On some sums involving primitive roots, II

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Abstract

Let $g \in \mathbb{Z}\backslash\{-1,0,1\}$. In part I a function $w_g(p)$ was introduced having the property that under the Generalized Riemann Hypothesis (GRH), we have $\sum_{p\le x} w_g(p) = \pi_g(x) + O(x \log\log x \log^{-2} x)$. Here $\pi_g(x)$ denotes the number of primes $p \le x$ such that g is a primitive root mod p. The function $w_g(p)$ can be thought of as the 'probability' that g is a primitive root mod p. In this note this result is generalized. Let $a(\text{mod } f)$ be a primitive residue class. Let $\pi_g(x; f, a)$ denote the number of primes $p \le x$ with $p \equiv a(\text{mod } f)$ such that g is a primitive root mod p. It is shown, under GRH, that

$$\sum_{\substack{p\le x \\ p\equiv a(\text{mod } f)}} w_g(p) = \pi_g(x; f, a) + O(x \log\log x \log^{-2} x).$$

1 Introduction

Let $g \in \mathbb{Z}\backslash\{-1,0,1\}$ and let h be the largest integer such that g is an hth power. Put $w_g(p) = 0$ if $(g/p) = 1$ (Legendre symbol) or $(p-1, h) > 1$ and $w_g(p) = 2\varphi(p-1)/(p-1)$ otherwise, with φ Euler's totient. In part I we made the point that $w_g(p)$ can be regarded as the 'probability' that g is a primitive root mod p. Working with the function $w_g(p)$ has the advantage that it takes out considerations of algebraic number theoretical nature from the evaluation of quantities involving primes p that are required to be such that a prescribed integer g is a primitive root mod p. The disadvantage of working with $w_g(p)$ is that one cannot be sure that the predicted result matches with reality. If $w_g(p)$ is indeed the 'probability' that g is a primitive root mod p, it should be true that

$$\sum_{p\le x} w_g(p) \sim \pi_g(x), \tag{1}$$

where $\pi_g(x)$ denotes the number of primes $p \le x$ such that g is a primitive root mod p, in case $\pi_g(x) \to \infty$ as $x \to \infty$. Indeed, in part I we proved, under GRH, that something strong than (1) is actually true.

Theorem 1 [3]. (GRH). *Let $g \in \mathbb{Z}\backslash\{-1,0,1\}$ and let h be the largest integer such that g is an hth power. Then*

$$2 \sum_{\substack{p\le x \\ (\frac{g}{p})=-1 \\ (h,p-1)=1}} \frac{\varphi(p-1)}{p-1} = \pi_g(x) + O\left(\frac{x \log\log x}{\log^2 x}\right). \tag{2}$$

The proof proceeds by showing, unconditionally, that the l.h.s. of (2) is $\pi(x)$-like, that is that there is a constant δ_g such that for every $C>0$ we have

$$\sum_{p\le x} w_g(p)=\delta_g \mathrm{Li}(x)+O_C(x\log^{-C}x),$$

with $\mathrm{Li}(x)$ the logarithmic integral. (As is well-known $\pi(x)$, the number of primes $p\le x$, is $\pi(x)$-like.) From the work of Hooley [1] it follows, under GRH, that $\pi_g(x)=\delta_g\mathrm{Li}(x)+O(x\log\log x\log^{-2}x)$. From this the result then follows.

Let $f\ge 1$ and $1\le a\le f$, $(a,f)=1$ and $g\in\mathbb{Z}$. Let $\mathcal{P}_{a,f,g}$ denote the set of primes p such that g is a primitive root mod p and $p\equiv a(\bmod f)$. Let $\pi_g(x;f,a)$ denote the number of primes $p\le x$ such that $p\in\mathcal{P}_{a,f,g}$. From the work of Lenstra [2] one knows that

Theorem 2 [2]. *Put $\zeta_m=e^{2\pi i/m}$. Let $f\ge 1$ and $1\le a\le f$, $(a,f)=1$. Let σ_a be the automorphism of $\mathbb{Q}(\zeta_f)$ determined by $\sigma_a(\zeta_f)=\zeta_f^a$. Let $c_a(n)$ be 1 if the restriction of σ_a to the field $\mathbb{Q}(\zeta_f)\cap\mathbb{Q}(\zeta_n,g^{1/n})$ is the identity and $c_a(n)=0$ otherwise. Put*

$$\delta(a,f,g)=\sum_{n=1}^{\infty}\frac{\mu(n)c_a(n)}{[\mathbb{Q}(\zeta_f,\zeta_n,g^{1/n}):\mathbb{Q}]},$$

with μ the Möbius function in case g is not -1 or a square and $\delta(a,f,g)=0$ otherwise. Then, assuming GRH, we have

$$\pi_g(x;f,a)=\delta(a,f,g)\frac{x}{\log x}+O\left(\frac{x\log\log x}{\log^2 x}\right).$$

This result might give the impression that there is no simple formula for $\delta(a,f,g)$. Hence one expects that the sum $\sum_{\substack{p\le x\\ p\equiv a(\bmod f)}} w_g(p)$ should have a complicated dependency on a,f and g. However, on making this computation, one is in for a surprise:

Theorem 3 *Let $f\ge 1$ and $1\le a\le f$, $(a,f)=1$ and $g\in\mathbb{Z}\backslash\{-1,0,1\}$ and h be the largest integer such that g is an hth power. Write $g=g_1g_2^2$ with g_1 squarefree. Let $\tilde{g}_1$, $\tilde{a}$ denote the odd part of g_1, respectively $\tilde{a}$. Let $\beta=g_1/(g_1,f)$ and $\epsilon\in\{-1,1\}$. Put*

$$A(a,f,h)=\prod_{p|(a-1,f)}(1-\frac{1}{p})\prod_{\substack{p\nmid f\\ p|h}}(1-\frac{1}{p-1})\prod_{\substack{p\nmid f\\ p\nmid h}}\left(1-\frac{1}{p(p-1)}\right)$$

if $(a-1,f,h)=1$ and $A(a,f,h)=0$ otherwise. Let $C>0$ be arbitrary. Then

$$2\sum_{\substack{p\le x,\ p\equiv a(\bmod f)\\ (\frac{g}{p})=\epsilon\\ (h,p-1)=1}}\frac{\varphi(p-1)}{p-1}=\nu_\epsilon(a,f,g)\mathrm{Li}(x)+O_C(\frac{x}{\log^C x}),$$

where

$$\nu_\epsilon(a,f,g)=\frac{A(a,f,h)}{\varphi(f)}\left(1+\epsilon(\frac{a}{(f,g_1)})\frac{(-1)^{\frac{\tilde{g}_1-1}{2}\frac{\tilde{a}-1}{2}}\mu(|\beta|)}{\prod_{p|\beta,\ p|h}(p-2)\prod_{p|\beta,\ p\nmid h}(p^2-p-1)}\right)$$

in case $g_1 \equiv 1 (\mathrm{mod}\ 4)$ *or* $g_1 \equiv 2 (\mathrm{mod}\ 4)$ *and* $8|f$ *or* $g_1 \equiv 3 (\mathrm{mod}\ 4)$ *and* $4|f$ *and*

$$\nu_\epsilon(a, f, g) = \frac{A(a, f, h)}{\varphi(f)},$$

otherwise. Here $(./.)$ *denotes the Jacobi symbol with the stipulation that* $(a/2) = (-1)^{(a^2-1)/8}$.

An alternative expression for $\nu_\epsilon(a, f, g)$ is

$$\nu_\epsilon(a, f, g) = \frac{A(a, f, h)}{\varphi(f)} \left(1 - \epsilon(\frac{\Delta}{a})(\frac{a}{|\beta|})\frac{\mu(2|\beta|)}{\prod_{p|\beta,\ p|h}(p-2)\prod_{p|\beta,\ p\nmid h}(p^2-p-1)}\right),$$

where (Δ/a) denotes the Kronecker symbol and $(a/|\beta|)$ the Jacobi symbol, cf. [4]. Although this expression is more compact, the one given in Theorem 3 is the one that arises from the proof given here.

Inspired by the relative simplicity of the formula for $\nu_\epsilon(a, f, g)$, I set out in [4] to calculate $\delta(a, f, g)$. It turns out that for a, f and g as in Theorem 3 we have $\delta(a, f, g) = \nu_{-1}(a, f, g)$. From this, Theorem 2 and Theorem 3 we deduce the following generalization of Theorem 1.

Theorem 4 (GRH). *Let* a, f, g *and* h *be as in the previous theorem. Then*

$$2 \sum_{\substack{p \le x,\ p \equiv a (\mathrm{mod}\ f) \\ (\frac{g}{p}) = -1 \\ (h, p-1) = 1}} \frac{\varphi(p-1)}{p-1} = \pi_{a,f,g}(x) + O\left(\frac{x \log\log x}{\log^2 x}\right). \tag{3}$$

This gives further corraboration to the fact that $w_g(p)$ should be considered as a probability.

The rest of this note is devoted to proving Theorem 3. In the next section we prove this result in the case $h = 1$ and use it in Section 3 as a building block in the proof of Theorem 3.

2 The case where $h = 1$

In this section we will prove Theorem 3 in the case $h = 1$. For convenience we write down Theorem 3 with $h = 1$ as a lemma.

Lemma 1 *Let* $g \in \mathbb{Z}\backslash\{-1, 0, 1\}$ *and suppose that* g *is not an hth power for any* $h \ge 2$. *Write* $g = g_1 g_2^2$ *with* g_1 *squarefree. Let* $f \ge 1$ *and* $1 \le a \le f$, $(a, f) = 1$. *Let* $\tilde{g}_1$, $\tilde{a}$ *denote the odd part of* $\tilde{g}_1$, *respectively* $\tilde{a}$. *Let* $\beta = g_1/(g_1, f)$ *and* $\epsilon \in \{-1, 1\}$. *Then*

$$2 \sum_{\substack{p \le x \\ p \equiv a (\mathrm{mod}\ f) \\ (\frac{g}{p}) = \epsilon}} \frac{\varphi(p-1)}{p-1}$$

is $\pi(x)$*-like with constant*

$$\frac{A(a, f, 1)}{\varphi(f)} \left(1 + \epsilon(\frac{a}{(f, g_1)})\frac{(-1)^{\frac{\tilde{g}_1 - 1}{2}\frac{\tilde{a}-1}{2}}\mu(|\beta|)}{\prod_{p|\beta}(p^2 - p - 1)}\right)$$

in case $g_1 \equiv 1(\text{mod } 4)$ *or* $g_1 \equiv 2(\text{mod } 4)$ *and* $8|f$ *or* $g_1 \equiv 3(\text{mod } 4)$ *and* $4|f$ *and* $\pi(x)$*-like with constant* $A(a,f,1)/\varphi(f)$*, otherwise. Here* $(./.)$ *denotes the Jacobi symbol with the stipulation that* $(a/2) = (-1)^{(a^2-1)/8}$.

The proof rests on the following three lemmata.

Lemma 2 *Let* r *be squarefree. Then*

$$\sum_{\delta|r} \frac{\mu(\delta)}{\delta}\varphi(\frac{r}{\delta}) = \frac{1}{r}\prod_{p|r}(p^2-p-1).$$

Proof. Denote the above sum by $h(r)$. It is a Cauchy product of two multiplicative functions and hence multiplicative. Notice that $h(p) = (p^2-p-1)/p$. Thus for squarefree r we have

$$h(r) = \prod_{p|r} h(p) = \frac{1}{r}\prod_{p|r}(p^2-p-1).$$

Lemma 3 *The sum*

$$\sum_{p\le x,\ p\equiv a(\text{mod } f)} \frac{\varphi(p-1)}{p-1}$$

is $\pi(x)$*-like with constant*

$$\frac{1}{\varphi(f)}\prod_{p|(a-1,f)}(1-\frac{1}{p})\prod_{p\nmid f}\left(1-\frac{1}{p(p-1)}\right).$$

Let $f_1,\cdots,f_s$ *be pairwise coprime integers and* $a_1(\text{mod } f_1),\cdots,a_s(\text{mod } f_s)$ *be primitive residue classes. The sum*

$$\sum_{p\le x,\ p\equiv a_i(\text{mod } f_i),\ 1\le i\le s} \frac{\varphi(p-1)}{p-1}$$

is $\pi(x)$*-like with constant*

$$\frac{1}{\varphi(f_1f_2\cdots f_s)}\prod_{p|(a_1-1,f_1)\cdots(a_s-1,f_s)}(1-\frac{1}{p})\prod_{p\nmid f_1\cdots f_s}\left(1-\frac{1}{p(p-1)}\right).$$

Proof. By the chinese remainder theorem there are integers n satisfying both $n \equiv a(\text{mod } f)$ and $n \equiv 1(\text{mod } d)$ iff $a \equiv 1(\text{mod } (f,d))$. Using this, it is easily seen (cf. [3]) that the sum under consideration is $\pi(x)$-like with constant

$$\sum_{\alpha|(a-1,f)}\sum_{(d,f)=\alpha}\frac{\mu(d)}{d\varphi(df/\alpha)}.$$

But this sum equals the Euler product in the statement of the lemma.

The last part of the assertion can be deduced from the first one. By the chinese remainder theorem there exists γ such that $\gamma \equiv a_i(\text{mod } f_i)$ for $1 \le i \le s$. Clearly $(\gamma-1,f_i) = (a_i-1,f_i)$. Now apply the first part with $a=\gamma$ and $f = f_1\cdots f_s$ and use that $(\gamma-1,f_1\cdots f_s) = (\gamma-1,f_1)\cdots(\gamma-1,f_s) = (a_1-1,f_1)\cdots(a_s-1,f_s)$. □

Lemma 4 *Let $t \geq 1$ be odd and squarefree. Put $b = t/(f,t)$. Then*

$$2 \sum_{\substack{p \leq x \\ p \equiv a (\mathrm{mod}\ f) \\ (p/t)=\epsilon}} \frac{\varphi(p-1)}{p-1}$$

is $\pi(x)$-like with constant

$$\left(1 + \epsilon(\frac{a}{(f,t)}) \frac{\mu(b)}{\prod_{p|b}(p^2-p-1)}\right) \frac{A(a,f,1)}{\varphi(f)}.$$

Proof. We have

$$\begin{aligned}
2 \sum_{\substack{p \leq x \\ p \equiv a (\mathrm{mod}\ f) \\ (p/t)=\epsilon}} \frac{\varphi(p-1)}{p-1} &= 2 \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ (\alpha/t)=\epsilon}} \sum_{\substack{p \leq x \\ p \equiv a (\mathrm{mod}\ f) \\ p \equiv \alpha (\mathrm{mod}\ t) \\ (p/t)=\epsilon}} \frac{\varphi(p-1)}{p-1} + O(1) \\
&= 2 \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ (\alpha/t)=\epsilon \\ \alpha \equiv a (\mathrm{mod}\ (f,t))}} \sum_{\substack{p \leq x \\ p \equiv a (\mathrm{mod}\ f) \\ p \equiv \alpha (\mathrm{mod}\ b)}} \frac{\varphi(p-1)}{p-1} + O(1),
\end{aligned}$$

where in the second step we used the chinese remainder theorem. Using Lemma 3 it follows that the sum in the statement of the lemma is $\pi(x)$-like with constant

$$\frac{2}{\varphi(fb)} \prod_{p|(a-1,f)} (1 - \frac{1}{p}) \prod_{p \nmid fb} \left(1 - \frac{1}{p(p-1)}\right) \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ (\alpha/t)=\epsilon \\ \alpha \equiv a (\mathrm{mod}\ (f,t))}} \sum_{d|(\alpha-1,b)} \frac{\mu(d)}{d},$$

that is with constant

$$\frac{1}{\varphi(fb)} \prod_{p|(a-1,f)} (1 - \frac{1}{p}) \prod_{p \nmid fb} \left(1 - \frac{1}{p(p-1)}\right) (I_1 + \epsilon I_2) \tag{4}$$

with

$$I_1 = \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ \alpha \equiv a (\mathrm{mod}\ (f,t))}} \sum_{d|(\alpha-1,b)} \frac{\mu(d)}{d}$$

and

$$I_2 = \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ \alpha \equiv a (\mathrm{mod}\ (f,t))}} (\frac{\alpha}{t}) \sum_{d|(\alpha-1,b)} \frac{\mu(d)}{d}$$

Now

$$\begin{aligned}
I_1 &= \sum_{d|b} \frac{\mu(d)}{d} \sum_{\substack{1 \leq \alpha \leq t,\ (\alpha,t)=1 \\ \alpha \equiv a (\mathrm{mod}\ (f,t)) \\ \alpha \equiv 1 (\mathrm{mod}\ d)}} 1 \\
&= \sum_{d|b} \frac{\mu(d)}{d} \varphi(\frac{b}{d}) \\
&= \frac{1}{b} \prod_{p|b} (p^2 - p - 1),
\end{aligned}$$

where we invoked Lemma 2. It remains to evaluate I_2. We have

$$\begin{aligned} I_2 &= \sum_{d|\frac{t}{(f,t)}} \frac{\mu(d)}{d} \sum_{\substack{1\le\alpha\le t,\ (\alpha,t)=1 \\ \alpha\equiv a(\text{mod } (f,t)) \\ \alpha\equiv 1(\text{mod } d)}} (\frac{\alpha}{t}) \\ &= (\frac{a}{(f,t)}) \sum_{d|b} \frac{\mu(d)}{d} \sum_{\substack{1\le\beta\le\frac{b}{d} \\ (\beta,\frac{b}{d})=1}} \left(\frac{\beta}{b/d}\right) \end{aligned}$$

The inner sum equals 1 if $d = b$ and is zero otherwise. Thus

$$I_2 = (\frac{a}{(f,t)})\frac{\mu(b)}{b}.$$

Inserting these expressions in (4), the result follows after some easy manipulations. □

Proof of Lemma 1. On using the law of quadratic reciprocity and its supplements, it is straightforward, but tedious due to case distinctions one has to make, to deduce Lemma 1 from Lemma 4. □

3 The general case

Here we will deduce Theorem 3 from Lemma 1, that is from Theorem 3 in the case where $h = 1$. We start with a lemma.

Lemma 5 *Let $t \ge 1$ be odd and squarefree. Let $\gamma(n) = \prod_{p|n} p$ denote the squarefree kernel of n. Then*

$$\sum_{\substack{1\le\alpha\le h \\ (\alpha,h)=1,\ (\alpha-1,h)=1}} (\frac{\alpha}{(h,t)}) = \mu((h,t))\frac{h}{\gamma(h)} \prod_{p|h,\ p\nmid t} (p-2)$$

Proof. Denote the l.h.s. by $\kappa(h,t)$. Using the chinese remainder theorem we see that $\kappa(h,t)$ is a multiplicative function in h. Suppose that $h = p^m$. If $p \nmid t$, then $\kappa(p^m,t) = p^{m-1}(p-2)$. If $p|t$, then

$$\kappa(p^m,t) = \sum_{\substack{1\le\alpha\le p^m \\ (\alpha,p)=1,\ (\alpha-1,p)=1}} (\frac{\alpha}{p}) = p^{m-1} \sum_{2\le\alpha\le p-1} (\frac{\alpha}{p}) = -p^{m-1}.$$

From this the result easily follows. □

Corollary 1 *Let $t \ge 1$ be an odd squarefree divisor of h. Then*

$$\sum_{\substack{1\le\alpha\le h \\ (\alpha,h)=1,\ (\alpha-1,h)=1}} (\frac{\alpha}{t}) = \mu(t)\frac{h}{\gamma(h)} \prod_{p|\frac{\gamma(h)}{t}} (p-2).$$

Proof of Theorem 3. Notice that Theorem 3 is true if h is even. Thus assume that h is odd. Since $(p-1,h)=1$ iff $(p-1,\gamma(h))=1$, we can assume w.l.o.g. that h is squarefree. Now

$$\begin{aligned} 2\sum_{\substack{p\le x,\ (\frac{g}{p})=\epsilon\\ p\equiv a(\mathrm{mod}\ f)\\ (h,p-1)=1}}\frac{\varphi(p-1)}{p-1} &= O(1)+2\sum_{\substack{1\le\alpha\le h\\ (\alpha,h)=1\\ (\alpha-1,h)=1}}\ \sum_{\substack{p\le x,\ p\equiv a(\mathrm{mod}\ f)\\ (\frac{g}{p})=\epsilon\\ p\equiv\alpha(\mathrm{mod}\ h)}}\frac{\varphi(p-1)}{p-1}\\ &= O(1)+2\sum_{\substack{1\le\alpha\le h,\ (\alpha,h)=1\\ (\alpha-1,h)=1\\ \alpha\equiv a(\mathrm{mod}\ (f,h))}}\ \sum_{\substack{p\le x,\ p\equiv a(\mathrm{mod}\ f)\\ (\frac{g}{p})=\epsilon\\ p\equiv\alpha(\mathrm{mod}\ h/(f,h))}}\frac{\varphi(p-1)}{p-1}. \qquad (5)\end{aligned}$$

Notice that the latter outer sum is empty in the case that $(a-1,(f,h))=(a-1,f,h)>1$. In this case Theorem 3 trivially follows. Thus we may assume that $(a-1,f,h)=1$. The latter inner sum can be evaluated using Lemma 1.

Put $h_1=h/(f,h)$. Notice that since by assumption h is squarefree, h_1 and f are coprime. For $\alpha\equiv a(\mathrm{mod}\ (f,h))$, let $1\le\beta_\alpha\le fh_1$ be the unique integer such that $\beta_\alpha\equiv a(\mathrm{mod}\ f)$ and $\beta_\alpha\equiv\alpha(\mathrm{mod}\ h_1)$. Now for $t\ge 1$, with t an odd squarefree divisor of h_1, put

$$S(t)=\sum_{\substack{1\le\alpha\le h\\ (\alpha,h)=1,\ (\alpha-1,h)=1\\ \alpha\equiv a(\mathrm{mod}\ (f,h))}}(\frac{\alpha}{t})\frac{A(\beta_\alpha,fh_1,1)}{\varphi(fh_1)}.$$

Now using Lemma 1 and (5), we find that the sum in the l.h.s. of (5) is $\pi(x)$-like with constant

$$S(1)+\epsilon(\frac{a}{(f,g_1)})\frac{(-1)^{\frac{\tilde g_1-1}{2}\frac{\tilde a-1}{2}}\mu(|\beta|)}{\mu((g_1,h_1))\prod_{p|\frac{\beta}{(g_1,h_1)}}(p^2-p-1)}S((h_1,g_1)), \qquad (6)$$

in case $g_1\equiv 1(\mathrm{mod}\ 4)$ or $g_1\equiv 2(\mathrm{mod}\ 4)$ and $8|f$ or $g_1\equiv 3(\mathrm{mod}\ 4)$ and $4|f$ and $\pi(x)$-like with constant $S(1)$ otherwise. Now

$$S(t)=\frac{1}{\varphi(f)\varphi(h_1)}\prod_{p|(a-1,f)}(1-\frac{1}{p})\prod_{\substack{p\nmid f\\ p\nmid h}}(1-\frac{1}{p(p-1)})\tilde S(t), \qquad (7)$$

with

$$\begin{aligned}\tilde S(t) &= \sum_{\substack{1\le\alpha\le h\\ (\alpha,h)=1,\ (\alpha-1,h)=1\\ \alpha\equiv a(\mathrm{mod}\ (f,h))}}(\frac{\alpha}{t})\prod_{p|(\alpha-1,h)}(1-\frac{1}{p})\\ &= \sum_{\substack{1\le\alpha\le h\\ (\alpha,h)=1,\ (\alpha-1,h)=1\\ \alpha\equiv a(\mathrm{mod}\ (f,h))}}(\frac{\alpha}{t})\\ &= \sum_{\substack{1\le\beta\le h_1\\ (\beta,h_1)=1,\ (\beta-1,h_1)=1}}(\frac{\beta}{t}),\end{aligned}$$

where in the derivation of the last equality we made use of the assumption that $(a-1,f,h)=1$ and that $t|h_1$. Since by assumption t is an odd squarefree divisor of h_1, we have, by the Corollary to Lemma 5, that $\tilde{S}(t) = \mu(t)\prod_{p|h_1/t}(p-2)$. Substituting this in (7), we obtain

$$S(t) = \frac{A(a,f,h)}{\varphi(f)}\frac{\mu(t)}{\prod_{p|t}(p-2)}. \tag{8}$$

On noting that $\prod_{p|(g_1,h_1)}(p-2) = \prod_{p|\beta,\ p|h}(p-2)$ and $\prod_{p|\beta/(g_1,h_1)}(p^2-p-1) = \prod_{p|\beta,\ p\nmid h}(p^2-p-1)$, the theorem follows on combining (6) with (8). □

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