ON RELATIONS AMONG DIRICHLET SERIES WHOSE COEFFICIENTS ARE CLASS NUMBERS OF BINARY CUBIC FORMS

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Abstract. We study the class numbers of integral binary cubic forms. For each $\mathrm{SL}_2(\mathbb{Z})$ invariant lattice L, Shintani introduced Dirichlet series whose coefficients are the class numbers of binary cubic forms in L. We classify the invariant lattices, and investigate explicit relationships between Dirichlet series associated with those lattices. We also study the analytic properties of the Dirichlet series, and rewrite the functional equation in a self dual form using the explicit relationship.

1. Introduction

Study of the class numbers of integral binary cubic forms was initiated by G. Eisenstein and developed by many mathematicians including C. Hermite, F. Arndt, H. Davenport and T. Shintani. Davenport [D] obtained asymptotic formulas for the sum of the class numbers of integral irreducible binary cubic forms of positive and negative discriminants. Shintani [S2] improved the error term by using the Dirichlet series whose coefficients are the class numbers of binary cubic forms introduced in [S1].

Let $V_{\mathbb{Q}}$ be the space of binary cubic forms over the rational number field $\mathbb{Q}$;

$$V_{\mathbb{Q}} = \{x(u,v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3 \mid x_1, \dots, x_4 \in \mathbb{Q}\}.$$

For $x \in V_{\mathbb{Q}}$, the discriminant $P(x)$ is defined by $P(x) = x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2$. The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $V_{\mathbb{Q}}$ by the linear change of variables and $P(x)$ is invariant under the action. Let L be a Γ-invariant lattice in $V_{\mathbb{Q}}$. We put $L_{\pm} = \{x \in L \mid \pm P(x) > 0\}$. For $x \in L$, let Γ_x be the stabilizer of x in Γ and $^{\#}\Gamma_x$ its order.

Definition 1.1. For each invariant lattice L and sign $\pm$, we put

$$\tilde{\xi}_{\pm}(L, s) := \sum_{x \in \Gamma \backslash L_{\pm}} \frac{(^{\#}\Gamma_x)^{-1}}{|P(x)|^s}.$$

This Dirichlet series was introduced by Shintani [S1] as an example of the zeta functions of prehomogeneous vector spaces. It is shown that this Dirichlet series has number of curious properties such as analytic continuation or functional equation. He treated when the invariant lattice is either $L_1 = \{x \in V_{\mathbb{Q}} \mid x_1, x_2, x_3, x_4 \in \mathbb{Z}\}$ or $L_2 = \{x \in V_{\mathbb{Q}} \mid x_1, x_4 \in \mathbb{Z}, x_2, x_3 \in 3\mathbb{Z}\}$, but the proof works for a general invariant lattice as we confirm in this paper. Note that L_1 and L_2 are the dual lattice to each other with respect to the alternating form $\langle x, y\rangle = x_1y_4 - 3^{-1}x_2y_3 + 3^{-1}x_3y_2 - x_4y_1$ on $V_{\mathbb{Q}}$.

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In 1997, the first author [O] conjectured that there are simple relations between $\tilde{\xi}_{\mp}(L_1,s)$ and $\tilde{\xi}_{\pm}(L_2,s)$. This was proved by Nakagawa [N].

Theorem 1.2 (Conjectured in [O], proved in [N])**.**

$$\tilde{\xi}_-(L_1,s)=3^{3s}\tilde{\xi}_+(L_2,s) \qquad \textit{and} \qquad \tilde{\xi}_+(L_1,s)=3^{3s-1}\tilde{\xi}_-(L_2,s).$$

The primary purpose of this paper is to classify the Γ-invariant lattices and investigate whether there are similar formulas for those lattices. In Section 3 we prove the following.

Theorem 1.3 (Theorem 3.3)**.** *There are* 10 *kinds of* Γ*-invariant lattices up to scaling. If we denote these lattices by* $L_1,\dots,L_{10}$ *as in Theorem 2.1, then for Dirichlet series associated with* $L_7,\dots,L_{10}$ *we have*

$$\tilde{\xi}_-(L_7,s)=3^{3s}\tilde{\xi}_+(L_8,s), \qquad \tilde{\xi}_+(L_7,s)=3^{3s-1}\tilde{\xi}_-(L_8,s),$$
$$\tilde{\xi}_-(L_9,s)=3^{3s}\tilde{\xi}_+(L_{10},s), \qquad \tilde{\xi}_+(L_9,s)=3^{3s-1}\tilde{\xi}_-(L_{10},s).$$

On the other hand, the Dirichlet series associated with $L_3,\dots,L_6$ *do not satisfy such simple relations as above. For example,* $\tilde{\xi}_-(L_3,s)$ *and* $3^{3s}\tilde{\xi}_+(L_4,s)$ *do not coincide with each other.*

These relations of the Dirichlet series are proved in Theorem 3.3 using Theorem 1.2. (In Section 3 we slightly modify the definition of the Dirichlet series.) It is likely that the relations among the Dirichlet series for $L_3,\dots,L_6$ are somewhat more complicated. If we take the arithmetic subgroup Γ smaller, there appears more invariant lattices and it may be an interesting problem to study Dirichlet series associated with those lattices. We hope these problems to be answered in the future.

Such a relation of the Dirichlet is expected to exist also for some other representations. Among them for the space of pairs of ternary quadratic forms $(G,V)=(\mathrm{GL}_3\times\mathrm{GL}_2,(\mathrm{Sym}^2\mathrm{Aff}^3)^*\otimes\mathrm{Aff}^2)$, this problem is considerably interesting and being studied by several mathematicians including Bhargava and Nakagawa. We note that there are only 2 types of $G_{\mathbb{Z}}$-invariant lattices for this case.

We explain a curious application of this theorem to the functional equation for $\tilde{\xi}_{\pm}(L_i,s)$. Let $a_1=a_2=0$ and $a_3=\cdots=a_{10}=2$. Following Datskovsky and Wright [DW] we put

$$\Lambda_{\pm}(L_i,s):=\frac{2^{(a_i+1)s}3^{3s/2}}{\pi^{2s}}\Gamma(s)\Gamma(\frac{s}{2}+\frac14\mp\frac13)\Gamma(\frac{s}{2}+\frac14\mp\frac16)\left(\sqrt{3}\tilde{\xi}_+(L_i,s)\pm\tilde{\xi}_-(L_i,s)\right)$$

for each sign. Then Shintani's functional equation between the vector valued functions $(\tilde{\xi}_+(L_i,1-s),\tilde{\xi}_-(L_i,1-s))$ and $(\tilde{\xi}_+(L_{i+1},s),\tilde{\xi}_-(L_{i+1},s))$ $(i=1,3,5,7,9)$ is diagonalized and symmetrized as

$$\Lambda_{\pm}(L_i,1-s)=\pm2^{a_i-b_i}3^{3s-1/2}\Lambda_{\pm}(L_{i+1},s)$$

where $b_1=0$, $b_3=1$, $b_5=3$ and $b_7=b_9=2$. Let i be either 1, 7 or 9. Then Theorems 1.2 and 1.3 state that $\Lambda_{\pm}(L_{i+1},s)=\pm3^{1/2-3s}\Lambda_{\pm}(L_i,s)$. Since $a_i=b_i$ holds also, we can write the functional equations above as follows.

Theorem 1.4 (Theorem 4.8)**.** *Let i be either* 1, 7 *or* 9*. Then*

$$\Lambda_{\pm}(L_i,1-s)=\Lambda_{\pm}(L_i,s).$$

A similar formula holds for $i=2$, 8 *or* 10.

The case $i = 1, 2$ is stated in [O, p.1088]. Unlike Shintani's original one, this functional equation is of the single Dirichlet series $\sqrt{3}\tilde{\xi}_+(L_i, s) \pm \tilde{\xi}_-(L_i, s)$ and also the equation is completely symmetric. We hope this equation might help us to know something on the real nature of the Dirichlet series. Note that the Dirichlet series $\sqrt{3}\tilde{\xi}_+(L_i, s) \pm \tilde{\xi}_-(L_i, s)$ does not have an Euler product for any L_i (see Proposition 4.7.)

This paper is organized as follows. In Section 2, we give the classification of the invariant lattices without a proof. The proof is given in Section 5. In Section 3, we study the explicit relationship of the Dirichlet series. In Section 4 we study the analytic properties of the Dirichlet series. In Theorem 4.3 we give functional equations explicitly and evaluate the residues of the poles. After that we study on the diagonalization of the functional equation and give a simple symmetric functional equation using the result of Section 3. We also give in Theorem 4.9 the density of the class numbers of the lattices. In Section 6, we give a table of about first fifty coefficients of the Dirichlet series.

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Notations. The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the set of rational, real and complex numbers and the rational integers. If V is a variety defined over a ring R and S is an R-algebra then V_S denotes its S-rational points. The 1-dimensional affine space is denoted by Aff.

2. Classification of invariant lattices

Let G be the general linear group of rank 2 and V the space of binary cubic forms;

$$G = \mathrm{GL}_2,$$
$$V = \{x = x(v_1, v_2) = x_1v_1^3 + x_2v_1^2v_2 + x_3v_1v_2^2 + x_4v_2^3 \mid x_i \in \mathrm{Aff}\}.$$

We identify V with Aff^4 via the map $x \mapsto (x_1, x_2, x_3, x_4)$. We define the action of G on V by

$$(gx)(v_1, v_2) = \frac{1}{\det(g)} \cdot x(pv_1 + rv_2, qv_1 + sv_2), \qquad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in G, \quad x \in V.$$

The twist by $\det(g)^{-1}$ is to make the representation faithful. For $x \in V$, let $P(x)$ be the discriminant;

$$P(x) = x_2^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 + 18x_1x_2x_3x_4 - 27x_1^2x_4^2.$$

Then we have $P(gx) = (\det g)^2P(x)$. We put $G^1 = \mathrm{SL}_2$. We assume these are defined over $\mathbb{Z}$.

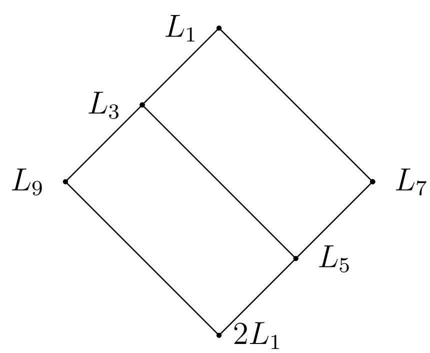
Let $\Gamma \subset G_{\mathbb{Q}}$ be an arithmetic subgroup. The zeta functions of the prehomogeneous vector space (G, V) over $\mathbb{Q}$ are defined for each Γ-invariant lattice in $V_{\mathbb{Q}}$. In this paper we consider the case $\Gamma = G^1_{\mathbb{Z}} = \mathrm{SL}_2(\mathbb{Z})$. To begin we need the classification of the invariant lattices. For a lattice L in $V_{\mathbb{Q}}$ and $q \in \mathbb{Q}^\times$, we put $qL = \{qx \mid x \in L\}$. Then if L is a Γ-invariant lattice, qL is Γ-invariant also. Up to such a scaling, $G^1_{\mathbb{Z}}$-invariant lattices are classified as follows.

Theorem 2.1. *Up to scaling, the following is a complete list of* $\mathrm{SL}_2(\mathbb{Z})$*-invariant lattices in* $V_{\mathbb{Q}}$*:*

$$\begin{aligned}
L_1 &= \{(a,b,c,d) \in \mathbb{Z}^4\} \\
L_2 &= \{(a,3b,3c,d) \in \mathbb{Z}^4 \mid b,c \in \mathbb{Z}\} \\
L_3 &= \{(a,b,c,d) \in L_1 \mid b+c \in 2\mathbb{Z}\} \\
L_4 &= \{(a,3b,3c,d) \in L_2 \mid a,d,b+c \in 2\mathbb{Z}\} \\
L_5 &= \{(a,b,c,d) \in L_1 \mid a,d,b+c \in 2\mathbb{Z}\} \\
L_6 &= \{(a,3b,3c,d) \in L_2 \mid b+c \in 2\mathbb{Z}\} \\
L_7 &= \{(a,b,c,d) \in L_1 \mid a+b+c, b+c+d \in 2\mathbb{Z}\} \\
L_8 &= \{(a,3b,3c,d) \in L_2 \mid a+b+d, a+c+d \in 2\mathbb{Z}\} \\
L_9 &= \{(a,b,c,d) \in L_1 \mid a+b+d, a+c+d \in 2\mathbb{Z}\} \\
L_{10} &= \{(a,3b,3c,d) \in L_2 \mid a+b+c, b+c+d \in 2\mathbb{Z}\}
\end{aligned}$$

We give a proof of this theorem in Section 5. Each of L_3, L_5, L_7, L_9 is a sublattice of L_1 and is containing $2L_1$. The relations of inclusions and their indices are given by

$$[L_1 : L_3] = [L_3 : L_9] = [L_7 : L_5] = [L_5 : 2L_1] = 2,$$
$$[L_1 : L_7] = [L_3 : L_5] = [L_9 : 2L_1] = 4.$$



There are similar relations for $L_2, \ldots, L_{10}$.

We define the alternating form on $V_{\mathbb{Q}}$ by $\langle x, y\rangle = x_1y_4 - 3^{-1}x_2y_3 + 3^{-1}x_3y_2 - x_4y_1$. Then L_i and $2^{-1}L_{i+1}$ are the dual lattices to each other for $i = 3, 5, 7, 9$.

Remark 2.2. We immediately see that all of the lattices in Theorem 2.1 are invariant under the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G_{\mathbb{Z}}$. Since the group $G_{\mathbb{Z}} = \mathrm{GL}_2(\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $G^1_{\mathbb{Z}}$, Theorem 2.1 also gives the list of $\mathrm{GL}_2(\mathbb{Z})$-invariant lattices.

3. Relations of the Dirichlet series

In this section, we define the Dirichlet series for each lattice and study their relations. Let $L_i^+ = \{x \in L_i \mid P(x) > 0\}$ and $L_i^- = \{x \in L_i \mid P(x) < 0\}$. For $x \in L_i$, we put $G^1_{\mathbb{Z},x} = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma x = x\}$ and denote by $^{\#}G^1_{\mathbb{Z},x}$ its order. We note that $^{\#}G^1_{\mathbb{Z},x}$ is either 1 or 3.

Definition 3.1. (1) For $i = 1, 3, 5, 7, 9$, we put

$$\xi_{\pm}(L_i, s) = \sum_{x \in G^1_{\mathbb{Z}} \backslash L_i^{\pm}} \frac{(^{\#}G^1_{\mathbb{Z},x})^{-1}}{|P(x)|^s}.$$

(2) For $i = 2, 4, 6, 8, 10$, we put

$$\xi_\pm(L_i, s) = 3^{3s} \sum_{x \in G^1_{\mathbb{Z}} \backslash L_i^\pm} \frac{(\# G^1_{\mathbb{Z},x})^{-1}}{|P(x)|^s}.$$

These Dirichlet series were introduced by Shintani [S1] as an example of the zeta functions of prehomogeneous vector spaces. This definition in (2) differs from that in [S1] by the factor of 3^{3s}. Note that if $x \in L_2$ then $P(x)$ is a multiple of 3^3. It is known that these Dirichlet series converges for $\Re(s) > 1$. The analytic properties are studied in Section 4.

In [O], the first author gave the following conjecture, and proved that if the conjecture is true then the Shintani's functional equation has a simple symmetric form. This conjecture was proved by Nakagawa [N].

Theorem 3.2 (Nakagawa)**.**

$$\xi_-(L_1, s) = \xi_+(L_2, s), \qquad 3\xi_+(L_1, s) = \xi_-(L_2, s).$$

In this section, we prove the following analogous relations. The simplification and symmetrization of Shintani's functional equation in terms of this theorem is given in Theorem 4.8.

Theorem 3.3.

$$\begin{aligned} \xi_-(L_7, s) &= \xi_+(L_8, s), \\ \xi_-(L_9, s) &= \xi_+(L_{10}, s), \\ 3\xi_+(L_7, s) &= \xi_-(L_8, s), \\ 3\xi_+(L_9, s) &= \xi_-(L_{10}, s). \end{aligned}$$

On the other side the table in Section 6 asserts that, for example, $\xi_-(L_3, s)$ and $\xi_+(L_4, s)$ do not coincide with each other. We will reduce Theorem 3.3 to Theorem 3.2. The proof is given after Proposition 3.8.

To prove this theorem, we study the relation between different lattices. Let $\mathcal{E}$ and $\mathcal{O}$ be the set of even integers and odd integers, respectively;

$$\mathcal{E} = \{2n \mid n \in \mathbb{Z}\}, \quad \mathcal{O} = \{2n+1 \mid n \in \mathbb{Z}\}.$$

We write elements of $L_1 = \mathbb{Z}^4$ as $x = (a, b, c, d)$ in this section. Hence

$$P(x) = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2.$$

We first consider the lattices in L_1. We put $\Delta = ac^3 + b^3d - a^2d^2$. Then

$$P(x) = (bc + ad)^2 - 4\Delta + 16(abcd - 2a^2d^2).$$

Definition 3.4. Let L be a lattice in L_1. For $l, N \in \mathbb{Z}$, $N \neq 0$, we put

$$L_{\equiv l\,(N)} = \{x \in L \mid P(x) \equiv l \mod N\}.$$

Proposition 3.5. *We have*

$$\begin{aligned} L_7 &= 2L_1 \amalg L_{1,\equiv 1\,(8)}, \\ L_9 &= 2L_1 \amalg L_{1,\equiv 5\,(8)}, \end{aligned}$$

We start with a lemma.

Lemma 3.6. *Let* $x = (a, b, c, d) \in L_1$.

(1) $P(x) \equiv 1 \mod 8$ *if and only if one of the following holds;*
  (a) $a, d \in \mathcal{E},\ b, c \in \mathcal{O}$,
  (b) $a, d \in \mathcal{O},\ b + c \in \mathcal{O}$.
(2) $P(x) \equiv 5 \mod 8$ *if and only if one of the following holds;*
  (a) $b, c \in \mathcal{E},\ a, d \in \mathcal{O}$,
  (b) $b, c \in \mathcal{O},\ a + d \in \mathcal{O}$.

Proof. Let $P(x) \equiv 1 \mod 4$. Then $ad + bc \in \mathcal{O}$ and $P(x) \equiv 1 + 4\Delta \mod 8$. Hence to know $P(x) \mod 8$, what we should see is $\Delta \mod 2$. Now the lemma follows from the observations below. In the following congruence expression means modulo 2.

(I) Assume $a + d \in \mathcal{O}$. Then $ad \in \mathcal{E}$, $bc \in \mathcal{O}$, $b, c \in \mathcal{O}$. Hence $\Delta \equiv ac^3 + bd^3 \equiv a + d \equiv 1$.
(II) Assume $a + d \in \mathcal{E}$. If $a, d \in \mathcal{O}$, then $bc \in \mathcal{E}$ and $\Delta \equiv b^3 + c^3 + 1 \equiv b + c + 1$. Hence either $(b, c \in \mathcal{E},\ \Delta \equiv 1)$ or $(b + c \in \mathcal{O},\ \Delta \equiv 0)$. If $a, d \in \mathcal{E}$, then $bc \in \mathcal{O}$ and hence $\Delta \equiv 0$. □

Proof of Proposition 3.5. We first show $L_7 = 2L_1 \amalg L_{1,\equiv 1\,(8)}$. Let $x = (a, b, c, d) \in L_{1,\equiv 1\,(8)}$. Then by the lemma above we have $a+b+c, b+c+d \in \mathcal{E}$ and so $x \in L_7$. Hence $L_7 \supset 2L_1 \amalg L_{1,\equiv 1\,(8)}$. We consider the reverse inclusion. Let $x = (a, b, c, d) \in L_7$. Then $a+b+c, b+c+d \in \mathcal{E}$, and so $a+d \in \mathcal{E}$. First assume $a, d \in \mathcal{O}$. Then $b+c \in \mathcal{O}$ and hence $x \in L_{1,\equiv 1\,(8)}$. Next assume $a, d \in \mathcal{E}$. Then $b + c \in \mathcal{E}$ and hence either $(a, b, c, d \in \mathcal{E})$ or $(a, d \in \mathcal{E},\ b, c \in \mathcal{O})$. This shows $x \in 2L_1 \amalg L_{1,\equiv 1\,(8)}$. Hence $L_7 \subset 2L_1 \amalg L_{1,\equiv 1\,(8)}$.

The equation $L_9 = 2L_1 \amalg L_{1,\equiv 5\,(8)}$ is proved similarly. □

We next consider the lattices in L_2. Recall that for $x \in L_2$, $P(x)$ is a multiple of 27. We put $Q(x) = P(x)/27$. Then $Q(x) \equiv 3P(x) \mod 8$.

Definition 3.7. Let L be a lattice in L_2. For $l, N \in \mathbb{Z}$, $N \neq 0$, we put

$$L_{\equiv' l\,(N)} = \{x \in L \mid Q(x) \equiv l \mod N\}.$$

Since $Q(x) \equiv 3P(x) \mod 8$, we have $L_{\equiv l\,(8)} = L_{\equiv' 3l\,(8)}$.

Proposition 3.8. *We have*

$$L_8 = 2L_2 \amalg L_{2,\equiv' 7\,(8)},$$
$$L_{10} = 2L_2 \amalg L_{2,\equiv' 3\,(8)},$$

Proof. The first one follows from $L_9 = 2L_1 \amalg L_{1,\equiv 5\,(8)}$ we proved in Proposition 3.5 and

$$L_9 \cap L_2 = L_8, \quad 2L_1 \cap L_2 = 2L_2, \quad L_{1,\equiv 5\,(8)} \cap L_2 = L_{2,\equiv 5\,(8)} = L_{2,\equiv' 7\,(8)}.$$

The second one is proved similarly. □

We now give a proof of Theorem 3.3.

Proof of Theorem 3.3. Let $\{a_n\}$ be the coefficients of $\xi_-(L_1, s)$;

$$\xi_-(L_1, s) = \sum_{n \geq 1} \frac{a_n}{n^s}.$$

Then by Proposition 3.5,

$$\xi_-(L_7,s)=\frac{1}{2^{4s}}\xi_-(L_1,s)+\sum_{n\geq 1, n\equiv 7\,(8)}\frac{a_n}{n^s},$$
$$\xi_-(L_9,s)=\frac{1}{2^{4s}}\xi_-(L_1,s)+\sum_{n\geq 1, n\equiv 3\,(8)}\frac{a_n}{n^s}.$$

If we put $\xi_+(L_2,s)=\sum_{n\geq 1} b_n/n^s$ then similarly by Proposition 3.8 we have

$$\xi_+(L_8,s)=\frac{1}{2^{4s}}\xi_+(L_2,s)+\sum_{n\geq 1, n\equiv 7\,(8)}\frac{b_n}{n^s},$$
$$\xi_+(L_{10},s)=\frac{1}{2^{4s}}\xi_+(L_2,s)+\sum_{n\geq 1, n\equiv 3\,(8)}\frac{b_n}{n^s}.$$

Hence the first two formulas follows from $\xi_-(L_1,s)=\xi_+(L_2,s)$ and $a_n=b_n$. The rests are proved similarly. □

We will give some properties on $\xi_\pm(L_i,s)$. These can be checked using the table of the coefficients of $\xi_\pm(L_i,s)$ given in Section 6.

Proposition 3.9. (1) *The Dirichlet series $\xi_\pm(L_i,s)$ does not have an Euler product.*

(2) *The linear relations of the twenty Dirichlet series $\{\xi_\pm(L_i,s)\}$ are exhausted by that given in Theorems 3.2 and 3.3. Namely, the $\mathbb{C}$-vector space spanned by Dirichlet series by $\{\xi_\pm(L_i,s)\}$ is of dimension* 14.

4. Analytic properties of the Dirichlet series

In this section, we study analytic properties of $\xi_\pm(L_i,s)$. We also separate the contributions of irreducible binary cubic forms and reducible binary cubic forms in the residue formulas. Let $V_{\mathbb{Z}}^{\mathrm{ird}}=\{x(v)\in V_{\mathbb{Z}}\mid x(v) \text{ is irreducible over } \mathbb{Q}\}$ and $V_{\mathbb{Z}}^{\mathrm{rd}}=V_{\mathbb{Z}}\setminus V_{\mathbb{Z}}^{\mathrm{ird}}$. They are $G_{\mathbb{Z}}$-invariant subsets.

Definition 4.1. (1) For $i=1,3,5,7,9$, we put

$$\xi_\pm^{\mathrm{ird}}(L_i,s)=\sum_{x\in G_{\mathbb{Z}}^1\backslash(L_i^\pm\cap V_{\mathbb{Z}}^{\mathrm{ird}})}\frac{(^\#G_{\mathbb{Z},x}^1)^{-1}}{|P(x)|^s},\qquad \xi_\pm^{\mathrm{rd}}(L_i,s)=\sum_{x\in G_{\mathbb{Z}}^1\backslash(L_i^\pm\cap V_{\mathbb{Z}}^{\mathrm{rd}})}\frac{(^\#G_{\mathbb{Z},x}^1)^{-1}}{|P(x)|^s}.$$

(2) For $i=2,4,6,8,10$, we put

$$\xi_\pm^{\mathrm{ird}}(L_i,s)=3^{3s}\sum_{x\in G_{\mathbb{Z}}^1\backslash(L_i^\pm\cap V_{\mathbb{Z}}^{\mathrm{ird}})}\frac{(^\#G_{\mathbb{Z},x}^1)^{-1}}{|P(x)|^s},\qquad \xi_\pm^{\mathrm{rd}}(L_i,s)=3^{3s}\sum_{x\in G_{\mathbb{Z}}^1\backslash(L_i^\pm\cap V_{\mathbb{Z}}^{\mathrm{rd}})}\frac{(^\#G_{\mathbb{Z},x}^1)^{-1}}{|P(x)|^s}.$$

By definition we have $\xi_\pm(L_i,s)=\xi_\pm^{\mathrm{ird}}(L_i,s)+\xi_\pm^{\mathrm{rd}}(L_i,s)$.

Definition 4.2. For $i=1,3,5,7,9$, we put $a_i=[\widehat{L_i}:L_{i+1}]$ and $2^{b_i}=[V_{\mathbb{Z}}:L_i]$, where $\widehat{L_i}$ is the dual lattice of L_i with respect to the bilinear form $\langle x,y\rangle$.

It is easy to see that (a_i,b_i) is $(0,0)$, $(2,1)$, $(2,3)$, $(2,2)$, $(2,2)$ for $i=1,3,5,7,9$, respectively. The analytic properties of these series are summarized as follows.

Theorem 4.3. (1) *The Dirichlet series $\xi_\pm(L_i,s)$ can be continued holomorphically to the whole complex plane except for simple poles at $s=1$ and $5/6$. Furthermore, they satisfy the following functional equations*

$$\begin{pmatrix}\xi_+(L_i,1-s)\\ \xi_-(L_i,1-s)\end{pmatrix} = \frac{2^{2a_is-b_i}3^{3s-2}}{2\pi^{4s}}\Gamma(s)^2\Gamma(s-\frac{1}{6})\Gamma(s+\frac{1}{6})\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin\pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\xi_+(L_{i+1},s)\\ \xi_-(L_{i+1},s)\end{pmatrix}$$

where $i=1,3,5,7,9$.

(2) *The Dirichlet series $\xi^{\rm ird}_\pm(L_i,s)$ and $\xi^{\rm rd}_\pm(L_i,s)$ have meromorphic continuations to the whole complex plane. The first one is holomorphic for $\Re(s)>1/2$ except for simple poles at $s=1$ and $s=5/6$. The second one is holomorphic for $\Re(s)>1/2$ except for a simple pole at $s=1$.*

(3) *Let*

$$\alpha_{i,\pm} = \mathrm{Res}_{s=1}\,\xi_\pm(L_i,s),\quad \beta_{i,\pm} = \mathrm{Res}_{s=5/6}\,\xi_\pm(L_i,s),$$
$$\alpha^{\rm ird}_{i,\pm} = \mathrm{Res}_{s=1}\,\xi^{\rm ird}_\pm(L_i,s),\quad \alpha^{\rm rd}_{i,\pm} = \mathrm{Res}_{s=1}\,\xi^{\rm rd}_\pm(L_i,s).$$

Then if we put

$$\alpha=\frac{\pi^2}{9},\qquad \beta=\frac{3^{1/2}(2\pi)^{1/3}}{18}\zeta\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)^{-1},$$

the values are given by Table 1.

Proof. For L_1 and L_2, Shintani [S1, S2] proved this theorem by establishing the theory of zeta functions associated with the space of binary cubic forms and the space of binary quadratic forms. His global theory was rewritten in the adelic language by Wright [W] and the second author [T]. We would like to mention that a quite simpler version of the global theory for the space of binary cubic forms [W] were given by Kogiso [K]. Let $\mathbb{A}$ and $\mathbb{A}_{\rm f}$ be the rings of adeles and finite adeles of $\mathbb{Q}$, respectively. Note that $\mathbb{A}_{\rm f}=\widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathbb{A}=\mathbb{A}_{\rm f}\times\mathbb{R}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Let $\mathcal{S}(V_{\mathbb{A}})$, $\mathcal{S}(V_{\mathbb{A}_{\rm f}})$ and $\mathcal{S}(V_{\mathbb{R}})$ be the spaces of Schwartz-Bruhat functions on each of the indicated domains. Let $\Phi_{\rm f}\in\mathcal{S}(V_{\mathbb{A}_{\rm f}})$ be the characteristic function of $L_i\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}\subset V_{\mathbb{A}_{\rm f}}$ and $\Phi_\infty\in\mathcal{S}(V_{\mathbb{R}})$ arbitrary. Then by considering the global zeta functions in [T, W] with the test function $\Phi_{\rm f}\otimes\Phi_\infty\in\mathcal{S}(V_{\mathbb{A}})$, we can prove the theorem the same way as [S1, S2]. Here we illustrate the proof of (3) with $i=3,5,7,9$. We fix a prime p. We fix any Haar measures du on $\mathbb{Q}_p$ and $d^\times t$ on $\mathbb{Q}_p^\times$. For $t\in\mathbb{Q}_p^\times$, we put $|t|_p=d(tu)/du$. For $\Phi\in\mathcal{S}(V_{\mathbb{Q}_p})$, we define

$$\mathcal{A}^{\rm ird}_p(\Phi)=\int_{\mathbb{Q}_p^4}\Phi(u_1,u_2,u_3,u_4)du_1du_2du_3du_4,$$
$$\mathcal{A}^{\rm rd}_p(\Phi)=\int_{\mathbb{Q}_p^\times\times\mathbb{Q}_p^2}|t|_p^2\Phi(0,t,u_1,u_2)d^\times t du_1du_2,$$
$$\mathcal{B}_p(\Phi)=\int_{\mathbb{Q}_p^\times\times\mathbb{Q}_p^3}|t|_p^{1/3}\Phi(t,u_1,u_2,u_3)d^\times t du_1du_2du_3.$$

Let Φ_i be the characteristic function of $L_i\otimes\mathbb{Z}_p$. Since $i=3,5,7,9$ we have $\Phi_i=\Phi_1$ unless $p=2$. Hence by [T, Proposition 8.6], we have

$$\frac{\alpha^{\rm ird}_{i,\pm}}{\alpha^{\rm ird}_{1,\pm}}=\frac{\mathcal{A}^{\rm ird}_2(\Phi_i)}{\mathcal{A}^{\rm ird}_2(\Phi_1)},\qquad \frac{\alpha^{\rm rd}_{i,\pm}}{\alpha^{\rm rd}_{1,\pm}}=\frac{\mathcal{A}^{\rm rd}_2(\Phi_i)}{\mathcal{A}^{\rm rd}_2(\Phi_1)},\qquad \frac{\beta_{i,\pm}}{\beta_{1,\pm}}=\frac{\mathcal{B}_2(\Phi_i)}{\mathcal{B}_2(\Phi_1)}.$$

i	1	3	5	7	9	2	4	6	8	10
$\alpha_{i,+}$	α	$\frac{\alpha}{2}$	$\frac{7}{32}\alpha$	$\frac{\alpha}{4}$	$\frac{\alpha}{4}$	$\frac{3}{2}\alpha$	$\frac{9}{32}\alpha$	$\frac{3}{4}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{8}\alpha$
$\beta_{i,+}$	β	$\frac{\beta}{2}$	$\frac{\beta}{4\sqrt[3]{2}}$	$\frac{\beta}{4}$	$\frac{\beta}{4}$	$\sqrt{3}\beta$	$\frac{\sqrt{3}}{4\sqrt[3]{2}}\beta$	$\frac{\sqrt{3}}{2}\beta$	$\frac{\sqrt{3}}{4}\beta$	$\frac{\sqrt{3}}{4}\beta$
$\alpha_{i,+}^{\mathrm{ird}}$	$\frac{1}{4}\alpha$	$\frac{1}{8}\alpha$	$\frac{1}{32}\alpha$	$\frac{1}{16}\alpha$	$\frac{1}{16}\alpha$	$\frac{3}{4}\alpha$	$\frac{3}{32}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$
$\alpha_{i,+}^{\mathrm{rd}}$	$\frac{3}{4}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{4}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$
$\alpha_{i,-}$	$\frac{3}{2}\alpha$	$\frac{3}{4}\alpha$	$\frac{9}{32}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{8}\alpha$	3α	$\frac{15}{32}\alpha$	$\frac{3}{2}\alpha$	$\frac{3}{4}\alpha$	$\frac{3}{4}\alpha$
$\beta_{i,-}$	$\sqrt{3}\beta$	$\frac{\sqrt{3}}{2}\beta$	$\frac{\sqrt{3}}{4\sqrt[3]{2}}\beta$	$\frac{\sqrt{3}}{4}\beta$	$\frac{\sqrt{3}}{4}\beta$	3β	$\frac{3}{4\sqrt[3]{2}}\beta$	$\frac{3}{2}\beta$	$\frac{3}{4}\beta$	$\frac{3}{4}\beta$
$\alpha_{i,-}^{\mathrm{ird}}$	$\frac{3}{4}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{32}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$	$\frac{9}{4}\alpha$	$\frac{9}{32}\alpha$	$\frac{9}{8}\alpha$	$\frac{9}{16}\alpha$	$\frac{9}{16}\alpha$
$\alpha_{i,-}^{\mathrm{rd}}$	$\frac{3}{4}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{4}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{8}\alpha$	$\frac{3}{16}\alpha$	$\frac{3}{16}\alpha$

TABLE 1

The computations of the right hand sides in the equations are easily carried out. For example,

$$\frac{\mathcal{A}_2^{\mathrm{ird}}(\Phi_3)}{\mathcal{A}_2^{\mathrm{ird}}(\Phi_1)} = \frac{1}{2}, \qquad \frac{\mathcal{A}_2^{\mathrm{rd}}(\Phi_5)}{\mathcal{A}_2^{\mathrm{rd}}(\Phi_1)} = \frac{1}{4}, \qquad \frac{\mathcal{B}_2(\Phi_7)}{\mathcal{B}_2(\Phi_1)} = \frac{1}{4}.$$

Since $\alpha_{1,\pm}^{\mathrm{ird}}$, $\alpha_{1,\pm}^{\mathrm{rd}}$ and $\beta_{1,\pm}$ are known, we obtain the value. Note that $\alpha_{i,\pm} = \alpha_{i,\pm}^{\mathrm{ird}} + \alpha_{i,\pm}^{\mathrm{rd}}$. The rest are proved similarly and we omit the detail. Note that $a_3 = a_5 = a_7 = a_9 = 2$ in (1) comes from the fact that for $i = 3, 5, 7, 9$ the dual lattice of L_i with respect to the alternating form on V is $2^{-1}L_{i+1}$. Also $b_3 = 1$, $b_5 = 3$ and $b_7 = b_9 = 2$ are because $[L_1 : L_3] = 2$, $[L_1 : L_5] = 8$ and $[L_1 : L_7] = [L_1 : L_9] = 4$, respectively. □

Remark 4.4. As in [O, Proposition 2.1], the functional equation in the theorem is compatible with Theorem 3.3. For example, from $\xi_-(L_7, s) = \xi_+(L_8, s)$ and Theorem 4.3 (1) for $i = 7$, we can deduce $\xi_-(L_8, s) = 3\xi_+(L_7, s)$.

We discuss on the diagonalization of the functional equation in Theorem 4.3 (1) following [DW, Proposition 4.1] and a related important observation given in [O, p.1088]. Let $a_{i+1} = a_i$ for $i = 1, 3, 5, 7, 9$.

Definition 4.5. For $1 \leq i \leq 10$ and each sign $\pm$, we put

$$\Lambda_\pm(L_i, s) = \frac{2^{(a_i+1)s}3^{3s/2}}{\pi^{2s}}\Gamma(s)\Gamma(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{6})\Gamma(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{3})\left(\sqrt{3}\xi_+(L_i, s) \pm \xi_-(L_i, s)\right).$$

As a corollary to Theorem 4.3 we have the following.

Corollary 4.6. (1) *For* $i = 1, 3, 5, 7, 9$,

$$\Lambda_\pm(L_i, 1-s) = \pm 3^{-1/2} 2^{a_i - b_i} \Lambda_\pm(L_{i+1}, s).$$

(2) *Let* $1 \le i \le 10$. *The function* $\Lambda_+(L_i, s)$ *is holomorphic except for simple poles at* $s = 0, 1/6, 5/6, 1$, *while* $\Lambda_-(L_i, s)$ *is holomorphic except for simple poles at* $s = 0, 1$.

(3) *Let* $1 \le i \le 10$. *The set of zeros of the Dirichlet series* $\sqrt{3}\xi_+(L_i, s) + \xi_-(L_i, s)$ *and* $\sqrt{3}\xi_+(L_i, s) - \xi_-(L_i, s)$ *in the negative real axis are respectively given by*

$$\{-n \mid n \in \mathbb{Z}_{\ge 1}\} \cup \{-2n + 1/6 \mid n \in \mathbb{Z}_{\ge 1}\} \cup \{-2n + 11/6 \mid n \in \mathbb{Z}_{\ge 1}\},$$
$$\{-n \mid n \in \mathbb{Z}_{\ge 1}\} \cup \{-2n + 5/6 \mid n \in \mathbb{Z}_{\ge 1}\} \cup \{-2n + 7/6 \mid n \in \mathbb{Z}_{\ge 1}\},$$

where we put $\mathbb{Z}_{\ge 1} = \{n \in \mathbb{Z} \mid n \ge 1\}$.

Proof. By a simple computation we can prove that the equalities in (1) are equivalent to the functional equation given in Theorem 4.3 (1). (2) follows from the values of residues given in Theorem 4.3 (3) and equalities (1) of this corollary. (3) follows from (2) and Definition 4.5. □

It is interesting that the poles of $\Lambda_-(L_i, s)$ at $s = 5/6$ vanishes. Taking the properties in Corollary 4.6 into account, it may be natural to ask that whether the Dirichlet series $\sqrt{3}\xi_+(L_i, s) \pm \xi_-(L_i, s)$ has an Euler product. The answer is negative.

Proposition 4.7. *None of the Dirichlet series* $\sqrt{3}\xi_+(L_i, s) + \xi_-(L_i, s)$, $\sqrt{3}\xi_+(L_i, s) - \xi_-(L_i, s)$ $(1 \le i \le 10)$ *has an Euler product.*

Proof. If a Dirichlet series $\sum_{n \ge 1} c_n/n^s$ has an Euler product, then $c_1 c_{pq} = c_p c_q$ for any distinct primes p and q. We can immediately confirm that any of our Dirichlet series does not satisfy this relation for $p = 3$ and $q = 5$ using the table given in Section 6. □

Now we assume $i = 1, 7, 9$. Then Theorems 3.2, 3.3 assert $\Lambda_\pm(L_{i+1}, s) = \pm\sqrt{3}\Lambda_\pm(L_i, s)$. Since $a_i = b_i$ also, the functional equation in Corollary 4.6 (1) turns out to be of a single function $\Lambda_\pm(L_i, s)$.

Theorem 4.8. *For* $i = 1, 2, 7, 8, 9, 10$,

$$\Lambda_\pm(L_i, 1-s) = \Lambda_\pm(L_i, s).$$

Namely, for $i = 1, 2, 7, 8, 9, 10$, the function $\Lambda_\pm(L_i, s)$ is invariant if we replace s by $1 - s$.

We conclude this section with deriving asymptotic behavior of some arithmetic functions. For $n \in \mathbb{Z}, n \ne 0$, let $h(L_i, n)$ be the number of $G^1_{\mathbb{Z}}$-orbit in $L_i \cap V^{\mathrm{ird}}_{\mathbb{Z}}$ with discriminant n. Applying Tauberian theorem, Shintani [S2, Theorem 4] obtained an asymptotic formula of the function $\sum_{0<\pm n<X} h(L_1, n)$. By the same argument, we have the following. Note that the functional equations of $\xi_\pm(L_i, s)$ and $\xi^{\mathrm{rd}}_\pm(L_i, s)$ are used in the proof.

Theorem 4.9. (1) *Let* i *be either* $1, 3, 5, 7$ *or* 9. *For any* $\varepsilon > 0$,

$$\sum_{0<\pm n<X} h(L_i, n) = \alpha^{\mathrm{ird}}_{i,\pm} X + \frac{\beta_{i,\pm}}{5/6} X^{5/6} + O(X^{2/3+\varepsilon}) \qquad (X \to \infty).$$

(2) *Let i be either* $2,4,6,8$ *or* 10. *For any* $\varepsilon>0$,

$$\sum_{0<\pm n<X} h(L_i,27n)=\alpha_{i,\pm}^{\mathrm{ird}}X+\frac{\beta_{i,\pm}}{5/6}X^{5/6}+O(X^{2/3+\varepsilon}) \qquad (X\to\infty).$$

5. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. We use an argument similar to [IS, Section 3]. Let L be a $\mathrm{SL}_2(\mathbb{Z})$-invariant lattice. By taking some constant multiple if necessary, we can assume that L is contained in L_1 and that there exists an element $x\in L$ such that $p^{-1}x\notin L_1$ for each prime p. Such an element x is called primitive for p. We put $(L)_p=L\otimes_{\mathbb{Z}}\mathbb{Z}_p$ for a prime p. In the following, we prove that $(L)_p=(L_1)_p$ $(p\neq 2,3)$ in Lemma 5.1, $(L)_3=(L_1)_3$ or $(L_2)_3$ in Lemma 5.2, and $(L)_2=(L_1)_2$, $(L_3)_2$, $(L_5)_2$, $(L_7)_2$ or $(L_9)_2$ in Lemma 5.4. It is easy to see that the lattices $L_1,L_2,\dots,L_{10}$ are $G^1_{\mathbb{Z}}$-invariant. Therefore we get Theorem 2.1 by these facts, because $L=\cap_{p:\mathrm{prime}}(V_{\mathbb{Q}}\cap(L)_p)$.

From now, we shall prove Lemmas 5.1, 5.2 and 5.4. Since $\mathrm{SL}_2(\mathbb{Z}_p)$ contains $\mathrm{SL}_2(\mathbb{Z})$ as a dense subgroup, $(L)_p$ is $\mathrm{SL}_2(\mathbb{Z}_p)$-invariant. We put

$$u(\alpha)=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix},\quad w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and $E_1=(1,0,0,0)$, $E_2=(0,1,0,0)$, $E_3=(0,0,1,0)$, $E_4=(0,0,0,1)$. The action of $u(\alpha)$ on L is given by

$$u(\alpha)\cdot x=(x_1+\alpha x_2+\alpha^2x_3+\alpha^3x_4,x_2+2\alpha x_3+3\alpha^2x_4,x_3+3\alpha x_4,x_4).$$

For $x\in L$, we put

$$\psi(x)=u(1)\cdot x-x=(x_2+x_3+x_4,2x_3+3x_4,3x_4,0)\in L.$$

Lemma 5.1. *If* $p\neq 2,3$, *then* $(L)_p=(L_1)_p$.

Proof. Let $x=(x_1,x_2,x_3,x_4)\in L$ be primitive for p.

First we assume that $x_1\in\mathbb{Z}_p^\times$ or $x_4\in\mathbb{Z}_p^\times$. By considering the action of w, we may assume $x_4\in\mathbb{Z}_p^\times$. Let $X_1=x_4^{-1}u(-3^{-1}x_4^{-1}x_3)\cdot x$. Then since X_1 is of the form $(*,*,0,1)$, we have $6^{-1}\psi(\psi(X_1))=(1,1,0,0)$. Since $E_2=u(-1)\cdot(1,1,0,0)$ and $E_1=\psi(E_2)$, we have E_1, E_2, E_3, $E_4\in(L)_p$. Hence $(L)_p=(L_1)_p$.

Second we assume $x_1,x_4\notin\mathbb{Z}_p^\times$. Then we have $x_2\in\mathbb{Z}_p^\times$ or $x_3\in\mathbb{Z}_p^\times$. We may assume $x_3\in\mathbb{Z}_p^\times$. Since the first component of $u(1)\cdot x+u(-1)\cdot x-2x$ is $2x_3\in\mathbb{Z}_p^\times$, by the argument above we have $(L)_p=(L_1)_p$. □

Lemma 5.2. $(L)_3=(L_1)_3$ *or* $(L_2)_3$.

Proof. Let $x=(x_1,x_2,x_3,x_4)\in L$ be primitive for 3.

First we assume $x_2\in\mathbb{Z}_3^\times$ or $x_3\in\mathbb{Z}_3^\times$. Taking the action of w into account, we may assume $x_3\in\mathbb{Z}_3^\times$. Let $X_1=(2x_3+3x_4)^{-1}\psi(x)=(x_1',1,x_3',0)$ and $X_2=(2x_3+6x_4)^{-1}\psi(\psi(x))=(1,x_2',0,0)$. Then $x_2',x_3'\in 3\mathbb{Z}_3$. Further we put $X_3=X_1-x_1'X_2=(0,1-x_1'x_2',x_3',0)$, $1-x_1'x_2'\in\mathbb{Z}_3^\times$, $X_4=(1-x_1'x_2')^{-1}(w\cdot X_3)=(0,x_2'',1,0)$, $x_2''\notin\mathbb{Z}_3^\times$, $X_5=u(-2^{-1}x_2'')\cdot X_4=(x_1'',0,1,0)$, $X_6=\psi(X_5)=(1,2,0,0)$. Then since $E_1=2^{-1}\psi(X_6)$ and $E_2=2^{-1}(X_6-E_1)$, we have $(L)_3=(L_1)_3$.

Second we assume $x_2,x_3\notin\mathbb{Z}_3^\times$. Then we have $x_1\in\mathbb{Z}_3^\times$ or $x_4\in\mathbb{Z}_3^\times$. We may assume $x_4\in\mathbb{Z}_3^\times$. We have $X_7=\psi(x)=(x_2+x_3+x_4,2x_3+3x_4,3x_4,0)$, $x_2+x_3+x_4\in\mathbb{Z}_3^\times$, $2x_3+3x_4\in 3\mathbb{Z}_3$, $3x_4\in 3\mathbb{Z}_3^\times$, $X_8=u(-2^{-1}x_4^{-1}\cdot 3^{-1}(2x_3+3x_4))\cdot X_7=(x_1',0,3x_4,0)$,

$x_1' \in \mathbb{Z}_3^\times$, $3x_4 \in 3\mathbb{Z}_3^\times$. Then since $x_4^{-1}\psi(X_8) = 3E_1 + 6E_2$, $2^{-1}\psi(3E_1 + 6E_2) = 3E_1$, $3E_2 = 2^{-1}((3E_1 + 6E_2) - 3E_1)$ and $E_1 = x_1'^{-1} \cdot (X_8 - 3x_4E_3)$, we get $(L_2)_3 \subset (L)_3$.

We see $(L_2)_3 \subset (L)_3 \subset (L_1)_3$ from the above results. Suppose $(L_2)_3 \neq (L)_3$. Since $(L_1)_3/(L_2)_3$ is represented by the set $\{aE_2 + bE_3 \ ; \ 0 \leq a, b \leq 2\}$, $(L)_3$ has an element of the form $aE_2 + bE_3$ for some $(a, b) \neq (0, 0)$. Hence we have $(L)_3 = (L_1)_3$. So we get this lemma. □

Lemma 5.3. *$(L)_2$ contains $(L_5)_2$ or $(L_9)_2$.*

Proof. Let $x = (x_1, x_2, x_3, x_4) \in L$ be primitive for 2.

(i) We assume $x_1 \in \mathbb{Z}_2^\times$ or $x_4 \in \mathbb{Z}_2^\times$. We may assume $x_4 \in \mathbb{Z}_2^\times$. Let $X_1 = u(-3^{-1}x_4^{-1}x_3) \cdot x = (*, *, 0, x_4)$, $X_2 = (3x_4)^{-1}\psi(X_1) = (x_1', 1, 1, 0)$. Then since $2E_1 + 2E_2 = \psi(X_2)$ and $2E_1 = \psi(\psi(X_2))$, we have $2E_1$, $2E_2$, $2E_3$, $2E_4 \in (L)_2$.
(i-a) We assume $x_1' \notin \mathbb{Z}_2^\times$. We have $E_2 + E_3 = X_2 - (2^{-1}x_1') \cdot (2E_1) \in (L)_2$. Since $L_5 = \mathbb{Z}(2E_1) + \mathbb{Z}(2E_4) + \mathbb{Z}(E_2 + E_3) + \mathbb{Z}(2E_2)$, we get $(L_5)_2 \subset (L)_2$.
(i-b) We assume $x_1' \in \mathbb{Z}_2^\times$. From $x_1' = 1 + x_1''$, $(x_1'' \in 2\mathbb{Z}_2)$, we have $X_2 - (2^{-1}x_1'') \cdot (2E_1) = E_1 + E_2 + E_3$. Since $L_9 = \mathbb{Z}(E_1 + E_2 + E_3) + \mathbb{Z}(E_2 + E_3 + E_4) + \mathbb{Z}(2E_1) + \mathbb{Z}(2E_2)$. we get $(L_9)_2 \subset (L)_2$.

(ii) We assume $x_1, x_4 \notin \mathbb{Z}_2^\times$.
(ii-a) We assume $x_2 + x_3 \in \mathbb{Z}_2^\times$. Since the first component of $\psi(x)$ is $x_2 + x_3 + x_4 \in \mathbb{Z}_2^\times$, we can reduce the case (ii-a) to the case (i).
(ii-b) We assume $x_2 + x_3 \notin \mathbb{Z}_2^\times$. Since x is primitive, we have x_2, $x_3 \in \mathbb{Z}_2^\times$. We have $X_3 = (x_3 + 3x_4)^{-1}\psi(\psi(x)) = (2, c, 0, 0)$, $c \in 4\mathbb{Z}_2$, $X_4 = w^{-1} \cdot X_3 = -cE_3 + 2E_4$. Furthermore we put $X_5 = x - (2^{-1}x_1) \cdot X_3 - (2^{-1}x_4) \cdot X_4 = (0, \alpha, \beta, 0)$. Then $\alpha = x_2 - 2^{-1}x_1c \in \mathbb{Z}_2^\times$, $\beta = x_3 + 2^{-1}x_4c \in \mathbb{Z}_2^\times$. Let $X_6 = \psi(X_5) - 2^{-1}(\alpha+\beta)X_3 = (0, 2\beta - 2^{-1}(\alpha+\beta)c, 0, 0)$. Then $2\beta - 2^{-1}(\alpha+\beta)c \in 2\mathbb{Z}_2^\times$. Hence we have $2E_2 = (\beta - 2^{-2}(\alpha+\beta)c)^{-1}X_6$, $2E_1 = X_3 - (2^{-1}c) \cdot (2E_2)$, $2E_3$, $2E_4 \in (L)_2$, $E_2 + E_3 = X_5 - 2^{-1}(\alpha - 1) \cdot (2E_2) - 2^{-1}(\beta - 1) \cdot (2E_3) \in (L)_2$. Therefore we get $(L_5)_2 \subset (L)_2$. □

Lemma 5.4. *$(L)_2 = (L_1)_2$, $(L_3)_2$, $(L_5)_2$, $(L_7)_2$ or $(L_9)_2$.*

Proof. Form Lemma 5.3, we know $(L_5)_2 \subset (L)_2 \subset (L_1)_2$ or $(L_9)_2 \subset (L)_2 \subset (L_1)_2$. Hence we have only to take all representation elements of $(L_1)_2/(L_5)_3$, $(L_1)_2/(L_9)_3$ and compute all cases for subspaces containing representation elements.

(I) We treat the case $(L_5)_2 \subset (L)_2 \subset (L_1)_2$. Let $(L)_2 \neq (L_5)_2$. Since $L_1 = \mathbb{Z}E_1 + \mathbb{Z}E_4 + \mathbb{Z}(E_2 + E_3) + \mathbb{Z}E_2$ and $L_5 = \mathbb{Z}(2E_1) + \mathbb{Z}(2E_4) + \mathbb{Z}(E_2 + E_3) + \mathbb{Z}(2E_2)$, $(L_1)_2/(L_5)_2$ is represented by the set $\{aE_1 + bE_4 + cE_2 \, ; \, 0 \leq a, b, c \leq 1\}$.
(I-1) $(L)_2$ contains one of E_1, E_4, $E_1 + E_4$. We easily see that $(L)_2$ contains $(L_3)_2$. Since $(L_1)_2/(L_3)_2 \cong \mathbb{Z}/2\mathbb{Z}$, $(L)_2$ is either$(L_1)_2$ or $(L_3)_2$.
(I-2) $(L)_2$ contains either E_2, $E_1 + E_2$ or $E_2 + E_4$. Since $E_1 = \psi(E_2) = \psi(E_1 + E_2) = \psi(w \cdot (E_2 + E_4)) - 2E_2$, we have $(L)_2 = (L_1)_2$.
(I-3) $(L)_2$ contains $E_1 + E_2 + E_4$. Since $L_7 = \mathbb{Z}(E_1 + E_2 + E_4) + \mathbb{Z}(E_1 + E_3 + E_4) + \mathbb{Z}(2E_1) + \mathbb{Z}(2E_4)$, we see $(L_7)_2 \subset (L)_2$. Furthermore $(L_1)_2/(L_7)_2$ is represented by $\{0, E_1, E_4, E_1 + E_4\}$. If $(L)_2$ contains one of this representation element, then we have $(L)_2 = (L_1)_2$. Therefore $(L)_2 = (L_1)_2$ or $(L_7)_2$.

(II) We treat the case $(L_9)_2 \subset (L)_2 \subset (L_1)_2$. Suppose $(L)_2 \neq (L_9)_2$. Since $L_1 = \mathbb{Z}E_1 + \mathbb{Z}E_2 + \mathbb{Z}(E_1 + E_2 + E_3) + \mathbb{Z}(E_2 + E_3 + E_4)$ and $L_9 = \mathbb{Z}(E_1 + E_2 + E_3) + \mathbb{Z}(E_2 + E_3 + E_4) + \mathbb{Z}(2E_1) + \mathbb{Z}(2E_2)$, $(L_1)_2/(L_9)_2$ is represented by $\{aE_1 + bE_2 \, ; \, 0 \leq a, b \leq 1\}$.
(II-1) $(L)_2$ contains E_1. We have $(L_3)_2 \subset (L)_2$. Hence we have $(L)_2 = (L_1)_2$ or $(L_3)_2$.
(II-2) $(L)_2$ contains E_2 or $E_1 + E_2$. Since $\psi(E_2) = \psi(E_1 + E_2) = E_1$, we have $(L_1)_2 = (L)_2$.

Form (I) and (II), we get this lemma. $\square$

6. Table of the coefficients

We give the table of about first fifty coefficients of the Dirichlet series $\xi_\pm(L_i, s)$. In the table, we give the value multiplied by 3 for the each coefficient except for $\xi_+(L_i, s)$, $i = 2, 4, 6, 8, 10$ where in which cases we give the exact value of the coefficients. Hence the table means, for example,

$$\xi_+(L_4, s) = \frac{1/3}{3^s} + \frac{1}{11^s} + \frac{1}{19^s} + \frac{4/3}{27^s} + \frac{1}{35^s} + \frac{1}{43^s} + \frac{1}{48^s} + \frac{1}{51^s} + \dots,$$
$$\xi_-(L_7, s) = \frac{1/3}{1^s} + \frac{1}{9^s} + \frac{1/3}{16^s} + \frac{1}{17^s} + \frac{1}{25^s} + \frac{1}{33^s} + \frac{1}{41^s} + \frac{5/3}{49^s} + \dots,$$
$$\xi_+(L_8, s) = \frac{1}{1^s} + \frac{3}{9^s} + \frac{1}{16^s} + \frac{3}{17^s} + \frac{3}{25^s} + \frac{3}{33^s} + \frac{3}{41^s} + \frac{5}{49^s} + \dots.$$

	L_1^-	L_2^+	L_3^-	L_4^+	L_5^-	L_6^+	L_7^-	L_8^+	L_9^-	L_{10}^+
3	3	3	3	1	0	1	0	0	3	3
4	3	3	0	0	0	0	0	0	0	0
7	3	3	3	0	3	3	3	3	0	0
8	3	3	0	0	0	0	0	0	0	0
11	3	3	3	3	0	3	0	0	3	3
12	3	3	0	0	0	0	0	0	0	0
15	3	3	3	0	3	3	3	3	0	0
16	6	6	3	0	0	3	0	0	0	0
19	3	3	3	3	0	3	0	0	3	3
20	3	3	0	0	0	0	0	0	0	0
23	9	9	3	0	3	9	9	9	0	0
24	3	3	0	0	0	0	0	0	0	0
27	6	6	6	4	0	4	0	0	6	6
28	9	9	0	0	0	6	0	0	0	0
31	9	9	3	0	3	9	9	9	0	0
32	6	6	3	0	0	3	0	0	0	0
35	3	3	3	3	0	3	0	0	3	3
36	3	3	0	0	0	0	0	0	0	0
39	3	3	3	0	3	3	3	3	0	0
40	3	3	0	0	0	0	0	0	0	0
43	3	3	3	3	0	3	0	0	3	3
44	9	9	6	0	0	0	0	0	0	0
47	3	3	3	0	3	3	3	3	0	0
48	6	6	3	3	3	3	3	3	3	3
51	3	3	3	3	0	3	0	0	3	3

	L_1^+	L_2^-	L_3^+	L_4^-	L_5^+	L_6^-	L_7^+	L_8^-	L_9^+	L_{10}^-
1	1	1	1	0	1	1	1	1	0	0
4	3	3	0	0	0	2	0	0	0	0
5	3	3	3	1	0	1	0	0	3	3
8	3	3	0	0	0	0	0	0	0	0
9	3	3	3	0	3	3	3	3	0	0
12	3	3	0	0	0	0	0	0	0	0
13	3	3	3	1	0	1	0	0	3	3
16	4	4	1	1	1	3	1	1	1	1
17	3	3	3	0	3	3	3	3	0	0
20	3	3	0	0	0	0	0	0	0	0
21	3	3	3	1	0	1	0	0	3	3
24	3	3	0	0	0	0	0	0	0	0
25	3	3	3	0	3	3	3	3	0	0
28	3	3	0	0	0	0	0	0	0	0
29	3	3	3	1	0	1	0	0	3	3
32	6	6	3	0	0	3	0	0	0	0
33	3	3	3	0	3	3	3	3	0	0
36	9	9	0	0	0	6	0	0	0	0
37	3	3	3	3	0	3	0	0	3	3
40	3	3	0	0	0	0	0	0	0	0
41	3	3	3	0	3	3	3	3	0	0
44	3	3	0	0	0	0	0	0	0	0
45	3	3	3	1	0	1	0	0	3	3
48	6	6	3	0	0	3	0	0	0	0
49	5	5	3	0	3	5	5	5	0	0

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