Bundles over moduli space of Bohr - Sommerfeld lagrangian cycles

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Introduction

In the present paper we study some geometrical properties of the moduli space of Bohr - Sommerfeld lagrangian cycles which belongs to a new subject, built on the border (or on a neighborhood of the border) of algebraic geometry and lagrangian geometry. This new subject was called ALAG — abelian lagrangian algebraic geometry — by its inventors, A. N. Tyurin and A. Gorodentsev (see [6]). ALAG - programme could be seen as a programme indeed: as the input for the programme one takes any smooth compact symplectic simply connected manifold (finite dimensional) (M, ω) with integer symplectic form (so $[\omega] \in H^2(M, \mathbb{Z})$), and as the output of the programme one gets an infinite dimensional algebraic manifold $\mathcal{B}_S^{hw,r}$ which is the moduli space of half weighted Bohr - Sommerfeld lagrangian cycles of fixed topological type and volume. The construction is essentially universal - one doesn't use other ingredients than a symplectic manifold naturally has, and this universality implies an important property of the construction which was called "dynamical correspondence" (see [17]). It means that every hamiltonian action on the given symplectic manifold M induces a natural action on the moduli space $\mathcal{B}_S^{hw,r}$ and this action is *hamiltonian* with respect to the Kaehler form on $\mathcal{B}_S^{hw,r}$. Moreover, one can write down explicitly the corresponding Hamiltonian $F_f \in C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$ for any "classical" Hamiltonian $f \in C^\infty(M, \mathbb{R})$ (see [16], [17]). This dynamical property leads one to apply the ALAG - programme for a generalization of geometric quantization. And a solution for the problem of nonlinear version of geometric quantization is given by the moduli space $\mathcal{B}_S^{hw,r}$ (a survey on the subject can be found in [18]).

It doesn't look unexpected since the inventors of ALAG had in mind the following main background idea. The point is that the group of symplectomorphisms of M acts in a natural way on the moduli space $\mathcal{B}_S^{hw,r}$, and this action preserves the Kaehler structure. It means that one could try to perform the factorization which would give a finite dimensional quotient space which should be an algebraic variety (finite dimensional). Leaving aside all possible details and difficulties (such as the transitivity problem of the action etc.) one concludes with the following ideal picture: for any given compact symplectic manifold with integer symplectic form there is a canonical partner — finite dimensional algebraic variety. This correspondence now would be highly desired because of Mirror Symmetry conjecture. Indeed, Mirror Symmetry today in the broad context is understood as a duality between Algebraic Geometry and Symplectic Geometry. This means that if M, W are mirror partners then the algebraic geometry of M corresponds to the symplectic geometry of W and vice versa. The main problem is to find the meaning of term "correspond" in the previous phrase. For example, in Homological Mirror Symmetry, proposed by M. Kontsevich, this one is understood as follows: in the framework of Algebraic Geometry one constructs some category over M (namely, the derived category of coherent sheaves) while in the symplectic setup one takes some category over W (which is the Fukaya - Floer - Oh - Ohta - Ono category) and the correspondence means that these two categories are equivalent as triangulated categories. Another example is given in more general framework for Fano varieties by D. Orlov, L. Karzarkov and D. Auruox where one compares the derived category of coherent sheaves with so- called Fukaya - Seidel category for the corresponding Landau - Ginzburg model, see [2].

Unfortunately the straightforward background idea of ALAG - programme couldn't be realized since as it was shown in [19] if the quotient space

$$\mathcal{B}_S^{hw,r}/\mathrm{SymM}$$

does exist it should be zero dimensional (and the number of points is equal to $h^0(\mathcal{B}_S, \mathbb{Z})$, see below). Zero dimensional complex manifolds are interesting only if they are naturally included into some ambient algebraic space (f.e.

the last author of [6] is extremely skilfull in the recognizing of some geometric objects under different views including the case of points). But in our case a priori there is no any natural ambient space for the factor. Despite of this the situation is not pathologic: suppose one constructs a natural vector bundle (or some other fibered object) of finite rank over the moduli space $\mathcal{B}_S^{hw,r}$ and factorize it with respect to SymM - action, then it would give a quotient object of finite (non zero) dimension. If this natural bundle over $\mathcal{B}_S^{hw,r}$ is holomorphic then the resulting quotient space should be algebraic.

This enforces us to study some geometrical properties of the moduli spaces. In the present text we follow this reason, starting with some natural objects over the moduli space of Bohr - Sommerfeld Lagrangian cycles (without weights). Even on this level (real analytic) there are examples of the objects. It's not hard to describe how the group of symplectomorphisms acts on these ones. At the same time the basic construction with half-weighting in ALAG can be slightly deformed and this would lead to extensions of the objects to the weighted case. This needs some geometric interpretation of the Maslov class which we discuss below.

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1 Induced dynamics

Every symplectic manifold (M, ω) where ω is the symplectic form, can be regarded as the phase space of some classical mechanical system. We will deal with compact case when M is a compact real even dimensional manifold and moreover we will assume that it is simply connected. Then every hamiltonian vector field (= infinitesimal symplectomorphism) is generated by a smooth function which is called the Hamiltonian. Every Hamiltonian defines (or induces) dynamics of the system so if one considers a point then it is moved by the hamiltonian vector field and its motion gives us some trajectory which is an integral curve, tangent at each point to the hamiltonian vector field. To study the induced motion one has to linearize the phase space near this distinguished point (= take a chart in some atlas, which defines the smooth structure) and solve the corresponding differential equation (the Hamilton equation). It's the subject of classical mechanics — an old and very important part of mathematics. But at the same time there is an aspect of the situation above leading to some more generic picture.

Indeed, let us consider any submanifold $S \subset M$ inside of M. Then for any Hamiltonian $f \in C^\infty(M, \mathbb{R})$ one has the corresponding infinitesimal deformation of S, and if we continue the infinitesimal deformation, given by X_f, to the induced 1- parameter family of symplectomorphisms ϕ_t (perhaps for sufficiently small t) we get some motion of S. This motion could be divided into two parts:

Inner part: the deformation of S itself (so how the points of S move with respect to each other and change the location)

and

Outer part: the deformation of S inside of M as a "body".

Generalizing we see that on the space of all sub objects of M one has some induced action of any Hamiltonian and it's quite natural to speak in this case about some induced dynamics: dynamics of objects which are subsets inside of M.

But even the simplest case when one takes as this subset an arbitrary chosen set of points already shows that this induced dynamics can be extremely complicated. So in the full range of possible sub objects we must find an appropriate type (if it exists) of submanifolds for which we could describe the induced dynamics. Indeed, to do this we need

1) some good notion of the space of such sub objects including the question of what is the corresponding smooth structure (to introduce some coordinates),

2) some natural splitting of the Hamiltonian action into "inner" and "outer" parts compatible with the choice of coordinates,

3) this splitting should give us some reasonable equations which would be solvable,

4) some relationship between classical mechanical setup and the new one, given by the investigations.

Of course, any such realization is interesting for us if it can be understood as a solution or a generalization of some problem. ALAG - programme gives us an example of such induced dynamics.

In this case we take as the sub objects some lagrangian submanifolds (or more generally some cycles) of fixed topological and homological types which satisfy so called Bohr - Sommerfeld condition with respect to some natural additional data. The meaning of this condition is just that local deformations of any such lagrangian submanifold inside of our set are generated by hamiltonian vector fields. So we distinguish in the space of all lagrangian submanifolds of fixed topological and homological types the ones which are hamiltonian equivalent. Let us recall briefly the main points of ALAG - construction (see [6]) taking in mind our requirements above for induced dynamics.

Let (M, ω) be a compact smooth simply connected symplectic manifold of dimension $2n$ with integer symplectic form ω so $[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$. Since the class $[\omega]$ is integer and our manifold M is simply connected there exists unique hermitian line bundle L with the first Chern class $c_1(L) = [\omega]$ together with unique up to gauge transformations hermitian connection a such that $F_a = 2\pi i \omega$. The data (L, a) are defined uniquely (so they are intrinsic for our symplectic manifold) and called "prequantization data". One takes a smooth oriented n - dimensional manifold S_{par} as the parameter space considering smooth maps (or more generally immersions)

$$\phi : S_{par} \to M,$$

which satisfy the properties

fixed homological type $[\phi(S_{par})] = [S] \in H_n(M, \mathbb{Z})$ — a fixed class,

lagrangian condition $\phi^* \omega = 0$,

Bohr - Sommerfeld condition $\phi^*(L, a)$ is a trivial line bundle with a trivial connection (so the pair $\phi^*(L, a)$ is gauge equivalent to the pair $(C^\infty(S_{par}, \mathbb{C}), d)$ where d is the ordinary differential).

The space of all such maps is factorized with respect to the reparametrization group which is $Diff_0^+ S_{par}$ and the resulting quotient space is called the moduli space of Bohr - Sommerfeld lagrangian cycles and denoted as $\mathcal{B}_S$. Forgetting about the parameter space S_{par} we can think about $\mathcal{B}_S$ as consisting of the images of ϕs so of Bohr - Sommerfeld lagrangian submanifolds in M (we will deal with smooth Bohr - Sommerfeld lagrangian submanifolds so we will speak about $\mathcal{B}_S$ near smooth points).

Now we list the main properties of the moduli space $\mathcal{B}_S$ (see [6]) with respect to the requirements 1) - 4) above.

1) The moduli space $\mathcal{B}_S$ is smooth real infinite dimensional manifold. The tangent space $T\mathcal{B}_S$ at a smooth point $S \in \mathcal{B}_S$ is modeled by the space $C^\infty(S, \mathbb{R})$ modulo constants. The smooth structure is given by the Darboux - Weinstein theorem which says that for every $S \subset M$ there exists a neighborhood $\mathcal{O}(S) \subset M$ symplectomorphic to an ϵ- neighborhood of the zero

section in T^*S endowed with the canonical symplectic structure. Thus the choice of a covering of M by a number of Darboux - Weinstein neighborhoods (sufficiently small) gives us an atlas on $\mathcal{B}_S$ and at each its chart we have the corresponding local coordinates — smooth functions on S modulo constants.

2) As it was pointed out in [20] any choice of smooth structure gives us some canonical decomposition of any hamiltonian vector field on any Bohr - Sopmmerfeld lagrangian cycle. It's very crucial point since in general one could not decompose canonically vector field at points of some submanifold without additional data (f.e. some compatible riemannian metric). But the choice of some local coordinates (an appropriate atlas) gives the splitting which is very simple to describe in the coordinates (see [17]). At the same time this decomposition is compatible with the corresponding description in terms of local Darboux - Weinstein coordinates: the outer part of the hamiltonian vector field X_f, defined by function $f \in C^\infty(M, \mathbb{R})$ responses for the motion of S in $\mathcal{B}_S$ and the last one is given infinitesimally by $\psi \in C^\infty(S, \mathbb{R})$ equals to

$$\psi = f|_S \text{ modulo constants.}$$

In general, it is a special feature of lagrangian submanifolds that X_f is tangent to S if and only if f is constant on S.

3) The splitting gives us quite simple and natural equation of motion over $\mathcal{B}_S$: it's not hard to see that the hamiltonian dynamics of any function $f \in C^\infty(M, \mathbb{R})$ induces the corresponding transformation of $\mathcal{B}_S$, generated by some smooth vector field. We've almost described it in the previous item — every function $f \in C^\infty(M, \mathbb{R})$ being restricted to any cycle $S \in \mathcal{B}_S$ gives a tangent vector $\psi = f|_S$, and generalizing the picture over whole $\mathcal{B}_S$ one gets a smooth vector field A_f. This vector field induces a 1- parameter family of smooth transformations of $\mathcal{B}_S$ and it is exactly what the family ϕ_t generates. So the equation of motion is very close to the classical one.

4) The desired relationship is absolutely clear. The point is that here we have some dynamical correspondence: the induced dynamics coincides with the dynamics, described by some natural equation including the vector field A_f induced by f. This dynamical naturality shows that for any pair of smooth functions f, g the Poisson bracket $\{f, g\}$ induces a vector field which is exactly the commutator $[A_f, A_g]$. So the picture is quite compatible and self contained. Let us remark that a function $f \in C^\infty(M, \mathbb{R})$ really induces not just a vector field on $\mathcal{B}_S$ but something more: at the points where the field degenerates we have some exact numerical values. Indeed, A_f degenerates at

point $S \in \mathcal{B}_S$ iff $f|_S = const$, and this constant gives us the numerical value. We've discussed what it looks like at the last part of [18].

Now one wants to compexify the story. The reason comes from the quantization problem stated as follows (see f.e. [8]). One has the maximal symmetry group of our phase space (M, ω) which is the infinite-dimensional group of symplectomorphisms $SymM$. The construction given above realizes some representation of SymM as a subgroup of $Diff \mathcal{B}_S$. But this group is essentially real. Can we extend the construction such that it would give us a realization of SymM in complex (hermitian) setup? On the other hand, we have used only one part of the splitting of hamiltonian vector fields which was declared as of very importance. It means that the picture with $\mathcal{B}_S$ is a bit degenerated since if f_1 and f_2 coincide on S and doesn't coincide on any neighborhood of S in M they nevertheless give the same deformation vector at $S \in \mathcal{B}_S$. To activate the inner parts of hamiltonian vector field we need to add some objects over our Bohr - Sommerfeld lagrangian cycles. One try to exploited either flat connections (supercycle story) or densities or forms over these cycles — anyway it should lead to some complexification of the construction. In ALAG one takes halfweights (see [6]) so the "halfweighted" moduli space $\mathcal{B}_S^{hw,r}$ is fibered over $\mathcal{B}$ such that

$$\begin{aligned} &\pi : \mathcal{B}_S^{hw,r} \to \mathcal{B}_S, \\ &\pi^{-1}(S) = \{\theta | \textstyle\int_S \theta^2 = r\}, \end{aligned}$$

where θ s are halfweights over S. The moduli space of halfweighted Bohr - Sommerfeld cycles of fixed volume is an infinite dimensional algebraic manifold (see [6]) and as it was shown in [19] the component of unity of the symplectomorphism group $\mathrm{Sym_0M} \subset \mathrm{SymM}$ is irreducibly represented as a subgroup of the symmetry group of the algebraic manifold. In particular it means that the induced infinitesimal action of X_f (for any smooth function $f \in C^\infty(M, \mathbb{R})$) on the moduli space $\mathcal{B}_S^{hw,r}$ is hamiltonian with respect to the Kaehler form Ω and as it was shown in [17] it is strictly hamiltonian with hamiltonian function $F_f \in C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$. This function has extremely simple dependence on f namely

$$F_f(S, \theta) = \tau \int_S f|_S \theta^2$$

where τ is a real parameter. Although the story with this new kind of quantization is not finished yet ALAG - programme itself looks like a appropriate tool for investigations in lagrangian geometry.

The applicability of ALAG in quantization problem hints that one could try to exploite the programme in another mathematical problem closely related to physics — in mirror symmetry. It is the main background idea of this paper (and as we hope of some future investigations). Some idealogical approach to such an application is given in the next section.

2 Extended mirror symmetry

It's a natural idea built over some standard concept of mirror symmetry independently in [21] and [15], [14]. Here we follow the last source since it's more suitable for our aims.

We recall roughly the usual setup: one considers Calabi - Yau 3 - manifolds (originally one deals with quintics in $\mathbb{CP}^4$, see [11]). Then two such manifolds M and W are partners if

(**numerical level**) $h^{1,1}(W) = h^{1,2}(M)$ and vice versa;

(**geometrical level**) the complex deformations of M naturally corresponds to the Kaehler deformations of W and vice versa.

The geometrical level is higher than the numerical one since the complex deformations of M form (infinitesimally) the space of dimension $h^{1,2}(M)$ while the number $h^{1,1}(W)$ responses for the Kaehler deformations of W.

Let $D(M), D(W)$ are the Hodge diamonds of M and W. An extension of the numerical mirror symmetry would read as the coincidence of $D(M)$ and $\hat{D}(W)$, where the last one is given by $D(W)$, transformed by rotation on $\pi/2$. More generally, it could be extended as the condition that there exists some isomorphism

$$\phi : H^{\text{even}}(M, \mathbb{Z}) \to H^{\text{odd}}(W, \mathbb{Z})$$

(which could be given by the rotation f.e.). But such a coincidence again should be pure numerical — to "geometrize" it we must to attach to the cohomological classes some geometrical realizations which would be appropriate in the corresponding context. F.e. if we are speaking about M and complex geometry of M then H^{even} should be endowed by some complex realizations, and in the case of W, symplectic geometry comes to forefront. An element from $H^{\text{even}}(M, \mathbb{Z})$ reads as (r, c_1, c_2, c_3) and it is clear that one takes the realization by holomorphic vector bundles of the topological type given by rank r and the Chern classes c_1, c_2, c_3. The fixed Kaehler form on our Calabi - Yau manifold M which is a necessary entire defines the corresponding polarization and one can ask about the realizations by stable holomorphic vector bundles of the fixed topological type. Thus every couple $v = (r, c_1, c_2, c_3) \in H^{\text{even}}$ is endowed with a finite dimensional complex moduli space $\mathcal{M}_v$. The local theory of the moduli space says that at a smooth point $E \in \mathcal{M}_v$ the tangent space is modeled by the kernel of a map

$$\psi : H^1(M, adE) \to H^2(M, adE)$$

but due to the Calabi - Yau condition (the canonical class is trivial) both the spaces have the same dimension so one could expect that $\mathcal{M}_v$ is zero dimensional so one has a number of points.

On the other hand $H^{\text{odd}}(W, \mathbb{Z})$ just coincides with $H^3(W, \mathbb{Z})$ by the definition of Calabi - Yau manifold. The group $H^3(W, \mathbb{Z})$ is isomorphic due to the Poincare duality to $H_3(W, \mathbb{Z})$ so one should look for any reasonable realizations of classes from $H_3(W, \mathbb{Z})$. Such realization could be given by some lagrangian submanifolds representing a fixed class. But the space of all lagrangian submanifolds is too big thus one should impose some natural conditions to derive some good finite dimensional moduli space. One choice comes with special lagrangian submanifolds (see [7]). The local theory was investigated by McLean in [10] and it ensures that the moduli space at a smooth point $S \subset M$ has dimension $b_1(S)$ — the first Betty number. So as an appropriate realization in the symplectic setup the moduli space $\mathcal{M}_{[S]}$ of special lagrangian submanifolds would be a natural realization of class $[S] \in H_3(W, \mathbb{Z})$. These moduli spaces are essentially real but one can switch on complex geometry adding some supercycle structure to the story (see [7], [**?**]). Then the moduli space of special lagrangian super cycles is a complex moduli space.

The logical consequence is clear now: the Calaby - Yau manifolds M and W are mirror dual if there exists an isomorphism

$$\phi : H^{\text{even}}(M, \mathbb{Z}) \to H^{\text{odd}}(W, \mathbb{Z})$$

such that for any $v \in H^{\text{even}}(M, \mathbb{Z})$ the moduli space $\mathcal{M}_v$ is isomorphic to the moduli space $\mathcal{M}_{[S]=\phi(v)}$.

As an example we could mention the case where the SYZ - construction (see [13]) works. This is the case of elliptic curves, which is worked out in details in [14]. In this case the moduli space of stable vector bundles of type (r, d) where r is the rank and d is the degree is isomorphic to the curve itself.

At the same time (see [14]) the moduli space of special lagrangian cycles on the mirror curve always has dimension 2 being generated by translations in one direction. Adding some supercycle structure to the consideration one avoids this divergence getting a direct correspondence between holomorphic bundles and special lagrangian supercycles, see [14]. We will discuss the situation more concretely in Section 5. At the same time we will show that the correspondence carries some hidden parts which hints that it could be lifted to a correspondence between lagrangian submanifolds and connections (not only that flat). Thus our background idea is to extend the extended mirror symmetry picture to be some general correspondence between lagrangian geometry and gauge geometry.

For example we can understand the special lagrangian condition as a minimality condition on lagrangian submanifolds or on the moduli space of Bohr - Sommerfled cycles. As we will show in Section 5 in general the intersection of notions is "finite", it means that the space of special Bohr - Sommerfeld lagrangian cycles in general is a finite set of points thus it's a kind of the usual story when one takes some special representatives for an infinte space (harmonic forms, instantons etc).

So it's a natural idea to combine together the special lagrangian condition and the Bohr - Sommerfeld condition to one spBS - condition and consider the moduli space of spBS lagrangian cycles. Then we can rearrange the order and consider the story in the opposite way. Indeed, one could consider the special lagrangian condition as a condition imposed on the moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian cycles. Then it is not hard to see that in the presence of the corresponding riemannian metric on W one has a map

$$V_g : \mathcal{B}_S \to \mathbb{R}_+,$$

attaching the riemannian volume to each Bohr - Sommerfeld cycle. Then the set of spBS - lagrangian cycles is given by the absolute minima of this functional. And the story turns to be very familiar if we remember that there is a very good paradigm for the theory of stable vector bundles namely the gauge theory.

3 Infinite dimensional geometry of $\mathcal{B}_S$

SpBS cycles can be regarded as solutions of some equations over the space of Bohr - Sommerfeld lagrangian cycles $\mathcal{B}_S$. The last one is an infinite dimensional smooth real manifold. To study the geometry one needs a pattern to work with infinite dimensional manifolds. We would like to use the gauge theory as the pattern since the theory is highly developed and at the same time the gauge theory is a good and natural framework for the theory of stable holomorphic vector bundles. The main example is given by the Donaldson theory which is closely related to stable holomorphic vector bundles over algebraic surfaces.

The battle field of the gauge theory is presented by a based manifold X, which is smooth real, a principle G - bundle $P \to X$ over X, the space $\mathcal{A}_P$ of G - connections on P and the gauge group $\mathcal{G}$ of fiberwise G - transformations of P. The space of connections is just an affine space, but the real configuration space is given by the set of equivalence classes

$$\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}.$$

This space is usually extremely complicated so one derives some regular subset $\mathcal{B}^*_P \subset \mathcal{B}_P$ which consists of classes of irreducible connections (with minimal centralizer $C(G)$). This is an infinite dimensional real space locally modeled at a regular point $[a] \in \mathcal{B}^*_S$ by

$$\Omega^1_X(adG)/d_a\Omega^0_X(adG),$$

where d_a is the covariant derivative of a connection a which represents $[a]$. In presence of a riemannian metric g on X the slice near $[a]$ can be described as the kernel of formal adjoint operator d^*_a

$$\mathrm{ker d^*_a} \subset \Omega^1_{\mathrm{X}}(\mathrm{adG}).$$

So the quotient space $\mathcal{B}^*_P$ can be taken as a possible cousin of the moduli space $\mathcal{B}_S$ (at least it would explain why the moduli space of Bohr - Sommerfeld lagrangian cycles was denoted by the same letter $\mathcal{B}$ in the original paper [6]).

But the quotient space $\mathcal{B}^*_P$ possesses more remarkable properties (see [4]), namely

(1) it is orientable,

(2) it admits a universal object,

(3) all rational cohomologies of $\mathcal{B}^*_P$ are induced by this universal object.

An orientation in the infinite dimensional case is understood as follows. Suppose that there exists a real line bundles $\Lambda \to \mathcal{B}$ such that if $M \subset \mathcal{B}$ is any regular finite dimensional submanifold then the restriction $\Lambda|_M$ is isomorphic to $\Lambda^{\max} TM$. Then $\mathcal{B}$ is orientable if this Λ is trivial, and an orientation is given by a class of trivializations of Λ. In gauge theory it follows from the topological properties of $\mathcal{B}^*_P$ being a classifying space. In our case we can not apply the classical arguments from [4]. But nevertheless one has the following

Proposition 1 *The moduli space $\mathcal{B}_S$ is orientable.*

The proof is a little bit tricky: let us lift the story to the space of Bohr - Sommerfeld lagrangian maps $\phi : S_{par} \to M$, then the tangent space of these maps is given by the direct sum $Vect(S_{par}) \oplus C^\infty(S_{par}, \mathbb{R})/consts$. This space carries a natural skew symmetric operation:

$$< (v_1, f_1), (v_2, f_2) >= ([v_1, v_2], \mathcal{L}_{v_1} f_2 - \mathcal{L}_{v_2} f_1).$$

The existence of this nondegenerated operator ensures that the space of Bohr - Sommerfeld maps is orientable, and the factorization by an infinite Lie group which is $\mathrm{Diff}^+ S_{par}$ doesn't destroy this property.

At the same time one can see that the choice of an orientation on $\mathcal{B}_S$ depens on the choice of an orientation on S_{par}.

The second important property of $\mathcal{B}^*_P$ listed above concerns with some universal object. In the gauge theory it is some universal adG - bundle constructed as follows.

On the direct product $X \times \mathcal{A}_P$ one has a universal bundle $\mathbb{P}$ with universal connection $\mathbb{A}$ such that for every slice $X \times \{A\}$ the restriction of the universal objects is presented just by (P, A). The gauge group $\mathcal{G}$ acts in a natural way both on the base and on the universal bundle what gives us the corresponding adG bundle

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ X \times \mathcal{B}_P \end{array}$$

The same picture takes place for irreducible connections therefore

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & & \downarrow & & \\ & & X \times \mathcal{B}^*_P & & \\ & \swarrow p & & \searrow q & \\ X & & & & \mathcal{B}^*_P \end{array}$$

The universal adG - bundle $\mathcal{P}$ is of great importance in the story. Using this universal object one could transport objects from X to $\mathcal{B}_S^*$. For example (and it belongs to item (3)), we can transport homological classes of X to cohomological classes of $\mathcal{B}_P^*$. To do this one takes any characteristic class c of G, say, degree d, this class gives us the corresponding class $c(\mathcal{P}) \in H^d(X \times \mathcal{B}_P^*)$ and then substituting any homological class $[\Sigma] \in H_i(X, \mathbb{Z})$ we get a class from $H^{d-i}(\mathcal{B}_P^*, \mathbb{Q})$ (if our characteristic class is rational). Generally it defines a homomorphism

$$\mu : H_i(X, \mathbb{Z}) \to H^{d-i}(\mathcal{B}_P^*, \mathbb{Q}),$$

and in the known cases these μ - classes generate whole the ring $H^*(\mathcal{B}_P^*, \mathbb{Q})$.

For example, in the Donaldson theory X is a compact 4 - dimensional manifold, $G = SU(2)$, then the characteristic class is the Pontryagin class degree 4 of $SO(3)$ universal bundle $\mathcal{P}$, this class is rational but nevertheless as it was established the corresponding μ - classes $\mu([\Sigma_1]), ..., \mu([\Sigma_k]), [\Sigma_i] \in H_2(X, \mathbb{Z})$ are integer and the ring $H^*(\mathcal{B}_P^*, \mathbb{Z})$ in this case is generated by these two dimensional classes and one 4 -dimensional class η which came from the class of point on X (see [4]).

Our main observation for today is that in the case of the moduli space of Bohr - Sommerfeld lagrangian cycles $\mathcal{B}_S$ we always have a universal object which makes it possible to repeat (at least formally) the constructions recalled above.

On the direct product $M \times \mathcal{B}_S$ consider the universal cycle

$$M \times \mathcal{B}_S \supset \mathcal{U} = \{(x, S) | \text{ where } x \in M, S \in \mathcal{B}_S \text{ and } x \in S\}.$$

In the classical language it is called the incidence cycle. It's not hard to see that this cycle is smooth at smooth points and has codimension n. Then one takes the Poincare dual class $[\mathcal{U}]^* \in H^n(M \times \mathcal{B}_S^*)$ (our $\mathcal{B}_S$ is orientable so formally one could do this) and substituting homological classes from $H_i(M, \mathbb{Z})$ one gets a homomorphism

$$\mu_L : H_i(M, \mathbb{Z}) \to H^{n-i}(\mathcal{B}_S, \mathbb{Z}).$$

It is natural to understand (and call) the images as the generalized μ - classes.

The homology and cohomology classes are something well defined in the finite dimensional situation and it's hard to imagine what are they in the infinite dimensional case. Again we follow here [4] and understand these classes in the spirit of the Donaldson theory. There one chooses a riemannian metric on the based manifold X which defines the Yang - Mills functional on $\mathcal{B}^*_P$. The absolute minima of the functional are anti self dual connection classes or instantons which form the moduli space $\mathcal{M}^g_{asd} \subset \mathcal{B}^*_P$. For a generic metric this moduli space is smooth finite dimensional (but generally non compact). Thus we could "test" our μ - classes on a finite dimensional subobject in $\mathcal{B}^*_P$: if $d = v.dim \mathcal{M}^g_{asd}$ is even then

$$< \mu([\Sigma_1]).....\mu([\Sigma_{d/2}]); [\mathcal{M}^g_{asd}] >\in \mathbb{Z},$$

where $[\mathcal{M}^g_{asd}] \in H_d(\mathcal{B}^*_P, \mathbb{Z})$ is the fundamental class of the moduli space (perhaps after a small deformation of g). Moreover, if $b^+_2(X) > 1$ the topology of $\mathcal{M}^g_{asd}$ for a generic metric doesn't depend on it so one gets some polynomial map

$$\gamma^d : S^d H_2(X, \mathbb{Z}) \to \mathbb{Z}$$

invariant with respect to the $Diff^+ X$ - action. Such a map is the celebrated Donaldson polynomial (see [4]); these polynomials distinguish possible non equivalent smooth structures on a given topological base X. Thus the nontriviality of the μ - classes is reflected by the nontriviality of the polynomials (in the case of nontrivial Donaldson invariants).

Following these ideas, we would like to show that the generalized μ - classes are not trivial computing them on some appropriate finite subspaces of $\mathcal{B}_S$ (but it would not mean any usefulness of the classes, while we expect that it is). The simplest possible case is just a point — some fixed Bohr - Sommerfeld cycle $S_0 \in \mathcal{B}_S$. To pair with one should take classes from $H^0(\mathcal{B}_S, \mathbb{Z})$ so it leads to considerations of $H_n(M, \mathbb{Z})$ and the corresponding μ_L - classes. Let us take any class $[S_1] \in H_n(M, \mathbb{Z})$ and compute $< \mu_L([S_1]); [pt] >\in \mathbb{Z}$ where the point class is represented by S_0. It's easy to see that topologically this integer number is given by the intersection points of S_1 and S_0 in M (perhaps after a small perturbation of S_1) counting with signs. Therefore

$$< \mu_L([\Sigma_1]; [pt] >= Q_M([S_1], [S_0])$$

where Q_M is the topological intersection form of M. This shows that if $\mathcal{B}_S$ is not connected there would be no difference between pairings $< \mu_L([S_1]); [pt_i] >$ for $i = 1, ..., b_1(\mathcal{B}_S)$ and consequently in the good cases the moduli space $\mathcal{B}_S$ is connected.

To test other μ_L - classes we need other finite dimensional subsets in $\mathcal{B}_S$. Repeating the gauge theory story one could fix an additional data on M f.e.

some compatible riemannian metric g. Then it defines a functional

$$V_g : \mathcal{B}_S \to \mathbb{R}_+, \quad V_g(S) = \int_S Vol_{g|_S} \in \mathbb{R}_+,$$

attaching the riemannian volume to any cycle S (recall that our cycles are oriented). Then, one can ask about the extrema of this functional. The choice of a riemannian metric compatible with our symplectic form is equivalent to the choice of an almost complex structure J compatible with ω. In good cases the set of minimal Bohr - Sommerfeld cycles is finite dimensional and smooth and we can take the corresponding subset as the testing subspace in $\mathcal{B}_S$. It's well known (see [**?**]) that if J is integrable then S is minimal if the Ricci tensor of g identically vanishes on S. It implies that topologically the restriction of the canonical bundle K_J must be trivial since in the integrable case the Ricci tensor represents the canonical class. In the cases of main interest (Calabi - Yau and Fano varieties) this topological condition is satisfied automatically and one should expect that the moduli space of special Bohr - Sommerfeld lagrangian cycles is finite dimensional and smooth. We will denote it as $\mathcal{M}_{spBS} \subset \mathcal{B}_S$. Let us consider a couple of examples.

Projective line. It is the toy example from [6] so we already know that for S^2 with the standard symplectic form the moduli space $\mathcal{B}_S$ consists of unicursal real curves which divide the sphere into two equal parts with the same symplectic volume (really one has a double cover of the space since we take the curves with orientations and because $H_1(S^2, \mathbb{Z})$ is trivial we consider only one homological class as $[S]$). Let us take the standard Fubini - Study metric on S^2; then it's not hard to see that $\mathcal{M}_{spBS}$ consists of big circles, it is smooth oriented manifold of dimension 2. But in this case we have only one potentially nontrivial μ_L - class which comes from the class of point in $H_0(S^2, \mathbb{Z})$, and it is easy to see that the class $\mu_L([pt])$ has degree 1 and we can't compute it on $[\mathcal{M}_{spBS}]$, getting some new symplectic invariants. It doesn't look suprizely since for S^2 there is unique symplectic invariant — due to the celebrated Moser theorem — and it is the symplectic volume of S^2.

Elliptic curve. This case is not simply connected but one could modify the definitions adding the choice of an admissible hermitian connection on the prequantization bundle L. It's not hard to see that one instead can choose a oriented cycle $S \subset E$, representing any nontrivial fixed class $[S] \in H_1(E, \mathbb{Z})$, and declair that it is Bohr - Sommerfeld. Then each other oriented cycle from the same homology class is Bohr - Sommerfeld if and only if the symplectic volume of the film connecting them equals to zero (or more generally if we change the prequantization level from 1 to $k \in \mathbb{N}$, then the symplectic volume must be $\frac{l}{k} \int_E \omega, l \in \mathbb{Z}$). F.e. for level 1 every two Bohr - Sommerfeld must intersect each other. On the other hand, the space of translations naturally splits (so E itself does) into two parts: the first one contains the translations which preserves the Bohr - Sommerfeld condition while the second part is transversal to the first one. The special Bohr - Sommerfeld cycle is defined then as a Bohr - Sommerfeld cycle invariant with respect to admissible translations. It is clear that there is only one such a cycle.

Hirzebruch surfaces. A complex quadric with the standard Kaehler form from the symplectic point of view is just the product of two projective lines endowed with the standard symplectic forms (and the standard Fubini - Study metrics)

$$Q = \mathbb{CP}^1_+ \times \mathbb{CP}^1_-,$$

so the lagrangian geometry for this case is completely described by the product structure together with the known facts about the projective line. Again one could represents by lagrangian submanifolds the trivial middle homology class only due to the numerical properties of $H_2(Q, \mathbb{Z})$; it is not hard to see that $S \subset Q$ is lagrangian if and only if both the projections $pr_{\pm} S$ to the factors $\mathbb{P}^1_{\pm}$ are smooth one dimensional and any such S is Bohr - Sommerfeld if and only if both the projections $pr_{\pm} S$ are Bohr - Sommerfeld in $\mathbb{P}^1_{\pm}$. The subset $\mathcal{M}_{spBS} \subset \mathcal{B}_S$ is exactly the product of two 2- dimensional spaces coming from the factors $\mathbb{P}^1_{\pm}$. So the story is essentially the same as for the projective line. For another topological type of the Hirzebruch surfaces (note that symplectically there are only two types of the surfaces — quadric and complex projective plane with one blown - up point) we have the same results.

Algebraic K3 - surface. This is the first case when there are non trivial middle homology classes which can be realized by lagrangian submanifolds. The point is (see f.e. [7], [14]) that if $\omega_I, \omega_J, \omega_K$ are the standard symplectic forms, compatible with a fixed Ricci flat metric g on our K3 - surface X then the holomorphic with respect to complex structure I riemann surfaces are special lagrangian with respect to (ω_J, g) and (ω_K, g). This gives the dimension of the moduli space of special lagrangian cycles representing a given class from $\operatorname{Pic} X$. But we are interested in the moduli space of Bohr - Sommerfeld lagrangian cycles and its subset $\mathcal{M}_{spBS}$. Leaving aside the concrete description of $\mathcal{B}_S$ just mention that if the genus of a Lagrangian submanifold equals to zero then it is automatically Bohr - Sommerfeld. If such a submanifold Σ represents a fixed class $D \in PicX$ then the moduli space has the same dimension as $H^1(\Sigma, \mathbb{Z})$

$$g(\Sigma) = \frac{1}{2}D^2 + 1 = h^0(D) - 1,$$

and consequently the virtual dimension of $\mathcal{M}_{spBS}$ is equal to zero. F.e. it is true for an elliptic pencil since as it was pointed out in [17] each lagrangian fiberation with compact fibers can contain just discrete set of Bohr - Sommerfeld lagrangian fibers.

Therefore the known examples do not ensure that the generalized μ-classes are nontrivial.

4 Floer bundles

The extended mirror symmetry implies that one could deal with lagrangian submanifolds as with vector bundles (since they are dual with respect to the symmetry). But the vector bundles are much more familiar to mathematicians to work with so any representation of a lagrangian submanifold (or a class of lagrangian submanifolds) by a vector bundle or a sheaf would be quite useful in the mirror story. This section is devoted to a such representation.

Let (M, ω) is as above a simply connected symplectic manifold with integer symplectic form ω. Let $\mathcal{B}_S$ is the moduli space of Bohr - Sommerfeld lagrangian cycles of fixed topological type $[S] \in H_n(M, \mathbb{Z})$. Consider any other oriented lagrangian submanifold $S_1 \subset M$. Then for each Bohr - Sommerfeld lagrangian submanifold $S \in \mathcal{B}_S$ we have the Floer cohomology $FCH(S, S_1, \mathbb{C})$ defined as follows (see [5], [12]). Let $p_1, ..., p_m$ be the set of the intersection points (perhaps after a small hamiltonian perturbation of S_1). Let us fix a generic almost complex structure I, compatible with ω. Then for each pair $p_i, p_j \in S \cap S_1$ one can take the space of holomorphic films, compatible with the orientations of S and S_1 and linking p_i and p_j (due to the orientations this operation is not symmetric). Then it gives the following operator

$$\delta : \mathbb{C}_{p_i} \to \mathbb{C}_{p_j}$$

which is trivial if the space of the films has positive or nagative dimension and is multiplication by d if it is zero dimensional where d is the number of the films. Totally it is generalized to operator

$$\delta : \sum_1^m \mathbb{C}_{p_i} \to \sum_1^m \mathbb{C}_{p_i},$$

which possesses $\delta^2 = 0$ and the cohomology of the complex induced by the operator δ is called the Floer cohomology of S and S_1

$$FCH(S, S_1, \mathbb{C}) = \ker \delta / \operatorname{Im} \delta.$$

The main property of the cohomology space is that it is stable with respect to the hamiltonian deformations and is equivariant with respect to symplectomorphisms. In particular (as it was pointed out by Floer and proved by Oh), if S_1 is given by a hamiltonian deformation of S then $FCH(S, S_1, \mathbb{C}) = H^*_{dR}(S, \mathbb{C})$ (if some topological assumption holds, namely $\pi_2(M, S, \mathbb{Z})$ is trivial).

Globalizing the definition over whole $\mathcal{B}_S$, one gets an object:

$$\begin{array}{ccc} FCH(S, S_1, \mathbb{C}) & & \mathcal{F}_{S_1} \\ \downarrow & & \downarrow \\ S & \in & \mathcal{B}_S \end{array}$$

which looks like a sheaf or a $\mathbb{C}$ - bundle over $\mathcal{B}_S$. The stability of the Floer cohomology with respect to hamiltonian deformations immediately leads to the following

Proposition 2 *Totally $\mathcal{F}_{S_1}$ is a $\mathbb{C}$ - bundle over $\mathcal{B}_S$.*

Indeed, one has a distinguished trivialization for any choice of local coordinates (charts) since these local coordinates are given by hamiltonian deformations of point $S \in \mathcal{B}_S$.

On the other hand it is clear that this bundle doesn't change under hamiltonian deformations of S_1. This means that we can vary S_1 along the moduli space $\mathcal{B}_{S_1}$ getting the same bundle. But there is an object which depends exactly on S_1.

Proposition 3 *This bundle $\mathcal{F}_{S_1}$ carries a natural singular connection A_{S_1}, which depends on S_1.*

This connection is given by the following arguments. If S (as it was in the definition) has only transversal intersections with S_1 then there exists a small neighboorhod of S in $\mathcal{B}_S$ which contains the Bohr - Sommerfeld cycles with the same property. Then for each point $p_i \in S \cap S_1$ belongs to $\ker \delta / \operatorname{Im} \delta$ its graph with respect to any hamiltonian deformation gives a lifting of the corresponding tangent vector on the base. Thus for a neighborhood where the intersections are transversal it gives a natural lifting of paths from the base which is a connection. The singularities come there if the intersection turns to be nontransversal — but the type of a singular point is reflected by the corresponding singularity of the intersection with S_1. The singular set $\operatorname{Sing} A_{S_1} \subset \mathcal{B}_S$ is naturally stratified by the level of the intersection degeneration. Thus the geometry of this singular canonical connection is highly non trivial.

We call the bundle $\mathcal{F}_{S_1}$ the Floer bundle since its definition is inspired by the Floer construction. Let us repeat again that

the bundle $\mathcal{F}_{S_1}$ itself depends only on the class of Hamiltionian deformations of S_1,

the canonical connection A_{S_1} depends on S_1 itself.

Thus the changing of S_1 inside of the fixed class of hamiltonian deformations just leads to the perturbation of this connection A_{S_1} inside of a fixed class. Turning to S_1' which is slightly deformed S_1 in the same class of hamiltonian deformations gives one another connection $A_{S_1'}$ on the same bundle $\mathcal{F}_{S_1}$. This new connection $A_{S_1'}$ is again a singular one, but its singular set is different from $\operatorname{Sing} A_{S_1}$.

As an example one could take the case when S_1 belongs to $\mathcal{B}_S$. Then the fiber of the Floer bundle is given by $H^*(S, \mathbb{C})$ and the singular set $\operatorname{Sing} A_{S_1}$ has a "center" — the point S_1 itself, which belong to each stratum, and the geometry of this singular set is dictated by the singularity theory of lagrangian submanifolds (see f.e. [22]).

Another example of a related object induced over a moduli space of Bohr - Sommerfeld cycles by some vector bundles over M is given as follows.

Suppose that (M, ω) admits an integrable complex structure I, compatible with ω. Then (M, ω, I) becomes an algebraic manifold and one can consider the moduli space of stable vector bundles over it with respect to the principle polarization given by the Kaehler class $[\omega]$. Then for each stable bundle with the first Chern class proportional to $[\omega]$ one could take the corresponding Kobayshi - Hitchin connection a whose curvature form is just proportional to our symplectic form. Then the restriction of (E, a) to any lagrangian cycle should be a $SU(n)$ - bundle equipped with a flat connection. Then over each $S \in \mathcal{B}_S$ one has the corresponding deformed de Rahm complex:

$$\Omega^0_S(E) \xrightarrow{d_a} \Omega^1_S(E) \xrightarrow{d_a} ...\Omega^n_S(E). \qquad 1$$

The flatness condition ensures that it is a complex indeed so one could take the cohomology spaces. Every such a space can be globalized over whole $\mathcal{B}_S$ giving us a sheaf $\mathcal{F}_E$. But a priori it is really a sheaf since for a fixed index i the rank of the corresponding group $H^i_a(S, E)$ can jump when one deforms the starting point $S \in \mathcal{B}_S$ in a neighborhood. At the same time the total space $H^*_a(S, E)$ should give a bundle again and we call it the Floer bundle and denote it as $\mathcal{F}_E$ taking in mind that it is somehow (may be just formally) related to the Floer homology groups. Indeed (see [3]), it's a known way to define some special homology group for 3 - dimensional real manifolds via some external objects (which are self dual connections on smooth four dimensional manifold with ends, given by our 3 - dimensional manifold). If S has dimension 3 then there exists just a finite number of gauge classes of flat $SU(2)$ - connections and for this finite number of points $p_1, ..., p_m$ it could be defined an operator δ, quite analogous to the operator form the definition of the Floer cohomology. But there it was defined by some internal objects of our symplectic manifold, the holomorphic films. In the case of the Floer homology group this analogous operator δ is defined by the moduli space of anti self dual connections on $S \times \mathbb{R}$. In our case the space $H^*_a(S, E)$ is defined by an external object as well — some stable holomorphic bundle and we know how close are the notions of stable holomorphity and anti self duality in dimension 4. Suppose that S admits just a finite number of gauge equivalence classes of flat connections. Then the loop space of $\mathcal{B}_S$ should generate the Floer homology group of S. Moreover, there is a natural connection on $\mathcal{F}_E$ given by the following arguments. In presence of globally defined on M Kobayashi - Hitchin connection a we can compute how the complex (1) is deformed with respect to any hamiltonian vector field on the base. This means that for any path on $\mathcal{B}_S$ there exists the corresponding lifting to the fibers of $\mathcal{F}_E$. Of course, this lifting depends on the connection a itself so if we change it in the same equivalence class the induced lifting should change as well. This lifting is our induced connection on $\mathcal{F}_E$.

Notice that the descriptions of two vector bundles

$$\begin{array}{ccc} \mathcal{F}_S & & \mathcal{F}_E \\ & \searrow \quad \swarrow & \\ & \mathcal{B}_S & \end{array}$$

look quite similar and it is a reasonable idea to compare these two kinds of bundles. But the problem is that while $\mathcal{F}_S$ ia a complex bundle $\mathcal{F}_E$ has another structure group namely adG where G is the structure group of E. Anyway one could say that a vector bundle E over M is equivalent to a lagrangian cycle S if there exists the moduli space $\mathcal{B}_{S_0}$ over M such that

$$\mathcal{F}_S \equiv \mathcal{F}_E.$$

Here we can place just the simplest example of the correspondent objects, namely one has

Proposition 4 *The prequantization line bundle L is equivalent over the moduli space $\mathcal{B}_S$ to $S \in \mathcal{B}_S$ itself.*

The proof is obvious: by the definition the restriction of (L, a) to every $S \in \mathcal{B}_S$ is gauge equivalent to the pair $(C^\infty(S, \mathbb{C}), d)$ where d is the ordinary differential. In this case the complex (1) is exactly the de Rahm complex, and the fiber of $\mathcal{F}_L$ is given by $H^*_{dR}(S, \mathbb{C})$. On the other hand, according to the basic result of Floer, which we have mentioned above, the fiber of $\mathcal{F}_S$ over $\mathcal{B}_S$ is the same and it follows immediately that the bundles $\mathcal{F}_L$ and $\mathcal{F}_S$ over $\mathcal{B}_S$ are isomorphic.

At the same time the problem of comparison of connections over bundles $\mathcal{F}_L$ and $\mathcal{F}_S$ is much more complicated. But at the same time it is much more interesting since if we are lucky we can get in this way some correspondence between Bohr - Sommerfeld cycles from $\mathcal{B}_S$ and *connections* in a fixed gauge class.

Indeed, as we have seen every $S \in \mathcal{B}_S$ and every $a \in [a] \in \mathcal{A}(L)/\mathcal{G}$ define some connections A_S and A_a on the same bundle $\mathcal{F}_S = \mathcal{F}_L$. Then we would get a map

$$\mathcal{B}_S \to \mathcal{A}(L)$$

thus we could find a correspondence between Bohr - Sommerfeld cycles and connections on the prequantization bundle. Of course, it is just a rough idea, but such a correspondence is highly desired.

Let us mention another way to construct the bundle $\mathcal{F}_E$. To do this one could exploit the universal object

$$\mathcal{U} \subset M \times \mathcal{B}_S$$

in the following form: consider the diagram:

$$\begin{array}{ccc} & \mathcal{U} & \\ \swarrow & & \searrow \\ M & & \mathcal{B}_S \end{array}$$

with two natural projections $p : \mathcal{U} \to M$ and $q : \mathcal{U} \to \mathcal{B}_S$. Then any bundle (E, a) endowed with a connection can be lifted to $\mathcal{U}$ and then if the connection is flat along the fibers of q can be pushed down producing a set of sheaves $R^i q_* p^*(E, a)$. The direct sum of these sheaves gives us the bundle $\mathcal{F}_E$ above.

5 Special and Bohr - Sommerfeld lagrangian cycles on elliptic curve

Let Σ is a riemann surface of genus 1 so topologically it is torus T^2 endowed with a flat Kahler structure with integer Kahler form ω. Thus one has the prequantization bundle L with the first Chern class $c_1(L) = [\omega] \in H^2(\Sigma, \mathbb{Z})$. The first question we have to answer is the following: what is the meaning in this classical set up of the choice of a prequantization connection a such that $F_a = 2\pi i \omega$? In the simply connected case this connection is given automatically but in our case we have a real affine plane Π of noneqivalent solutions to the equation $F_a = 2\pi i \omega$. This affine space is associated to the vector space $H^1(\Sigma, \mathbb{R})$. The first fact is contained in the following

Proposition 5 *The choice of a prequantization connection is equivalent to the choice of a fixed point $o \in \Sigma$.*

This means that in this prequantization setup one needs not just a complex 1- dimensional Kahler manifold but an elliptic curve which is in addition an abelian group. The arguments are extremely simple — if we choose a prequantization connection a_0 then we get a holomorphic structure on the line bundle L, so one gets the bundle $L(o)$ for a point o. It's easy to see that the complete correspondence is given as a factorization

$$\Pi \to \Sigma$$

with respect to periods of Σ. On the other hand we can this correspondence can be explained via the fact that the moduli space of stable holomorphic bundles is isomorphic to Σ.

Thus one can see that imposing this prequantization setup we get an elliptic curve indeed.

Let us fix a prequantization connection $a \in \mathcal{A}_h(L)$ and some homology class $[S] \in H_1(\Sigma, \mathbb{Z}), [S] \neq 0$. Let S_0 be a Bohr - Sommerfeld lagrangian oriented cycle so $S_0 \in \mathcal{B}_S^a$, where symbol a in the notation reflects the dependence on the choice of a. Then we have the following simple proposition.

Proposition 6 *An oriented cycle S representing the same homology class $[S]$ belongs to the same moduli space $\mathcal{B}_S^a$ if and only if the volume of the area, restricted by oriented cycles S_0 and S_1, is integer.*

The proof is the same as in Toy Example from [6]: the characters for the restrictions of (L, a) on S_0 and S differ by exponent of $2\pi i Vol_\omega N$ where N is the area. Notice that there are exactly two choices of this area, bounded by the oriented cycles, but since the total volume of Σ is integer it doesn't matter which one is taken.

Our Kahler manifold Σ admits symmetries which are given by the straight lines on the universal covering of Σ. The symmetries are defined, f.e., by the corresponding nondegenerated hamiltonian vector fields, which are parametrized by $H^1(\Sigma, \mathbb{R})$ in the usual way. Then

Proposition 7 *There exists unique direction in $H^1(\Sigma, \mathbb{R})$, such that the corresponding vector field preserves the Bohr - Sommerfeld condition.*

This means that $\mathcal{B}_S^a$ can be equipped with a circle action, but this action is not smooth. To ensure let us recall (see [14]) that in the case of elliptic curves special lagrangian cycles are given by geodesic cycles so they correspond to straight lines on the universal covering. This means that if one fixes a homology class $[S] \in H_1(\Sigma, \mathbb{Z})$ then the moduli space $\mathcal{M}_{spLag}$ of special lagrangian cycles is isomorphic to S^1. On the other hand

Proposition 8 *The set $\mathcal{B}_S \cap \mathcal{M}_{spLag}$ consists of exactly one point. For any prequantization level k this set consists of k points.*

Ideed, according to the description of special Lagrangian cycles and Proposition 6, we get the statement. Now we can come back to the circle action. It is not hard to see that this circle action preserves exactly the special lagrangian cycles, which belong to $\mathcal{B}_S^a$. From this we see that

Proposition 9 *The circle action on $\mathcal{B}_S^a$, induced by the translations described above, has unique fixed point, which is given by the special lagrangian cycle.*

In the case of level k the distunguished set of isolated points $S_1, ..., S_k \in \mathcal{B}_S^a$ are the special Bohr - Sommerfeld lagrangian cycles. Thus, one could say, that "euler characteristic" of $\mathcal{B}_S^a$ is equal to some derivation of the set of these special Bohr - Sommerfeld cycles. Since the moduli space is orientable then this characteristic equals to the number of special Lagrangian cycles, counted with signs.

On the other hand, the distinguished direction in $H^1(\Sigma, \mathbb{R})$ is given by the cohomology class, dual to $[S] \in H_1(\Sigma, \mathbb{Z})$. To ensure one could consider the corresponding Hodge star operator, harmonic forms etc. At the same time the translation arguments show that

$$\mathcal{B}_S^a = \mathcal{B}_S^{a+\alpha\rho+k\tau},$$

where $\alpha \in \mathbb{R}, k \in \mathbb{Z}$ and ρ, τ correspond to the translations which preserve this a - Bohr - Sommerfeld condition and the orthogonal to it respectively. Thus it remains just a circle

$$H^1(\Sigma, \mathbb{R}) \to S_a^1,$$

which parametrizes different moduli spaces of Bohr - Sommerfeld lagrangian cycles.

Moreover, the translation arguments show that

Proposition 10 *For any $[a_0], [a_1] \in S_a^1$ the moduli spaces $\mathcal{B}_S^{a_0}$ and $\mathcal{B}_S^{a_1}$ are isomorphic. The isomorphism is given by some appropriate translation.*

Now let us discuss the construction from [14] attaching to any stable vector bundle on a given elliptic curve Σ some special lagrangian submanifold on the mirror curve Σ'. Recall that this mirror curve is constructed in this case following the SYZ - strategy (see [13]). Since one can fix a decomposition of

Σ into the direct product of groups $U(1)^+ \times U(1)^-$ there is the corresponding fibration

$$\Sigma \to S^1. \tag{2}$$

Then the mirror curve is given as dual fibration over the same base:

$$\Sigma' \to S^1, \tag{2'}$$

where the fibers are dual 1-dimensional torus for the fibers of (2) above. Then, f.e., if one has a holomorphic line bundle over Σ then choosing the corresponding connection a (or, more rigorius, a class of connections) and then restricting it to each fiber in (2), one should get a representation of its fundamental group so an element of the dual torus. The point is that the image of this correspondence is collected to a lagrangian cycle in Σ' and as it was shown in [14] this cycle is special lagrangian.

This picture with the mirror curves contains a lot of hidden degrees of freedom. Let us mention that real dimension 2 possesses the following specification:

— *every* 1- dimensional submanifold is lagrangian;

— *every* hermitian connection, restricted to a fiber, is flat.

Indeed, in the correspondence

$$\text{line bundles} \leftrightarrow \text{lagrangian cycles}$$

represented above, one can take *any* hermitian connection b on L over Σ, and it should be flat on the fibers thus it gives a lagrangian submanifold in Σ'. The point is that this resulting submanifold is not special lagrangian in general. Let us take two connection a (which is our prequantization connection) and b (choosen arbitrary) on the prequantization bundle L and compare the images under the procedure, namely lagrangian cycles S_a and S_b in Σ'. It's clear that these cycles belong to the same homology class. On the other hand

Proposition 11 *Two lagrangian cycles S_a and S_b belongs to the same moduli space of Bohr - Sommerfeld cycles on Σ' if and only if*

$$V_\Sigma(\frac{\delta}{2\pi i}) \in \mathbb{Z}. \tag{3}$$

Here $\delta = a - b$ is a pure imaginary 1 - form, defined as the difference between two hermitian connections, and V_Σ is a map

$$V_\Sigma : \Omega^1_\Sigma \to \mathbb{R},$$

defined as follows. Every real 1 - form $\rho \in \Omega^1_\Sigma$ can be intergated over the fibers of (2) which gives us a function on the base S^1, and this function can be integrated as well over S^1 with respect to the given volume form (all the data are given by the fixed Kahler structure). Thus one gets a real number and it is the value of $V_\Sigma(\rho)$.

The description of V_Σ given above is not quite elegant but it is suitable to explain the statement of the proposition. Indeed, the integration along the fibers gives us the difference in the characters which can be regarded as the length of the arc in the corresponding fiber of (2') ended at points of S_a and S_b. Then the integration of the symplecitc form ω' on Σ' over the film between S_a and S_b is given by the integration of the arc lengthes over the base. If we change the prequantization level on Σ' then one should correct (3) as follows

$$V_\Sigma(\frac{\delta}{2\pi i}) \cdot k \in \mathbb{Z}. \qquad 3'$$

Now the expression in (3) is not quite clear so let us correct it in more invariant way. It's not hard to see that

$$V_\Sigma(\rho) = \int_\Sigma \rho \wedge \alpha,$$

where α is the harmonic form which represents the cohomology class $P.D.[S] \in H^1(\Sigma, \mathbb{Z}) \subset H^1(\Sigma, \mathbb{R})$. Here $[S]$ is the homology class of the fibers in (2). The last one can be rearranged accoeding to the Hodge theory:

$$\int_\Sigma \rho \wedge \alpha = \int_\Sigma < \rho, *\alpha > \omega = \int_\Sigma < \rho_H, *\alpha > \omega,$$

where

$$\rho = d\rho_1 + \rho_H + d^* \rho_2$$

— is the orthogonal Hodge decomposition of ρ into three parts: exact, harmonic and co- exact. Thus we see that S_a and S_b belong to the same Bohr - Sommerfeld class if and only if the harmonic projection of the difference form ρ is equal to $r\alpha + m * \alpha$, where $r \in \mathbb{R}, m \in \mathbb{Z}$. On the other hand we have that the resulting images S_b depends only on the gauge class of b (which is not surprising) and we could consider the space $\mathcal{A}(L)/\mathcal{G} = d\Omega^1_\Sigma$ since the gauge class of hermitian connections in this case is completely defined by their curvature form. At the same time the admissible harmonic part of ρ

doesn't change the image so we can understand $d\Omega^1_\Sigma$ as the parameter space for the images $S_b \subset \Sigma'$.

But the problem is that the images S_b for any $b \in \mathcal{A}(L)$ form just a special subset in the space of all lagrangian submanifolds in Σ'. This subset is distinguished by the condition that every S_b projects properly on the base S^1 in (2'). Thus imposing the condition (3) we can cover just the Bohr - Sommerfeld lagrangian cycles with *trivial Maslov class* (see [9], [1]) with respect to the real polarization given by (2'). The idea coming in this way says that we could proceed as it was done in [9], constructing first some Bohr - Sommerfeld submanifolds which project properly on S^1 and then deform these ones using some global hamiltonian vector fields on Σ'.

So the strategy is:

1. We can take some connection $b \in \mathcal{A}(L)$. It is defined by some 1- form ρ and we take such b that the harmonic part of ρ is trivial. Then b is defined uniquelly by two exact froms: $d\rho_1, dd^*\rho_2$.

2. For $dd^*\rho_2 \in d\Omega^1_\Sigma$ we have the corresponding image $S_b \subset \Sigma'$.

3. Since ρ_1 is a function we could take its hamiltonian vector field on Σ and produce the corresponding vector field on Σ'. It should be hamiltonian as well and we can act by the corresponding symplectomorphism $\phi_t, t = 1$ to S_b. Totally it gives some Bohr - Sommerfeld submanifold S'_b.

There remain two problem:

Problem 1. Can we produce in this way *any* Bohr - Sommerfeld lagrangian cycle in Σ'?

Problem 2. What is the fiber of this correspondence so can we completely describe the (not gauge) class of hermitian connections in $\mathcal{A}(L)$ which give us the same Bohr - Sommerfeld cycle?

We hope to solve the problems in the nearest future. Here we just mention that the correspondence, given by the strategy steps above is suitable to work with Floer bundles and to compare the induced connections on the Floer bundles described in the previous section.

6 Geometric interpretation of the Maslov class

The classical definition of the Maslov class (see [1]) supposes that we are dealing with a flat symplectic space hence the lagrangian grasmannization is given by the direct product of the standard grasmannian $Gr\mathbb{R}^{2n}$ and the based space (say, $\mathbb{R}^{2n}$). Then (see [1]) for any lagrangian submanifold $S \subset$

$\mathbb{R}^{2n}$ one can get a map

$$i : S \to S^1,$$

combining the Gauss map and the determinant map. Then the Maslov class is given by the lifting of the fundamental class of S^1 (or of the dual class). This classical definition can be naturally extended to the case of non-flat based space. It could be done for any Fano variety since in this case the canonical class is poporptional to the class of the symplectic (Kahler) form. This generalization makes it possible to extend the notion of special lagrangian submanifold to the case when

$$K_M = k[\omega],$$

and then the Maslov class is presented as an obstruction for the existence of the minimal (special) lagrangian submanifolds in a contineous family of lagrangian submanifolds (f.e. for any connected component of the moduli space of Bohr - Sommerfeld lagrangian cycles). Moreover, as we will show the Maslov class can be defined in absolutely general situation.

To start with let us remark the following geometrical fact.

Proposition 12 *The intersection of any lagrangian submanifold with any symplectic divisor is always nontransversal. It is smooth and has dimension* $n - 1$.

It's clear that under this "symplectic divisor" one means a smooth symplectic submanifold of real codimension 2. Therefore in the Kahler case a divisor $D \subset M_I$ is a symplectic divisor.

The reason is obvious: in the transversal case the intersection should have dimension

$$\dim_{\mathbb{R}} S \cap D = n - 2,$$

if $\dim_{\mathbb{R}} M = 2n, \dim_{\mathbb{R}} D = 2n - 2, \dim_{\mathbb{R}} S = n$ (we don't suppose that the intersection is topologically nontrivial — just adding suggestion "if the intersection is not empty"). But the real dimension is bigger than in the transversal case since D is simplectic and S is lagrangian, namely it is $n - 1$. Indeed, at a point $p \in S \cap D$ one can choose two vectors $v_1, v_2 \in T_pM$ such that $\omega_p(v_1, v_2) \neq 0, v_i \notin T_pD$ (because of the fact that T_pD is a symplectic subspace of T_pM). In the transversal case these vectors could be choosen in T_pL, but it is impossible due to the lagrangian condition. On the other hand, at each point $p \in S \cap D$ the dimension of $T_p(S \cap D) \subset T_pM$ can not be bigger than $n-1$. Indeed, the intersection $S \cap D$ should be isotropic inside of $(D, \omega|_D)$, and the maximal dimension of an isotropical submanifold is $n-1$. Therefore the intersection has the same dimension at each point, equals to $n-1$, and it follows that $S \cap D$ is smooth.

Corollary 1 *The intersection* $(S \cap D)$ *is always a smooth lagrangian submanifold of* $(D, \omega|_D)$.

Following this "non- transversality" argument one could deduce the existence of a topological characterization of this too large intersection property. Namely suppose that some line bundle L over our symplectic manifold (M, ω) admits a section whose zeros form a symplectic divisor $D \in M$. Then for any oriented lagrangian submanifold $S \subset M$ one defines the following class $m \in H^1(M, \mathbb{Z})$:

$$m = P.D.[D \cap S].$$

Proposition 13 *In the setup this class is correctly defined.*

Indeed, both S, D are oriented so the smooth intersection $S \cap D$ is oriented too in S. Thus the Poincare dual class of $[(S \cap D)] \in H_{n-1}(L, \mathbb{Z})$ is correctly defined.

Example. Consider the direct product of two elliptic curves $\Sigma_1 \times \Sigma_2$ with Kahler structures ω_1, ω_2. We can take a decomposition for each Σ_i so

$$\Sigma_i = S_i^+ \times S_i^-.$$

Then the fundamental group of $Y = \Sigma_1 \times \Sigma_2$ is presented as the product of these 4 circles, and the cohomology group of Y as a 4- dimensional torus can be expressed in terms of the circles. Then let us take a holomorphic line bundle with first Chern class equals to ω_1. Then it has unique up to scaling holomorphic section s with the zero set, given by $p \times \Sigma_2 \subset Y$. This smooth 2 - dimensional submanifold is a symplectic divisor and it is our D. On the other hand if we take

$$S = p_1 \times S_1^+ \times S_2^+ \times p_2 \subset Y$$

where $p_1 \in S_1^-$ is arbitrar point and $p_2 \in S_2^-$ corresponds to $p \in \Sigma_2$, then it's clear that S is lagrangian. At the same time we see that the intersection

$$S \cap D = p_1 \times S_1^+ \times p \subset S$$

has dimension 1 and represents a nontrivial homology class in 2 - torus S. The Poincare dual class is nontrivial as well.

On this example we can see an interesting effect: it's a possible situation when the topological intersection of S and D is trivial while the defined lagrangian class is non trivial.

Suppose now that we have an algebraic variety M_I, defined over an integer symplectic manifold (M, ω) such that ω is the Kahler form. Suppose additionally that either $K_M > 0$ or $K_M < 0$. It means that either the canonical or anticanonical class admits holomorphic sections. Then for a section $s \in H^0(M_I, K_M^{\pm})$ the zero set $(s)_0 = D$ is a holomorphic divisor and consequently a symplectic divisor so we can proceed in the way represented above. Namely for any lagrangian submanifold $S \subset M_I$ we have the class

$$m = \pm P.D.[(D \cap S)] \in H^1(S, \mathbb{Z}).$$

where the choice of the sign is dictated by the choice of the sign in $K_M^{\pm}$ to get holomorphic sections.

Proposition 14 *This class $m \in H^1(S, \mathbb{Z})$ doesn't depend on the choice of holomorphic structure I and on the choice of holomorphic section $s \in H^0(M_I, K_M^{\pm})$.*

1 The class $m \in H^1(S, \mathbb{Z})$ is called the universal Maslov class of lagrangian submanifold S.

The definition given above is algebro - geometrical. We can use the main idea of the construction above to proceed in the case of any symplectic manifold. For this generalization we need just the following:

Proposition 15 *The definition can be extended to the case when a given symplectic manifold (M, ω) admits symplectic divisors, which represent either canonical or anticanonical bundle.*

Indeed, what we need is just a symplectic realization of the canonical or anticanonical bundle so a symplectic divisor whose homology class is Poincare dual to either K_M or K_M^{-1}. Then for any lagrangian submanifold S we can apply the procedure above getting the universal Maslov class via the intersection. It's not too hard to establish that the class doesn't depend on the realization by symplectic divisors.

The description of the universal Maslov class implies that

Theorem 1 *The universal Maslov class is invariant with respect to the group of symplectomorphisms of (M, ω). In particular it is stable with respect to hamiltonian deformations.*

At the same time this class can be defined for any line bundle over our symplectic manifold (M, ω) as it was mentioned above. The problem is to realize the corresponding first Chern class by a symplectic submanifold. In general symplectic setup there is just unique improved case when we know that such a realization exists. Namely let (as usual in this paper) ω is an integer symplectic form and the prequantization bundle L exists. Then as it was proved by S. Donaldson there is an integer number $k \in \mathbb{Z}_+$ such that L^k admits a section whose zeros form a symplectic divisor. In this case for any lagrangian submanifold $S \subset M$ we can compute the class m for L^k and then define

$$m = \frac{1}{k} P.D.[(D \cap S)] \in H^1(S, \mathbb{Q}).$$

A conjecture in this case states that m belongs to $H^1(S, \mathbb{Z})$.

Why we understand these classes as the universal Maslov classes? The point is that one understands the Maslov index as a correction term in some approximation of a solution to the Schroedinger equation (see [9]). On the other hand the Maslov class is invariant of lagrangian submanifolds with respect to hamiltonian action which preserves some additional structure and is an obstruction for the lagrangian submanifold to be deformed to a minimal one under this hamiltonian action.

Indeed, in [1] one introduces this Maslov class using the lagrangian grassmanization of (M, ω) but to proceed with one needs some special identification for the fibers of the grassmanization over whole M which can be non flat. In order to get such an identification one can use some real polarization and then the identification takes place. But the price one must pay doing this is that any invariant, getting in this way, is not invariant under the whole group of symplectomorphisms. One should restrict himself to the hamiltonian vector fields (and hence the functions) which preserve this real polarization.

F.e. in the main case from [9] one deals with the cotangent bundle of a real manifold S. Then the corresponding polarization which is implicitly used is just the canonical fiberation

$$T^*S \to S,$$

and the freedom is just to apply hamiltonian trasnformations which preserve the fibers. Thus the Maslov index (or Maslov class more generally, see [1]) is invariant under this "translations along the fibers". But if this class is nontrivial for a lagrangian submanifold $S' \subset T^*S$ from the homology class of the base then this S' couldn't be transported using this "translations along the fibers" to the minimal one (so to S itself).

We would like to show that the universal Maslov class defined here is an obstruction as well. But in our case it is invariant under any hamiltonian deformation. The reason is that any symplectic divisor remains to be a symplectic divisor under any hamiltonian deformation.

Let (M_I, ω) is an algebraic variety such that $K_M = k[\omega]$, viewed as a symplectic manifold M with an integer symplectic form ω, which admits a compatible integrable complex structure I. Then it is defined the corresponding riemannian metric g such that the triple (g, I, ω) is a Kahler triple. It's well known that this Kahler structure defines

1. a canonical hermitian structure on the canonical line bundle K_M;

2. a canonical hermitian connection a_{LC} on K_M, which is called the Levi - Civita connection such that its curvature form F_{a_LC} is equal to $2\pi i \rho$ where ρ is the Ricci tensor of the Kahler metric. Due to the equality

$$K_M = k[\omega] \tag{4}$$

it follows that $\rho = k\omega$. Ona can take the associated $U(1)$ - bundle $S^1(K_M)$ and then the corresponding connection on it can be as well denoted as a_{LC}.

Consider now a lagrangian submanifold $S \subset M_I$ and restrict the pair $(S^1(K_M), a_{LC})$ to it. According to (4), the restriction $a_{LC}|_S$ should be flat so it defines a character:

$$\chi : \pi_1(S) \to U(1).$$

If this character is trivial, then S is minimal. This character is not an integer data so it can change in a contineous family of lagrangian submanifolds. As a family we would take the space of Bohr - Sommerfeld cycles $\mathcal{B}_S$. Then for this family we have a universal class on the sheaf

$$\mathcal{H}^1 \to \mathcal{B}_S, \quad H^1(S, \mathbb{Z}) \mapsto S,$$

given by the universal Maslov class. Thus we can see that

Theorem 2 *Over any Fano or Calabi - Yau variety the moduli space $\mathcal{B}_S$ contains a minimal Bohr - Sommerfeld cycle only if the universal Maslov class is trivial.*

This statement should be completed by

Conjecture. The implication in the previous statement is "if and only if".

On the other hand we introduce this universal Maslov class in another perspective. Since we've mentioned that it plays the role of a correction in a quantization programme, one expects that this class should play the corresponding correction role in ALG(a) - quantization and ALAG - programme at all. At the same time what one needs to extend the Floer bundles to the weighted version of the moduli space of Bohr - Sommerfeld lagrangian cycles, which is $\mathcal{B}_S^{hw,r}$, is again to realize the geometrical ideas underlying to the definition of the universal Maslov class as a correspondence between holomorphic n forms on M_I and real n - forms on S.

At the end one can report that in both the ways work is in progress.

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