Abstract

This paper is dedicated to an explicit description of the category of effective geometric motives (see [23]). We construct explicitly a category $\mathfrak{H}$ that is isomorphic to the motivic category DM (whose idempotent completion is Voevodsky's DM_{gm}^{eff}). We use *twisted complexes* (defined by Bondal and Kapranov) over a certain differential graded category J; the objects of J are the Suslin complexes of smooth projective varieties. For any motivic complex M (i.e. an object of DM_{-}^{eff} that comes from DM_{gm}^{eff}; for instance, the Suslin complex of an arbitrary variety) there exists a quasi-isomorphic complex M' 'constructed from' the Suslin complexes of smooth projective varieties; M' is unique up to a homotopy. This fact is important for the study of cohomology of motives (i.e. realizations, see below). This gives a 'differential graded' description of DM; it is similar to the motivic category of Hanamura and yet works on the integral level.

As an application we give a general description of any subcategory of DM that is generated by a fixed set of objects and of all localizations of DM. We study realizations of motives. We construct a family of canonical exact *truncation* functors $t_N : \mathfrak{H} \to \mathfrak{H}_N$ for $N > 0$ for certain triangulated categories $\mathfrak{H}_N$. $\mathfrak{H}_0$ is the homotopy category of $\mathfrak{Chow}$ (that is 'almost' the category of Chow motives). We prove that $K_0(DM_{gm}^{eff}) \cong K_0(Chow)$ answering the question of Gillet and Soulé. The N-th weight filtration of the étale and de Rham realizations can be factorized through t_N. The weight complex of Gillet and Soulé is the restriction of t_0 to motives with compact support of varieties. Besides the motif of a smooth variety is a mixed Tate one whenever its weight complex is.

We describe a vast generalization of the method of constructing cohomological functors that was described in [11]. For any realization D obtained by our method we get a large family of 'truncated realizations'; in particular, this could be applied to 'standard' realizations and motivic cohomology; an interesting new family of realizations is obtained this way. We construct a spectral sequence S converging to the cohomology of $D(X)$ for an arbitrary motive X. S is the *spectral sequence of motivic descent* (note that the usual cohomological descent spectral sequences compute cohomology of varieties only). Its E_1-terms are cohomology of smooth projective varieties; its E_n-terms have a nice description in terms of $t_{2n-4}(X)$, $n \geq 2$. S is 'motivically functorial'. S gives a canonical weight filtration on the cohomology of $D(X)$; for the 'standard' realizations this filtration coincides with the usual one. For the motivic cohomology this weight filtration is non-trivial and appears to be quite new.

Weight filtrations and motivic descent spectral sequence for differential graded realizations of the Voeovodsky motives; enhancement and truncations for the category of motives

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Introduction

This paper is dedicated to an explicit description of the category of effective geometric motives defined by Voevodsky (see [23]) and its realizations. The main classification result is a new description of the motivic category DM. Here DM is the full triangulated subcategory of the category of effective geometric motives (defined in [23]) generated by motives of smooth varieties (we do not add the kernels of projectors). It is proved that DM can be embedded into the homotopy category of abelian Nisnevich sheaves over the category of smooth correspondences (smooth correspondences were defined in [23] also). For any motivic complex M (i.e. an object of DM_{-}^{eff} that comes from DM_{gm}^{eff}; in particular, the Suslin complex of an arbitrary variety) there exists a quasi-isomorphic complex M' 'constructed from' the Suslin complexes of smooth projective varieties; M' is unique up to a homotopy. The image of this embedding has a description that is similar to the motivic category of Hanamura (see [13]). It follows that all questions on motives can be reduced to questions about smooth projective varieties and *higher morphisms* between them. Here a higher morphisms $X \to Y$ is a smooth correspondence $X \times \mathbb{A}^n \to Y$, $\mathbb{A}^n$ is the affine space.

Our motivic category $\mathfrak{H}$ is defined as the category of *twisted complexes* over a certain differential graded category whose objects are cubical Suslin complexes; we construct an equivalence $m : \mathfrak{H} \to DM$. Differential graded

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categories and twisted complexes over them were considered in the paper [4]. It was stated there that most of reasonable triangulated categories come from differential graded categories; exact functors come from functors of differential graded categories. In terms of [4] our description of DM gives an *enhancement* of this category. One should think of twisted complexes as of the results of repetitive computation of cones of morphisms in an 'enhanced' triangulated category. In particular, this result allows to describe any subcategory of $\mathfrak{H}$ that is generated by a fixed set of objects. One can apply this method to the triangulated category of (mixed effective) Tate motives and obtain a description of this category on the integral level similar to the description on the rational level constructed in [17] (also see [22]). Besides, any localization of $\mathfrak{H}$ can be described explicitly using the construction of Drinfeld (see [8]).

As an application we consider the problem of constructing exact functors from DM (i.e. realizations) in terms of cubical Suslin motivic complexes. Our method uses the formalism of differential graded categories. The most simple and yet quite interesting of those are the *truncation functors* t_N from $\mathfrak{H}$ to certain triangulated categories $\mathfrak{H}_N$; t_N correspond to the canonical filtration of the Suslin complex.

The target of t_0 is just the homotopy category of $\mathfrak{Chow}$ (that is 'almost' the category of Chow motives, see Notation). We prove that t_0 induces an isomorphism $K_0(DM_{gm}^{eff}) \cong K_0(Chow)$ thus answering the question of [11].

We show that the N-th weight filtration of any 'standard' realization of motives could be factorized through t_N for any $N \geq 0$. Moreover, if W denotes the weight filtration on $H^i(X)$ then $W_{l+N}H^i(X)/W_{l-1}H^i(X)$ factorizes through t_N; a morphism f of motives induces a zero morphism on cohomology (modulo torsion) if $t_0(f)$ is zero.

We prove that a motif X belongs to a triangulated category $M \subset \mathfrak{H}$ generated by motives of a given set of smooth projective varieties P_i whenever the same is true for $t_0(X)$ (as a complex of Chow motives). In particular, the motif of a smooth variety is a mixed Tate one iff its weight complex is.

For any realization D of motives that belongs to a wide class of 'differential graded' realizations we construct a family of 'truncated realizations'. This gives a canonical spectral sequence S converging to the cohomology of $D(X)$ of an arbitrary motive X. S could be called the *spectral sequence of motivic descent* (note that the usual cohomological descent spectral sequences compute cohomology of varieties only). The E_1-terms of S are cohomology of smooth projective varieties, the E_n-terms of S have a nice description in terms of $t_{2n-4}(X)$, $n \geq 2$. S is 'motivically functorial', it gives a canonical non-trivial weight filtration for 'differential graded' realizations of motives; for the 'standard' realizations this filtration coincides with the usual one.

Since motivic cohomology is also a 'differential graded' realization, we obtain a canonical 'weight' filtration and the corresponding 'weight' spectral sequence for it. The simplest case of this spectral sequence is the Bloch's long exact localization sequence for higher Chow groups (see [3]). This 'weight' filtration appears to be non-trivial and not mentioned in the literature. Using the spectral sequence relating algebraic K-theory to the motivic cohomology (see [9]) we get a new filtration on the K-theory of a smooth variety X. The author plans to study these matters in more detail in future.

We also study motives with compact support (M^c_{gm} in the notation of Voevodsky). We give an explicit description of M^c_{gm} for a variety Z and prove that the weight complex of Gillet and Soulé can be described as $t_0(m^{-1}(M^c_{gm}(Z)))$ (with inverted arrows); the functor h of Guillen and Navarro Aznar is (essentially) $t_0(m^{-1}(M_{gm}(Z)))$.

We define the 'length' of a motive; this is a natural motivic analogue of the length of weight filtration for a mixed Hodge structure. For a smooth variety X the length of $M_{gm}(X)$ lies between the length of the weight filtration of the Hodge cohomology of X and the dimension of X.

The main distinction of the motivic category $\mathfrak{H}$ described here from those defined by Hanamura is that we consider homological motives. Instead of the Bloch complex of a smooth projective variety (used by Hanamura) we use its cubical Suslin complex; we never have to choose distinguished subcomplexes for our constructions. Note also that our definition works on the integral level in contrast with those of [13]. Another difference with the category of Hanamura is that we don't add formal Tate twists of motives. This simplifies the construction; yet it makes difficult to calculate the inner Hom functor and to write a formula for the motive of a non-projective variety. It seems probable that one can solve these problems in a certain category that has a similar (but more complicated) description.

We note that in the current paper we apply several results of [23] that use resolution of singularities, so we assume that the characteristic of the ground field k is 0. Yet to the knowledge of the author most of the results of the paper (at least with rational coefficients) could be obtained using recent results of F. Deglise (see [6] and [7]) along with h-motives and de Jong's alterations (see [10] and [5]); cf. also Appendix B of [15].

We describe the contents of the paper.

In the first section we describe cubical Suslin complexes and their properties. Most of the proofs are postponed till §4 since they are not important for the understanding of main results.

We start section 2 by recalling the formalism of differential graded categories and twisted complexes. First we prove some general results for certain categories constructed from an arbitrary abelian (or just additive) category

A. Next we use this formalism to construct our main objects of study: a triangulated category $\mathfrak{H}$ and a functor h from $\mathfrak{H}$ into the homotopy category of Nisnevich sheaves with transfers. We prove that $\mathfrak{H}$ is generated by 'motives' of smooth projective varieties. For the convenience of the reader we also describe $\mathfrak{H}$ and h explicitly. We show that the category of morphisms of complexes over A has a simple description in terms of twisted complexes over the category of morphisms of A. We describe (three types of) truncation functors for the category of complexes; we study of an important subclass of differential graded categories called *negative* categories. These constructions will be used in section 6.

In section 3 we prove that h composed with the natural functor from the homotopy category of sheaves with transfers to the derived category gives an equivalence $m : \mathfrak{H} \to DM$.

In section 4 we verify the properties of cubical Suslin complexes. The reader not interested in the proof of the auxiliary results of §1 may omit it. Our main tool is the projection of the derived category of presheaves with transfers onto the derived category of complexes with homotopy invariant cohomology; this functor is similar to the functor RC constructed in 3.2 of [23].

In section 5 using the canonical filtration of the (cubical) Suslin complex we define the 'truncation' functors $t_N : \mathfrak{H} \to \mathfrak{H}_N$. $\mathfrak{H}_0$ (i.e. the target of t_0) is just the homotopy category of $\mathfrak{Chow}$. These functors are new though certain (very) partial cases were (essentially) considered in [11] and [12] (and were shown to be quite important). We prove that t_0 induces an isomorphism $K_0(DM_{gm}^{eff}) \cong K_0(Chow)$. We define the 'length' of a motive. We prove that motives of smooth varieties of dimension N have length $\leq N$; besides $t_N(X)$ contains all information on motives of length $\leq N$. The length of a motif is a natural motivic analogue of the length of weight filtration for a mixed Hodge structure. In the end of the section we calculate $m^{-1}(M_{gm}^c(X))$ explicitly. Using this result as well as cdh-descent we prove that the weight complex of Gillet and Soulé for X/k can be described as $t_0(m^{-1}(M_{gm}^c(X)))$ (with arrows inverted). Moreover, we verify that the weight complex construction of [11] in fact gives a well-defined motive $GS(X)$ (i.e. an object of DM). Besides, $t_0(m^{-1}(M_{gm}(X)))$ essentially coincides with the functor h described in Theorem 5.10 of [12].

In section 6 we study *realizations* of the category of motives and their connections with (certain) weight filtrations. For any given family S of higher morphisms we describe a canonical triangulated functor from DM that maps elements of S into 0. We also describe a general recipe of constructing covariant realizations using the differential graded categories formalism. It is very easy to determine which of those *differential graded* realizations can be factorized through a given truncation functor.

Every complex of (étale) sheaves over smooth correspondences gives a differential graded realization. In particular, this is true for any complex that calculates étale cohomology (for example, with coefficients in $\mathbb{Z}/n\mathbb{Z}$ for $n > 0$). This result could be extended to all other 'standard' realizations.

By means of the spectral sequence of a filtered complex we construct a spectral sequence S converging to the given differential graded realization of a motive Y. Its E_n-terms have a nice description in terms of $t_{2n-4}(Y)$, $n \geq 2$; in particular, E_2-terms depend only on $t_0(Y)$. S is the *spectral sequence of motivic descent*. Note that the usual cohomological descent spectral sequences compute cohomology of varieties only. Besides we never need the realization to be torsion as one does for étale cohomology (instead we need other restrictions). S gives a canonical integral weight filtration for differential graded realizations of motives; for the 'standard' realizations this filtration coincides with the usual one. We verify that the N-th weight filtration of 'standard' realizations can be factorized through t_N. In fact, we prove much more: if W denotes the weight filtration on $H^i(X)$ then $W_{l+N}H^i(X)/W_{l-1}H^i(X)$ factorizes through t_N; a morphism f induces a zero morphism on cohomology if $t_0(f)$ is zero (cf. subsection 6.3).

As a partial case, we also obtain a canonical 'weight' filtration on the motivic cohomology of any variety and the corresponding 'weight' spectral sequence for it. This filtration induces a new filtration on the K-theory of a smooth variety.

We conclude the section by the discussion of qfh-descent and motives of singular varieties. It turns out that a wide class of realizations are 'qfh-representable' (hence they are 'differential graded' realizations of DM); moreover, the qfh-motif of a (not necessarily smooth) variety has 'right values of standard realizations'.

In section 7 we apply the general theory of [4] to describe any subcategory of $\mathfrak{H}$ that is generated by a fixed set of objects. In particular, this method can be used to obtain the description of the triangulated category of effective Tate motives (i.e. the full triangulated subcategory of DM generated by $\mathbb{Z}(n)$ for $n > 0$).

We describe the construction of 'localization of differential graded categories' (due to Drinfeld). This gives us a description of localizations of $\mathfrak{H}$. All such localizations come from differential graded functors. As an application, we prove that the motif of a smooth X/k is a mixed Tate one whenever the weight complex of X (defined in [11]) is.

We note that one could easily extend $\mathfrak{H}$ by adjoining certain direct summands of its objects. Next we give an explicit description of a tensor product on DM and prove that t_N are tensor functors. We also study certain functors

$m_N : \mathfrak{H}_N \to DM_-^{eff}$; t_N and m_N could be related to the (yet conjectural) weight filtration on the category of motives themselves. We conclude the section by some remarks on the internal Hom functor.

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Contents

Notation. In this paper k will denote the ground field; we will assume that the characteristic of k is zero. pt is a point, $\mathbb{A}^n$ is the n-dimensional affine space (over k), $x_1, \dots, x_n$ are the coordinates, $\mathbb{P}^n$ is the projective space of dimension n.

$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over k, resp. of smooth varieties, resp. of smooth projective varieties.

For an additive category A we denote by $C^-(A)$ the category of complexes over A bounded from above; $C^b(A) \subset C^-(A)$ is the subcategory of bounded complexes; $K^-(A)$ is the homotopy category of $C^-(A)$ i.e. the morphisms of complexes are considered up to homotopy equivalence; K^b denotes the homotopic category of bounded complexes; sometimes we will also need the unbounded categories $C(A)$ and $K(A)$; Ab is the category of abelian groups.

For a category C, $A, B \in ObjC$, we denote by $C(A,B)$ the set of A-morphisms from A into B. Note that complexes of sheaves could be considered as objects of different categories (for example, of $C^-(Shv(SmCor))$, $K^-(Shv(SmCor))$, and $D^-(Shv(SmCor))$, see the definition of $Shv(SmCor)$ below).

For categories C, D we write $C \subset D$ if C is a full subcategory of D.

We use much notation from [23]. We recall it for the convenience of the reader.

$SmCor$ is the category of 'smooth correspondences' i.e $ObjSmCor = SmVar$, $SmCor(X,Y) = \sum_U \mathbb{Z}$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X.

$Shv(SmCor) = Shv(SmCor)_{Nis}$ is the abelian category of additive cofunctors $SmCor \to Ab$ that are sheaves in the Nisnevich topology (when restricted to the category of smooth varieties); these sheaves are usually called 'sheaves with transfers'. Moreover, by default all sheaves will be sheaves in Nisnevich topology. By an abuse of notation we will also denote by $Shv(SmCor)$ the set of all Nisnevich sheaves with transfers.

$D^-(Shv(SmCor))$ is the derived category of $Shv(SmCor)$; $p : K^-(Shv(SmCor)) \to D^-(Shv(SmCor))$ is the natural projection.

For $Y \in SmVar$ (more generally, for $Y \in Var$, see §4.1 of [23]) we consider $L(Y) = SmCor(-,X) \in Shv(SmCor)$. $L^c(X)(Y) \supset L(X)(Y)$ denotes the group whose generators are the same as for $L(X,Y)$ except that U is only required to be quasi-finite over X. $L(X) = L^c(X)$ for proper X. Note that $L^c(X)$ is also a sheaf.

$M_{gm}(X) = \underline{C}(L(X)) \cong C(L(X))$ is the Suslin complex of $L(X)$, see subsection 1.1 below; $M^c_{gm}(X) = \underline{C}(L^c(X)) \cong C(L^c(X))$.

$S \in Shv(SmCor)$ is called homotopy invariant if for any $X \in SmVar$ the projection $\mathbb{A}^1 \times X \to X$ gives an isomorphism $S(X) \to S(\mathbb{A}^1 \times X)$.

$DM^{eff}_- \subset D^-(Shv(SmCor))$ is the subcategory of complexes whose cohomology sheaves are homotopy invariant. It was proved in [23] that for any $F \in Shv(SmCor)$ we have $\underline{C}(F) \in DM^{eff}_-$. $RC : D^-(Shv(SmCor)) \to DM^{eff}_-$ is given by taking total complexes of the Suslin bicomplex of a complex of sheaves, see §3.2 of [23] for details.

DM will denote the full triangulated subcategory of DM^{eff}_- generated by $M_{gm}(X)$ for $X \in SmVar$ (we do not add the kernels of projectors). DM has a natural tensor structure that could be defined using the relation $M_{gm}(X) \otimes M_{gm}(Y) = M_{gm}(X \times Y)$; tensor multiplication of morphisms is defined by means of a similar relation.

$DM^{eff}_{gm} \supset DM$ is obtained by adding all kernels of projectors. Voevodsky also gave a definition of DM^{eff}_{gm} in terms of $SmCor$; we don't need it here. DM_{gm} in [23] was obtained from DM^{eff}_{gm} (considered as an abstract category i.e. not as a category of DM^{eff}_-) by the formal inversion of $\mathbb{Z}(1)$ (see subsection 3.1 below) with respect to $\otimes$. DM_{gm} is a rigid tensor triangulated category. We only need DM_{gm} for a few calculations that use duality; the reader interested in those calculations should consult §4.3 of [23] for details.

$\mathfrak{Chow}$ will denote a (slightly) modified version of the classical additive category of homological Chow motives. Its objects are smooth projective varieties; the morphisms are morphisms in $SmCor$ up to homotopy equivalence. The category $Chow$ is obtained from $\mathfrak{Chow}$ by adding all kernels of projectors; it was shown in Proposition 2.1.4 of [23] that $Chow$ is naturally isomorphic to the usual category of homological Chow motives.

We note that for categories of geometric origin (for example, $\mathfrak{Chow}$ and $SmCor$) the addition of objects is induced by the disjoint union of varieties.

We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a for a functor that preserves the structures of triangulated categories).

We list the main definitions of this paper. $C(X)$ will be defined in 1.1; g^l will be defined in 1.2; differential graded categories, $H(C)$ for a differential graded category C, $S(A)$, $S_N(A)$, $B^-(A)$, $B^b(A)$, $B(A)$, and $C(A)$ for an additive category A will be defined in 2.1; the categories of twisted complexes (Pre-Tr(C), $Tr(C)$, Pre-Tr$^+(C)$, $Tr^+(C)$), $[P]$ and $P[i]$ for $P \in ObjC$ will be defined in 2.2 and 2.3; $Tr(F)$, Pre-Tr(F), $Tr^+(F)$, and Pre-Tr$^+(F)$ for a differential graded functor F will be defined in Remark 2.3.3; J, $\mathfrak{H}$, and h will be defined in 2.4; $\mathfrak{H}'$, h', j, and J' will be defined in 2.5; C_- and different types of truncations of complexes ($\tau_{\le b}$, $\tau_{[a,b]}$ and the canonical $[a,b]$-truncation) will be defined in 2.6, m will be defined in 3.1; $C^N(P)$ and t_N will be defined in 5.1, t will be defined in 5.2, truncated realizations will be defined in 6.3, m_N will be defined in 7.5.

1 Cubical Suslin complexes

In this paper instead of the simplicial Suslin complex $\underline{C}(L(P))$ we consider its cubical version $C(P)$. In this section we prepare for the proof of the following fact: there exists a differential graded category J (it will be defined in §2) whose objects are the (cubical) Suslin complexes of smooth projective varieties, while its morphisms are related to the morphisms between those complexes in DM_-^{eff}.

1.1 The definition of the cubical complex

For any $P \in SmVar$ we consider the sheaves

$$C_i'(P)(Y) = SmCor(\mathbb{A}^{-i} \times Y, P),\ Y \in SmVar;\ C_i = 0 \text{ for } i > 0.$$

We will usually consider projective P.

By Ioneda's lemma

$$C_i'(P)(Y) \cong Shv(SmCor)(L(Y), L(P)) = Shv(SmCor)(C_0'(Y), C_i'(P)).$$

For all $1 \leq j \leq -i$, $x \in k$, we define $d_{ijx} = d_{jx} : C'_i \to C'_{i+1}$ as $d_{jx}(f) = f \circ g_{jx}$, where $g_{jx} : \mathbb{A}^{-i-1} \times Y \to \mathbb{A}^{-i} \times Y$ is induced by the map $(x_1, \ldots, x_{-1-i}) \to (x_1, \ldots, x_{j-1}, x, x_j, \ldots, x_{-1-i})$. We define $C_i(P)(Y)$ as $\cap_{1 \leq j \leq -i} \operatorname{Ker} d_{j0}$. One may say that $C_i(P)(Y)$ consists of correspondences that 'are zero if one of the coordinates is zero'. The boundary maps $\delta^i : C_i \to C_{i+1}$ are defined as $\sum_{1 \leq j \leq -i} (-1)^j d_{j1}$.

Since $C'_0 = C_0$, we have $C_i(P)(Y) \cong Shv(SmCor)(C_0(Y), C_i(P))$. We formulate the main property of C.

Remark 1.1.1. 1. The definition of the cubical Suslin complex can be easily extended to an arbitrary complex D over $Shv(SmCor)$ (or over a slightly different abelian category). One should consider the total complex of the double complex whose terms are

$$D_{ij}(X) = \cap_{1 \leq l \leq -i} \operatorname{Ker} g^*_{jl0} : D_j(\mathbb{A}^{-i} \times X) \to D_j(\mathbb{A}^{-i-1} \times X),$$

the boundaries are induced by δ^i.

2. In the usual (simplicial) Suslin complex one defines $\underline{C}_i(F)(X) = F(D^{-i} \times X)$, $D^{-i} \subset \mathbb{A}^{1-i}$ is given by $\sum_{1 \leq l \leq 1-i} x_i = 1$; the boundaries come from restrictions to $x_l = 0$.

Proposition 1.1.2. *For any $j \in \mathbb{Z}$, $Y \in SmVar$, and $P \in SmPrVar$ there is a natural isomorphism $H^j C(P)(Y) \cong A_{0,-j}(Y, P) \cong DM^{eff}_{-}(\underline{C}(Y), \underline{C}(P)[j])$.*

Proof. $A_{0,-j}(Y, P) \cong DM^{eff}_{-}(\underline{C}(Y), \underline{C}^c(P))$ by Proposition 4.2.3 [23]. Since P is projective, by Proposition 4.1.5 [23] we have $\underline{C}^c(P) = \underline{C}(P)$.

The first isomorphness will be proved in §4 below.

All isomorphisms are natural. □

In particular the cohomology presheaves of $C(P)$ are homotopy invariant.

We denote the initial object of $SmCor$ by 0. We define $C_i(0) = 0$ for all $i \in \mathbb{Z}$. We obtain

$$p(C(P)) \in Obj DM^{eff}_{-} \subset Obj D^{-}(Shv(SmCor)).$$

1.2 The assignment $g \to (g^l)$

Let $P, Y \in SmVar$. We construct a family of morphism $C(Y) \to C(P)[i]$.

For any $f \in C'_i(P)(Y)$, $l \leq 0$, we define $f^l : C'_l(Y) \to C'_{l+i}(P)$ as follows. To the element $h \in SmCor(Z \times \mathbb{A}^{-l}, Y)$, $Z \in SmVar$, we assign $(-1)^{li} f \circ (id_{\mathbb{A}^{-i}} \otimes h)$. It is easily seen that the same formula also defines for $f \in C_i(P)(Y)$ the maps $f^l : C_l(Y) \to C_{l+i}(P)$.

Proposition 1.2.1. *1. The assignment $g \to G = (g^l)$ defines a homomorphism* $\operatorname{Ker} \delta^i(P)(Y) \to K^-(Shv(SmCor))(C(Y), C(P)[i])$.

2. The assignment $g \to G = (g^l)$ induces an isomorphism $H^i(C(P)(Y)) \cong DM_-^{eff}(C(Y), C(P)[i])$.

Proof. 1. For any $f \in C_i'(P)(Y), h \in C_l'(Y)(Z),\ Z \in SmVar$ we have an equality

$$\delta^{i+l} f^l(h) = (-1)^i f^{l+1} \delta^l(h) + (\delta^i f)^l(h). \tag{1}$$

Hence if $\delta^i g = 0,\ g \in C_i(P)(Y)$ then G defines a morphism of complexes $C(Y) \to C(P)[i]$.

2. Using (1) we obtain that the elements of $\delta^{i+1}(C_{i+1}(P)(Y))$ give homomorphisms $C(Y) \to C(P)[i]$ that are homotopy equivalent to 0. Hence we obtain a homomorphism $H^i C(P)(Y) \to DM_-^{eff}(p(C(Y)), p(C(P)[i]))$. The bijectivity of this homomorphism will be proved in §4 below. □

2 Differential graded categories; the description of h

Categories of *twisted complexes* (defined in subsections 2.2 and 2.3) were first considered in [4]. Yet our notation differs slightly from those of [4]; some of the signs are also different.

In subsections 2.4 and 2.5 we define and describe our main categories: $J, \mathfrak{H}, J'$, and $\mathfrak{H}'$.

We will not need the material of subsection 2.6 till section 6.

2.1 The definition of differential graded categories

Recall that an additive category C is called graded if for any $P, Q \in ObjC$ there is a canonical decomposition $C(P,Q) \cong \oplus_i C_i(P,Q)$ defined; this decomposition satisfies $C_i(*,*) \circ C_j(*,*) \subset C_{i+j}(*,*)$. A differential graded category (cf. [4] or [8]) is a graded category endowed with an additive operator $\delta : C_i(P,Q) \to C_{i+1}(P,Q)$ for all $i \in \mathbb{Z}, P, Q \in ObjC$. δ should satisfy the equalities $\delta^2 = 0$ (so $C(P,Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in ObjC,\ f \in C_i(P,Q)$, $g \in C(Q,R)$. In particular, $\delta(id_P) = 0$.

We denote δ restricted to morphisms of degree i by δ^i.

For an additive category A one can construct the following differential graded categories. The notation introduced below will be used throughout the paper.

We denote the first one by $S(A)$. We set $ObjS(A) = ObjA$; $S(A)_i(P,Q) = A(P,Q)$ for $i = 0$; $S(A)_i(P,Q) = 0$ for $i \neq 0$. We take $\delta = 0$.

We also consider the category $B^-(A)$ whose objects are the same as for $C^-(A)$ whence for $P = (P_i)$, $Q = (Q_i)$ we define $B^-(A)(P,Q)_i = \oplus_{j\in\mathbb{Z}} A(P_j, Q_{i+j})$. Obviously $B^-(A)$ is a graded category.

We denote by $B^b(A)$ the full subcategory of $B^-(A)$ whose objects are bounded complexes. $B(A)$ and $C(A)$ will denote the corresponding categories whose objects are unbounded complexes.

We set $\delta f = d_Q \circ f - (-1)^i f \circ d_P$, where $f \in B_i(P,Q)$, d_P and d_Q are the differentials in P and Q. Note that the kernel of $\delta^0(P,Q)$ coincides with $C(A)(P,Q)$ (the morphisms of complexes); the image of δ^{-1} are the morphisms homotopic to 0.

For any $N \geq 0$ one can define a full subcategory $S_N(A)$ of $B^b(A)$ whose objects are complexes concentrated in degrees $[0,N]$. We have $S(A) = S_0(A)$.

$B^b(A)$ can be obtained from $S(A)$ (or any $S_N(A)$) by means of the category functor Pre-Tr described below.

Other examples of differential graded categories and differential graded functors could be found in subsection 2.6 below.

For any differential graded C we define a category $H(C)$; its objects are the same as for C; its morphisms are defined as

$$H(C)(P,Q) = \operatorname{Ker} \delta^0_C(P,Q) / \operatorname{Im} \delta^{-1}_C(P,Q).$$

2.2 Categories of twisted complexes (Pre-Tr(C) and $Tr(C)$)

Having a differential graded category C one can construct two other differential graded categories Pre-Tr(C) and Pre-Tr$^+$(C) as well as triangulated categories $Tr(C)$ and $Tr^+(C)$. The simplest example of these constructions is Pre-Tr$(S(A)) = B^b(A)$.

Definition 2.2.1. The objects of Pre-Tr(C) are

$$\{(P_i),\ P_i \in ObjC, i \in \mathbb{Z}, q_{ij} \in C_{i-j+1}(P_i, P_j)\};$$

here almost all P_i are 0; for any $i, j \in \mathbb{Z}$ we have $\delta q_{ij} + \sum_l q_{lj} \circ q_{il} = 0$. For $P = \{(P_i), q_{ij}\}$, $P' = \{(P'_i), q'_{ij}\}$ we set

$$\text{Pre-Tr}_l(P, P') = \bigoplus_{i,j\in\mathbb{Z}} C_{l+i-j}(P_i, P'_j).$$

For $f \in C_{l+i-j}(P_i, P'_j)$ we define the differential of the corresponding morphism in Pre-Tr(C) as

$$\delta_{\text{Pre-Tr}(C)} f = \delta_C f + \sum_m (q'_{jm} \circ f - (-1)^{(i-m)l} f \circ q_{mi}).$$

It can be easily seen that Pre-Tr(C) is a differential graded category (see [4]). There is also an obvious translation functor on Pre-Tr(C). Note also that the terms of the complex Pre-Tr$(C)(P, P')$ do not depend on q_{ij} and q'_{ij} whence the differentials certainly do.

We denote by $Q[j]$ the object of Pre-Tr(C) that is obtained by putting $P_i = Q$ for $i = -j$, all other $P_j = 0$, all $q_{ij} = 0$. We will write $[Q]$ instead of $Q[0]$.

Immediately from definition we have Pre-Tr$(S(A)) \cong B^b(A)$.

A morphism $h \in \operatorname{Ker} \delta^0$ (a closed morphism of degree 0) is called a *twisted morphism*. For a twisted morphism $h = (h_{ij}) \in$ Pre-Tr$((P_i, q_{ij}), (P'_i, q'_{ij}))$, $h_{ij} \in C(P_i, P'_j)$ we define $Cone(h) = P_i", q_{ij}"$, where $P"_i = P_{i+1} \oplus P'_i$,

$$q"_{ij} = \begin{pmatrix} q_{i+1,j+1} & 0 \\ h_{i+1,j} & q'_{ij} \end{pmatrix}$$

We have a natural triangle of twisted morphisms

$$P \xrightarrow{f} P' \to Cone(f) \to P[1]. \tag{2}$$

This triangle induces a triangle in the category $H($Pre-Tr$(C))$.

Definition 2.2.2. For distinguished triangles in $Tr(C)$ we take the triangles isomorphic to those that come from the diagram (2) for $P, P' \in$ Pre-Tr(C).

We summarize the properties of Pre-Tr and Tr of [4] that are most relevant for the current paper. We have to replace bounded complexes by complexes bounded from above. Part 4 is completely new.

Proposition 2.2.3. *I $Tr(C)$ is a triangulated category.*

II For any additive category A there are natural isomorphisms

1. Pre-Tr$(B^-(A)) \cong B^-(A)$.

2. $Tr(B^-(A)) \cong K^-(A)$.

3. Pre-Tr$(B(A)) \cong B(A)$.

4. $Tr(S_N(A)) \cong B^b(A)$

Proof. I See Proposition 1 §2 of [4].

II 1. By example 1 §3 of [4] a similar fact holds for $B^b(A)$. The corresponding isomorphism is provided by the convolution of twisted complexes, see §2 of [4]. it is easily seen that convolution is continuous with respect to the stupid filtration for $B^b(A)$. Hence passing to the limit we obtain our assertion.

2. Immediate from assertion II1.

3. The proof is very similar to those of II1.

4. We have natural full embeddings $S_0(A) \subset S_N(A) \subset B^b(A)$. Since $Tr(S_0(A)) \cong Tr(B^b(A)) \cong B^b(A)$, we obtain the assertion. □

2.3 The categories $\text{Pre-Tr}^+(C)$ and $Tr^+(C)$

In [4] $\text{Pre-Tr}^+(C)$ was defined as a full subcategory of $\text{Pre-Tr}(\tilde{C})$, where $\tilde{C}$ was obtained from C by adding formal shifts of objects. Yet it can be easily seen that the category defined in [4] is canonically equivalent to the category defined below (see also [8]). So we adopt the notation $\text{Pre-Tr}^+(C)$ of [4] for the category described below.

The definition of $\text{Pre-Tr}^+(C)$ and $Tr^+(C)$ could also be found in subsection 2.4 of [8]; there those categories were denoted by C^{pre-tr} and C^{tr}.

Definition 2.3.1. 1. $\text{Pre-Tr}^+(C)$ is defined as a full subcategory of $\text{Pre-Tr}(C)$. $A = \{(P_i), q_{ij}\} \in Obj\,\text{Pre-Tr}^+(C)$ if there exist $m_i \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ we have $q_{ij} = 0$ for $i + m_i \geq j + m_j$.

2. $Tr^+(C)$ is defined as $H(\text{Pre-Tr}^+(C))$.

The following statement is an easy consequence of definitions.

Proposition 2.3.2. *1. $Tr^+(C)$ is a triangulated subcategory of $Tr(C)$.*

2. $Tr^+(C)$ is generated by as a triangulated category by the image of the natural map $ObjC \to ObjTr^+(C) : P \to [P]$.

3. There are natural embeddings of categories $i : C \to \text{Pre-Tr}^+(C)$ and $H(C) \to Tr^+(C)$ sending P to $[P]$.

4. $\text{Pre-Tr}(i)$, $Tr(i)$, $\text{Pre-Tr}^+(i)$, and $Tr^+(i)$ are equivalences of categories.

It can be also easily seen that assertion 2 characterizes $Tr^+(C)$ as a full subcategory of $Tr(C)$

Proof. 1. It is sufficient to check that the cone of a map in $\text{Pre-Tr}^+(C)$ belongs to $\text{Pre-Tr}^+(C)$. This is easy. Also see §4 of [4].

2. See Theorem 1 §4 of [4].

3. By definition of $\text{Pre-Tr}^+(C)$ (resp. of $Tr^+(C)$) there exists a canonical isomorphism of bifunctors $C(-,-) \cong \text{Pre-Tr}^+(C)([-],[-])$ (resp. $HC(-,-) \cong Tr^+(C)([-],[-])$). In remains to note that both of these isomorphisms respect addition and composition of morphisms; the first one also respects differentials.

4. The proof was given in §3 and §4 of [4]. □

Remark 2.3.3. 1. Since Pre-Tr, Pre-Tr^+, Tr, and Tr^+ are functors on the category of differential graded categories, any differential category functor $F : C \to C'$ naturally induces functors $\text{Pre-Tr}F$, Pre-Tr^+F, TrF, and Tr^+F. We will use this fact throughout the paper.

For example, for $X = (P_i, q_{ij}) \in Obj\,\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}F(X) = (F(P_i), F(q_{ij}))$; for a morphism $h = (h_{ij})$ of $\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}F(h) =$

$(F(h_{ij}))$. Note that the definition of Pre-TrF on morphisms does not involve q_{ij}; yet Pre-TrF certainly respects differentials for morphisms.

2. Let F : Pre-Tr$^+(C) \to D$ be a differential graded functor. Then the restriction of F to $C \subset$ Pre-Tr$^+(C)$ (see part 3 of Proposition 2.3.2) gives a differential graded functor $FC : C \to D$. Moreover, since $FC = F \circ i$, we have Pre-Tr$^+(FC) =$ Pre-Tr$^+(F) \circ$ Pre-Tr$^+(i)$; therefore Pre-Tr$^+(FC) \cong$ Pre-Tr$^+(F)$.

2.4 Definition of $\mathfrak{H}$ and h

For $X, Y, Z \in SmPrVar$, $i, j, l \leq 0$, $f \in C_i(X)(Y)$, $g \in C_j(Y)(Z)$ we have the equality

$$(f^j(g))^l = f^{j+l}(g^l). \tag{3}$$

Hence we can define a (non-full!) subcategory J of $B^-(Shv(SmCor))$ whose objects are $[P] = C(P)$, $P \in SmPrVar$, the morphisms are defined as

$$J_i(C(P), C(Q)) = \{\bigoplus_{l \leq 0}(g^l) : \ g \in C_i(Q)(P)\},$$

the composition of morphisms and the boundary operators are the same as for $B^-(Shv(SmCor))$. There is an obvious addition defined for morphisms; the operation of disjoint union of varieties gives us the addition on objects. It follows immediately from (1) that J is a differential graded subcategory of $B^-(Shv(SmCor))$.

Note that $J_i(-,-) = 0$; this is a very important property! In particular, for any $i < 0$, $X, Y, Z \in Obj J$, it implies that $dJ_i(Y, Z) \circ J_0(X, Y) \subset dJ_i(X, Z)$ and $J_0(Y, Z) \circ dJ_i(X, Y) \subset dJ_i(X, Z)$. This is crucial for the construction of truncation functors t_N (see subsection 5.1 below). We call categories that have no morphisms of positive degrees *negative* differential graded categories; this property will be discussed in subsection 2.6 below.

We define $\mathfrak{H}$ as $Tr(J)$. Since $C_l = 0$ for $l > 0$, we have $\mathfrak{H} = Tr^+(J)$ (we can take $m_i = 0$ for any object of $Tr(J)$ in Definition 2.3.1). Now Proposition 2.3.2 implies the following statement immediately.

Proposition 2.4.1. *$\mathfrak{H}$ is generated by $[P]$, $P \in SmPrVar$, as a triangulated category. Here $[P]$ denotes the object of $\mathfrak{H}$ that corresponds to $[P] = C(P) \in Obj J$.*

We consider the functor $h : \mathfrak{H} \to K^-(Shv(SmCor))$ that is induced by the inclusion $J \to B^-(Shv(SmCor))$.

We also note that any differential graded functor $J \to A$ induces a functor $\mathfrak{H} \to Tr^+(A)$.

The definition of $\mathfrak{H}$ implies immediately that $\mathfrak{H}([P], Q[i]) = H^i(C(Q)(P))$ for $P, Q \in SmPrVar$.

2.5 The explicit description of $\mathfrak{H}$ and h

For the convenience of the reader we describe $\mathfrak{H}$ and h explicitly. Since in this subsection we just describe the category of twisted complexes over J explicitly, we don't need any proofs here. Yet one can check directly that $\mathfrak{H}$ (as described in this section) is a triangulated category.

We define $J' = \text{Pre-Tr}^+(J)$. J' is an *enhancement* of $\mathfrak{H}$ (in the sense of [4]). The idea is that taking cones of (twisted) morphisms becomes a well-defined operation in J' (in $\mathfrak{H}$ it is only defined up to a non-canonical isomorphism).

We describe an auxiliary category $\mathfrak{H}'$. $Obj\mathfrak{H}' = ObjJ' = Obj\mathfrak{H}$ whence $\mathfrak{H}'(X, Y) = \operatorname{Ker} \delta^0_{J'}(X, Y)$ for $X, Y \in Obj\mathfrak{H}'$.

Hence the objects of $\mathfrak{H}'$ are $(P_i,\ i \in \mathbb{Z}, f_{ij},\ i < j)$, where (P_i) is a finite sequence of (not necessarily connected) smooth projective varieties (we assume that almost all P_i are 0), $f_{ij} \in C_{i-j+1}(P_j)(P_i)$ for all $m, n \in \mathbb{Z}$ satisfy the condition

$$\delta^{m-n+1}(P_n)(f_{mn}) + \sum_{m<l<n} f_{ln}^{m-l+1}(f_{ml}) = 0. \tag{4}$$

The morphisms $g : A = (P_i, f_{ij}) \to B = (P'_i, f'_{ij})$ can be described as sets $(g_{ij}) \in C_{i-j}(P'_j)(P_i), i \leq j$, where g_{ij} satisfy

$$\delta^{i-j}_{P'_j}(g_{ij}) + \sum_{j \geq l \geq i} f'^{i-l}_{lj}(g_{il}) = \sum_{j \geq l \geq i} g^{i-l+1}_{lj}(f_{il})\ \forall i, j \in \mathbb{Z}. \tag{5}$$

We will assume that $g_{ij} = 0$ for $i > j$.

Note that $g_{ij} = 0$ if $P_i = 0$ or $P_j = 0$. Hence the morphisms for any pair of objects in $\mathfrak{H}'$ are defined by means of a finite set of equalities.

The composition of $g = (g_{ij}) : A \to B$ with $h = (h_{ij}) : B \to C = (P'_i, f'_{ij})$ is defined as

$$l_{ij} = \sum_{i \leq r \leq j} h^{i-r}_{rj}(g_{ir}).$$

$\mathfrak{H}'$ has a natural structure of an additive category. The direct sum of objects is defined by means of disjoint union of varieties.

The morphisms $g, h : A = (P_i, f_{ij}) \to B = (P'_i, f'_{ij})$ are called homotopic ($g \sim h$) if there exist $l_{ij} \in C_{i-j-1}(P'_j)(P_i), i \leq j$ such that

$$g_{ij} - h_{ij} = \delta^{i-j-1}_{P'_j} l_{ij} + \sum_{i \leq r \leq j} (f'^{i-r-1}_{rj}(l_{ir}) + l^{i-r+1}_{rj}(f_{ir})).$$

Now $\mathfrak{H}$ can be described as a category whose objects are the same as for $\mathfrak{H}'$ whence $\mathfrak{H}(A,B) = \mathfrak{H}'(A,B)/\sim$. The translation on $\mathfrak{H}$ is defined by shifts of indices (for P_i, f_{ij}). For $g = (g_{ij}) \in \mathfrak{H}'(A,B)$ its cone is defined as $C = Cone(g) \in Obj\mathfrak{H}'$, the i-th term of C is equal to $P"_i = P_{i+1} \oplus P'_i$ whence

$$h_{ij} \in C_{i-j+1}(P_{j+1} \oplus P'_j)(P_{i+1} \oplus P'_i) = \begin{pmatrix} f_{i+1,j+1} & 0 \\ g_{i+1,j} & f'_{ij} \end{pmatrix}, i < j.$$

It is easily seen that $\mathfrak{H}$ coincides with the category defined in 2.4. We denote the projection $\mathfrak{H}' \to \mathfrak{H}$ by j.

Moreover, as in Proposition 3.3 of [13] one can check (without using the formalism described above) that $\mathfrak{H}$ with the structures defined is a triangulated category. The direct proof is similar to the usual proof of the fact that the homotopy category of an abelian category is triangulated. One can also check directly that $P[0]$ for $P \in SmPrVar$ generate $\mathfrak{H}$ as a triangulated category.

For $A = (P_i, f_{ij}) \in Obj(\mathfrak{H}')$ we define $h'(A) \in C^-(Shv(SmCor))$ as $(C_{Aj}, \delta^j_A : C_{Aj} \to C_{Aj+1}) \in C^-(Shv(SmCor))$. Here $C_{Aj} = \sum_{i \le j} C_{i-j}(P_j)$, the component of δ^j_A that corresponds to the morphism of $C_{i-j}(P_j)$ into $C_{i-j'+1}(P'_j)$ equals $\delta^{i-j}_{P_j}$ for $j = j'$ and equals $f^{i-j}_{jj'}$ for $j' \neq j$.

Note that the condition (4) implies $d^2_{h'(A)} = 0$.

Now we define h' on morphisms. For $(l_{ij}) : A \to B$, $s \in \mathbb{Z}$, we set $h'(l)_s = \oplus_{i,j} l^{s-i}_{ij}$.

One can check explicitly that h' induces an exact functor $h : \mathfrak{H} \to K^-(Shv(SmCor))$.

By abuse of notation we denote by h' also the functor $J' \to B^-(Shv(SmCor))$.

Remark 2.5.1. P_i should be thought about as of 'stratification pieces' of the motif $A = (P_i, f_{ij})$. In particular, let Z be closed in X, $Z, X \in SmPrVar$, $Y = X - Z$; suppose that Z is everywhere of codimension c in X. If we adjoin $Z(c)[2c]$ to $Obj J$ (see subsection 7.3) then $M_{gm}(Y)$ could be presented in $\mathfrak{H}$ as $((X, Z(c)[2c]), g_Z)$, where g_Z is the Gysin morphism (see Proposition 3.5.4 of [23]). See also Proposition 5.4.1 for a nice explicit description of the motif with compact support of any smooth quasi-projective X.

The main distinction of $\mathfrak{H}$ from the motivic category defined by Hanamura (see [13])) is that we consider homological motives. As a result the Bloch cycle complexes are replaced by the Suslin complexes; we never have to choose distinguished subcomplexes for our constructions (in contrast with [13]). Note also that our definition works on the integral level in contrast with those of [13].

2.6 Other generalities on differential graded categories

We describe some new differential graded categories and differential graded functors. We will need them in section 6 below.

2.6.1 Differential graded categories of morphisms

For an additive category A we denote by $MS(A)$ the category of morphisms of $S(A)$. Its objects are $\{(X,Y,f): X,Y \in Obj A, f \in A(X,Y)\}$;

$$MS_0((X,Y,f),(X',Y',f') = \{(g,h): g \in A(X,X'),\ h \in A(Y,Y'),\ f'\circ g = h\circ g\}.$$

As for $S(A)$, there are no morphisms of non-zero degrees in $MS(A)$; hence the differential for morphisms is zero.

We denote Pre-Tr$(MS(A))$ by $MB^b(A)$. We recall that a twisted morphism is a closed morphism of degree 0, i.e. an element of the kernel of δ^0.

Proposition 2.6.1. *1. $MB^b(A)$ is the category of twisted morphisms of $B^b(A)$. That means that its objects are $\{(X,Y,f): X,Y \in Obj B(A), f \in \operatorname{Ker}\delta^0(B(A)(X,Y))\}$,*

$$MB^b_i((X,Y,f),(X',Y',f') = \{(g,h): g \in B(A)_i(X,X'),\ h \in B(A)_i(Y,Y'),\ f'\circ g = h\circ f\}.$$

2. Let $MB(A)$ denote the unbounded analogue of $MB^b(A)$. Then Pre-Tr$(MB(A)) \cong MB(A)$.

3. Let $Cone : MB(A) \to B^b(A)$ denote the natural cone functor. Then the functor Pre-Tr$(Cone)$ *in naturally isomorphic to $Cone$.*

Proof. 1. Easy direct verification.

2. The proof is very similar to those of part II1 of Proposition 2.2.3. First we note that Pre-Tr$(MB^bB(A)) \cong MB^b(A)$, then extend this to the unbounded analogue.

3. Obviously, $Cone$ is a differential graded functor. Hence it remains to apply part 2 of Remark 2.3.3. □

We have obvious differential graded functors $p_1, p_2 : MB(A) \to B(A)$: $p_1(X,Y,f) = X$, $p_2(X,Y,f) = Y$.

Corollary 2.6.2. *1. Let $F : J \to MB(A)$ be a differential graded functor. Then* Pre-Tr(F) *gives a functorial system of closed morphisms* Pre-Tr$(p_1 \circ F)(X) \to$ Pre-Tr$(p_2 \circ F)(X)$ *in $B(A)$ for $X \in Obj J' = Obj \mathfrak{H}$.*

2. Let A be an abelian category. Suppose that for any $P \in SmPrVar$ the complex $F([P])$ is exact. Then there exists a natural quasi-isomorphism $Tr^+(p_1 \circ F)(X) \sim Tr^+(p_2 \circ F)(X)$ for $X \in Obj \mathfrak{H}$.

Proof. 1. Obvious.

2. We have to show that Pre-Tr$^+(Cone(F))(X)$ is quasi-isomorphic to 0 for any $X \in Obj J'$. We consider the exact functor $G = Tr^+(Cone(F))(X)$; it suffices to show that $G = 0$. Recall that $[P]$, $P \in SmPrVar$, generate $\mathfrak{H}$ as a triangulated category. Hence $G([P]) = 0$ for any $P \in SmPrVar$ implies $G = 0$. □

2.6.2 Negative differential graded categories; truncation functors

We recall that a differential graded category C is called *negative* if $C_i(X,Y) = 0$ for any $i > 0$, $X, Y \in Obj C$. Certainly in this case all morphisms of degree zero are closed (i.e. satisfy $\delta f = 0$). This notion is very important for us since J is negative.

For any differential graded C there exist a unique 'maximal' negative subcategory C_- (it is not full!). The objects of C_- are the same as for C whence $C_{-,i}(X,Y) = 0$ for $i > 0$, $= C_i(X,Y)$ for $i < 0$, $= \ker \delta^0(C(X,Y))$ for $i = 0$.

Obviously, if $F : D \to C$ is a differential graded functor, D is negative, then F factorizes through the faithful embedding $C_- \to C$.

Suppose that A is an abelian category.

Then zeroth (or any other) cohomology defines a functor $B_-(A) \to S(A)$.

More generally, we define two versions of the canonical truncation functor for $B_-(A)$. We will need these functors in subsection 6.3 below.

Let X be a complex over A, $a, b \in \mathbb{Z}$, $a \le b$. We define $\tau_{\le b}$ as the complex

$$\cdots \to X_{b-2} \to X_{b-1} \to \operatorname{Ker}(X_b \to X_{b+1}),$$

here $\operatorname{Ker}(X_b \to X_{b+1})$ is put in degree b. $\tau_{[a,b]}(X)$ is defined as $\tau_{\le b}(X)/\tau_{\le a}(X)$ i.e. it is the complex

$$X_{a-1}/\operatorname{Ker}(X_{a-1} \to X_a) \to X_a \to X_{a+1} \to \dots X_{b-1} \to \operatorname{Ker}(X_b \to X_{b+1}).$$

The canonical $[a,b]$-truncation of X is defined as

$$X_{[a,b]} = X_a/dX_{a-1} \to X_{a+1} \to \dots X_{b-1} \to \operatorname{Ker}(X_b \to X_{b+1}),$$

again $\operatorname{Ker}(X_b \to X_{b+1})$ is put in degree b. Recall that truncations preserve homotopy equivalence of complexes.

Proposition 2.6.3. *1. $\tau_{\le b}$, $\tau_{[a,b]}$ and the canonical $[a,b]$-truncation define differential graded functors $B_-(A) \to B_-(A)$.*

2. Let $F : J \to B(A)$ be a differential graded functor; we can assume that its target is $B_-(A)$. We consider the functors $\tau_{[a,b]}F$ and $F_{[a,b]}$ that are *obtained from F by composing it with the corresponding truncations. Then there exists a functorial family of quasi-isomorphisms $Tr^+(\tau_{[a,b]}F)(X) \to Tr^+(F_{[a,b]})(X)$ for $X \in Obj\mathfrak{H}$.*

Proof. 1. Note that all truncations give idempotent endofunctors on $C(A)$.

Hence it suffices extend truncations to all morphisms of $B_-(A)$ and prove that truncations respect δ.

The definition of truncations on morphisms of negative degree is very easy. The only morphisms in $B_-(A)$ of degree zero are twisted ones i.e. morphisms coming from $C(A)$.

It remains to verify that if a given truncation τ of a morphism $f = (f_i) : (X_i) \to (Y_i)$ in $B_-(A)$ is zero then $\tau(\delta f) = 0$.

First we check this for $\tau = \tau_{\leq b}$. $\tau_{\leq b} f = 0$ means that $f(\tau_{\leq b} X) = 0$ (i.e. the corresponding restrictions of f_i are zero). Since the boundary maps $\tau_{\leq b} X$ into itself, the definition of δ for $B(A)$ gives the result.

Now we consider the case $\tau = \tau_{[a,b]}$. $\tau_{[a,b]}(f) = 0$ means that $f(\tau_{\leq b} X) \subset \tau_{\leq a} Y$. Again it suffices to note that the boundary maps $\tau_{\leq b} X$ and $\tau_{\leq a} Y$ into themselves.

The case of canonical truncation could be treated in the same way.

2. The natural morphism $m([P]) : \tau_{[a,b]}F([P]) \to F_{[a,b]}([P])$ gives a functor $H : J \to MB(A)$ such that $p_1(H) = \tau_{[a,b]}F$ and $p_1(H) = F_{[a,b]}$. It remains to note that $m([P])$ is a quasi-isomorphism for any $P \in SmPrVar$ and apply part 2 of Corollary 2.6.2. □

Remark 2.6.4. Another way to obtain differential graded categories is to take 'tenzor products' of differential graded categories. In particular, one could consider the categories $J \otimes J$ and $Tr(J \otimes J)$ (which could be denoted by $\mathfrak{H} \otimes \mathfrak{H}$).

3 The main classification result

In this section we prove the equivalence of $\mathfrak{H}$ and DM. it follows that the presentation of a motif as $m(X)$ for $X \in \mathfrak{H}$ could be thought about as of a 'motivic injective resolution'; here the Suslin complexes of smooth projective varieties play the role of injective objects.

3.1 The equivalence of categories

We denote the natural functor $K^-(Shv(SmCor)) \to D^-(Shv(SmCor))$ by p, denote $p \circ h$ by m.

Theorem 3.1.1. *m is a full exact embedding of triangulated categories; its essential image is DM.*

Proof. Since h is an exact functor, so is m. Now we check that m is a full embedding. By part 2 of Proposition 1.2.1, m induces an isomorphism $DM_-^{eff}(m([P]), m(Q[i])) \cong \mathfrak{H}([P], Q[i])$ for $P, Q \in SmPrVar$, $i \in \mathbb{Z}$. Since $[R]$, $R \in SmPrVar$, generate $\mathfrak{H}$ as a triangulated category (see Proposition 2.4.1), the same is true for any pair of objects of $\mathfrak{H}$.

We explain this argument in more detail.

First we verify that for any smooth projective P/k and arbitrary $B \in \mathfrak{H}$ the functor m gives an isomorphism

$$DM_-^{eff}(m([P]), m(B)) \cong \mathfrak{H}([P], B). \qquad (6)$$

By Proposition 1.2.1, (6) is fulfilled for $B = P'[j]$, $j \in \mathbb{Z}$, $P' \in SmPrVar$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{H}$ the functor m defines a morphism of long exact sequences

$$\begin{array}{ccccc} \to \mathfrak{H}([P], Y) & \longrightarrow & \mathfrak{H}([P], Z) & \longrightarrow & \mathfrak{H}([P], X[1]) \to \\ \downarrow & & \downarrow & & \downarrow \\ \to DM_-^{eff}(m([P]), m(Y)) & \longrightarrow & DM_-^{eff}(m([P]), m(Z)) & \longrightarrow & DM_-^{eff}(m(P), m(X)[1]) \to \end{array} \qquad (7)$$

Thus if m gives an isomorphism in (6) for $B = X[i]$ and $B = Y[i]$ for $i = 0, 1$, then m gives an isomorphism for $B = Z$. Since objects of the form $B = [P']$ generate $\mathfrak{H}$ as a triangulated category, (6) is fulfilled for any $B \in \mathfrak{H}$. Hence for all $i \in \mathbb{Z}$, $B \in \mathfrak{H}$, we have $DM_-^{eff}(m(P[i]), m(B)) \cong \mathfrak{H}(P[i], B)$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{H}$ the functor m defines a morphism of long exact sequences $(\cdots \to \mathfrak{H}(Z, B) \to \dots) \to (\cdots \to DM_-^{eff}(m(Z), m(B)) \to \dots)$ similar to (7). Now the same argument as above proves that m is a full embedding.

It remains to calculate the image of the map that is induced by m on $Obj(\mathfrak{H})$. According to Proposition 4.2.2 below we have $m([P]) = C(P) \cong M_{gm}(P)$.

Since $\mathfrak{H}$ is generated by $[P]$ for $P \in SmPrVar$ as a triangulated category, $M = m(\mathfrak{H})$ is the triangulated subcategory of DM_-^{eff} that is generated by all $M_{gm}(P)$. Since the tensor structure on DM_-^{eff} is defined by means of the relation $M_{gm}(X) \otimes M_{gm}(Y) = M_{gm}(X \times Y)$ for $X, Y \in SmVar$, M is a tensor subcategory of DM_-^{eff}. Since $M_{gm}(P) \in DM$ for any $P \in SmPrVar$, we have $M \subset DM$. It remains to prove that M contains DM. By definition (cf. subsection 2.1 in [23]) $\mathbb{Z}(1)[2] \in DM_-^{eff}$ can be presented as the cone of the natural map $M_{gm}(pt) \to M_{gm}(\mathbb{P}^1)$ (we will identify $\mathbb{Z}(1)$ with $M_{gm}(\mathbb{Z}(1))$).

Hence $\mathbb{Z}(n) \in ObjM$ for any $n > 0$. Since M_{gm} is a tensor functor, if $M_{gm}(Z) \in ObjM$ for $Z \in SmVar$, then $Z(c)[2c] \in ObjM$ for all $c > 0$. Now we apply Proposition 3.5.4 of [23] as well as the Mayer-Viertoris triangle for motives (§2 of [23]); similarly to Corollary 3.5.5 of [23] we conclude that $ObjM$ contains all $M_{gm}(X)$ for $X \in SmVar$ (cf. the remark in [23] that precedes Definition 2.1.1). A more detailed version of this argument will be used in the proof of Theorem 5.3.1 below.

Since DM is the smallest triangulated subcategory of DM_{-}^{eff} containing motives of all smooth varieties, we prove the claim. □

Remark 3.1.2. In order to calculate $DM_{-}^{eff}(M, M')$ for $M, M' \in DM$ (using the explicit description of $\mathfrak{H}$ given in subsection 2.5) in terms of cycles one needs to know $m^{-1}(M)$ and $m^{-1}(M')$ (or the preimages of their duals). See subsection 5.4 for a nice result in this direction.

3.2 On motives of (possibly) singular varieties

Recall (see §4.1 of [23]) that for any $X \in Var$ there were certain objects $M_{gm}(X)$ and $M_{gm}^{c}(X)$ of DM_{-}^{eff} defined. $M_{gm}(X)$ was called the motif of X; $M_{gm}^{c}(X)$ was called the motif of X with compact support.

The following statement follows easily.

Corollary 3.2.1. *For any (not necessarily smooth) variety X/k there exist $Z, Z' \in \mathfrak{H}$ such that $m(Z) \cong M_{gm}(X)$, $m(Z') \cong M_{gm}^{c}(X)$.*

Proof. It is sufficient to verify that $M_{gm}(X), M_{gm}^{c}(X) \in DM$. The proof of this fact is the same as for Corollaries 4.1.4 and 4.1.6 in [23]. Indeed, for the proofs in [23] one doesn't need to add the kernels of projectors.

□

Remark 3.2.2. We obtain that for the Suslin complex of an arbitrary variety X there exists a quasi-isomorphic complex M 'constructed from' the Suslin complexes of smooth projective varieties; $M \in h_*(Obj\mathfrak{H}) \subset ObjK^{-}(Shv(SmCor))$ is unique up to a homotopy.

Moreover, as we have noted in the proof of Lemma 4.2.1, the cohomology of $C(P)$ as a complex of presheaves for any $P \in SmPrVar$ coincides with its hypercohomology (the corresponding fact for $\underline{C}(P)$ was proved in [23]). Hence the same is true for any $M \in h_*(Obj\mathfrak{H})$. Hence for $X \in SmVar$ the quasi-isomorphism $C(X) \to M$ is given by an element of $H^0(M)(X)$.

This result shows that the presentation of a motif as $m(X)$ for $X \in \mathfrak{H}$ could be thought about as of a 'motivic' analogue of taking an injective resolution; here the Suslin complexes of smooth projective varieties play the role of injective objects.

4 The properties of cubical Suslin complexes

The main result of this section is that the cubical complex $C(X)$ is quasi-isomorphic (as a complex of presheaves) to the simplicial complex $\underline{C}(X)$ that was used in [23]. This fact was mentioned by Levine (Theorem 2.25 of [18]) yet no proof was given. One of the possible methods of the proof is the use of a bicomplex similar to those that was considered in §4 of [19]; we use another method. The reader not interested in the details of the proof could omit this section.

4.1 Certain adjoint functor for the derived category of presheaves

We denote by $PreShv(SmCor)$ the category of presheaves (of abelian groups) on $SmCor$, by $D^-(PreShv(SmCor))$ the derived category of $PreShv(SmCor)$ (complexes are bounded from above), by DPM^{eff} we denote a full subcategory of $D^-(PreShv(SmCor))$ whose objects are complexes with homotopy invariant cohomology.

Lemma 4.1.1. $C(P) \in DPM^{eff}$.

Proof. The scheme of the proof is the same as for Proposition 3.6 in [24]. First we check that for $Y \in SmVar$ and $P \in SmPrVar$ the maps $i_0^*, i_1^* : C(P)(Y \times \mathbb{A}) \to C(P)(Y)$ are homotopic; here i_0^*, i_1^* are induced by the embeddings $i_x : Y \times \{x\} \to Y \times \mathbb{A},\ x = 0, 1$. We consider the maps $pr_i : C'(P)_i(Y) \to C'(P)_i(Y \times \mathbb{A})$ induced by the projections $Y \times \mathbb{A} \to Y$. We consider the maps $h'_i : C'(P)_i(Y \times \mathbb{A}) \to C'(P)_{i-1}(Y)$ induced by isomorphisms $Y \times \mathbb{A} \times \mathbb{A}^{-i} \cong Y \times \mathbb{A}^{-i+1}$, and also $h_i : C(P)_i(Y \times \mathbb{A}) \to C(P)_{i-1}(Y)$, $h_i = h'_i - pr_{i-1} \circ i_0^*$. We have

$$\delta_*^{i-1} h_i + h_{i+1} \delta_*^i = (i_1^* - i_0^*)_i,$$

i.e. h_i gives the homotopy needed. Then i_0^*, i_1^* induce coinciding maps on cohomology. Let $Y = U \times \mathbb{A}$, $U \in SmVar$. We consider the morphism $H = id_U \times \mu : U \times \mathbb{A}^2 \to U \times \mathbb{A}$, where μ is given by multiplication. We proved that the maps induced by $\mu \circ i_0$ and $\mu \circ i_1$ on the cohomology of $C(P)(Y)$ coincide. Hence the composition $U \times \mathbb{A} \to U \stackrel{id_U \times i_0}{\to} U \times \mathbb{A}$ induces an isomorphism on the cohomology of $C(P)(U \times \mathbb{A})$ for any $U \in SmVar$, i.e. the cohomology presheaves of $C(P)$ are homotopy invariant. □

Now we formulate an analogue of Proposition 3.2.3 [23]. By (F) we denote a complex concentrated in degree 0 whose non-zero term is F.

Proposition 4.1.2. *1. There exists an exact functor* $R : D^-(PreShv(SmCor)) \to DPM^{eff}$ *right adjoint to the embedding* $DPM^{eff} \to D^-(PreShv(SmCor))$. *Besides* $R((F)) \cong \underline{C}(F)$ *(see the definition of* $\underline{C}(F)$ *in 3.2 [23]).*

2. In $D^-(PreShv(SmCor))$ *we have* $R(\underline{C}(L(P))) \cong C(P) \cong R((L(P)))$.

Proof. 1. The proof is similar to the proof of existence of the projection $RC : D^-(Shv(SmCor)) \to DM_-^{eff}$ in 3.2 of [23]. We consider the localising subcategory $\mathcal{A}$ in $D^-(PreShv(SmCor))$ that is generated by all complexes $L(X\times\mathbb{A}) \to L(X)$ for $X \in SmVar$. As in [23] we have $D^-(PreShv(SmCor))/\mathcal{A} \cong DPM^{eff}$ (cf. Theorem 9.32 of [21]).

Now as in the proof of Proposition 3.2.3 in [23] we should verify the following statements.

1. For any $F \in PreShv(SmCor)$ the natural morphism $\underline{C}(F) \to (F)$ is an isomorphism in $D^-(PreShv(SmCor))/\mathcal{A}$.

2. For all $T \in DPM^{eff}$ and $B \in \mathcal{A}$ we have $D^-(PreShv(SmCor))(B,T) = 0$.

The proof of the first assertion may be copied word for word from the similar statement in 3.2.3 of [23].

As in [23], for the second assertion we should check for any $X \in SmVar$ the bijectivity of the map

$$D^-(PreShv(SmCor))((L(X)),T) \to D^-(PreShv(SmCor))(((L(X\times\mathbb{A})),T)$$

induced by the projection $X \times \mathbb{A} \to X$. Since representable presheaves are projective in $PreShv(SmCor)$ (obvious from Ioneda's lemma, cf. 2.7 in [21]), this follows immediately from the homotopy invariance of the cohomology of $\underline{C}(F)$.

2. From part 2 of Lemma 4.1.3 below we obtain that the morphism $(L(P)) \to C(P)$ induces an isomorphism $R((L(P))) \cong R(C(P))$ in the category $D^-(PreShv(SmCor))$. Using assertion 1 we obtain that the map $\underline{C}(P) \to (L(P))$ induces an isomorphism $R((L(P)) \cong R(\underline{C}(P))$. Since R is right adjoint to an embedding of categories, it remains to note that $\underline{C}(L(P)), C(P) \in DPM^{eff}$. □

Lemma 4.1.3. *1.* $R(C_j(P)) = 0$ *for* $j < 0$.

2. The morphism $i_P : (L(P)) \to C(P)$ *induces an isomorphism* $R((L(P))) \cong R(C(P))$ *in* $D^-(PreShv(SmCor))$.

Proof. 1. We consider the same maps $h_i : C(P)_i(Y \times \mathbb{A}) \to C(P)_{i-1}(Y)$ as in the proof of Lemma 4.1.1. Obviously h_i is epimorphic, besides $\operatorname{Ker} h_i \cong C(P)_i(Y)$. We obtain an exact sequence

$$0 \to C(P)_i(Y) \to C(P)_i(Y \times \mathbb{A}) \to C(P)_{i-1}(Y) \to 0. \qquad (8)$$

We prove the assertion by induction on j. The case $j = -1$ follows immediately from (8) applied for the case $i = 0$. If $R(C_j(P)) = 0$ for $j = m$, then R maps $C(P)_m$ and $C"(P)_m$ to 0, where $C"(P)_m(Y) = C(P)_m(Y \times \mathbb{A})$. Applying (8) for $i = m$ we obtain $R(C(P)_{m-1}) = 0$ (recall that R is an exact functor).

2. Follows from part 1 immediately. □

Now we recall (Theorem 8.1 of [25]) that the cohomology groups of $\underline{C}(L(P))(Y)$ are exactly $A_{0,-i}(Y,P)$. Hence we completed the proof of Proposition 1.1.2.

Remark 4.1.4. 1. Applying Proposition 3.2.3 [23] itself we could prove that $\mathbb{H}^i(C(P))(-) \cong A_{0,-i}(P,-)$. Next we would have had to prove an analogue of Theorem 8.1 of [25] for the complex $C(P)$.

2. The same reasoning shows that $C(F) \cong \underline{C}(F)$ in DPM^{eff} for any presheaf F (or a complex of presheaves).

3. Lemma 4.1.1 could be proved in a different way. Indeed, one can obtain the complex $C(P)$ step-by-step. On each step we should complete a complex of sheaves

$$C_i = \cdots \to 0 \to F_{-i} \to F_{-i+1} \to \cdots \to F_0 \to 0 \to 0 \to \dots,$$

whose cohomology are homotopy invariant presheaves expect for dimension $-i$ by the map $F(- \times \mathbb{A}) \to F_{-i}$. Here $F(Y) = \mathrm{Ker}(F_{-i} \to F_{-i+1})$, the map $F(- \times \mathbb{A}) \to F_{-i}$ is induced by $(i_1^* - i_0^*)F(- \times \mathbb{A}) \to F$, where i_0, i_1 are the embeddings $Y \times \{x\} \to Y \times \mathbb{A},\ x = 0, 1$.

4.2 Proof of Proposition 1.2.1

Lemma 4.2.1. *For all $i \in \mathbb{Z}$, $P, Y \in SmPrVar$, the obvious homomorphism*

$$K^-(Shv(SmCor))((L(Y)), C(P)[i]) \to D^-(Shv(SmCor))((L(Y)), C(P)[i])$$

is bijective.

Proof. By definition the homomorphism considered in the map from the cohomology of $C(P) = C^c(P)$ into its hypercohomology. By theorem 8.1 of [25] for $\underline{C}(P)(= \underline{C}^c(P))$ the corresponding map is bijective. Hence the assertion follows from $\underline{C}(P) \cong C(P)$ in $D^-(PreShv(SmCor))$. □

Proposition 4.2.2. *$i_P : (L(P)) \to C(P)$ induces an isomorphism $RC((L(P))) \cong C(P)$ in DM_-^{eff}.*

Proof. Literally repeating the argument of the proof of part 1 of Lemma 4.1.3 we obtain $RC(C_j(P)) = 0$ for $j < 0$. Therefore $RC((L(P))) \cong RC((C(P))$. It remains to note that $C(P) \in ObjDM_-^{eff}$. $\square$

Now we finish the proof of Proposition 1.2.1. The assignment $g \to G = (g^l)$ defines a homomorphism

$$K^-(Shv(SmCor))((L(Y)), C(P)[i]) \to K^-(Shv(SmCor))(C(Y), C(P)[i]).$$

Hence it is sufficient to verify that the map

$$DM_-^{eff}(RC((L(Y))), C(P)[i]) \to DM_-^{eff}(C(Y), C(P)[i])$$

induced by this homomorphism coincides with the homomorphism induced by the map $i_{P*} : RC(L(Y)) \cong RC((C(Y)))$. Since $G \circ i_{P*} = g$, we are done.

5 Truncation functors and motives of limited length

In this section using the canonical filtration of the (cubical) Suslin complex we define the *truncation functors* t_N. These functors are new though certain very partial cases were (essentially) considered in [11] and [12] (there another approaches were used).

The target of t_0 is just $K^b(\mathfrak{Chow})$ (the 'modified' category of Chow motives, see Notation). We prove that t_0 induces an isomorphism $K_0(DM_{gm}^{eff}) \cong K_0(Chow)$ thus answering the question of 3.2.4 of [11].

We define the 'length' of a motive. We prove that motives of smooth varieties of dimension N have length $\leq N$; besides $t_N(X)$ contains all information on motives of length $\leq N$. The length of a motif is a natural motivic analogue of the length of weight filtration for a mixed Hodge structure; in particular, the length of the weight filtration of the Hodge cohomology of a motif X is not greater than the length of X.

For a smooth quasi-projective variety X we calculate $m^{-1}(M_{gm}^c(X))$ explicitly. Using this result we prove that the weight complex of Gillet and Soulé for a smooth quasi-projective variety X can be described as $t_0(m^{-1}(M_{gm}^c(X)))$. Next we recall the *cdh*-topology of Voevodsky and prove this statement for arbitrary $X \in Var$. Besides, $t_0(m^{-1}(M_{gm}(X)))$ essentially coincides with the functor h described in Theorem 5.10 of [12].

In the next section we will verify that the weight filtration of 'standard' realizations is closely related to t_N.

In section 7 we will also show that all t_N respect tensor products.

5.1 Truncation functors of level N

For $N \geq 0$ we denote the $-N$-th canonical filtration of $C(P)$ as a complex of presheaves (i.e. $C_{-N}(P)/d_P C_{-N-1}(P) \to C_{-N+1}(P) \to \dots \to C_0(P) \to 0$) by $C^N(P)$.

We denote by J_N the following differential graded category. Its objects are the symbols $[P]$ for $P \in SmPrVar$ whence $J_N([P],[Q])_i = C_i^N(Q)(P)$. The composition of morphisms is defined similarly to those in J. For morphisms in J_N presented by $g \in C_i(Q)(P)$, $h \in C_j(R)(Q)$, we define their composition as the morphism presented by $h^i(g)$ for $i+j \geq -N$ and zero for $i+j < -N$. Note that for $i+j=-N$ we take the class of $h^i(g) \mod d_R C_{-N-1}(R)(P)$; for $i=-N$, $j=0$, and vice versa, g is only defined up to an element of $d_Q C_{-N-1}(Q)(P)$ (resp. h is defined up to an element of $d_R C_{-N-1}(R)(Q)$) yet the composition is well-defined. The boundary on morphisms is also defined as in J i.e. for $g \in J_N(P,Q)$ we define $\delta g = d_Q g$. Certainly, all J_N are negative (i.e. there are no morphisms of degree > 0).

We have an obvious functor $J \to J_N$. As noted in remark 2.3.3, this gives canonically a functor $t_N : \mathfrak{H} \to Tr(J_N)$. We denote $Tr(J_N) = Tr^+(J_N)$ by $\mathfrak{H}_N$; note that $\mathfrak{H}_0$ is precisely $K^b(\mathfrak{Chow})$.

For any $m \leq N$ we also have an obvious functor $J_N \to J_m$. It induces a functor $t_{Nm} : \mathfrak{H}_N \to \mathfrak{H}_m$ such that $t_m = t_{Nm} \circ t_N$.

Obviously, one can give a description of $\mathfrak{H}_N$ that is similar to the description of $\mathfrak{H}$ given in subsection 2.5. Hence the objects of $\mathfrak{H}_N$ could be presented as certain $(P_i, f_{ij} \in C_{i-j+1}^N(P_j)(P_i), i < j \leq i+N+1)$, the morphisms between (P_i, f_{ij}) and (P'_i, f'_{ij}) are represented by certain $g_{ij} \in C_{i-j}^N(P'_j, P_i)$, $i \leq j \leq i+N$, etc. The functor t_N 'forgets' all elements of $C_m([P],[Q])$ for $P, Q \in SmPrVar$, $m < -N$, and factorizes $C_{-N}([P],[Q])$ modulo coboundaries. In particular, for $N=0$ we get ordinary complexes over $\mathfrak{Chow}$.

5.2 The study of $K_0(DM_{gm}^{eff})$

We recall some standard definitions (cf. 3.2.1 of [11]). We define the Grothendieck group $K_0(Chow)$ as a group whose generators are of the form $[A]$, $A \in ObjChow$; the relations are $[A]=[B]$ if $A \cong B$ in $Chow$, $[A \oplus B] = [A]+[B]$. The K_0-group of a triangulated category T is defined as the group whose generators are $[t]$, $t \in ObjT$; $[t]=[t']$ if $t \cong t'$ in T; if $A \to B \to C \to A[1]$ is a distinguished triangle then $[B] = [A]+[C]$.

The existence of t_0 allows to calculate $K_0(DM_{gm}^{eff})$ easily.

Proposition 5.2.1. *1. $t_0(m^{-1})$ can be extended to an exact functor $t : DM_{gm}^{eff} \to K^b(Chow)$.*

2. t induces an isomorphism $K_0(DM_{gm}^{eff}) \cong K_0(Chow)$.

Proof. 1. Recall that DM^{eff}_{gm} is the idempotent completion of DM. Hence $t_0(m^{-1})$ can be canonically extended to an exact functor from DM^{eff}_{gm} to the idempotent completion of $K_b(\mathfrak{Chow})$. It remains to note that the idempotent completion of $K_b(\mathfrak{Chow})$ is exactly $K_b(Chow)$ (see, for example, Corollary 2.12 of [1]).

2. Since t is an exact functor, it gives an abelian group homomorphism $a : K_0(DM^{eff}_{gm}) \to K_0(K^b(Chow))$. By Lemma 3 of 3.2.1 of [11], there is a natural isomorphism $b : K_0(K^b(Chow)) \to K_0(Chow)$. The embedding $Chow \to DM^{eff}_{gm}$ (see Proposition 2.1.4 of [23]) gives a homomorphism $c : K_0(Chow) \to K_0(DM^{eff}_{gm})$. The definitions of a, b, c imply immediately that $b \circ a \circ c = id_{K_0(Chow)}$. It remains to note that c is surjective by Corollary 3.5.5 of [23]. □

5.3 Motives of length $\leq N$

It was proved in [23] that the functor M_{gm} gives a full embedding of $\mathfrak{Chow} \to DM^{eff}_{-}$. In this subsection we prove a natural generalization of this statement.

We will say that $P = (P_i, f_{ij}) \in Obj\mathfrak{H}'$ is concentrated in degrees $[l, m]$, $l, m \in \mathbb{Z}$, if $P_i = 0$ for $i < l$ and $i > m$. We denote the corresponding additive set of objects of $\mathfrak{H}'$ by $\mathfrak{H}'_{[a,b]}$. We denote $j(\mathfrak{H}'_{[a,b]}) \subset Obj\mathfrak{H}$ (see subsection 2.5) by $\mathfrak{H}_{[a,b]}$.

Obviously, if $A \to B \to C \to A[1]$ is an exact triangle, $A, B \in \mathfrak{H}_{[a,b]}$, then $C \in \mathfrak{H}_{[a-1,b]}$.

Theorem 5.3.1. *1. For any smooth variety Y/k of dimension $\leq N$ we have $m^{-1}(M_{gm}(Y)) \in \mathfrak{H}_{[0,N]}$.*

2. For any smooth variety Y/k of dimension $\leq N$ we have $m^{-1}(M^c_{gm}(Y)) \in \mathfrak{H}_{[-N,0]}$.

3. If $A \in \mathfrak{H}_{[a,b]}$, $B \in \mathfrak{H}_{[c,d]}$, $N \geq d - a$, $N \geq 0$, then $\mathfrak{H}(A, B) \cong \mathfrak{H}_N(t_N(A), t_N(B))$.

4. For $Y \in \mathfrak{H}$ we have $Y = 0 \iff t_0(Y) = 0$.

5. $f : A \to B$ is an isomorphism whenever $t_0(f)$ is.

6. Let $t : DM^{eff}_{gm} \to K^b(Chow)$ be the functor defined in Proposition 5.2.1. Then for $X \in ObjDM^{eff}_{gm}$ we have $t(X) = 0 \iff X = 0$.

7. Let $A \in \mathfrak{H}_{[a,b]}$, $B \in \mathfrak{H}_{[c,d]}$, $N + 1 \geq d - a$, $N \geq 0$. Then $A \cong B$ iff $t_N(A) \cong t_N(B)$.

Proof. 1. Obviously, the statement is valid for smooth projective Y. We prove the general statement by induction on dimension.

By the projective bundle theorem (see Proposition 3.5.4 of [23]) for any $c \geq 0$ we have a canonical isomorphism $\mathbb{P}^c \cong \oplus_{0 \leq i \leq c}\mathbb{Z}(i)[2i]$. Hence $\mathbb{Z}(c)[2c]$

can be presented as a cone of the natural map $M_{gm}(\mathbb{P}^{c-1}) \to M_{gm}(\mathbb{P}^c)$. Therefore $\mathbb{Z}(c)[2c] \in m(\mathfrak{H}_{[-1,0]})$.

One could easily show that for any $X \in \mathfrak{H}_{[e,f]}$, $e, f \in \mathbb{Z}$, $c > 0$ we have $X(c)[2c] \in \mathfrak{H}_{[e-1,f]}$. This could be done using Proposition 7.4.1 below or by an easier reasoning involving only tensor products of the form $\mathbb{P}^n \otimes X$ and $\mathbb{P}^{n-1} \otimes X$.

We recall the Gysin exact triangle (see Proposition 3.5.4 of [23]). For a closed embedding $Z \to X$, Z is everywhere of codimension c, it has the form

$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}(Z)(c)[2c] \to M_{gm}(X - Z)[1]. \qquad (9)$$

Suppose that the assertion is always fulfilled for $\dim Y = N' < N$.

Let X/k be smooth quasi-projective. Since k admits resolution of singularities, X can be presented as a complement to a $P \in SmPrVar$ of a divisor with normal crossings $\cup_{i\geq 0} Q_i$. Then using (9) one proves by induction on j that the assertion is valid for all $Y_j = P \setminus (\sqcup_{0\leq i\leq j} Q_i)$. To this end we check by the inductive assumption for $j \geq 0$ that

$$M_{gm}(P\setminus(\cup_{0\leq i\leq j}Q_i)\setminus P\setminus(\cup_{0\leq i\leq j+1}Q_i) = M_{gm}(Q_{i+1}\setminus(\cup_{0\leq i\leq j}Q_j)) \in m(\mathfrak{H}_{[0,N]}).$$

Hence $M_{gm}(X) \in m(\mathfrak{H}_{[0,N]})$.

If X is not quasi-projective we can still choose closed $Z \subset X$ such that $X - Z$ is quasi-projective. Hence the assertion follows from the inductive assumption by applying (9).

2. The proof is similar to those of the previous part. The difference is that we don't have to twist and should use the exact triangle of Proposition 4.1.5 of [23]

$$M^c_{gm}(Z) \to M^c_{gm}(X) \to M^c_{gm}(X - Z) \to M^c_{gm}(Z)[1] \qquad (10)$$

instead of (9).

3. We can assume (by increasing d if needed) that $N = d - a$.

Let $A = (P_i, f_{ij})$, $B = (P'_i, f'_{ij})$. As we have seen in subsection 2.5, any $g \in \mathfrak{H}(A, B)$ is given by a certain set of $g_{ij} \in C_{i-j}(P'_j)(P_i), i \leq j$. The same is valid for $h = (h_{ij}) \in \mathfrak{H}_N(t_N(A), t_N(B))$; the only difference is that h_{ad} is given modulo $d_{P'_d} C_{-N-1}(P'_d)(P_a)$. Both (g_{ij}) and (h_{ij}) should satisfy the conditions

$$\delta^{i-j}_{P'_j}(m_{ij}) + \sum_{j\geq l\geq i} f'^{i-l}_{lj}(m_{il}) = \sum_{j\geq l\geq i} m^{i-l+1}_{lj}(f_{il}) \ \forall i, j \in \mathbb{Z}. \qquad (11)$$

First we check surjectivity. We note that the conditions (11) for g depend only on g_{ij} for $(i, j) \neq (a, d)$ and on $d_{P'_d} g_{ad}$. Hence if (h_{ij}) satisfies the conditions (11) then $h = t_N(r)$, where $r_{ij} = h_{ij}$ for all $(i,j) \neq (a,d)$, r_{ad} is an arbitrary element of $C_{-N}(P'_d)(P_a)$ satisfying $r_{ad} \mod d_{P'_d}C_{-N-1}(P'_d)(P_a) = h_{ad}$.

Now we check injectivity. Let $t_N(g) = 0$ for $g = (g_{ij}) \in \mathfrak{H}'(A,B)$. Note that $C_N(P)$ is a factor-complex of $C(P)$ for any $P \in SmPrVar$. Hence similarly to subsection 2.5 one can easily check that there exist $l_{ij} \in C_{i-j-1}(P'_j)(P_i), i \leq j$, such that

$$g_{ij} = \delta_{P'_j}^{i-j-1} l_{ij} + \sum_{i \leq r \leq j} ({f'}_{rj}^{i-r-1}(l_{ir}) + l_{rj}^{i-r+1}(f_{ir})) \qquad (12)$$

for all $(i,j) \neq (a,d)$, for $i = a$, $j = d$ the equality (12) is fulfilled modulo $d_{P'_d}q$ for some $q \in C_{-N-1}(P'_d)(P_a)$. We consider (l'_{ij}), where $l'_{ij} = l_{ij}$ for all $(i,j) \neq (a,d)$, $l'_{ad} = l_{ad} + q$. Obviously, if we replace (l_{ij}) by (l'_{ij}) then (12) would be fulfilled for all i,j. Therefore $g = 0$ in $\mathfrak{H}(A,B)$.

In fact, surjectvity (but not injectivity) is also valid for $d - a = N + 1$. The proof is similar to those of the case $d - a = N + 1$. We should choose r_{ad}, $r_{a+1,d}$, and $r_{a,d-1}$; the classes of $r_{a+1,d}$ and $r_{a,d-1}$ modulo coboundaries are fixed. This choice affects the equality (11) only for $i = a$, $j = d$. Note also that this equality only depends on $d_{P'_d}r_{ad}$. One can choose arbitrary values of $r_{a+1,d}$ and $r_{a,d-1}$ in the corresponding classes. Then the equality (11) with $r_{ad} = 0$ will be satisfied modulo $d_{P'_d}q$ for some $q \in C_{-N-1}(P'_d)(P_a)$. Therefore if we take $r_{ad} = q$ then $t_N(r) = h$.

4. Obviously, $t_0(0) \cong 0$. The converse implication will be proved in subsection 6.3.

5. Follows immediately from part 5 (recall that a morphism is an isomorphism whenever its cone is zero).

6. Certainly $t(0) = 0$.

We check the converse implication. By definition of DM_{gm}^{eff} any $X \in ObjDM_{gm}^{eff}$ can be presented as the 'formal kernel' for some $p \in DM(Y,Y)$, $Y \in ObjDM$, $p^2 = p$. If $t(X) = 0$ then $t(p)$ is an automorphism of $t(Y)$. Assertion 5 implies that p is an automorphism also. Hence $X = 0$.

7. If $f : A \to B$ is an isomorphism then $t_N(f)$ also is.

Conversely, let $f_N : t_N(A) \to t_N(B)$ be an isomorphism. Then, as was noted in the proof of assertion 3, there exists an $f \in \mathfrak{H}(A,B)$ such that $f_N = t_N(f)$ (f is not necessarily unique). Since $t_N(F)$ is an isomorphism, $t_0(f) = t_{N0}(t_N(f))$ also is. >From assertion 5 we obtain that f gives an isomorphism $A \cong B$. □

In fact, for a smooth quasi-projective X one can compute $M_{gm}^c(X)$ explicitly (see Proposition 5.4.1 below).

We say that $X \in \mathfrak{H}$ has length $\leq N$ if for some $l \in \mathbb{Z}, m \leq l+N$, the motive X is concentrated in degrees $[l, m]$.

Remark 5.3.2. 1. Immediately from part 7 of the theorem we obtain that two objects $A, B \in \mathfrak{H}$ of length $\leq N+1$ are isomorphic whenever $t_N(A) \cong t_N(B)$.

2. One could define $\mathfrak{H}_{N,[0,N]} \subset \mathfrak{H}_N$ similarly to $\mathfrak{H}_{[0,N]}$. Then t_N would give an equivalence of categories $\mathfrak{H}_{[0,N]} \to \mathfrak{H}_{N,[0,N]}$. Indeed, this restriction of t_N is surjective on objects; it is an embedding of categories by part 3 of Theorem 5.3.1.

3. The length of a motif is a natural motivic analogue of the length of weight filtration of a mixed Hodge structure or of a geometric representation (i.e. of a representation coming from the étale cohomology of a variety). The results of the next section easily imply that the length of the weight filtration of the Hodge cohomology of a motif X is not greater than the length of X. Parts 1,2 of the theorem above are motivic analogues of the corresponding statements for Hodge cohomology.

5.4 Explicit calculation of $m^{-1}(M^c_{gm}(X))$; the weight complex of Gillet and Soulé of smooth quasi-projective varieties

Proposition 5.4.1. *For smooth quasi-projective X/k let $j : X \to P$ be an embedding for $P \in SmPrVar$, let $P \setminus X = \cup Y_i, 1 \leq i \leq m$, be a divisor with normal crossings. Let $U_i = \sqcup_{(i_j)} Y_{i_1} \cap Y_{i_2} \cap \dots \cap Y_{i_r}$ for all $1 \leq i_1 \leq \dots \leq i_r \leq m$, $U_0 = P$. We have r natural maps $U_r \to U_{r-1}$. We denote by d_r their alternated sum (as a smooth correspondence). We consider $Q = (Q_i, f_{ij})$, where $Q_i = U_{-i}$ for $0 \leq i \leq -m$, $P_i = 0$ for all other i; $f_{ij} = d_i$ for $0 > i \geq -m$, $j = i+1$, and $f_{ij} = 0$ for all other (i,j). Then $M^c_{gm}(X) \cong m(Q)$.*

Proof. Let j^* denote the natural morphism $M^c_{gm}(P) = M_{gm}(P) \to M^c_{gm}(X)$ (see 4.1 of [23]). Then by Proposition 4.1.5 loc. cit. the cone of j^* is naturally isomorphic to $M^c_{gm}(R_0) = M_{gm}(R_0)$, where $R_0 = \cup Y_i$ (note that R_0 is proper). Hence our assertion is equivalent to the statement that $C(R) \cong 0$, here C denotes the Suslin complex of R,

$$R = L(U_m) \stackrel{d_{m*}}{\to} L(U_{m-1}) \stackrel{d_{m-1*}}{\to} \dots \stackrel{d_{2*}}{\to} L(U_1) \to L(R_0).$$

The acyclity of $C(R)$ could be called 'multi-Mayer-Viertoris'. Its proof is quite similar to the corresponding part of the proof of Theorem 3.2.6 of [23]. By Theorem 5.9 of [24] it suffices to check that R is acyclic. We can verify this by applying Proposition 3.1.3 of [23] for the covering $\{Y_i \to R_0\}$. $\square$

We also get an explicit presentation of $M^c_{gm}(X)$ as a complex over $SmCor$ (this corresponds to the first description of DM^{eff}_{gm} in [23]). The terms of the complex re (motives of) smooth projective varieties.

Applying Proposition 5.4.1 along with the statements of subsection 6.3 below we get a nice machinery for computing cohomology with compact support.

Using Proposition 5.4.1 along with Theorem 3.1.1 one could write an explicit formula for $DM^{eff}_-(M^c_{gm}(X), M^c_{gm}(Y))$ for smooth quasi-projective $X, Y/k$.

Using section 4.3 of [23] one could also calculate $DM^{eff}_-(M_{gm}(X), M_{gm}(Y))$. Indeed, if $\dim X = m$, $\dim Y = n$, X, Y are smooth equidimensional, then (in the category of geometric motives DM_{gm})

$$\begin{aligned} DM^{eff}_-(M_{gm}(X), M_{gm}(Y)) &= DM_{gm}(M_{gm}(Y)^*, M_{gm}(X)^*) \\ &= DM_{gm}(M^c_{gm}(Y)(-n)[-2n], M^c_{gm}(X)(-m)[-2m]) \\ &= DM_{gm}(M^c_{gm}(Y)(m)[2m], M^c_{gm}(X)(n)[2n]), \end{aligned}$$

see subsection 7.3 for the discussion on $\mathbb{Z}(r)[2r]$.

Let $M^c_{gm}(X)$ for $X \in SmCor$ denote the motive of X with compact support (cf. 2.2 or 4.1 of [23]), m^{-1} is the equivalence of $DM \subset DM^{eff}_-$ with $\mathfrak{H}$ inverse to m.

In the paper [11] for any X/k a certain *weight complex* of Chow motives was defined. In order to make the notation of [11] compatible with ours we invert arrows in the category of Chow motives. Thus we consider homological Chow motives instead of cohomological ones considered in [11].

Corollary 5.4.2. *For smooth quasi-projective X/k the complex $t_0(m^{-1}(M^c_{gm}(X)) \in K^b(\mathfrak{Chow})$ is isomorphic (up to inversion of arrows) to the weight complex defined in [11].*

Proof. By Proposition 2.8 of [11] the weight complex of X is the image in $K^b(\mathfrak{Chow})$ of the complex U defined in Proposition 5.4.1 (with arrows inverted). □

5.5 cdh-coverings; the weight complex of Gillet and Soulé for arbitrary varieties

We recall one of the main tools of [23] (cf. Definition 4.1.9); it allows to do computations with motives of non-smooth varieties.

Definition 5.5.1. cdh-topology is the smallest Grothendieck's topology such that both Nisnevich coverings and coverings of the form $X' \coprod Z \to X$ are

cdh-coverings; here $p : X' \to Z$ is a proper morphism, $i : Z \to X$ is a closed embedding, and the morphism $p^{-1}(X - i(Z)) \to X - i(Z)$ is an isomorphism.

By Lemma 12.26 of [21], proper cdh-coverings are exactly envelopes in the sense of [11]. Therefore, hyperenvelope in the sense of [11] is exactly the same thing as a proper cdh hypercovering. We recall that a cdh hypercovering is an augmented simplicial variety $X.$ such that each $X_i \to (cosk_{i-1}sk_{i-1}(X))_i$ is a cdh-covering.

We introduce the category Sch^{prop}. Its objects are varieties over k, its morphisms are proper morphisms of varieties.

In the paper [11] the weight complex was defined as a functor $W : Sch^{prop} \to K^b(\mathfrak{Chow})$ in the following way. The weight complex $W(T)$ for a simplicial smooth projective variety T was defined (up to the inversion of arrows) as $T_0 \to T_1 \to T_2 \dots$; the boundary maps were given by alternated sums of face maps.

For $X \in Var$ a proper $Y \supset X$ was chosen; $Z = Y - X$. It was shown in [11] that there exist hyperenvelopes $Z.$ of Z, $Y.$ of Y, and a simplicial closed embedding $Z. \to Y.$ extending the map $Z \to Y$ whence the terms of $Z.$ and $Y.$ are smooth projective varieties. Then $W(X)$ was defined as the cone of $W(Z.) \to W(Y.)$. By means of comparing different hyperenvelopes Gillet and Soule showed that $W(X)$ is well-defined as an object of $K^b(\mathfrak{Chow})$ and gives a functor $Sch^{prop} \to K^b(\mathfrak{Chow})$.

Proposition 5.5.2. *The functor $t_0(m^{-1}(M^c_{gm}(X)) : Sch^{prop} \to K^b(\mathfrak{Chow})$ coincides (up to inversion of arrows) with the functor W.*

Proof. Since $Y. \to Y$ is a cdh-hypercovering, the cdh-sheafication of the corresponding complex $L(Y.) \to L(Y)$ is quasi-isomorphic to 0. Then Theorem 5.5 of [25] shows that $C(Y.) \cong C(Y)$. Hence $C(Y.)$ calculates $M_{gm}(Y)$. The same is true for Z.

By Proposition 4.1.5 of [23] there exists an exact triangle

$$M_{gm}(Z)(= M^c_{gm}(Z)) \to M_{gm}(Y)(= M^c_{gm}(Y)) \xrightarrow{j} M^c_{gm}(X) \to M_{gm}(Z)[1]$$

in DM^{eff}_{gm}. Hence we obtain that $t_0(m^{-1}(M^c_{gm}(X)) \in K^b(\mathfrak{Chow})(X) \cong W(X)$.

The definitions easily imply that W and $t_0(m^{-1}(M^c_{gm}(X))$ coincide as functors on the category of proper varieties. Moreover, the morphism j is given by a functorial morphism $L^c(Y) \to L^c(X)$; hence the functoriality of M^c_{gm} implies the coincidence of functors in general.

□

Remark 5.5.3. 1. In Theorem 5.10 of [12] also a certain functor $h : Sch_k \to K^b(Chow)$ was constructed (Sch_k is the category of varieties over k). It can be shown that h coincides with the restriction of $t : DM^{eff}_{gm} \to K^b(Chow)$ to motives of varieties (see subsection 5.2 for the definition of t).

2. In §2 of [11] it was shown that any two different representatives W_i of $W(X)$ (considered as complexes over $SmCor$) could be connected by a chain of certain homomorphisms h_i of complexes of smooth projective varieties. Gillet and Soule proved that h_i induce isomorphisms on the level of $K^b(\mathfrak{Chow})$. The main technical tools were Proposition 2 and Theorem 1 of §1 of [11] showing that hyperenvelopes give quasi-isomorphisms of complexes of Chow motives.

To any such W_i we can associate an object of $\mathfrak{H}$. Since $t_0(h_i)$ is an isomorphism, the corresponding map of motives will be an isomorphism too, see part 5 of Theorem 5.3.1.

Hence one can prove that the method of [11] gives a well-defined motive $GS(X)$ without using the *cdh*-descent reasoning above.

6 Realizations of motives; weight filtration; the spectral sequence of motivic descent

One of the main parts of the theory of motives is the problem of constructing and studying different *realizations* i.e. exact functors $DM \to T$ for T being a triangulated category. Some authors consider functors from the category of (smooth) varieties to T, yet usually those functors can be factorized through DM (cf. [14] and [20]).

In this section we describe a general method for constructing realizations of motives that is a vast generalization of the method described in 3.1.1 of [11]. We call realizations that could be constructed using this method *differential graded* realizations; this class (essentially) contains all 'standard' realizations as well as all representable functors for the category of motives. For any differential graded realization D we define a family of *truncated realizations*. In particular, for 'standard' realizations and motivic cohomology one obtains an interesting new family of realizations this way.

Truncated realizations of *length* N could be factorized through t_N; they give a filtration on the natural complex that computes D. We obtain a spectral sequence S converging to $D(Y)$ for a motif Y. S could be called the *spectral sequence of motivic descent* (note that the usual cohomological descent spectral sequences compute cohomology of varieties only). For the cohomology with compact support of a variety S is very similar to the spectral sequence considered in 3.1.2 of [11]; yet the origin of S is substantially different from those of the mentioned one. Besides we don't need the sheaves to be torsion as one does for étale cohomology. E_1-terms of S are cohomology of smooth projective varieties; E_n-terms of S have a nice description in terms of $t_{2n-4}(Y)$ (see 14), $n \geq 2$; in particular, E_2-terms depend only on $t_0(Y)$. S gives a canonical weight filtration on a wide class of cohomological functors; for the 'standard' realizations this filtration coincides with the usual one.

We verify that the N-th weight filtration of the étale and de Rham realizations can be factorized through t_N. In fact, we prove much more: if W denotes the weight filtration on $H^i(X)$ then $W_{l+N}H^i(X)/W_{l-1}H^i(X)$ factorizes through t_N; a morphism f induces a zero morphism on cohomology if $t_0(f)$ is zero (cf. subsection 6.3).

We note that (as an easy partial case of our results) we get a canonical 'weight' filtration on the motivic cohomology of any variety and the corresponding 'weight' spectral sequence for it. This filtration induces a new filtration on the K-theory of a smooth variety.

We conclude the section by the discussion of qfh-descent cohomology theories and qfh-motives of (possibly) singular varieties. It turns out that a wide class of realizations (including 'standard' ones) are 'qfh-representable' (hence they are 'differential graded' realizations of DM). Moreover, the qfh-motif of a (not necessarily smooth) variety gives 'right values of standard realizations'.

6.1 Realizations coming from differential graded functors; other 'truncation-like' functors

We consider the problem of constructing and studying different *realizations* of motives i.e. exact functors $DM \to T$ for T being a triangulated category. Our description of DM gives us a simple recipe for constructing realizations. Any differential graded functor $F : J \to X$ for a differential graded category X gives an exact functor $Tr^+(F) : \mathfrak{H} \to Tr^+(X)$ (and hence also a functor $\mathfrak{H} \to Tr(X)$), cf. Remark 2.3.3. It can be easily seen that $Tr^+(F)$ can be factorized through t_N if $t(J_l([Y],[Z])) = 0$ for any $Y, Z \in SmPrVar$, $l < -N$. This is always valid if $X^l = 0$ for $l < -N$. One can also note that all functors factorizing through t_N could be reduced (in a certain sense) to functors of such sort.

Note that for $N = 0$, X being equal to $S(A)$ for A an abelian category (see the definition of $S(A)$ in 2.1), our construction of $Tr^+(F)$ essentially generalizes to motives the recipe proposed in 3.1.1 of [11] for cohomology of varieties with compact support (also cf. [12]).

One can also define a more general class of 'truncation' functors from $\mathfrak{H}$. For the convenience of the reader we remark that this generalization is not needed for the understanding of the main results of this section.

Let S be any family of morphisms in J such that any $f \in S$ is concentrated in one degree (i.e. belongs to some $J_i(X,Y)$). We denote the closure of S with respect to $+, -, \oplus, \delta$, and composition by S' (i.e. we demand $f \in S' \implies f \circ g, h \circ f \in S'$ for any morphisms g, h in J that can be composed with f). S' could be called the *'differential ideal'* generated by S.

Next we define the category J/S'. Its objects are the same as for J, $J/S'(X,Y) = J(X,Y)/\{f \in J(X,Y) \cap S'\}$. It can be easily seen that there is a unique natural differential category structure on J/S' such that the natural functor $a_{S'} : J \to J/S'$ is a differential graded functor.

$a_{S'}$ induces an exact functor $t_{S'} : \mathfrak{H}(J) \to Tr(J/S')$ (cf. Remark 2.3.3).

It is easily seen that for S equal to $\cup_{i<-N;P,Q\in SmPrVar} J_i([P],[Q])$ the functor $t_{S'}$ coincides with t_N.

More generally, let $f \geq g$ be functions $\mathbb{N} \times \mathbb{N} \to \mathbb{Z}$. Then one can consider $S = \cup_{P,Q,i>-g(l,m),i<-f(l,m)} J([P],[Q])$ for all $P, Q \in SmPrVar$ being indecomposable of dimensions l, m respectively. For example, for the cohomological reasons it seems very natural to consider $g = 0, f(l,m) = 2m$ or $g(l,m) = f(l,m) = 2m - 2l$.

If a differential graded functor $F : J \to X$ maps all $f \in S$ to zero then $Tr^+(F)$ can be factorized through $t_{S'}$.

6.2 'Representable' contravariant realizations; étale and de Rham cohomology

We describe a a large family of contravariant differential graded functors from J. Let A be a Grothendieck topology stronger than Nisnevich topology (for example, étale topology). We consider the category $Shv(SmCor)_A$ (i.e. the morphisms are those of $SmCor$, coverings are those of A); let $C(Shv(SmCor_A))$ denote the category of (unbounded) complexes over $Shv(SmCor_A)$. We suppose that for any $X \in SmCor$ the representable presheaf $L(X) = SmCor(-,X)$ is a sheaf. We denote by $D(Shv(SmCor_A))$ and DM_A^{eff} the categories of unbounded complexes over $Shv(SmCor_A)$ that are similar to the corresponding categories of [23] (i.e. derived category of complexes of sheaves, resp. derived category of complexes of sheaves with homotopy invariant cohomology). We also consider the categories $K(Shv(SmCor_A))$ and $B(Shv(SmCor_A))$ that are unbounded analogues of $K^-(Shv(SmCor_A))$ and $B^-(Shv(SmCor_A))$ (cf. 2.1) respectively.

Let $Y \in C^+(Shv(SmCor_A))$ be a complex of injective sheaves with transfers bounded from below with homotopy invariant hypercohomology (we need hypercohomology if $A \neq Nis$). Now we consider $C(L(X))$ for $X \in SmVar$. Since $C(L(X)$ is quasi-isomorphic to $\underline{C}(L(X))$, for any $i \in \mathbb{Z}$ we have

$$D(Shv(SmCor_A))(C(L(X)), Y[i]) = D(Shv(SmCor_A))(\underline{C}(L(X)), Y[i]).$$

Since the correspondence $(F) \to \underline{C}(F)$ defines a functor RC_A which is left adjoint to the embedding $DM_A^{eff} \to D(Shv(SmCor_A))$ (cf. Proposition 3.2.3 of [23]), $Y \in D(Shv(SmCor_A))$, we have

$$D(Shv(SmCor_A))(C(L(X)), Y[i]) = D(Shv(SmCor_A))(L(X), Y[i]).$$

Let $Z \in Obj\mathfrak{H} = Obj J'$ satisfy $m(Z) \cong M_{gm}(X) = \underline{C}(L(X))$ (in $D^-(Shv(SmCor))$ and so also in $D(Shv(SmCor_A))$, cf. Corollary 3.2.1). Since the terms of Y are injective sheaves, we conclude that

$$H^{-i}(Y)(X) = K(Shv(SmCor)_A)(L(X), Y[i]) \cong K(Shv(SmCor)_A)(h(Z), Y).$$

Moreover, the complex $B(Shv(SmCor_A))(h'(Z'), Y)$ computes the complex $Y(X)$ up to a quasi-isomorphism (see the definitions of 2.5).

Now we describe how the formalism of §2 can be applied to the computation of $B(Shv(SmCor_A))(h'(Z), Y)$. We have a contravariant functor $Y^* : J \to C(Ab)$ that maps $[P] \in Obj J$ to $B(Shv(SmCor_A))(C(P), Y)$. Let $a : J' = \text{Pre-Tr}\, J \to B(Shv(SmCor_A))$ denote the differential graded functor induced by the embedding $J \to B^-(Shv(SmCor_A))$, cf. Remark 2.3.3 and Proposition 2.2.3. Since $Y^* = B(Shv(SmCor_A))(-, Y) \circ a$, we obtain that

$$B(Shv(SmCor_A))(h'(Z), Y) \cong \text{Pre-Tr}(Y^*)(Z).$$

Here $\text{Pre-Tr}(Y^*)$ denotes the extension of Y^* to J', cf. Remark 2.3.3.

For example, we can take A being the étale site. Y could be injective resolution of $\mathbb{Z}/n\mathbb{Z}$ (or a resolution of any other étale complex C with transfers with homotopy invariant hypercohomology) by means of étale sheaves with transfers. By Proposition 3.1.8 and Remark 2 preceding Theorem 3.1.4 in [23] the cohomology of $Y(L(X))$ for $X \in SmVar$ will compute the 'usual' étale hypercohomology of C restricted to X. Hence $Tr(Y^*)$ gives the corresponding realization of motives. We obtain that in order to compute the étale realization of motives (with coefficients in $\mathbb{Z}/n\mathbb{Z}$ for any $n > 0$) it suffices to know the restriction of the corresponding 'representable functor' to the subcategory of $Shv(SmCor)_A$ consisting of sheaves of the form $C_i(P)$ for $P \in SmPrVar$. Note also that we can compute morphisms in the category of presheaves with transfers.

If C is a complex of cdh-sheaves (see subsection 5.5) with transfers then the cohomology of $Y(L(X))$ for $X \in SmVar$ will compute the cdh-hypercohomology of C restricted to X for any X/k. Yet cdh-topology is not subcanonical; cdh-hypercohomology does not necessarily coincides with "usual" hypercohomology of C. For the computation of "usual" cohomology for singular varieties the qfh-topology seems to be more useful; see subsection 6.4 below.

One could check that Galois action on $H^i_{et}(X \times_{Spec\, k} Spec\, \overline{k}, \mathbb{Z}/n\mathbb{Z})$, where $\overline{k}$ is the algebraic closure of k, n is prime to the characteristic of k, could be expressed in terms of our formalism. Indeed, isomorphisms in the derived category corresponding to the Galois action can be extended to an injective resolution of $\mathbb{Z}/n\mathbb{Z}$. Therefore all statements of this section are valid for étale realization with values in Galois modules.

A similar method for constructing the étale realization (without using the formalism of differential graded categories) was described in [14] (see the reasoning following Proposition 2.1.2). One could check that the tensor product on cohomology is graded commutative and associative using Remark 2.6.4; doing this (without applying the differential graded category formalism) was stated to be difficult in [14]. In the next subsection we describe a general method of obtaining weight filtrations for realizations.

It can be easily seen that the same method can be applied to the de Rham realization. Using a similar method one could compute other standard realizations (singular and Hodge cohomology, for example). This would give also the 'mixed realization' functor for motives. Note that in contrast to [14] all these realizations would be constructed on the integral level.

6.3 The spectral sequence of motivic descent; weight filtration of realizations; the connection with t_N

Using the method described above one could verify that 'standard' realizations could be described using functors of the type $Tr(F)$ (see also subsection 6.4 below). Here F is a contravariant differential graded functor $J \to C$ for some differential graded category C. Note also that all 'representable' functors on $\mathfrak{H}$ come from differential graded $F : J \to B(Ab)$; in particular, this is true for motivic cohomology (and homology if we invert the arrows). We will assume that $C = B(A)$ (or its bounded version) for some abelian category A.

In order to obtain all 'standard' realizations (including the 'mixed' ones) one should replace $Tr(F)$ by a functor obtained from certain $Tr(F_i)$ by 'gluing'. Here 'gluing' means the (suitably modified) formalism described in detail in the book [16]. In our setting we should have described certain transformations 'connecting' the functors F_i; transformations should satisfy certain relations on the level of $Tr(F_i)$.

For simplicity here we will only consider a single contravariant functor $F : J \to B(A)$. We denote the functor Pre-Tr$(F) : J' \to B(A)$ by G, denote $tr(F) : \mathfrak{H} \to K(A)$ by E. In practice it usually suffices to consider functors whose targets are categories of complexes bounded (at least) from one side.

The constructions of this subsection use the results of subsection 2.6 heavily.

We recall that for a complex X over A, $a, b \in \mathbb{Z}$, $a \le b$, its canonical $[a, b]$-truncation is the complex

$$X_a/dX_{a-1} \to X_{a+1} \to \dots X_{b-1} \to \operatorname{Ker}(X_b \to X_{b+1}),$$

here $\operatorname{Ker}(X_b \to X_{b+1})$ is put in degree b. We also consider truncations of the type $\tau_{\le b}$ (i.e. truncations from above).

For any $b \ge a \in \mathbb{Z}$ we consider the following functors (see subsection 2.6). By $F_{\tau_{\le b}}$ we denote the functor that that sends $[P]$ to $\tau_{\le b}(F([P]))$. By $F_{\tau_{[a,b]}}$ we denote the functor that that sends $[P]$ to $\tau_{\le b}(F([P]))/\tau_{\le a}(F([P]))$. For $N = a - b$ we consider the functor $F_{b,N}$ that sends $[P]$ to the $[a, b]$-th canonical truncation of $F([P])$. These functors are differential graded; hence they extend to $G_b = \text{Pre-Tr}(F_{\tau_{\le b}}) : J' \to B^-(A)$, $G_N^b = \text{Pre-Tr}(F_{b,N}) : J' \to B^b(A)$, and $G_{a,b} = \text{Pre-Tr}(F_{\tau_{[a,b]}}) : J' \to B^b(A)$. We recall that $G_{a,b}$ and G_N^b are connected by a canonical functorial quasi-isomorphism, see part 2 of Proposition 2.6.3. The reason for considering both of them is that the functors $G_{a,b}$ are more closely related to the spectral sequence (13) below whence G_N^b behave better with respect to t_N.

We denote $tr(F_{b,N}) : \mathfrak{H} \to K^b(A)$ by F_N^b. Since $F_{b,N}$ is concentrated in degrees $[b - N, b]$, $F_{b,N}$ maps all $J_m(X, Y)$ for $X, Y \in Obj J$, $m < -N$, to 0. Hence one can present F_N^b as $b_{b,N} \circ t_N$ for a unique $b_{b,N} : \mathfrak{H}_N \to K(A)$. The set of F_N^b could be called *truncated realizations* for the realization E; N is the *length* of the realization. These realizations appear to be new even in the case when E is the étale cohomology. Note that for $X = [P]$, $P \in SmVar$, the truncated realizations give exactly the corresponding truncations of $E([P])$ (i.e. of the corresponding 'cohomology' of P); that is what one usually expects from the weight filtration.

The complexes $G_b(X)$ give a filtration of $G(X)$ for any $X \in Obj J'$; moreover $G_{a,b}(X) = G_b(X)/G_a(X)$.

Let $X = (P_i, q_{ij}) \in Obj J' = Obj \mathfrak{H}$. We obtain a spectral sequence of a filtered complex

$$S : E_2^{ij}(S) = H^{i+j}G_{j,j}(X) = H^{i+j}(F_0^j(X)) \to H^{i+j}(G(X)) \qquad (13)$$

we call it the *spectral sequence of motivic descent*. Note that $H^{i+j}(G(X)) = H^{i+j}(E(X))$, in the right hand side we consider X as an object of $\mathfrak{H}$. All $G_b(X)$ are $\mathfrak{H}'$-contravariantly functorial with respect to X. Besides starting from E_2 the terms of S depend only on the homotopy classes of $G_b(X)$. Hence starting from E_2 the terms of S are functorial with respect to X (considered as on object of $\mathfrak{H}$).

Moreover, if $h : F \to F'$ is a transformation of functors then the corresponding map of spectral sequences depends only on $Tr(h)$ (starting from E_1-terms). In particular, for the étale and de Rham realizations the spectral sequence does not depend on the choice of an injective resolution for the corresponding complex (see the previous subsection).

We have $E_1^{ij} = H^j(F([P_{-i}]))$. S is similar to the spectral sequences that come from hypercoverings (and hyperenvelopes). Yet its terms are 'much more functorial'; it computes cohomology of any motive (not necessarily of a motive of a variety). Moreover, one could 'glue' (in the sense of [16]) spectral sequences corresponding to different realizations (using well-known comparison isomorphisms).

By definition, $E_2^{ij}(S) = H^{i+j}(F_0^j(Y))$ is the $-i$-th homology group of the chain complex $H^j(P_l)$. Hence the E_2-terms are functorial in the complex $(P_l) \in C^b(\mathfrak{Chow})$ i.e. in $t_0(Y)$. S is convergent since for X of length N only N rows could be non-zero. Besides if all $F_0^j(Y)$ are acyclic then $E(Y)$ is acyclic. We denote the filtration on $H^s(E(Y))$ given by S by W_l; we call it the weight filtration of H^s.

For any b, N we also have a 'spectral subsequence'

$$S_N^b : E_2^{ij}(S_N^b) = H^{i+j}(F_0^j(Y))_{b-N\le j\le b} \to H^{i+j}(G_{a,b}(X)) = H^{i+j}(G_N^b(X)) = H^{i+j}(F_N^b(Y)).$$

Its E_2-terms form a subset of the E_2-terms of S, the (non-zero) boundary maps are the same. We also have weight filtration on $H^s(F_N^b(Y))$.

For any $0 \le l \le N$ we have an obvious spectral sequence morphism $S_{N-l}^{b-l} \to S_N^b$. It induces an epimorphism

$$\alpha_{l,b,N}^s : H^s(F_{N-l}^{b-l}(Y)) \to W_{b-l}(H^s(F_N^b(Y))).$$

Now we use an argument that could be applied to any filtered complex. For any $N \ge 0$, $n \ge 2$, one easily sees that $E_n^{ij}(S_N^b) = E_\infty^{ij}(S_N^b)$ if $b+1-n < j < b-N+n-1$. Moreover, if $b+2-n \ge j \ge b-N+n-2$ then $E_n^{ij}(S_N^b) = E_n^{ij}(S)$. Therefore we have

$$\begin{aligned} E_n^{ij}(S) = Gr_{n-2}^W H^{i+j}(F_{2n-4}^{j+n-2}(Y)) &= W_{n-2}(H^{i+j}(F_{2n-4}^{j+n-2}(Y)))/W_{n-3}(H^{i+j}(F_{2n-4}^{j+n-2}(Y))) \\ &= \operatorname{Im}\alpha_{n-2,j+n-2,2n-4}^{i+j}/\operatorname{Im}\alpha_{n-1,j+n-2,2n-4}^{i+j} \end{aligned} \quad (14)$$

i.e. it is the middle term of the weight filtration of $H^{i+j}(F_{2n-4}^{j+n-2}(Y))$. A similar equality can be written for $E_n^{ij}(S_N^b)$ for any $b \in \mathbb{Z}, N \geq 0, n \geq 2$. Hence for any $n \geq 2$ the E_n-terms of S and all S_N^b depend only on $t_{2n-4}(Y)$.

Suppose that $X \in \mathfrak{H}'_{[c,d]}$. It can be easily verified (for example, using the spectral sequence (13)) that for any $j \in \mathbb{Z}, b-N-c \leq j \leq b-d$, the j-th cohomology group of $F_N^b(Y)$ coincides with $H^j(E(Y))$. Besides for any $j \in \mathbb{Z}$ the weights of $H^j(E(Y))$ lie between $j+c$ and $j+d$. In particular, by Theorem 5.3.1 the weights of $H^j(X)$ for $X \in SmVar$ of dimension N lie between j and $j+N$, the weights of $H_c^j(X)$ (the cohomology with compact support) lie between $j-N$ and j.

Hence all 'cohomological information' of the motive of length $\leq N$ could be factorized through t_N. This statement can be considered as the 'realization version' of Theorem 5.3.1.

Suppose now that there are no maps between different weights i.e. for any $P, P', Q, Q' \in SmPrVar, f \in SmCor(P, P'),\ g \in SmCor(Q, Q'),\ i \neq j$, we have

$$A(\mathrm{Ker}(H^i(E([P])) \xrightarrow{f^*} H^i(E([P']))), (\mathrm{Coker}(H^j(E([Q])) \xrightarrow{g^*} H^j(E([Q']))) = 0.$$

Note that this condition is fulfilled for the étale and Hodge realizations with rational coefficients. Then S and all S_N^b degenerate at E_2. Therefore $H^j F_N^b(X) = W_b(H^j(E(Y)))/W_{b-N-1}(H^j(E(Y)))$. Besides, for the étale and Hodge realizations our weight filtration coincides with the usual one. Hence t_N can be called the weight functors.

Moreover, for any morphism $f: Y \to Z$ for $Y, Z \in \mathfrak{H}$ the morphisms $H^l(f): H^l(E(Y)) \to H^l(E(Z))$ for $Y, Z \in \mathfrak{H}$ are strictly compatible with the weight filtration. Therefore $H^l(f)$ is zero if and only if the corresponding map of E_2-terms in $S(Y) \to S(Z)$ is zero. Hence the map $\mathfrak{H}(Y, Z) \to A(H^l(E(X)), H^l(E(Y))$ factorizes through t_{0*}.

For cohomology with integral coefficients one may apply the previous statement for rational cohomology to obtain that f^* is zero on $H^l \otimes \mathbb{Q}$ if $t_0(f) = 0$. Hence if $t_0(f) = 0$ then f^* is zero on cohomology modulo torsion.

Using the results of subsection 5.5 one can compute $\mathfrak{H}(Z, T)$ for $Z = m^{-1}(M_{gm}(X)), T = m^{-1}(M_{gm}(Y))$, and also compute $M_N(X, Y) = \mathrm{Im}\, t_{N*}(\mathfrak{H}(Z, T) \to \mathfrak{H}_N(t_N(Z), t_N(T))$. M_0 and M_2 are closely connected with standard realizations.

Now we conclude the proof of Theorem 5.3.1.

Suppose that $Y = j(X)$ for some $X \in \mathfrak{H}'$.

For an object $U \in ObjJ' = Obj\mathfrak{H}' = Obj\mathfrak{H}$ (recall that $J' = \text{Pre-Tr}(J)$) we consider the differential (contravariant) graded functor $J_U: J \to B(Ab)$ that maps $[P]$ to $J'([P], U)$. Again for any $X = (P_i, q_{ij}) \in \mathfrak{H}'$ we have a spectral sequence $E_1^{ij} = H^j(J_U([P_{-i}])) \to H^{i+j}(J'(X,U))$. Again we note that its E_2-terms depend only on $t_0(X)$. Suppose that $t_0(X) = 0$. Then $H^0(J'(X,U)) = 0$ for any U. Since $H^0(J'(X,U)) = \mathfrak{H}(j(X),U)$, we obtain that $j(X) \cong 0$.

Part 4 of Theorem 5.3.1 is proved.

Remark 6.3.1. 1. Note that we can take $U = \mathbb{Z}(n)$ for $n \geq 0$. Hence we obtain canonical 'weight' filtration on the motivic cohomology of any variety and the corresponding 'weight' spectral sequence for it. The simplest case of this spectral sequence is the Bloch's long exact localization sequence for higher Chow groups (see [3]). Hence the filtration is non-trivial in general; it appears not to be mentioned in the literature. Using the spectral sequence relating algebraic K-theory to the motivic cohomology (see [9]) we get a new filtration on the K-theory of a smooth variety X.

2. It seems to be interesting to study the truncations and the weight spectral sequence for the cohomology of the sheaf $Y \to G_m(X \times Y)$ for a fixed variety Y (G_m is the multiplicative group). These things seem to be related with the Deligne's one-motives of varieties as they were described in [2].

More generally, for any $X \in DM_{gm}^{eff}$ the functor $\underline{Hom}(-,X) : DM_{gm}^{eff} \to DM_{-}^{eff}$ has an enhancement.

3. Another important source of differential graded functors from J are those induced by localizations of $\mathfrak{H}$ (or of DM which is the same thing). It will be discussed in subsection 7.2 below.

6.4 qfh-descent cohomology theories; motives of singular varieties

Some 'standard' cohomology theories are difficult to represent by a complex of sheaves with transfers (in the way described in subsection 6.2). One of the ways to do this is to use qfh-topology.

We recall that the qfh-topology is the topology on the set of all varieties whose coverings are quasi-finite universal topological coverings (see [26] for a precise definition). In particular, the qfh-topology is stronger than the flat topology and the cdh-toplogy. There is a natural functor from DM_{-}^{eff} to the derived category of qfh-sheaves with homotopy invariant cohomology (it is denoted as $DM_{qfh}(k)$); this functor is surjective on objects. Note also that any 'ordinary' topological sheaf restricted to Var gives a qfh-sheaf. Moreover, qfh-descent follows from proper descent combined with Zarisky descent (see section 2 of [26]).

Let C be a complex of presheaves (possibly without transfers) whose cohomology satisfy qfh-descent. Then the qfh-hypercohomology of the qfh-sheafification of C coincides with the hypercohomology of C (for examples, a similar statement was proved in the proof of theorem 5.5 of [25]). Therefore if the cohomology of C is homotopy invariant then it could be presented by means of 'representable' functors on DM as it was described in subsection 6.2 above. In particular, this shows that Betti and Hodge cohomology theories could be enhanced to differential graded realizations.

Now let C be a complex of qfh-sheaves. It was proved in [26] (see Theorems 3.4.1 and 3.4.4) that the qfh-hypercohomology of a variety X with coefficients in C coincides with the étale hypercohomology of C in the cases when either C is a $\mathbb{Q}$-vector space sheaf complex and X is normal or C is a locally constant étale sheaf complex. Hence in this cases the étale hypercohomology of C also gives a 'representable' realization.

Note that these realization compute qfh-hypercohomology with coefficient in C of any (not necessarily smooth) variety X. Hence $M_{gm}(X)_{qfh}$ (i.e. the image of $M_{gm}(X)$ in $DM_{qfh}(k)$) seems to be the natural choice for the qfh-motive of a (possibly) singular variety. In particular, its 'standard' realizations have the 'right' values of $H^i_{et}(X, \mathbb{Z}_l(m))$.

Yet realizations described above do not fix motives up to an isomorphism. Indeed, different motives (i.e. objects of DM or DM^{eff}_{-}) could give isomorphic qfh-motives. Note that by Theorem [23] the category of qfh-motives becomes equivalent to DM^{eff}_{-} when tenzored with $\mathbb{Q}$; hence all these motives are isogenious. In order to choose between the different possible motives for X in DM one could either take $M_{gm}(X)$ of Voevodsky or apply a suitably modified version of the method of 2.3 of [2].

7 Concluding remarks

In this section we give a general description of a subcategory of $\mathfrak{H}$ that is generated by a fixed set of objects. In particular, this method can be used to obtain the description of the category of effective Tate motives (i.e. the full triangulated subcategory of DM generated by $\mathbb{Z}(n)$ for $n > 0$).

We describe the construction of 'localization of differential graded categories' (due to Drinfeld). This gives us a description of localizations of $\mathfrak{H}$. All such localizations come from differential graded functors. As an application, we prove that the motif of a smooth variety is a mixed Tate one whenever its weight complex (defined in [11], see subsection 5.4 and 5.5) is.

We remark that one can easily add direct summands of objects to J. In particular, one could include $[P][2i](i)$ into J.

We describe an explicit construction of the tensor product on $\mathfrak{H}$ and $\mathfrak{H}_N$, and prove that t_N are tensor functors.

We consider a functor $m_N : \mathfrak{H}_N \to DM_-^{eff}$ that maps $[P]$ into the N-th canonical truncation of $C(P)$ (as a complex of sheaves).

We conclude the section by some remarks on the internal Hom functor.

7.1 Subcategories of $\mathfrak{H}$ that are generated by a fixed set of objects

Let B be a set of objects of $\mathfrak{H}$; we assume that B is closed with respect to direct sums.

Let B' denote some full additive subcategory of $J' = \text{Pre-Tr}(J)$ such that the corresponding objects of $\mathfrak{H}$ are exactly elements of B (up to isomorphism).

Let $\mathfrak{B}$ denote the smallest triangulated category of $\mathfrak{H}$ containing B.

Proposition 7.1.1. *$\mathfrak{B}$ is canonically isomorphic to $Tr^+(B')$.*

Proof. Follows immediately from Theorem 1 §4 of [4]. □

Remark 7.1.2. 1. It follows immediately that in order to calculate the smallest triangulated category of containing an *arbitrary* fixed set of objects in $\mathfrak{H}$ it is sufficient to know morphisms between the corresponding objects in J' (i.e. certain complexes) as well as the composition rule for those morphisms.

2. We obtain that for any triangulated subcategory of $D \subset \mathfrak{H}$ the embedding $D \to \mathfrak{H}$ is isomorphic to $Tr^+(E)$ for some differential graded functor $E : F \to G$. Here G is usually equal to J' (though sometimes it suffices to take $G = G'$); F depends on D.

It follows that for any $h \in Obj\mathfrak{H}$ the representable contravariant functor $h^* : D \to Ab : d \to \mathfrak{H}(d, h)$ can be presented as $H^0(u)$ for some contravariant differential graded functor $u : F \to B^-(Ab)$. See part 2 Proposition 7.2.1 below for a similar statement for localization functors.

3. Using this statement one can easily calculate the derived category of (mixed effective) Tate motives (cf. [18]). It is sufficient to take $B = \sum_{a_i \geq 0}[(\mathbb{P}^1)^{a_i}]$, i.e. the additive category generated by motives of non-negative powers of the projective line. This gives a certain extension of the description of [22] to the case of integral coefficients. Note that the description of the category of effective Tate motives immediately gives a description of the whole category of Tate motives since $\mathbb{Z}(1)$ is quasi-invertible with respect to $\otimes$.

7.2 Localisations of $\mathfrak{H}$

Let C be a differential graded category satisfying the homotopical flatness condition i.e. for any $X, Y \in ObjC$ all $C_i(X, Y)$ are torsion-free. Note that both J and $J' = \text{Pre-Tr}^+(J)$ satisfy this condition.

In [8] V. Drinfeld has proved (modifying a preceding result of B. Keller) that for C satisfying the homotopical flatness condition and any full differential graded subcategory B of C there exists a differential graded quotient C/B of C modulo B. This means that there exists a differential graded $g : C \to C/B$ that is surjective on objects such that $Tr^+(g)$ induces an equivalence $Tr^+(C)/Tr^+(B) \to Tr^+(C/B)$ (i.e. $Tr^+(C)/Tr^+(B) \cong Tr^+(C/B)$, $Tr^+(g)$ is zero on $Tr^+(B)$ and induces this equivalence).

The objects of C/B are the same as for C whence for $C_1, C_2 \in ObjC = Obj(C/B),\ i \in \mathbb{Z}$, we define

$$(C/B)_i(C_1, C_2) = C_i(C_1, C_2) \bigoplus$$

$$\bigoplus_{j \geq 0} \bigoplus_{B_1, \dots, B_j \in ObjB, \sum a_i = i+j} C(C_1, B_1) \otimes \varepsilon_{B_1} C(B_1, B_2) \otimes \varepsilon_{B_2} \dots \otimes \varepsilon_{B_j} \otimes C(B_j, C_2). \tag{15}$$

Here $\varepsilon_b \in (C/B)_{-1}(b, b)$ for each $b \in ObjB \in Obj(C/B)$ is a 'canonical new morphism' such that $d_b \varepsilon_b = id_b$; ε_b spans a canonical direct summand $\mathbb{Z}\varepsilon_b \subset (C/B)_{-1}(b, b)$. From this condition one recovers the differential on morphisms of C/B.

For example, this construction (for $C = J$) gives an explicit description of the localization of $\mathfrak{H}$ by the triangulated category generated by all $[Q], Q \in SmPrVar, \dim Q < n$, for a fixed n (and hence also of the corresponding localization of DM).

For the statement below we only use the following obvious property of the construction: if $C_i(-,-) = 0$ for $i > 0$ then the same is true for C/B. Note also that in this case $C_0(-,-) = (C/B)_0(-,-)$.

More generally, for localizations of $\mathfrak{H}$ modulo some $A \subset \mathfrak{H}$ it is sufficient to know the complexes $C/B([P],[Q])$ and the composition law for a certain B and all $P, Q \in SmPrVar$; here either $C = J$ or $C = J'$. In the case when A is not generated by objects of length 0 we are forced to take $C = J'$; this makes the direct sum in (15) huge.

Proposition 7.2.1. *1. If $F : \mathfrak{H} \to T$ is a certain localization functor (T is a triangulated category) then $F \cong Tr^+(G)$ for a certain differential graded functor G from J.*

2. For any $t \in T$ the contravariant functor $t^* : \mathfrak{H} \to Ab : X \to T(F(x), t)$ can be presented as $H^0(u)$ for some contravariant differential graded functor $u : J \to B^-(Ab)$.

3. Let B be a full additive subcategory of J, $ObjB = T \subset SmPrVar$. Let $\mathfrak{B}$ denote the smallest triangulated subcategory of $\mathfrak{H}$ containing the objects of B. Then for $M \in Obj\mathfrak{H}$ we have $M \in Obj\mathfrak{B}$ whenever $t_0(M)$ is homotopy equivalent to a complex all whose terms have the form $[Q]$, $Q \in T$.

Proof. 1. Let $A = \{X \in Obj\mathfrak{H}, F(X) = 0\} \subset ObjJ' = Obj\mathfrak{H}$, we denote the corresponding full subcategory of J' by B'. By Proposition 7.1.1, $Tr^+(B')$ is isomorphic to the categorical kernel of F. Let $H : J' \to J'/B'$ denote the functor given by Drinfeld's construction. Since $Tr^+(J') = \mathfrak{H}$, we obtain $Tr^+(H) \cong F$. Hence by part 2 of Remark 2.3.3 we can take G being the restriction of H to $J \subset \mathfrak{H}$.

2. Let w denote some element that corresponds to t in $\text{Pre-Tr}^+(J'/B')$. Then it suffices to take $u([P]) = \text{Pre-Tr}^+(J'/B')(\text{Pre-Tr}^+(G)([P]), w)$.

3. If $M \in Obj\mathfrak{B}$ then $t_0(M)$ belongs to the triangulated subcategory of $\mathfrak{H}_0 = K^b(\mathfrak{Chow})$ generated by $[Q]$, $Q \in T$. This subcategory consists exactly of complexes described in the assertion 3.

We prove the converse implication.

Let $t_0(M)$ be homotopy equivalent to a complex all whose terms have the form $[Q]$, $Q \in T$.

Let I denote J/B, let $K = Tr^+(I)$. Let $S : J \to J/B$ be the localization functor of [8]; let $u = Tr^+(S)$. Since $I_i(-,-) = 0$ for $i > 0$, we can define I_0 similarly to J_0; we have natural functors $I \to I_0$ and $J_0 \to I_0$. Applying Tr^+ we get a category $K_0 = Tr^+(I_0)$; we get functors $u_0 : \mathfrak{H}_0 \to K_0$ and $v : K \to K_0$ such that $u_0 \circ t_0 = v \circ u$. We obtain $v(u(M)) = 0$. Exactly the same spectral sequence reasoning as the one used in the proof of part 4 of Theorem 5.3.1 (see the end of subsection 6.3) shows that $u(M) = 0$. Since u is the localization functor, we obtain that $M \in Obj\mathfrak{B}$. □

Part 3 is a generalization of part 4 of Theorem 5.3.1 (there $T = \{0\}$).

Corollary 7.2.2. *Let $X \in SmVar$. Then $M_{gm}(X)$ is a mixed Tate motif (as described in part 3 of Remark 7.1.2) in DM_{gm}^{eff} (i.e. we add direct summands) whenever the complex $t_0(U) \in K^b(Chow)$ is.*

Proof. We apply part 3 of Proposition 7.2.1 for $T = \{\sqcup_{a_i \geq 0}(\mathbb{P}^1)^{a_i}\}$.

We obtain that $M_{gm}^c(X)$ is a mixed Tate motif whenever $t_0(M_{gm}^c(X)) = W(X)$ is mixed Tate as an object of $K^b(Chow)$.

On the category of geometric of Voevodsky motives $DM_{gm} \supset DM_{gm}^{eff}$ (see §4.3 of [23]) we have a well-defined duality such that $\mathbb{Z}(n)^* = \mathbb{Z}(-n)$. Therefore the category of mixed Tate motives is a self-dual subcategory of DM_{gm}. Since $M_{gm}(X)^* = M_{gm}^c(X)(-n)[-2n]$, $n = \dim X$ (we can assume hat X is equidimensional), we obtain that $M_{gm}(X)$ is a Tate motif if and only if $M_{gm}^c(X)$ is. □

One can translate this statement into a certain condition on the motives $M_{gm}(Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_r})$.

7.3 Adding kernels of projectors to J

The description of the derived category of Tate motives would be nicer if $\mathbb{Z}(i),\ i \geq 0$ would be motives of length 0 (see also subsection 5.3). To this end we show that one can easily add direct summands of objects to J.

Indeed, if the cohomology of a complex of sheaves coincides with its hypercohomology, the same it true for any direct summand of this complex. Therefore, if D, D' are direct summands (in $C^-(Shv(SmCor))$) of $C(P), C(P')$ respectively, $P, P' \in SmPrVar$, then the natural analogue of Proposition 1.2.1 will be valid for $DM_-^{eff}(p(D), p(D'))$. Hence any such D can be naturally added to J; then an analogue of Theorem 3.1.1 would be valid with DM extended by adding the corresponding direct summands of objects.

In particular, let $P \subset Q \in SmPrVar$ and let there exist a section of the inclusion $j : Q \to P$. Then one can add the cone of j to J (note that it is isomorphic to $m^{-1}(M_{gm}^c(Q-P))$.

For example, one can present $[\mathbb{P}^1]$ as $[pt] \oplus [\mathbb{Z}(1)[2]]$. Hence for $P \in SmPrVar,\ i \geq 0$ one could include $[P][2i](i)$ into J (cf. the reasoning in the proof of part 1 of Theorem 5.3.1 and also the next subsection).

7.4 Tensor product on $\mathfrak{H}$ and $\mathfrak{H}_N$

For $f \in C'_i(P)(Y)$, $g \in C'_j(Q)(Z)$ we denote by $f \otimes g \in C'_{i+j}(P \times Q)(Y \times Z)$ the natural product of f and g. Obviously, $C_i(P) \otimes C_j(Q) \subset C_{i+j}(Q \times P)$.

Now we define the natural tensor product on $\mathfrak{H}$. For $X = (P_i, f_{ij}),\ Y = (Q_i, g_{ij})$ we define $X \otimes Y = (U_i, h_{ij})$, where $U_i = \sqcup_{j \in \mathbb{Z}} P_j \times Q_{i-j}$,

$$h_{ij} = \bigoplus_{l \in \mathbb{Z}} (f_{i-l,j-l} \otimes id_{Q_l *} \oplus (-1)^l id_{P_l *} \otimes g_{i-l,j-l}).$$

The tensor product for morphisms is defined as the direct sum of the tensor products of their components.

One can also define the tensor structure on all $\mathfrak{H}_N$ in a similar way.

Proposition 7.4.1. *1. Tensor product defines on $\mathfrak{H}$ the structure of a tensor triangulated category.*

2. The same is true for $\mathfrak{H}_N$ for all $N \geq 0$. Besides, t_N is a tensor functor.

3. m is a functor of tensor triangulated categories.

Proof. 1. Easy (though long) direct verification. One could avoid it by developing a theory of tensor differential graded categories. The central fact is that the map $C(P) \otimes C(Q) \to C(P \times Q)$ defined below is a morphism of complexes.

2. It suffices to note that the morphism $C(P) \otimes C(Q) \to C(P \times Q)$ induces also a morphism $C^N(P) \otimes C^N(Q) \to C^N(P \times Q)$ for all $N \geq 0$.

3. Since m is a triangulated category functor it is sufficient to prove that $m(f \otimes g) = m(f) \otimes m(g)$ for morphisms of generators of $\mathfrak{H}$ i.e. objects of the form $P[i]$.

We have $m(P[i] \otimes Q[j]) = m(P[i]) \otimes m(Q[j])$. It remains to verify that $m(f \otimes g) = m(f) \otimes m(g)$ for $f \in \mathfrak{H}([P], Q[i])$, $g \in \mathfrak{H}(T, U[j])$, where $P, Q, U, T \in SmPrVar$, $i, j \leq 0$.

Now recall (see §3.2 [23]) that the tensor product on DM_{-}^{eff} is defined by means of the projection $RC : D^{-}(Shv(SmCor)) \to DM_{-}^{eff}$. Hence to prove the equality it is sufficient to find $X, Y \in Obj D^{-}(Shv(SmCor))$,

$$f' \in D^{-}(Shv(SmCor))((L(P)), X[i]),\ g' \in D^{-}(Shv(SmCor))((L(T)), Y[j]),$$

such that $RC(X) = M_{gm}(Q),\ RC(Y) = M_{gm}(U)$ whence

$$RC(f') = m(f),\ RC(g') = m(g),\ RC(f' \otimes g') = m(f \otimes g).$$

By Proposition 4.2.2 we can take $X = C(Q),\ Y = C(U)$. Now, f and g are represented by some $f" \in \operatorname{Ker} \delta^i(Q)(P)$ and $g" \in \operatorname{Ker} \delta^i(U)(T)$ respectively. Then $f \otimes g$ is represented by $f" \otimes g"$. We take $f' = (f"^l), g' = (g"^l)$. We also consider $(f \otimes g)' = ((f" \otimes g")^l)$.

For $A = Q, U, Q \times U$ we consider natural morphisms $i_A : (L(A)) \to C(A)$ and also the auxiliary $t : C(Q) \otimes C(U) \to C(Q \times U)$. We have $t(i_Q \otimes i_U) = i_{Q \times U}$. Then by Proposition 4.2.2 $RC_*(t)$ coincides with the canonical morphism $M_{gm}(Q) \otimes M_{gm}(U) \to M_{gm}(Q \times U)$. It remains to note that $t \circ (f' \otimes g') = (f \otimes g)'$. □

7.5 The functors m_N

We consider the functor $J_N \to B^{-}(Shv(SmCor))$ that maps $[P]$ into $SC^N(P)$. Here $SC_i^N(P)$ is the Nisnevich sheafification of the presheaf $C_i^N(P)(-)$ (they coincide for $i \neq -N$). We consider the corresponding functor $h_N : \mathfrak{H}_N \to K^{-}(Shv(SmCor))$ and $m_N = p \circ h_N$. Note that for any $X \in Obj \mathfrak{H}_N$ we have $m_N(X) \in DM_{-}^{eff}$.

By part 1 of Proposition 2.6.2 the natural morphisms $C_P \to SC^N(P)$ in $K^{-}(Shv(SmCor))$ induce a transformation of functors $tr_N : m \to m_N$. Besides tr_N for any $X \in \mathfrak{H}$ is induced by a canonical map in $K^{-}(Shv(SmCor))$.

It seems that no nice analogue of Theorem 5.3.1 is valid for m_N. Yet for low-dimensional varieties m_N coincides with m.

Proposition 7.5.1. *Suppose that the Beilinson-Soulé conjecture holds over* k*. Then* $m_{2n}(X) \cong m(X)$ *in* DM_-^{eff} *if the dimension of* X *is* $\leq n$*.*

Proof. We check by induction on n that tr_{2n} is the identity for $m(X)$. This is obviously valid for $n = 0$.

The same reasoning as in the proof of part 1 of Theorem 5.3.1 shows that it is sufficient to prove the assertion for smooth projective X of dimension $\leq n$.

It is sufficient to check that $DM_-^{eff}(M_{gm}(Y)[N], M_{gm}(X)) = 0$ for any $Y \in SmVar$, $N \geq 2n$.

By Theorem 4.3.2 of [23] if the dimension of X equals n then we have $\underline{Hom}_{DM_-^{eff}}(M_{gm}(X), \mathbb{Z}(n)[2n]) \cong M_{gm}(X)$. Hence

$$\begin{aligned} DM_-^{eff}(M_{gm}(Y)[N], M_{gm}(X) &= DM_-^{eff}(M_{gm}(Y \times X)[N], \mathbb{Z}(n)[2n]) \\ &= DM_-^{eff}(M_{gm}(Y \times X)[N-2n], \mathbb{Z}(n)). \end{aligned}$$

It remains to note that the by the Beilinson-Soulé conjecture $DM_-^{eff}(M_{gm}(Y \times X), \mathbb{Z}(n)[i]) = 0$ for $i < 0$. □

Remark 7.5.2. 1. We also see that for any N there exist $P, Q \in SmPrVar$ such that $\mathfrak{H}(P[N], [Q]) \neq 0$. Hence none of t_N and m_N are full functors.

2. It could be also easily checked that tr_{2n} being identical for all X of dimension $\leq n$, $n \in \mathbb{Z}$, implies the Beilinson-Soulé conjecture.

7.6 Some remarks on <u>Hom</u>

As in [23], we could define the category $\mathfrak{H}$" obtained from $\mathfrak{H}$ by inverting $\mathbb{Z}(1)$ with respect to $\otimes$. On $\mathfrak{H}$ one could define a bifunctor $\underline{Hom}$ (in particular, this would give duality). It would satisfy all properties mentioned in [23] (see Theorem 4.3.7 in [23]). Possibly $\underline{Hom}$ could be described explicitly (not mentioning DM_-^{eff}); yet maybe to this end one should replace $\mathfrak{H}$ by a category whose description is similar but more complicated. One could also prove the existence of $\underline{Hom}$ using the reasoning of [20], Part I, Ch. IV, §§1.4 and 1.5, see also Appendix B of [15].

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