fact, mentioned in Section 2.3, that the complement M_X is obtained (up to homotopy) from M_X^∞ by adding cells of dimension $\geq n$ and the fact that twisted homology groups are homotopy invariants. □

Definition. Given a manifold M and a class $\xi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ we say that ξ is *fibered* if there exists a bundle map $p\colon M \to S^1$ such that $p_* = \xi\colon \pi_1(M) \to \mathbb{Z}$.

Proposition 3.3. *Let M be a manifold. Let $(\psi\colon \pi_1(M) \to \Gamma, \xi \in H^1(M;\mathbb{Z}))$ be an admissible pair and let $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,S)$ be a representation over a domain S. If ξ is fibered, then for any i we have $H_i^\alpha(M;\mathcal{R}_\xi S[\Gamma]^k) = 0$.*

This proposition is well-known to the experts, it can be proved along the lines of [11, Theorem 4.2] or alternatively [2, 10, 8, 7]. Since we could not find a result in the literature which gives precisely the statement desired we sketch a proof.

Proof. In the following, given a manifold X and a map $\varphi\colon X \to X$ we denote by $T(X,\varphi) = (X \times [0,1])/(x,0) \sim (\varphi(x),1)$ the corresponding mapping torus. We refer to the induced map $\pi_1(T(X,\varphi)) \to \pi_1([0,1]/0 \sim 1) = \mathbb{Z}$ as the canonical homomorphism. We can identify the manifold M with a mapping torus $T := T(X,\varphi)$ such that $\xi \in \mathrm{Hom}(\pi_1(M),\mathbb{Z}) = \mathrm{Hom}(T,\mathbb{Z})$ agrees with the canonical homomorphism. Following [8, Section 3] there exists a Meyer–Vietoris sequence

$$\begin{aligned}\cdots \to H_i(X \times [0,1]; S[\Gamma]^k) \otimes_{S[\Gamma]} \mathcal{R}_\xi S[\Gamma] &\xrightarrow{\mathrm{id} - t\varphi_*} H_i(X \times [0,1]; S[\Gamma]^k) \otimes_{S[\Gamma]} \mathcal{R}_\xi S[\Gamma] \\ &\to H_i(M;\mathcal{R}_\xi S[\Gamma]^k) \to \dots\end{aligned}$$

where t is an element with $\xi(t) = 1$. All the maps $\mathrm{id} - t\varphi_*$ are invertible over $\mathcal{R}_\xi S[\Gamma]$. It follows that the the homology groups $H_i(M;\mathcal{R}_\xi S[\Gamma]^k)$ vanish. □

3.2. **Novikov homology for positive integral cohomology classes.**

Definition. Let $X \subset \mathbb{C}^n$ be a hypersurface with complement M_X. A cohomology class $\xi \in H^1(M_X;\mathbb{R})$ is called *positive* if the corresponding group homomorphism $\xi\colon \pi_1(M_X) \to \mathbb{R}$ takes strictly positive values on each positively oriented meridian generator γ_i about an irreducible component of X_{red}.

The following theorem takes care of Theorem 1.1 for *integral* cohomology classes.

Theorem 3.4. *Let $X \subset \mathbb{C}^n$ be a hypersurface with complement M_X. We assume that X is in general position at infinity. Let $(\psi\colon \pi_1(M_X) \to \Gamma, \xi \in H^1(M_X;\mathbb{Z}))$ be an admissible pair and let $\alpha\colon \pi_1(M_X) \to \mathrm{GL}(k,S)$ be a representation over a domain S. If ξ is positive, then*

$$b_i^\alpha(M_X,\psi,\xi) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n k \cdot \chi(M_X), & \text{for } i = n. \end{cases} \tag{5}$$

In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0. \tag{6}$$

Moreover

$$q_i^\alpha(M_X,\psi,\xi) = 0, \text{ for all } i \geq 0. \tag{7}$$

Proof. Since M_X has the homotopy type of a finite CW complex, the homology groups $H_i^\alpha(M_X;\mathcal{R}_\xi S[\Gamma]^k)$ are finitely generated $\mathcal{R}_\xi S[\Gamma]$-modules.

Let f be a square-free polynomial defining X_{red}, the reduced hypersurface underlying X. We denote the factors of f by $f_1,\dots,f_s$. Let $\xi \in H^1(M_X;\mathbb{Z})$ be a positive integral cohomology class, with $(n_1,\cdots,n_s) \in \mathbb{N}^s$ the vector of values of $\xi\colon \pi_1(M_X) \to \mathbb{Z}$ on the positive meridians γ_i, $i = 1,\cdots,s$, about the irreducible components of X_{red} corresponding to the factors $f_1,\dots,f_s$. We consider the polynomial $g = {f_1}^{n_1}\cdots {f_s}^{n_s}$ on $\mathbb{C}^n$. Clearly, the underlying reduced hypersurface $\{g=0\}_{red}$ coincides with X_{red} and, moreover, the homomorphism $g_*\colon \pi_1(M_X) \to \mathbb{Z}$ induced by g coincides with ξ (cf. [3, p.76-77]). By Section 2.3, the element $\xi = g_* \in H^1(M_X^\infty;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M_X^\infty),\mathbb{Z})$ is fibered. It follows from Proposition 3.3 that $H_i^\alpha(M_X^\infty;\mathcal{R}_\xi S[\Gamma]^k) = 0$ for all i.

The theorem now follows from the combination of Proposition 2.2 (a), Lemma 3.1 and Proposition 3.2. □

Remark. The statement about the vanishing of the classical Novikov-Betti numbers in Theorem 3.4 has also been obtained implicitely in [13] by using Alexander modules. Furthermore it can be also derived by using the corresponding vanishing statement for the L^2-Betti numbers of such complements, see [14, Theorem 1.1]. Indeed, it follows from [9, Proposition 2.4] that we have the identification:

$$b_i(M_X;\xi) = b_i^{(2)}(M_X,\xi\colon \pi_1(M_X) \to \mathrm{Im}(\xi)) \tag{8}$$

between the Novikov-Betti numbers and the L^2-Betti numbers corresponding to ξ. However, to our knowledge, Novikov torsion numbers do not have such interpretation in terms of L^2-invariants.

3.3. **Novikov homology for positive real cohomology classes: The proof of Theorem 1.1.**

Definition.

1. A *lattice* in an n-dimensional real vector space V is an additive subgroup L of V of rank n such that L generates V as a real vector space.

(2) Let V be a vector space with lattice L.
   (a) An *open integral half-space* of V is a subset of the form $f^{-1}(\mathbb{R}_{>0})$ where $f\colon V \to \mathbb{R}$ is a homomorphism that takes integral values on L.
   (b) A *closed integral half-space* of V is a subset of the form $f^{-1}(\mathbb{R}_{\geq 0})$ where $f\colon V \to \mathbb{R}$ is a homomorphism that takes integral values on L.
   (c) The intersection of finitely many open and closed integral half-spaces is called an *integral cone*.
   (d) A finite union of integral cones is called an *integral subset* of V.

The following elementary lemma summarizes some properties of integral subsets.

Lemma 3.5. *Let V be a vector space together with a lattice L.*

(1) *The complement of an integral subset is again an integral subset.*
(2) *The intersection of finitely many integral subsets is again an integral subset.*
(3) *The union of finitely many integral subsets is again an integral subset.*
(4) *Any non-empty integral subset contains at least one lattice point.*

Let Γ be a free abelian group of rank n. In the following we always view $\operatorname{Hom}(\Gamma, \mathbb{R})$ as equipped with the lattice $\operatorname{Hom}(\Gamma, \mathbb{Z})$. We can now formulate the following technical proposition that will be proved in the next section.

Proposition 3.6. *Let S be a domain, let Γ be a free abelian group and let C_* be a chain complex of finite free $S[\Gamma]$-modules. Then*

$$\left\{\xi \in \operatorname{Hom}(\Gamma, \mathbb{R}) \,\middle|\, H_*(\mathcal{R}_\xi S[\Gamma] \otimes_{S[\Gamma]} C_*) = 0\right\}$$

is an integral subset of $\operatorname{Hom}(\Gamma, \mathbb{R})$.

The statement of Proposition 3.6 is closely related to work of Pajitnov, see e.g. [15, Theorem 2.2][17, Corollary 2.7]. But to the best of our knowledge the statement of Proposition 3.6 cannot be found in the literature. More precisely, all results that we could found that have similar statements are dealing only with ξ's in $\operatorname{Hom}(\Gamma, \mathbb{R})$ that are monomorphisms.

Assuming Proposition 3.6 we are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $X \subset \mathbb{C}^n$ be a complex hypersurface in general position at infinity, with complement M_X. Let $\psi\colon \pi_1(M_X) \to \Gamma$ be an epimorphism onto a free abelian group Γ and let $\alpha\colon \pi_1(M_X) \to \operatorname{GL}(k, S)$ be a representation over a domain S. Using the notations from Section 2.3, let X^∞ be the link at infinity of X, with complement M_X^∞. As per our convention, we also denote by ψ and α the induced epimorphism $\pi_1(M_X^\infty) \to \Gamma$ and, respectively, the representation $\pi_1(M_X^\infty) \to \operatorname{GL}(k, S)$. Clearly, an admissible pair (ψ, ξ) for M_X induces an admissible pair for M_X^∞.

As in the proof of Theorem 3.4, it follows from Lemma 3.1 and Proposition 3.2 that it suffices to extend the vanishing $H_*^\alpha(M_X^\infty; \mathcal{R}_\xi S[\Gamma]^k) = 0$ to all positive real cohomology classes ξ, with (ψ, ξ) admissible.

We denote by $\widetilde{M_X^\infty}$ the universal cover of M_X^∞ and we write

$$C_* := S[\Gamma]^k \otimes_{\mathbb{Z}[\pi_1(M_X^\infty)]} C_*(\widetilde{M_X^\infty}).$$

In the following, given $\xi \in \mathrm{Hom}(\Gamma, \mathbb{R})$ we denote the induced composite homomorphism $\pi_1(M_X^\infty) \to \pi_1(M_X) \to \Gamma \to \mathbb{R}$ by ξ as well. Note that for any $\xi\colon \Gamma \to \mathbb{R}$ we have

$$\begin{aligned} H_*\big(\mathcal{R}_\xi S[\Gamma] \otimes_{S[\Gamma]} C_*\big) &= H_*\big(\mathcal{R}_\xi S[\Gamma] \otimes_{S[\Gamma]} S[\Gamma]^k \otimes_{\mathbb{Z}[\pi_1(M_X^\infty)]} C_*(\widetilde{M_X^\infty})\big) \\ &\cong H_*\big(\mathcal{R}_\xi S[\Gamma]^k \otimes_{\mathbb{Z}[\pi_1(M_X^\infty)]} C_*(\widetilde{M_X^\infty})\big) \\ &= H_*(M_X^\infty; \mathcal{R}_\xi S[\Gamma]^k). \end{aligned}$$

Combining this observation with Proposition 3.6 and with Lemma 3.5 (1) we see that

$$V := \big\{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,\big|\, \text{there exists an } i \text{ with } H_i(M_X^\infty; \mathcal{R}_\xi S[\Gamma]^k) \neq 0\big\}$$

is an integral subset of $\mathrm{Hom}(\Gamma, \mathbb{R})$.

Now let $\mu_1, \dots, \mu_s$ be the generators of Γ that correspond to the meridians of the s irreducible components of X_{red}. Recall that for an admissible pair (ψ, ξ) we say $\xi \in \mathrm{Hom}(\Gamma, \mathbb{R})$ is positive if $\xi(\mu_i) > 0$ for $i = 1, \dots, s$. We denote by $\mathrm{Hom}^+(\Gamma, \mathbb{R})$ the set of all positive homomorphisms.

Clearly $\mathrm{Hom}^+(\Gamma, \mathbb{R})$ is an integral subset of $\mathrm{Hom}(\Gamma, \mathbb{R})$. From Lemma 3.5 (2) we deduce that $\mathrm{Hom}^+(\Gamma, \mathbb{R}) \cap V$ is an integral subset. By Theorem 3.4 we know that $\mathrm{Hom}^+(\Gamma, \mathbb{Z}) \cap V = \emptyset$. Put differently, $\mathrm{Hom}^+(\Gamma, \mathbb{R}) \cap V$ does not contain a lattice point. It follows from Lemma 3.5 (4) that $\mathrm{Hom}^+(\Gamma, \mathbb{R}) \cap V = \emptyset$. But that means exactly that $H_*(M_X^\infty; \mathcal{R}_\xi[\Gamma]^k) = 0$ for all $\xi \in \mathrm{Hom}^+(\Gamma, \mathbb{R})$. □

3.4. **Proof of Proposition 3.6.** Before we can give the proof of Proposition 3.6 we need to formulate two more lemmas.

Definition. Let Γ be a free abelian group and let S be a domain. Given $p \in S[\Gamma]$ we write

$$M(p) := \{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,|\, p \text{ is invertible in } \mathcal{R}_\xi S[\Gamma]\}$$

and given a matrix A over $S[\Gamma]$ we write

$$M(A) := \{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,|\, A \text{ is invertible over } \mathcal{R}_\xi S[\Gamma]\}.$$

Furthermore, given a chain complex C_* over $S[\Gamma]$ we write

$$M(C_*) := \big\{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,\big|\, H_*(\mathcal{R}_\xi S[\Gamma] \otimes_{S[\Gamma]} C_*) = 0\big\}.$$

Lemma 3.7. *Let S be a domain and let Γ be a free abelian group. For any $p \in S[\Gamma]$ and for any matrix A over $S[\Gamma]$ the sets $M(p)$ and $M(A)$ are integral subsets of $\mathrm{Hom}(\Gamma, \mathbb{R})$.*

Proof. Clearly it suffices to prove the lemma for any non-zero $p \in S[\Gamma]$. By Lemma 2.1 we have

$$M(p) = \{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,|\, t_\xi(p) \text{ is a monomial}\}.$$

We write $p = \sum_{i=1}^n a_i g_i$ with $a_1, \dots, a_n \in S \setminus \{0\}$ and where $g_1, \dots, g_n$ are pairwise disjoint elements of Γ. Furthermore we can arrange that there exists an m such that $a_1, \dots, a_m$ are units in S and such that $a_{m+1}, \dots, a_n$ are not units in S. It follows from the above description of $M(p)$ that

$$M(p) = \bigcup_{i=1}^m \big\{\xi \in \mathrm{Hom}(\Gamma, \mathbb{R}) \,|\, \xi(a_i) > \xi(a_j) \text{ for all } j \neq i\big\}.$$

This shows that $M(p)$ is the disjoint union of finitely many open integral cones, in particular $M(p)$ is an integral subset. □

Definition. Let R be a domain.

(1) We say that a chain complex C_* of free R-modules is *based* if each chain module C_i is equipped with a basis.
(2) Let C_* be a based finite chain complex of length m of finitely generated free R-modules. We denote by $A_i = (a^i_{jk})$, $i = 0, \dots, m-1$ the corresponding boundary matrices. (Here following the convention of [19], we think of elements in R^k as row vectors and we think of the matrices as multiplying on the right.) Following [19, p. 8] we define a *matrix chain for* C_* to be a collection of sets $\alpha = (\alpha_0, \dots, \alpha_m)$ where $\alpha_i \subset \{1, 2, \dots, \dim C_i\}$ so that $\alpha_0 = \emptyset$. We denote by $A_i(\alpha)$ the submatrix of A_i formed by the entries a^i_{jk} with $j \in \alpha_{i+1}$ and $k \notin \alpha_i$. The matrix chain α is called a *τ-chain* if $A_0(\alpha), \dots, A_{m-1}(\alpha)$ are square matrices.

The following lemma is precisely [19, Lemma 2.5].

Lemma 3.8. *Let R be a domain and let C_* be a based finite chain complex of finitely generated free R-modules. We denote by A_* the corresponding boundary matrices. Then $H_i(C_*) = 0$ if and only if there exists a τ-chain α such that $\det(A_i(\alpha))$ is invertible over R for all i.*

Proof of Proposition 3.6. Let S be a domain, let Γ be a free abelian group and let C_* be a chain complex of finite free $S[\Gamma]$-modules of length m. We pick a basis for each chain module C_i. We denote by A_i the corresponding boundary matrices of the chain complex. It follows from Lemma 3.8 that

$$M(C_*) = \bigcup_{\alpha} \{\xi \in \operatorname{Hom}(\Gamma, \mathbb{R}) \mid \det(A_i(\alpha)) \text{ is invertible over } \mathcal{R}_\xi S[\Gamma] \text{ for all } i\},$$

where we take the union over all τ-chains. Put differently, we have

$$M(C_*) = \bigcup_{\alpha} \bigcap_{i=0}^{k-1} M(A_i(\alpha)).$$

Each $M(A_i(\alpha))$ is by Lemma 3.7 an integral subset of $\operatorname{Hom}(\Gamma, \mathbb{R})$. It follows from Lemma 3.5 (2) and (3) that $M(C_*)$ is also an integral subset. □

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