LOCAL DENSITIES OF 2-ADIC QUADRATIC FORMS

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Abstract. In this paper, we give an explicit from formula for the local density number of representing a two by two 2-integral matrix T by a quadratic 2-integral lattice L over $\mathbb{Z}_2$. The non-dyadic case was dealt in a previous paper. The special case when L is a (maximal) lattice in the space of trace zero elements in a quaternion algebra over $\mathbb{Q}_2$ yields a clean and interesting formula, which matches up perfectly with the non-dyadic case in terms of the Gross-Keating invariants. In this special case, The derivatives (when value is zero) matches up with certain local intersection numbers on Shimura curves.

1. Introduction and general set-up.

Local density of a quadratic form representing another quadratic form was first introduced by C. L. Siegel ([Si]) as the local building block of the Fourier coefficients of an Eisenstein series in 1935. Its connection with arithmetic geometry was discovered much later by Gross and Keating ([GK]), and by Kudla ([Ku]). It is very important to compute the local density and its derivative for arithmetic applications. We refer to [Ki], [HS], [Ka], [Ki], and [Ya] among others for introduction and reference on this subject. In fact, this paper is a complement to [Ya], where we considered the case $p \ne 2$.

In general, let (L, q) be a non-degenerate even integral quadratic lattice over $\mathbb{Z}_p$ of rank n, i,e., $q(x) \in \mathbb{Z}_p$ for every $x \in L$. Let T be a $m \times m$ symmetric matrix over $\mathbb{Z}_p$, we define the quadratic local density of L representing T as

$$W(T, L) = \int_{\mathrm{Sym}_m(\mathbb{Q}_p)} \int_{L^m} \psi(\ \mathrm{tr} b q(x)) dx \psi(-\ \mathrm{tr} T b) db. \tag{1.1}$$

Here the additive character $\psi : \mathbb{Q}_p \longrightarrow \mathbb{C}^1$ is given by $\psi(x) = e^{-2\pi i \lambda(x)}$ where $\lambda \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. In fact, the local density does not depend on the choice of

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ψ, we choose it for convenience. It depends of course on the choice of the Haar measures on L^m and $\mathrm{Sym}_m(\mathbb{Q}_p)$. For any integer $s \ge 0$, let $L_s = L \oplus H^s$ where H is the standard hyperplane $H = \mathbb{Z}_p^2$ with the quadratic form $q(x_1, x_2) = x_1 x_2$. We define the quadratic local density polynomial $W(X; T, L)$ via

$$W(p^{-s}; T, L) = W(T, L_s). \tag{1.2}$$

Let K be an open compact subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$ acting on $\mathrm{Sym}_m(\mathbb{Q}_p)$ via $k.\gamma = k\gamma{}^t k$. We define

$$A(\gamma, L) = \int_{L^m} \mathrm{tr}(\gamma q(x)) dx, \tag{1.3}$$

for $\gamma \in \mathrm{Sym}_m(\mathbb{Q}_p)$, and

$$B(T, \gamma) = \int_K \psi(-\,\mathrm{tr} k.T\gamma) dk. \tag{1.4}$$

Finally, for every $\gamma \in \mathrm{Sym}_m(\mathbb{Q}_p)$ with $\det \gamma \neq 0$, there is a number $C(\gamma)$ such that

$$C(\gamma) \int_K \phi(k.\gamma) dk = \int_{O_\gamma} \phi(x) dx \tag{1.5}$$

for every locally constant function ϕ on O_γ, where O_γ is the K-orbit of γ in $\mathrm{Sym}_m(\mathbb{Q}_p)$. Here we define the Haar measure dk on $\mathrm{GL}_m(\mathbb{Z}_p)$ as the restriction of the standard Haar measure on $M_m(\mathbb{Z}_p)$. In particular, $\mathrm{vol}(M_m(\mathbb{Z}_p)) = 1$ and

$$\mathrm{vol}(\mathrm{GL}_m(\mathbb{Z}_p)) = p^{-\frac{m(m+1)}{2}} [\mathrm{GL}_m(\mathbb{Z}_p) : K_0(p^r)], \tag{1.6}$$

where

$$K_0(p^r) = \{(g_{ij}) \in \mathrm{GL}_m(\mathbb{Z}_p) : g_{ij} \in 2^r \mathbb{Z}_p, \text{ for } i < j\}.$$

Let $\mathcal{A}$ be a complete set of representatives of the K-orbits in $\mathrm{Sym}_m(\mathbb{Q}_p) \cap \mathrm{GL}_m(\mathbb{Q}_p)$, then we have as in [Ya, Section 7], and [HS]

$$W(T, L) = \sum_{\gamma \in \mathcal{A}} A(\gamma, L) B(T, \gamma) C(\gamma). \tag{1.7}$$

As observed in [HS], the key is to choose K so that $\mathcal{A}$ is easy to describe and that $A(\gamma, L)$, $B(T, \gamma)$, and $C(\gamma)$ are computable. Another problem is how to put things together to get a reasonable formula. When $n = 1$ or $n = 2$ but $p \neq 2$, this was done by the author in [Ya]. Y. Hironaka and F. Sato extend the work to general n but $p \neq 2$. The case $p = 2$ was left out in [Ya] and [HS] mainly because that $\mathcal{A}$ is more complicated, though completely determined by Watson in the case $K = \mathrm{GL}_m(\mathbb{Z}_2)$ ([Wa]), and that putting things together might be too complicated.

In this paper, we deal with the case $n = p = 2$, and obtain explicit formulae for the local density polynomial $W(X, T, L)$ in sections 2-4, according to the shape of the Jordan form of T (see (1.16)). They are complicated, I have to admit, and we have to use a computer program written by Song Guo to compute examples (http://www.math.wisc.edu/∼thyang/programs). However, in the case where B is a quaternion algebra over $\mathbb{Q}_2$ with maximal order $\mathcal{O}_B$, and L is the lattice of trace zero elements in $\mathcal{O}_B$ with the reduced norm as its quadratic form, our formula for $W(X, T, L)$ turns out to be surprisingly clean and simple in terms of the Gross-Keating invariants and to be of independent of the shape of the Jordan form of T (Theorems 5.5). In fact, it matches up with non-dyadic case ([Ya]) perfectly too (Theorem 5.7). From these formulae, it becomes clear (Corollaries 5.8 and 5.9), as first observed by Kudla ([Ku]), that when $B = M_2(\mathbb{Q}_2)$ and $W(X, T, L) = 0$, $W'(1, T, L)$ is equal to the local intersection numbers computed by Gross and Keating ([GK, Proposition 5.4]). When B is a division algebra over $\mathbb{Q}_2$, a similar identity is expected to be true as Kudla and Rapoport finish the geometric calculation soon ([KR2]). The same is known to be true for $p \neq 2$ using the same method ([KR1]). Sections 6-9 are technical computations of the functions $A(\gamma, L)$, $B(T, \gamma)$, and $C(\gamma)$. In Appendix A, we study the structure of $U(1)$ in the 2-adic case for lack of literature reference,and give an explicit filtration of $U(1)$ needed for computing $C(\gamma)$. The ramified case was first computed by J. S. Li ([Li]). Appendix B is based on Kudla's note and is essentially due to him. Here we computed the Gross-Keating invariants in terms of Jordan form. Both appendices should be of independent interest. In Appendix C, we take this opportunity to correct some minor mistakes in [Ya] and reformulate some of the results there in a more unified way.

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Now we set up the general framework and notation for the paper. Let c and $r \geq -2$ be integers, we define the sets $\mathcal{A}_r(c)$ as follows. When $r \geq 3$,

$$\mathcal{A}_r(c) = \{diag(\beta_1 2^c, \beta_3 2^{c+r}) : \beta_1, \beta_2 \in (\mathbb{Z}/8)^*\}, \tag{1.8}$$

When $r=2$, we define $\mathcal{A}_2(c)$ the same way except that $\beta_1 \in (\mathbb{Z}/4)^*$, noticing that $\mathrm{diag}(1, 4\beta)$ is $\mathrm{GL}_2(\mathbb{Z}_2)$ equivalent to $\mathrm{diag}(1+4\beta, 4+\beta)$ in this case.

When $r=1$, define

$$\mathcal{A}_1(c) = \{2^c \,\mathrm{diag}(1, 2\beta), 2^c \,\mathrm{diag}(-3, 2\beta), \beta \in (\mathbb{Z}/8)^*\}. \tag{1.9}$$

Notice that $\mathrm{diag}(1, 2\beta)$ is $\mathrm{GL}_2(\mathbb{Z}_2)$ equivalent to $\mathrm{diag}(1+2\beta, 2\beta(1+2\beta))$.

When $r=0$, we define $\mathcal{A}_0(c) = \mathcal{A}_0^+(c) \cup \mathcal{A}_0^-(c)$ with

$$\mathcal{A}_0^+(c) = \{\pm 2^c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm 2^c \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\}, \tag{1.10}$$

and

$$\mathcal{A}_0^-(c) = \{2^c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 2^c \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\}. \tag{1.11}$$

Here we remark that $\mathrm{diag}(1, \beta)$ is $\mathrm{GL}_2(\mathbb{Z}_2)$ equivalent to $\mathrm{diag}(\beta, 1)$ and to $\mathrm{diag}(1+4\beta, 4+\beta)$.

We further define

$$\mathcal{A}_{-1}(c) = \{2^c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}, \text{ and } \mathcal{A}_{-2}(c) = \{2^c \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\}. \tag{1.12}$$

Then a theorem of Watson ([Wa, Theorem 3]) asserts that $\mathcal{A} = \coprod A_r(c)$, i.e., every 2×2 nonsingular symmetric matrix over $\mathbb{Q}_2$ is $\mathrm{GL}_2(\mathbb{Z}_2)$-equivalent to exactly one matrix in $\coprod_{r \ge -2, c \in \mathbb{Z}} \mathcal{A}_r(c)$. So one has

$$W(T, L) = \sum_{r \ge -2, c \in \mathbb{Z}} S_r(c, T, L), \tag{1.13}$$

with

$$S_r(c, T, L) = \sum_{\gamma \in \mathcal{A}_r(c)} C(\gamma) A(\gamma, L) B(T, \gamma). \tag{1.14}$$

Let $L = \mathbb{Z}_2^n$ with quadratic form

$$q = \sum_{i=1}^{H} \epsilon_i 2^{l_i - 1} x_i^2 + \sum_{j=1}^{M} 2^{m_j} y_{j1} y_{j2} + \sum_{k=1}^{N} 2^{n_k} (z_{k1}^2 + z_{k1} z_{k2} + z_{k2}^2). \tag{1.15}$$

Here $H + 2M + 2N = n$, $\epsilon_i \in \mathbb{Z}_2^*$, and $l_i \ge 1, m_j, n_k \ge 0$ are integers. We reorder l_i, m_j, m_j, n_k, n_k into an increasing sequence $0 \le l_1' \le l_2' \le \cdots \le l_n'$. We choose the standard Haar measure on $\mathbb{Z}_2^n$ as the Haar measure on L. Let

$$T = 2^a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2^a \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \text{ or } 2^a \,\mathrm{diag}(\alpha_1, \alpha_2 2^t) \tag{1.16}$$

with $a \geq -1$ in the first two cases and $a, t \geq 0$ in the third case. Our purpose in this paper is to give an explicit formula for $W(X, T, L)$ when L and T are given as in (1.15) and (1.16).

First notice that

$$\sum_{c \geq 0, r \geq -2} S_r(c, T, L) = \int_{\mathrm{Sym}_2(\mathbb{Z}_2)} \int_{L^m} \psi(\ \mathrm{tr} bq(x)) dx \psi(-\ \mathrm{tr} Tb) db = 1 \tag{1.17}$$

Secondly, one sees from definition that, for $\gamma = \mathcal{A}_r(-c)$ with $c > 0$,

$$A(\gamma, L_s) = A(\gamma, L) \begin{cases} X^{2c} & \text{if } r < 0, \\ X^{2c-r} & \text{if } 0 \leq r < c, \\ X^c & \text{if } r \geq c. \end{cases} \tag{1.18}$$

Here $X = 2^{-s}$. We will write for convenience

$$S_{<a} = \sum_{r<c} S_r \text{ and } S_{\geq a} = \sum_{r \geq a} S_r.$$

Combining (1.13) $-$ (1.14) with (1.17) $-$ (1.18), we obtain the following general proposition.

Proposition 1.1. *Let the notation be as above. Then*

$$\begin{aligned} W(X, T, L) &= 1 + \sum_{c>0} S_{<0}(-c, T, L) X^{2c} \\ &\quad + \sum_{c>0} \sum_{0 \leq r < c} S_r(-c, T, L) X^{2c-r} + \sum_{c>0} S_{\geq c}(-c, T, L) X^c \\ &= 1 + \sum_{c>0} R(c, T, L) X^c. \end{aligned}$$

Here

$$R(c, T, L) = S_{\geq c}(-c, T, L) + \sum_{0 < l \leq \frac{m}{2}} S_{c-2l}(l - c, T, L) + S_{<0}(-\frac{c}{2}, T, L).$$

In this paper, we will use the following notations.

$$f_1(a, c) = \begin{cases} 3 & \text{if } 1 \leq c \leq a+1, \\ 1 & \text{if } c = a+2, \\ 0 & \text{if } c > a+2, \end{cases} \tag{1.19}$$

$$f_2(a,c) = \begin{cases} 1 & \text{if } 0 < c \le a+1, \\ -1 & \text{if } c = a+2, \\ 0 & \text{if } c > a+2, \end{cases} \tag{1.20}$$

$$g_1(c) = g_1(c, L) = \sum_{1 \le i \le n} \min(0, l_i' - c), \tag{1.21}$$

and

$$g_2(c) = g_2(c, L) = (-1)^{\sum_{1 \le i \le H} \min(0, l_i - c)}. \tag{1.22}$$

We will prove the following lemma in Section 7

Lemma 1.2. *One has*

$$A(\gamma, L) = \begin{cases} 2^{g_1(c)} & \text{if } \gamma \in \mathcal{A}_{-1}(-c), \\ 2^{g_1(c)} g_2(c) & \text{if } \gamma \in \mathcal{A}_{-2}(-c), \\ A(\beta_1 2^{-c}, L) A(\beta_2 2^{-c+r}, L) & \text{if } \gamma = 2^{-c} \ \operatorname{diag}(\beta_1, 2^r \beta_2) \in \mathcal{A}_r(-c), \end{cases}$$

where

$$A(\beta 2^c, L) = \prod_{1 \le i \le H} I_1(\epsilon_i \beta 2^{c+l_i}) \prod_{1 \le j \le M} 2^{\min(0, c+m_j)} \prod_{1 \le k \le N} (-2)^{\min(0, c+n_k)},$$

where

$$I_1(\alpha 2^c) = \int_{\mathbb{Z}_2} \psi(\alpha 2^{c-1} x^2) dx = \begin{cases} 1 & \text{if } c > 0, \\ 0 & \text{if } c = 0, \\ 2^{\frac{c}{2}} \psi(\frac{\alpha}{8})(2, \alpha)^c & \text{if } c < 0. \end{cases}$$

In particular, $A(\beta 2^c, L) = 1$ *whenever* $c \ge 0$.

Lemma 1.3. *Set*

$$\tilde{A}(c, L) = \sum_{\beta \in (\mathbb{Z}/8)^*} A(\beta 2^c, L). \tag{1.}$$

Then

$$2 \sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma, L) + \sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma, L) = \big(A(2^{-c}, L) + A(-2^{-c}, L)\big) \tilde{A}(-c, L)$$

and

$$\sum_{\gamma \in \mathcal{A}_r(-c)} A(\gamma, L) = \begin{cases} (A(2^{-c}, L) + A(-3 \cdot 2^{-c}, L))\, \tilde{A}(1-c, L) & \text{if } r = 1, \\ (A(2^{-c}, L) + A(-2^{-c}, L))\, \tilde{A}(2-c, L) & \text{if } r = 2, \\ \tilde{A}(-c, L)\tilde{A}(r-c, L) & \text{if } r \geq 3. \end{cases}$$

Proof. The last equality follows from definitions of $\mathcal{A}_r(c)$ and $\tilde{A}$, and the fact

$$A(\ \mathrm{diag}(\gamma_1, \gamma_2), L) = A(\gamma_1, L)A(\gamma_2, L).$$

The same reason gives

$$\begin{aligned} &2 \sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma, L) + \sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma, L) - \big(A(2^{-c}, L) + A(-2^{-c}, L)\big)\, \tilde{A}(-c, L) \\ &= A(2^{-c}, L)A(3 \cdot 2^{-c}, L) - A(-2^{-c}, L)A(-3 \cdot 2^{-c}, L). \end{aligned}$$

Lemma 1.2 implies

$$A(2^{-c}, L)A(3 \cdot 2^{-c}, L) - A(-2^{-c}, L)A(-3 \cdot 2^{-c}, L) = 0.$$

The following lemma will be proved in Section 9.

Lemma 1.4. *For $\gamma \in \mathcal{A}_r(-c)$, one has*

$$C(\gamma) = 2^{3c-2} \begin{cases} \frac{1}{3} & \text{if } r = -2, \\ 1 & \text{if } r = -1, \\ 1 & \text{if } \gamma \in \mathcal{A}_0^-(-c), \\ 2^{-1} & \text{if } \gamma \in \mathcal{A}_0^+(-c), \\ 2^{-r-1} & \text{if } 1 \leq r \leq 2, \\ 2^{-r-2} & \text{if } r \geq 3. \end{cases}$$

2. The main formula I.

In this section, we consider the cases $T = 2^a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{A}_{-1}(a)$ and $T = 2^a \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathcal{A}_{-2}(a)$ with $a \geq -1$. The following lemma will be proved in Section 8.

Lemma 2.1. *Let $\gamma \in \mathcal{A}_r(-c)$ with $c > 0$.*

(1) *When $T \in \mathcal{A}_{-2}(a)$ with $a \ge -1$, one has*

$$B(T,\gamma) = B(\gamma, T) = \begin{cases} \frac{3}{8} f_2(a,c) & \text{if } r \neq 0, \\ \frac{3}{8}\ char(\mathbb{Z}_2)(2^{a-c+2}) & \text{if } r = 0. \end{cases}$$

(2) *When $T \in \mathcal{A}_{-1}(a)$ with $a \ge -1$, one has*

$$B(T,\gamma) = B(\gamma, T) = \begin{cases} \frac{3}{8} f_2(a,c) & \text{if } r < 0, \\ \frac{1}{8} f_1(a,c) & \text{if } r > 0, \\ \frac{1}{8} f_1(a,c) f_2(a,c) & \text{if } r = 0. \end{cases}$$

Theorem 2.2. *Let $T = 2^a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $a \ge -1$. Then*

$$W(X,T,L) = 1 + \sum_{0<c\le a+2} S_{<0} X^{2c} + \sum_{0<c\le a+2} \sum_{0\le r<c} S_r X^{2c-r} + \sum_{0<c\le a+2} S_{\ge c} X^c,$$

where

$$S_{<0} = 2^{3c-5+g_1(c)}(3+g_2(c)) f_2(a,c), \tag{2.1}$$

$$S_0 = 2^{3c-6} f_1(a,c) f_2(a,c) \tilde{A}(-c,L) \left(A(2^{-c},L) + A(-2^{-c},L) \right), \tag{2.2}$$

and

$$S_r = 2^{3c-6-r} f_1(a,c) \tilde{A}(r-c,L) \begin{cases} A(2^{-c},L) + A(3\cdot 2^{-c},L) & \text{if } r = 1, \\ A(2^{-c},L) + A(-2^{-c},L) & \text{if } r = 2, \\ \frac{1}{2}\tilde{A}(-c,L) & \text{if } r \ge 3. \end{cases} \tag{2.3}$$

and

$$S_{\ge c} = \begin{cases} 2^{2c-4} f_1(a,c) \tilde{A}(-c,L) & \text{if } c \ge 3, \\ \sum_{c\le r<3} S_r + 2^{3c-7} f_1(a,c)\tilde{A}(-c,L) & \text{if } c < 3. \end{cases} \tag{2.4}$$

Here $f(r)$, $f_1(a,c)$, and $g_i(c)$ are given by formulae (1.19) – (1.22) above. $S_r = S_r(-c,T,L)$.

Proof. It follows from Proposition 1.1, Lemmas 1.2-1.4 and 2.1, and formula (1.14), which we recall here:

$$S_r(-c,T,L) = \sum_{\gamma \in \mathcal{A}_r(-c)} C(\gamma)B(T,\gamma)A(\gamma,L).$$

By Lemma 2.1, $S_r(-c,T,L) = 0$ unless $c \le a+2$. Formula (2.1) follows from Lemmas 1.2, 1.4, and 2.1 immediately. So does (2.3) with $r > 0$, since $C(\gamma)$ and $B(T,\gamma)$ depend only on r. When $r = 0$, one has by Lemmas 1.2-1.4 and 2.1

$$\begin{aligned} S_0(-c,T,L) &= \sum_{\gamma \in \mathcal{A}_0^-(-c)} 2^{3c-2}A(\gamma,L)\frac{1}{8}f_1(a,c) + \sum_{\gamma \in \mathcal{A}_0^+(-c)} 2^{3c-3}A(\gamma,L)\frac{1}{8}f_1(a,c) \\ &= 2^{3c-6}f_1(a,c)\left(2\sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma,L) + \sum_{\gamma \in \mathcal{A}_0^+(-c)} A(\gamma,L)\right) \\ &= 2^{3c-6}f_1(a,c)\tilde{A}(-c,T,L)(A(2^{-c},L) + A(-2^{-c},L)) \end{aligned}$$

as claimed by (2.2).

When $r \ge c \ge 3$, one has $\tilde{A}(r-c,L) = 2^2$ by Lemmas 1.2-1.3. So for any $r_0 \ge \max(3,c)$, one has by (2.3)

$$\begin{aligned} S_{\ge r_0} &= 3^{3c-5}f_1(a,c)\tilde{A}(-c,L)\sum_{r \ge r_0} 2^{-r} \\ &= 2^{3c-4-r_0}f_1(a,c)\tilde{A}(-c,L). \end{aligned}$$

Now (2.4) is clear.

Theorem 2.3. *Let* $T = 2^a \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ *with* $a \ge -1$. *Then*

$$\begin{aligned} W(X,T,L) = 1 &+ \sum_{0<c\le a+2} S_{<0}X^{2c} \\ &+ \sum_{0<c\le a+2}\sum_{0\le r<c} S_r X^{2c-r} + \sum_{0<c\le a+2} S_{\ge c}X^c, \end{aligned}$$

where

$$S_{<0} = 2^{3c-5+g_1(c)}(3+g_2(c))f_2(a,c), \tag{2.5}$$

and

(2.6)

$$S_r = 3\cdot 2^{3c-6-r}\tilde{A}(r-c,L)\begin{cases} A(2^{-c},L) + A(-2^{-c},L) & \text{if } r=0, \\ f_2(a,c)(A(2^{-c},L) + A(3\cdot 2^{-c},L)) & \text{if } r=1, \\ f_2(a,c)(A(2^{-c},L) + A(-2^{-c},L)) & \text{if } r=2, \\ \frac{1}{2}f_2(a,c)\tilde{A}(-c,L) & \text{if } r\ge 3. \end{cases}$$

and

$$S_{\geq c} = \begin{cases} 3\cdot 2^{2c-4} f_2(a,c)\tilde{A}(-c,L) & \text{if } c \geq 3, \\ \sum_{c\leq r<3} S_r + 3\cdot 2^{3c-7} f_2(a,c)\tilde{A}(-c,L) & \text{if } c < 3. \end{cases} \tag{2.7}$$

Here $f(r)$, $f_1(a,c)$, and $g_i(c)$ are given by formulae (1.19) – (1.22) *above.* $S_r = S_r(-c,T,L)$.

The proof is the same as that of Theorem 2.2, and is left to the reader.

3. The Main Formula II.

In this section, we consider the case $T = 2^a\ \mathrm{diag}(\alpha_1,\alpha_2) \in \mathcal{A}_0(a)$ with $0 \leq a$. Denote $\alpha = \alpha_1+\alpha_2$. The following will be proved in Section 8.

Lemma 3.1. *Let $\gamma = 2^{-c}\ diag(\beta_1, \beta_2 2^r) \in \mathcal{A}_r(-c)$ with $r \geq 0$ and $c > 0$.*

(1) When $r \geq 3$, $B(T,\gamma)$ is independent of r or β_2, and is equal to $2^{-3}B(T,\beta_1 2^{-c})$, where

$$B(T,\beta 2^{-c}) = \begin{cases} 0 & \text{if } c > a+3, \\ \psi(-\frac{\alpha\beta}{8}) & \text{if } c = a+3, \\ \psi(-\frac{\alpha\beta}{4}) + \psi(-\frac{\alpha_1\beta}{4}) + \psi(-\frac{\alpha_2\beta}{4}) & \text{if } c = a+2, \\ -1 & \text{if } c = a+1, \\ 3 & \text{if } c \leq a. \end{cases}$$

(2) When $r = 2$,

$$B(T,\gamma) = 2^{-3}B(T,\beta_1 2^{-c}) \begin{cases} -1 & \text{if } c = a+3, \\ 1 & \text{if } c \neq a+3. \end{cases}$$

(3) When $r = 1$, one has

$$B(T,\gamma) = 2^{-3} \begin{cases} 0 & \text{if } c > a+3, \\ \frac{1}{2}\psi(-\frac{\alpha\beta_1}{8}) \left(\psi(-\frac{\alpha_1\beta_2}{4}) + \psi(-\frac{\alpha_2\beta_{-2}}{4})\right) & \text{if } c = a+3, \\ (-1)^{\frac{\alpha_1-\alpha_2}{2}} & \text{if } c = a+2, \\ -1 & \text{if } c = a+1, \\ 3 & \text{if } c \leq a. \end{cases}$$

(4) When $r = 0$, one has

$$B(T,\gamma) = \frac{1}{8} \begin{cases} 0 & \text{if } c > a+2, \\ \psi(-\frac{\alpha\beta_1}{4}) - \psi(-\frac{\alpha_1\beta_1}{4}) - \psi(-\frac{\alpha_2\beta_1}{4}) & \text{if } c = a+2, \beta_2 \equiv \beta_1 \bmod (4), \\ \psi(-\frac{(\alpha_1-\alpha_2)\beta_1}{4}) & \text{if } c = a+2, \beta_2 \equiv -\beta_1 \bmod (4), \\ -1 & \text{if } c = a+1, \\ 3 & \text{if } c \leq a. \end{cases}$$

In particular, when $c \le a+1$, one has always

$$B(T,\gamma) = 2^{-3} f_1(a,c+1) f_2(a,c+1).$$

Theorem 3.2. *For $T = 2^a\ \mathrm{diag}(\alpha_1,\alpha_2)$ with $a \ge 0$, one has*

(3.1)
$$W(X,T,L) = 1 + \sum_{0<c\le a+2} S_{<0}(-c,T,L) X^{2c} + \sum_{0<c\le a+3} \sum_{0\le r<c} S_r(-c,T,L) X^{2c-r} + \sum_{0<c\le a+3} S_{\ge c}(-c,T,L) X^c.$$

Here $S_{<0}$, S_r, and $S_{\ge c}$ are defined as follows. First,

$$(3.2) \quad S_{<0}(-c,T,L) = \begin{cases} 2^{3c-5+g_1(c)}(f_1(a,c)f_2(a,c)+g_2(c)) & \text{if } c \le a+2, \\ 0 & \text{if } c > a+2, \end{cases}$$

Secondly,

$$(3.3) \qquad S_r(-c,T,L) = \sum_{\gamma \in \mathcal{A}_r(-c)} C(\gamma) B(T,\gamma) A(\gamma,L)$$

with $A(\gamma,L)$, $B(T,\gamma)$, and $C(\gamma)$ given by Lemmas 1.2, 1.4, and 3.1. In particular, when $0 < c < a+2$,

(3.4)
$$S_r = 2^{3c-6-r} f_1(a,c+1) f_2(a,c+1) \tilde{A}(r-c,L) \begin{cases} A(2^{-c},L) + A(3\cdot 2^{-c},L) & \text{if } r = 1, \\ A(2^{-c},L) + A(-2^{-c},L) & \text{if } r = 0,2, \\ \frac{1}{2}\tilde{A}(-c,L) & \text{if } r \ge 3. \end{cases}$$

Finally,

$$(3.5) \qquad S_{\ge c}(-c,T,L) = 2^{2c-4} \sum_{\beta \in (\mathbb{Z}/8)^*} B(T,\beta 2^{-c}) A(\beta 2^{-c},L)$$

for $c \ge 3$, and

$$(3.6) \quad S_{\ge c}(-c,T,L) = 2^{3c-7} \sum_{\beta \in (\mathbb{Z}/8)^*} B(T,\beta 2^{-c}) A(\beta 2^{-c},L) + \sum_{c \le r < 3} S_r(-c,T,L)$$

for $0 < c < 3$.

The proof is the same as that of Theorem 4.4 below, and is left to the reader.

4. The Main formula III.

In this section, we consider the case $T = 2^a \operatorname{diag}(\alpha_1, \alpha_2 2^t) = \operatorname{diag}(\alpha_1 2^{a_1}, \alpha_2 2^{a_2})$ with $a = a_1 \geq 0$ and $t = a_2 - a_1 > 0$. Corollaries 8.5 and 8.6 give

Lemma 4.1. *When $a_2 > a_1$, one has*

$$B(T, \gamma) = B(\gamma, T) = B_1(\gamma, T) + B_2(\gamma, T) + B_3(\gamma, T),$$

where

$$\begin{aligned} B_1(\gamma, T) = \frac{1}{8}\ & char(\mathbb{Z}_2)(2^{a_1+3-c}) I_1(-2^{a_2+1-c}\alpha_2\beta_1) I_1(2^{a_1+3-c+r}) \\ & \cdot \psi\left(-2^{a_1-c}\alpha_1\beta_1 - 2^{a_2-c+r}\alpha_2\beta_2)\right), \end{aligned}$$

$$\begin{aligned} B_2(\gamma, T) = \frac{1}{8}\ & char(\mathbb{Z}_2^2)(2^{a_2+3-c}, 2^{a_1+3-c+r}) I_1(-2^{a_1+1-c}\alpha_1\beta_1) I_1(-2^{a_2+3-c+r}) \\ & \cdot \psi(-2^{a_2-c}\alpha_2\beta_1 - 2^{a_1-c+r}\alpha_1\beta_2) \end{aligned}$$

and

$$\begin{aligned} B_3(\gamma, T) = \frac{1}{8}\ & char(\mathbb{Z}_2^2)(2^{a_2+3-c}, 2^{a_1+3-c+r}) I_1\left(-2^{a_1+1-c}\alpha_1\beta_1 - 2^{a_2+1-c}\alpha_2\beta_1\right) \\ & \cdot \psi\left(-2^{a_2-c}\alpha_1\alpha_2\alpha\beta_1 - 2^{a_1-c+r}\alpha_1\beta_2 - 2^{a_2-c+r}\alpha_2\beta_2\right). \end{aligned}$$

Here $\alpha = \alpha_1 + 2^t\alpha_2$. In particular, $B(T, \gamma) = 0$ unless $c \leq a + t + 3 = a_2 + 3$. When $0 < c \leq a + 1$, one has

$$B(T, \gamma) = 2^{-3} f_1(a, c+1) f_2(a, c+1).$$

A couple of special cases are worth recording.

Corollary 4.2. *Assume that $t = a_2 - a_1 \geq 3$.*

(1) When $r = 0$, then

$$\begin{aligned} B(T, \gamma) = & \frac{1}{8}\ char(\mathbb{Z}_2)(2^{a_1+2-c})\psi(-2^{a_1-c}\alpha_1\beta_1) \\ & + \frac{1}{4}\ char(\mathbb{Z}_2)(2^{a_1+3-c}) I_1(-2^{a_1+1-c}\alpha_1\beta_1)\psi(-2^{a_2-c}\alpha_2\beta_1 - 2^{a_1-c}\alpha_1\beta_2) \end{aligned}$$

(2) When $r \geq 1$, one has

$$\begin{aligned} B(T, \gamma) = & \frac{1}{8}\ char(\mathbb{Z}_2)(2^{a_1+3-c})\psi(-2^{a_1-c}\alpha_1\beta_1) \\ & + \frac{1}{4}\ char(\mathbb{Z}_2^2)(2^{a_2+3-c}, 2^{a_1+3-c+r}) I_1(-2^{a_1+1-c}\alpha_1\beta_1) \\ & \cdot \psi(-2^{a_2-c}\alpha_2\beta_1 - 2^{a_1-c+r}\alpha_1\beta_2) \end{aligned}$$

Corollary 4.3. *Assume $t = a_2 - a_1 > 0$ and $r \geq t+3$. Then $B(T,\gamma) = \frac{1}{8}B(T, \beta_1 2^{-c})$ is independent of r or β_2, and*

$$\begin{aligned} B(T,\beta 2^{-c}) = \; & char(\mathbb{Z}_2)(2^{a_1+3-c}) I_1(-2^{a_2+1-c}\alpha_2\beta)\psi(-2^{a_1-c}\alpha_1\beta) \\ & + \; char(\mathbb{Z}_2)(2^{a_2+3-c}) I_1(-2^{a_1+1-c}\alpha_1\beta)\psi(-2^{a_2-c}\alpha_2\beta) \\ & + \; char(\mathbb{Z}_2)(2^{a_2+3-c}) I_1(-2^{a_1+1-c}\alpha\beta)\psi(-2^{a_2-c}\alpha_1\alpha_2\alpha\beta). \end{aligned}$$

Here $\alpha = \alpha_1 + 2^t\alpha_2$.

Theorem 4.4. *Let the notation be as above. Then*

(4.1)

$$\begin{aligned} W(X,T,L) = 1 + & \sum_{0<c\leq a+2} S_{<0}(-c,T,L)X^{2c} \\ & + \sum_{0<c\leq a+t+3}\sum_{0\leq r<c} S_r(-c,T,L)X^{2c-r} + \sum_{0<c\leq a+t+3} S_{\geq c}(-c,T,L)X^c. \end{aligned}$$

Here $S_{<0}$, S_r, and $S_{\geq c}$ are given as follows. First,

$$S_{<0}(-c,T,L) = 2^{3c-5+g_1(c)}(f_1(a,c) + g_2(c)f_2(a,c)), \tag{4.2}$$

Secondly,

$$S_r(-c,T,L) = \sum_{\gamma\in\mathcal{A}_r(-c)} C(\gamma)B(T,\gamma)A(\gamma,L) \tag{4.3}$$

with $A(\gamma, L)$, $B(T,\gamma)$, and $C(\gamma)$ given in Lemmas 1.2, 1.4, and 4.1. In particular, when $0 < c \leq a+1$, one has

(4.4)

$$\begin{aligned} S_r(-c,T,L) = 2^{3c-r-6} & f_1(a,c+1)f_2(a,c+1)\tilde{A}(r-c,L) \\ & \cdot \begin{cases} (A(2^{-c},L) + A(-3\cdot 2^{-c},L)) & \text{if } r=1, \\ (A(2^{-c},L) + A(-2^{-c},L)) & \text{if } r=0,2, \\ \frac{1}{2}\tilde{A}(-c,L) & \text{if } r\geq 3. \end{cases} \end{aligned}$$

Finally,

$$S_{\geq c}(-c,T,L) = 2^{2c-4}\sum_{\beta\in(\mathbb{Z}/8)^*} B(T,\beta 2^{-c})A(\beta 2^{-c},L) \tag{4.5}$$

for $c \geq t+3$, and

$$S_{\geq c}(-c,T,L) = \sum_{c\leq r<t+3} S_r(-c) + 2^{3c-t-7}\sum_{\beta\in(\mathbb{Z}/8)^*} B(T,\beta 2^{-c})A(\beta 2^{-c},L) \tag{4.6}$$

for $0 < c < t+3$. Here $B(T, \beta 2^{-c})$ is given by Corollary 4.3.

Proof. (sketch) When $r < 0$, one has by Lemma 2.1 (switching roles of a and $-c$, and so on)

$$B(T,\gamma) = B(\gamma, T) = \begin{cases} \frac{3}{8} f_2(a,c) & \text{if } r = -2, \\ \frac{1}{8} f_1(a,c) & \text{if } r = -1. \end{cases} \tag{4.7}$$

In particular, $S_{<0}(-c, T, L) = 0$ unless $c \le a+2$. By Lemma 4.1, $B(T,\gamma) = 0$ and thus $S_r(-c, T, L) = 0$ unless $c \le a + t + 3$. By Proposition 1.1, this proves (4.1) with formulas (4.2) – (4.4) to be verified. (4.3) is a general formula given in (1.14). Combining (4.3), (4.7) with Lemmas 1.2 and 1.4, one proves (4.2). When $0 < c \le a+1$, one has by Lemmas 4.1, 1.4, and 1.3

$$\begin{aligned} S_r(-c,T,L) &= 2^{-3} f_1(a,c+1) f_2(a,c+1) \sum_{\gamma \in \mathcal{A}_r(-c)} C(\gamma) A(\gamma, L) \\ &= 2^{3c-r-6} f_1(a,c+1) f_2(a,c+1) \\ &\quad \cdot \begin{cases} 2\sum_{\gamma \in \mathcal{A}_0^-(-c)} A(\gamma,L) + \sum_{\gamma\in\mathcal{A}_0^+(-c)} A(\gamma, L) & \text{if } r = 0, \\ \sum_{\gamma \in \mathcal{A}_r(-c)} A(\gamma, L) & \text{if } 1 \le r \le 2, \\ \frac{1}{2} \sum_{\gamma \in \mathcal{A}_r(-c)} A(\gamma, L) & \text{if } r \ge 3. \end{cases} \\ &= 2^{3c-r-6} f_1(a,c+1) f_2(a,c+1) \tilde{A}(r-c, L) \\ &\quad \cdot \begin{cases} (A(2^{-c}, L) + A(-3\cdot 2^{-c}, L)) & \text{if } r = 1, \\ (A(2^{-c}, L) + A(-2^{-c}, L)) & \text{if } r = 0, 2, \\ \frac{1}{2}\tilde{A}(-c, L) & \text{if } r \ge 3. \end{cases} \end{aligned}$$

This proves (4.4). Finally, when $r \ge t+3$, $B(T,\gamma) = 2^{-3} B(T, \beta_1 2^{-c})$ and $A(\gamma, L) = A(\beta_1 2^{-c}, L)$ are independent of r or β_2 by Corollary 4.3 and Lemma 1.2. So

$$\begin{aligned} S_r(-c,T,L) &= 2^{3c-r-4} 2^{-3} \sum_{\gamma \in \mathcal{A}_r(-c)} B(T, \beta_1 2^{-c}) A(\beta_1 2^{-c}, L) \\ &= 2^{3c-r-5} \sum_{\beta \in (\mathbb{Z}/8)^*} B(T, \beta 2^{-c}) A(\beta 2^{-c}, L). \end{aligned}$$

So for any $r_0 \ge t+3$, one has

$$S_{\ge r_0}(-c, T, L) = 2^{3c - r_0 - 4} \sum_{\beta \in (\mathbb{Z}/8)^*} B(T, \beta 2^{-c}) A(\beta 2^{-c}, L).$$

When $c \ge t+3$, taking $r_0 = c$, ones gets (4.5). When $c < t+3$, one obtains (4.6) by taking $r_0 = t+3$.

The main formulas in Section 2-4 are programmed by Song Guo using MAGMA, and can be downloaded from my web page http://www.math.wisc.edu/∼thyang/programs. One can get result with it for any given quadratic lattice of the form (1.15) and matrices of form (1.16). The next section is mainly a product of this program.

5. Examples.

Before going to the examples, we recall a proposition of Kudla. It tells when $W(1, T, L) = 0$, which is a good first test of whether a given formula for $W(X, T, L)$ is wrong.

Let (V, q) be a quadratic (non-degenerate) space over a local field F of dimension $n + 1$. Let S be an associated symmetric matrix in the following sense that with respect to some isomorphism $V \cong F^{n+1}$, one has

$$q(x) = {}^t x S x. \tag{5.1}$$

Let T be a $n \times n$ symmetric matrix over F. Its associated quadratic space is $V_T = F^n$ with quadratic form

$$q_T(x) = {}^t x T x.$$

We say V represent T if there is $x = {}^t(x_1, \cdots, x_n) \in V^n$, such that

$$\frac{1}{2}((x_i, x_j)) = T \tag{5.2}$$

where $(x, y) = q(x + y) - q(x) - q(y)$. We say that a lattice L represents T if x can be chosen in L^n.

It is known that $W(1, T, L) = 0$ if and only if L does not represent T. The following lemma is proved in [Ku, Proposition 1.3].

Lemma 5.1. *Let the notation be as above, and let $\epsilon_F(V) = \epsilon_F(S)$ be the Hasse invariant of V. Then V represents T if and only if*

$$\epsilon_F(V) = \epsilon_F(T)(\det T, -\det S). \tag{5.3}$$

Now coming to our special examples. Let B be a quaternion algebra over $\mathbb{Q}_2$, and let V be the trace zero elements in B with the reduced norm as the quadratic form. We use the superscript ra and s to stand for ramified and split case respectively when necessary. Notice that

$$\epsilon_2(V) = \begin{cases} -1 & \text{if } V = V^s, \\ 1 & \text{if } V = V^{ra}. \end{cases}$$

Set

(5.4)

$$\mu_2(T) = -\epsilon_{\mathbb{Q}_2}(T)(\det T, -1)$$
$$= \begin{cases} (-1)^{a+1} & \text{if } T \in \mathcal{A}_{-2}(a), \\ 1 & \text{if } T \in \mathcal{A}_{-1}(a), \\ -(\alpha_1, \alpha_2)(-1, \alpha_1\alpha_2)(2, \alpha_1\alpha_2)^a(2, \alpha_1)^t & \text{if } T = 2^a \operatorname{diag}(\alpha_1, 2^t\alpha_2) \in \mathcal{A}_t(a). \end{cases}$$

Then Lemma 5.1 asserts that V^s represents T if and only if $\mu_2(T) = 1$, and that V^{ra} represents T if and only if $\mu_2(T) = -1$. In particular, for any lattice L of V^s, $W(1, T, L) = 0$ unless $\mu_2(T) = 1$.

When $B = M_2(\mathbb{Q}_2)$, we take the following lattices of V for $i \geq 0$:

$$L^i = \{X = \begin{pmatrix} a & b \\ 2^i c & -a \end{pmatrix} \in M_2(\mathbb{Z}_2) \cap V : \, a, b, c \in \mathbb{Z}_2\}, \tag{5.5}$$

with quadratic form

$$Q(X) = \det X = -a^2 - 2^i bc. \tag{5.6}$$

When B is ramified, i.e., a division algebra, let $\mathcal{O}_B$ be the maximal order of B and $L^{ra} = V \cap \mathcal{O}_B$. Then $L^{ra} \cong \mathbb{Z}_2^3$ with quadratic form

$$Q(x) = x_1^2 + x_2^2 + x_3^2. \tag{5.7}$$

The following three theorems are obtained by Song Guo's computer program and tedious calculations.

Theorem 5.2. *Let* $T = 2^a \begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix}$ *with* $a \geq -1$ *and* $b = 0$ *or* 2. *Set*

$$\delta_i = \begin{cases} 1 & \text{if } i = 0, [\frac{a}{2}] + 1, \\ -1 & \text{if } 0 < i < [\frac{a}{2}] + 1. \end{cases}$$

(1) When a is even, then

$$\frac{W(X, T, L^0)}{1 - 2^{-2}X^2} = \sum_{0 \leq i \leq \frac{a}{2}} 2^i (X^{2i} + \mu_2(T) X^{2a+2-2i})$$

and

$$W(X, T, L^{ra}) = \sum_{0 \leq i \leq \frac{a}{2}+1} \delta_i 2^i (X^{2i} - \mu_2(T) X^{2a+4-2i}).$$

(2) *When a is odd,*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2} = \sum_{0 \le i \le \frac{a-1}{2}} 2^i (X^{2i} + \mu_2(T) X^{2a+2-2i}) + 2^{\frac{a+1}{2}} X^{a+1},$$

and

$$W(X,T,L^{ra}) = 1 - \mu_2(T) X^{2a+4} - \sum_{1 \le i \le \frac{a+1}{2}} 2^i (X^{2i} - \mu_2(T) X^{2a+4-2i})$$

Theorem 5.3. *Let $T = 2^a\ diag(\alpha_1, \alpha_2) \in \mathcal{A}_0(a)$ with $\alpha_i \in \mathbb{Z}_2^*$ and $a \ge 0$. Then*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2} = \sum_{0 \le i \le \frac{a}{2}} 2^i (X^{2i} + \mu_2(T) X^{2a+2-2i}),$$

and

$$W(X,T,L^{ra}) = \sum_{0 \le i \le \frac{a}{2}+1} \delta_i 2^i (X^{2i} - \mu_2(T) X^{2a+4-2i}).$$

Here δ_i is as in Theorem 5.2.

Theorem 5.4. *Let $T = 2^a\ diag(\alpha_1, 2^t\alpha_2) \in \mathcal{A}_t(a)$ with $t \ge 1$ and $a \ge 0$. Set $v_0 = (2, \alpha_1)_2$.*

(1) *When a is odd or $\alpha_1 \equiv 1 \bmod 4$, one has Then*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2} = \sum_{0 \le i \le \frac{a}{2}} 2^i (X^{2i} + \mu_2(T) X^{2a+t+2-2i}),$$

and

$$W(X,T,L^{ra}) = \sum_{0 \le i \le \frac{a}{2}+1} \delta_i 2^i (X^{2i} - \mu_2(T) X^{2a+t+4-2i}).$$

(2) *When a is even and $\alpha_1 \equiv -1 \bmod 4$, one has Then*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2} = \sum_{0 \le i \le \frac{a}{2}} 2^i (X^{2i} + \mu_2(T) X^{2a+t+2-2i}) + 2^{\frac{a}{2}+1} \sum_{0 \le k \le t-2} v_0^k X^{a+2+k},$$

and

$$W(X,T,L^{ra}) = 1 - \mu_2(T) X^{2a+t+4} - \sum_{1 \le i \le \frac{a}{2}+1} 2^i (X^{2i} - \mu_2(T) X^{2a+t+4-2i})$$
$$- v_0 2^{\frac{a}{2}+2} (X^{a+3} - \mu_2(T) X^{a+1+t}).$$

These formulae are almost the same. It can be unified using the Gross-Keating invariants of a quadratic lattice ([GK, Section 4]). In Appendix B, which is essentially due to Kudla, we recall the definition of the Gross-Keating invariants of a quadratic lattice and study the relation between the usual Jordan form and the Gross-Keating invariants when the rank is less than or equal to 3.

Theorem 5.5. *Let the Gross-Keating invariants of* $diag(1,T)$ *be* $(0,a_2,a_3)$, *and let* $\mu_2(T)$ *be as given by* (5.4). *Set*

$$\delta_i=\delta_i(T)=\begin{cases}1 & \textit{if } i=0,[\frac{a_2}{2}]+1,\\ -1 & \textit{if } 0<i<[\frac{a_2}{2}]+1.\end{cases}$$

(1) *When* a_2 *is odd, one has*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2}=\sum_{0\le i\le \frac{a_2-1}{2}}2^i(X^{2i}+\mu_2(T)X^{a_2+a_3-2i}),$$

and

$$W(X,T,L^{ra})=\sum_{0\le i\le \frac{a_2+1}{2}}\delta_i 2^i(X^{2i}-\mu_2(T)X^{a_2+a_3-2i}),$$

(2) *When* a_2 *is even, one has*

$$\frac{W(X,T,L^0)}{1-2^{-2}X^2}=\sum_{0\le i\le \frac{a_2}{2}-1}2^i(X^{2i}+\mu_2(T)X^{a_2+a_3-2i})+2^{\frac{a_2}{2}}\sum_{0\le k\le a_3-a_2}(\epsilon X)^{a_2+k},$$

and

$$\begin{aligned}W(X,T,L^{ra})=&\sum_{0\le i\le \frac{a_2}{2}}\delta_i 2^i(X^{2i}-\mu_2(T)X^{a_2+a_3+2-2i})\\ &-\epsilon 2^{\frac{a_2}{2}+1}(X^{a_2+1}-\mu_2(T)X^{a_3+1}).\end{aligned}$$

Here $\epsilon=\epsilon(T)$ *is the Gross-Keating's* ϵ*-constant given in [GK, page 236], see also Appendix B.*

Remark 5.6. When $T=2^a\,\mathrm{diag}(\alpha_1,2^t\alpha_2)\in\mathcal{A}_t(a)$ with $t\ge 2$ and a even, $\epsilon(T)=(2,\alpha_1)$ is the sign v_0 in Theorem 5.4. In other cases, $\epsilon(T)$ does not make any difference in the formula.

Proof of Theorem 5.5 (sketch). (1) When $T=2^a\begin{pmatrix}b&1\\1&b\end{pmatrix}\in\mathcal{A}_{<0}(a)$ with $a\ge -1$ and $b=0,2$, the Gross-Keating invariants of $\mathrm{diag}(1,T)$ is $(0,a+1,a+1)$ by Proposition B.4. When a is even $a_2=a+1$ is odd, Theorem 5.5(1) matches up perfectly with Theorem 5.2(1). When a is odd, a_2 is even, and $(\epsilon_v X)^{a_2}=X^{a_2}$. In this case, $\mu_2(T)=1$ and $a_2=a_3$, so $\epsilon 2^{a_2/2+1}(X^{a_2}-\mu_2(T)X^{a_3})=0$, Theorem 5.5(2) matches up with Theorem 5.2(2) perfectly too.

(2) When $T = 2^a \operatorname{diag}(\alpha_1, \alpha_2) \in \mathcal{A}_0(a)$, the Gross-Keating invariants of $\operatorname{diag}(1, T)$ is, by Proposition B.5,

$$(0, a_2, a_3) = \begin{cases} (0, a, a+2) & \text{if } a \text{ is odd,} \\ (0, a+1, a+1) & \text{if } a \text{ is even.} \end{cases}$$

So a_2 is always odd, and Theorem 5.5(1) matches up perfectly with Theorem 5.3.

(3) When $T = 2^a \operatorname{diag}(\alpha_1, 2\alpha_2) \in \mathcal{A}_1(a)$, the Gross-Keating invariants of $\operatorname{diag}(1, T)$ is, by Proposition B.5,

$$(0, a_2, a_3) = \begin{cases} (0, a, a+3) & \text{if } a \text{ is odd,} \\ (0, a+1, a+2) & \text{if } a \text{ is even.} \end{cases}$$

So a_2 is always odd, and Theorem 5.5(1) matches up perfectly Theorem 5.4(1) perfectly. When a is even and $\alpha_1 \equiv -1 \bmod 4$, we now check that Theorem 5.5(1) matches with Theorem 5.4(2). When $L = L^0$, it is clear since $t - 2 = -1 < 0$. When $L = L^{ra}$,

$$v_0 \mu_2 = -(2, \alpha_1)(\alpha_1, \alpha_2)(-1, \alpha_1\alpha_2)(2, \alpha_2) = -(-1, \alpha_1)(-\alpha_1, \alpha_2) = 1$$

since $\alpha_1 \equiv -1 \bmod 4$. So Theorem 5.4(2) gives

$$\begin{aligned} W(X, T, L^{ra}) &= 1 - \mu_2(T) X^{a_2+a_3+2} - \sum_{1 \le i \le \frac{a}{2}} 2^i (X^{2i} - \mu_2(T) X^{a_2+a_3+2-2i}) \\ &\quad - 2^{\frac{a}{2}+1}(X^{a+2} - \mu_2(T) X^{a+3}) + 2^{\frac{a}{2}+2}(\mu_2(T) X^{a+3} + X^{a+2}) \\ &= \sum_{0 \le i \le \frac{a_2+1}{2}} \delta_i 2^i (X^{2i} - \mu_2(T) X^{a_2+a_3+2-2i}). \end{aligned}$$

This matches up perfectly with Theorem 5.5(1).

(4) Finally, when $T = 2^a \operatorname{diag}(\alpha_1, 2^t\alpha_2) \in \mathcal{A}_t(a)$ with $t \ge 2$, the Gross-Keating invariants of $\operatorname{diag}(1, T)$ is, by Proposition B.5,

$$(0, a_2, a_3) = \begin{cases} (0, a, a+t+2) & \text{if } a \text{ is odd,} \\ (0, a+1, a+t+1) & \text{if } a \text{ is even and } \alpha_1 \equiv 1 \bmod 4, \\ (0, a+2, a+t) & \text{if } a \text{ is even and } \alpha_1 \equiv 1 \bmod 4. \end{cases}$$

Theorem 5.5 again matches up with Theorem 5.4 perfectly if we can verify $v_0 = \epsilon$, which is Remark 5.6. Indeed, assume that a_2 is even, i.e., a is even and $\alpha_1 \equiv -1 \bmod 4$. Let $e = \{e_1, e_2, e_3\}$ be a basis of $L = \mathbb{Z}_2^3$ with associated matrix $\operatorname{diag}(1, T)$, and let

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2 - \beta_1}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then ψ is an optimal basis of L by the proof of Lemma B.8. The subspace generated by ψ_1 and ψ_2, is the same as that generated by e_1 and e_2, and on which the quadratic form is $Q(x_1e_1 + x_2e_2) = x_1^2 + 2^a\epsilon_1 x_2^2$. This quadratic form is isotropic (i.e., $\epsilon = 1$ if and only if $(2, -\epsilon_1) = (2, \epsilon_1) = 1$, i.e., $v_0 = 1$. So $\epsilon = v_0$.

It is interesting to compare Theorem 5.5 with [Ya, Propositions 8.6 and 8.7]. For a prime number p, odd or even, let $\mu_p(T) = \pm 1$ according to whether $V_p^s = \{X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{Q}_p)$ with quadratic form det represents T or not. For $p = 2$, it is just our $\mu_2(T)$, and when $p \neq 2$, it is the $\mu_p(T)$ in [Ku, Lemma 8.3] and

$$\mu_p(T) = \begin{cases} v_0^b & \text{if } a \text{ is even,} \\ v_1 & \text{if } a \text{ is odd,} \end{cases}$$

in terms of the notation in [Ya, Propositions 8.6 and 8.7]. One sees that [Ya, Propositions 8.6 and 8.7] hold for $p = 2$ too if we replace a and b by a_2 and a_3. We record the unified formula in a slightly different form as the the following theorem

Theorem 5.7. *Assume that the Gross-Keating invariants of T is $(0, a_2, a_3)$, and let $\mu_p(T)$ be as above. $X = p^{-s}$ and*

$$\delta_{p,i} = \delta_{p,i}(T) = \begin{cases} 1 & \text{if } i = 0, [\frac{a_2}{2}] + 1, \\ 1 - p & \text{if } 1 \le i \le [\frac{a_2}{2}]. \end{cases}$$

(1) When a_2 is odd, one has

$$\frac{W(X, T, L^0)}{1 - p^{-2}X^2} = \sum_{0 \le i \le \frac{a_2 - 1}{2}} p^i (X^{2i} + \mu_p(T) X^{a_2 + a_3 - 2i}),$$

and

$$W(X, T, L^{ra}) = \sum_{0 \le i \le \frac{a_2+1}{2}} \delta_{p,i} p^i (X^{2i} - \mu_p(T) X^{a_2 + a_3 - 2i}),$$

(2) When a_2 is even, one has

$$\frac{W(X, T, L^0)}{1 - p^{-2}X^2} = \sum_{0 \le i \le \frac{a_2}{2} - 1} p^i (X^{2i} + \mu_p(T) X^{a_2 + a_3 - 2i}) + p^{\frac{a_2}{2}} \sum_{0 \le k \le a_3 - a_2} (\epsilon X)^{a_2 + k},$$

and

$$\begin{aligned} W(X, T, L^{ra}) = &\sum_{0 \le i \le \frac{a_2}{2}} \delta_{p,i} p^i (X^{2i} - \mu_p(T) X^{a_2 + a_3 + 2 - 2i}) \\ &- \epsilon p^{\frac{a_2}{2} + 1} (X^{a_2 + 1} - \mu_p(T) X^{a_3 + 1}). \end{aligned}$$

Here $\epsilon = \epsilon(T)$ is the Gross-Keating's ϵ-constant given in [GK, page 236], see also Appendix B.

Because of this theorem, [Ya, Propositions 8.9] and [Ya, Theorem 8.10] are all true for $p = 2$, when one replaces a, b, and v_0 by a_2, a_3, and ϵ respectively.

Corollary 5.8. *Assume $\mu_p(T) = -1$. Then $W(1, T, L^0) = 0$ and $W(1, T, L^1) = 2(p+1)$. Moreover,*

$$\frac{W'(1,T,L^0)}{1-p^{-2}} = \begin{cases} -\sum_{0\le i\le \frac{a_2-1}{2}} p^i(a_2+a_3-4i) & \text{if } a_2 \text{ is odd,} \\ -\sum_{0\le i\le \frac{a_2}{2}-1} p^i(a_2+a_3-4i) - (a_3-a_2-1)p^{\frac{a_2}{2}} & \text{if } a_2 \text{ is even.} \end{cases}$$

In particular, when $diag(1, T)$ is anisotropic,

$$W'(1,T,L^0) = -(1-p^{-2})\alpha_p(\ diag(1,T))$$

where $\alpha_p(\ diag(1, T))$ is the local intersection number defined by Gross and Keating ([GK, Proposition 5.4]).

Corollary 5.9. *Assume $\mu_p(T) = 1$.*

(1) *One has $W(1, T, L^{ra}) = 0$, and*

$$\frac{W(1,T,L^0)}{1-p^{-2}} = \begin{cases} 2\sum_{0\le i\le \frac{a_2-1}{2}} p^i & \text{if } a_2 \text{ is odd,} \\ 2\sum_{0\le i\le \frac{a_2}{2}-1} p^i + p^{\frac{a_2}{2}}(1+\frac{1+\epsilon}{2}(a_3-a_2)) & \text{if } a_2 \text{ is even.} \end{cases}$$

(2) *When a_2 is odd, one has*

$$\begin{aligned} W'(1,T,L^{ra}) &= \sum_{0\le i\le \frac{a_2}{2}+1} \delta_{p,i}p^i(4i-a_2-a_3-2) \\ &= 4\sum_{1\le i\le \frac{a_2+1}{2}} p^i - (p+1)(a_2+a_3+2). \end{aligned}$$

(3) *When a_2 is even, one has*

$$\begin{aligned} W'(1,T,L^{ra}) &= \sum_{0\le i\le \frac{a_2}{2}} \delta_{p,i}p^i(4i-a_2-a_3-2) - \epsilon p^{\frac{a_2}{2}+1}(a_2-a_3) \\ &= 4\sum_{1\le i\le \frac{a_2}{2}+1} p^i + p^{\frac{a_2}{2}+1}(2+(1+\epsilon)(a_3-a_2)) - (p+1)(a_2+a_3+2). \end{aligned}$$

(4) *One has*

$$W'(1,T,L^{ra}) = -(1+p)(a_2+a_3+2) + \frac{2p^3}{p^2-1}W(1,T,L^0).$$

Encouraged by Theorem 5.7, we tried the case $L = L^1$ and obtained the following theorem. It seems very interesting and possible to found a clean formula for $L = L^i$ for any $i \ge 0$.

Proposition 5.10. *Let*

$$L^1 = \{\begin{pmatrix} a & b \\ pc & -a \end{pmatrix} : \, a, b, c \in \mathbb{Z}_p\}$$

with the determinant as the quadratic form. Let T be a p-integral 2×2 symmetric matrix over $\mathbb{Q}_p$ with the Gross-Keating invariants $(0, a_2, a_3)$. Let ϵ and $\mu_p(T)$ be defined as above.

(1) When a_2 is odd, one has

$$W(X, T, L^1) = 1 - \mu_p(T) X^{a_2+a_3+2} + p^{\frac{a_2-1}{2}} \left((p-2) X^{a_2+1} + \mu_p(T) p X^{a_3+1} \right) + (p-1) \sum_{1 \le i \le \frac{a_2-1}{2}} p^{i-1} \left((p+2) X^{2i} + \mu_p(T) p X^{a_2+a_3+2-2i} \right).$$

(2) When a_2 is even, one has

$$W(X, T, L^1) = 1 - \mu_p(T) X^{a_2+a_3+2} + p^{\frac{a_2}{2}} \sum_{a_2 \le i \le a_3} C_k (\epsilon X)^{k+1} + (p-1) \sum_{1 \le i \le \frac{a_2}{2}} p^{i-1} \left((p+2) X^{2i} + \mu_p(T) p X^{a_2+a_3+2-2i} \right)$$

with

$$C_k = \begin{cases} p & \text{if } k = a_2, \\ 2(p-1) & \text{if } a_2 < k < a_3, \\ p-2 & \text{if } k = a_3. \end{cases}$$

Proof. (sketch) When $p \neq 2$, this follows from [Ya, Corollary 8.4] and [Ya, Proposition 8.7]. When $p = 2$, this follows from the Main formulae in Sections 2-4 plus tedious calculations, and can be verified via Song Guo's computer program.

Theorem 5.7. *Let $L = L^1$ be given by (5.5), and let ϵ and $\mu_p(T)$ as in Theorem 5.5. Let $(0, a_2, a_3)$ be the Gross-Keating invariants of $diag(1, T)$.*

(1) When a_2 is odd, one has

$$W(X, T, L^1) = 1 - \mu_2(T) X^{a_2+a_3+2} + 2\mu_2(T) X^{a_2+a_3} + 2 \sum_{1 \le i \le \frac{a_2-1}{2}} 2^i (X^{2i} + \mu_2(T) X^{a_2+a_3-2i}).$$

(2) When a_2 is even, one has

$$W(X, T, L^1) = 1 - \mu_2(T) X^{a_2+a_3+2} + 2^{\frac{a_2}{2}+1} \sum_{a_2 < k \le a_3} (\epsilon X)^k + \sum_{1 \le i \le \frac{a_2}{2}} 2^i (2X^{2i} + \mu_2(T) X^{a_2+a_3+2-2i}).$$

6. Basic Integrals.

First we recall from [Ya, Lemmas 4.3 and 4.4] the following basic formulas for $t = \alpha 2^a$ with $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$

$$I_1(t) = \int_{\mathbb{Z}_2} \psi(\frac{1}{2}tx^2)dx = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ 2^{\frac{a}{2}}\psi(\frac{\alpha}{8})(2,\alpha)^a & \text{if } a < 0, \end{cases} \tag{6.1}$$

Alternatively,

$$I_1(\alpha 2^a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ \frac{a}{2}\psi(\frac{\alpha}{8}) & \text{if } a < 0 \text{ is even}, \\ \frac{a-1}{2}(1+\psi(\frac{\alpha}{4})) & \text{if } a < 0 \text{ is odd}. \end{cases} \tag{6.2}$$

$$I_2(t) = \int_{\mathbb{Z}_2^2} \psi(txy)dxdy = 2^{\min(0,a)}, \tag{6.3}$$

and

$$I_3(t) = \int_{\mathbb{Z}_2^2} \psi(t(x^2+xy+y^2))dxdy = (-2)^{\min(0,a)} \tag{6.4}$$

Here (a, b) is the Hilbert symbol over $\mathbb{Q}_2$.

The following two lemmas will be needed later. We leave the proof of Lemma 6.1 to the reader.

Lemma 6.1. *For $t \in \mathbb{Q}_2$ and $\beta \in \mathbb{Q}_2$, let*

$$I_1(t,\beta) = \int_{\mathbb{Z}_2} \psi(t(x^2+\beta x))dx. \tag{6.5}$$

Then one has

$$I_1(t,\beta) = \begin{cases} \psi(-\frac{1}{4}t\beta^2)I_1(2t) & \textit{if } \beta \in 2\mathbb{Z}_2, \\ char(\mathbb{Z}_2)(2t) & \textit{if } \beta \in \mathbb{Z}_2^*, \\ char(\mathbb{Z}_2)(\beta t) & \textit{if } \beta \notin \mathbb{Z}_2. \end{cases}$$

Lemma 6.2. *For $t \in \mathbb{Q}_2$ and $a, b \in \frac{1}{2}\mathbb{Z}_2$, let*

$$I_3(t; a, b) = \int_{a+\mathbb{Z}_2} \int_{b+\mathbb{Z}_2} \psi(t(x^2 + xy + y^2)) dx dy. \tag{6.6}$$

Then

$$I_3(t; a, b) = \begin{cases} I_3(t) & \text{if } a, b \in \mathbb{Z}_2, \\ char(2\mathbb{Z}_2)(t)\psi(\frac{t}{4}) & \text{else.} \end{cases}$$

Proof. Of course, the formula depends only on $a, b \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. The case $a = b = 0$ follows from the definition. When $a = \frac{1}{2}$ and $b = 0$, one has, by Lemma 6.1

$$\begin{aligned} I_3(t; \frac{1}{2}, 0) &= \int_{\frac{1}{2}+\mathbb{Z}_2} \psi(tx^2) I_1(t, x) dx \\ &= \operatorname{char}(\mathbb{Z}_2)(\frac{1}{2}t)\psi(\frac{1}{4}t) \int_{\mathbb{Z}_2} \psi(t(x^2 + x)) dx \\ &= \operatorname{char}(2\mathbb{Z}_2)(t)\psi(\frac{1}{4}t). \end{aligned}$$

Finally, one has by Lemma 6.1 again

$$\begin{aligned} I_3(t; \frac{1}{2}, \frac{1}{2}) &= \int_{\frac{1}{2}+\mathbb{Z}_2} \psi(tx^2) dx \int_{\frac{1}{2}\mathbb{Z}_2} \psi(t(xy + y^2)) dy - I_3(t; \frac{1}{2}, 0) \\ &= 2\psi(\frac{1}{4}t) \int_{\mathbb{Z}_2} \psi(t(x^2 + x)) I(\frac{t}{4}, 2x) dx - I_3(t; \frac{1}{2}, 0) \\ &= 2 \operatorname{char}(2\mathbb{Z}_2)(t)\psi(\frac{1}{4}t) - \operatorname{char}(2\mathbb{Z}_2)(t)\psi(\frac{1}{4}t) \\ &= \operatorname{char}(2\mathbb{Z}_2)(t)\psi(\frac{1}{4}t). \end{aligned}$$

7. The integral $A(\gamma, L)$.

In this section, we will compute the integral

$$A(\gamma, L) = \int_{L^n} \psi(\operatorname{tr} \gamma q(x)) dx, \tag{7.1}$$

in a general situation. Here $\gamma \in \operatorname{Sym}_n(\mathbb{Q}_2)$ and (L, q) is a non-degenerate quadratic lattice over $\mathbb{Z}_2$, and for $x = (x_1, \cdots, x_n) \in L^n$, $q(x) = \frac{1}{2}((x_i, x_j)) \in \frac{1}{2}\operatorname{Sym}_n(\mathbb{Z}_2)$. We only need the case $n = 2$ in this paper. We first record some useful easy facts.

Lemma 7.1. *(1)* $A(k.\gamma, L) = A(\gamma, L)$ *for any* $k \in GL_n(\mathbb{Z}_2)$. *Here* $k.\gamma = k\gamma^t k$ *is the action of* GL_n *on* Sym_n.

(2) $A(\gamma, L_1 \oplus L_2) = A(\gamma, L_1)A(\gamma, L_2)$.

(3) $A(\ diag(\gamma_1, \gamma_2), L) = A(\gamma_1, L)A(\gamma_2, L)$.

(4) $A(a\gamma, L) = A(\gamma, aL)$ *for any* $a \in \mathbb{Q}_2^*$. *Here* aL *is the same lattice* L *with a new quadratic form* $q_a(x) = aq(x)$.

By this lemma and the classification of quadratic lattices, we need only to consider the following three cases for γ,

$$\gamma \in \mathbb{Q}_2, \quad \gamma = 2^c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{or} \quad \gamma = 2^c \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and three cases for (L, q):

$$\begin{aligned} L_1(t): &\quad L = \mathbb{Z}_2, \quad q(x) = \frac{1}{2}tx^2, \\ L_2(t): &\quad L = \mathbb{Z}_2^2, \quad q(x) = tx_1x_2, \\ L_3(t): &\quad L = \mathbb{Z}_2^2, \quad q(x) = t(x_1^2 + x_1x_2 + x_2^2). \end{aligned}$$

The case $\gamma \in \mathbb{Q}_2$ was dealt with in [Ya, Section 4], we record it here for convenience.

Lemma 7.2. *([Ya, Lemmas 4.3 and 4.4]) For* $\gamma = \beta 2^c$ *and* $t = \alpha 2^a \in \mathbb{Q}_2^*$, *with* $\alpha, \beta \in \mathbb{Z}_2^*$ *and* $a, c \in \mathbb{Z}$, *one has*

$$A(\gamma, L_i(t)) = I_i(t\gamma) = \begin{cases} I_1(t\gamma) & \text{if } i = 1, \\ 2^{\min(0,a+c)} & \text{if } i = 2, \\ (-2)^{\min(0,a+c)} & \text{if } i = 3. \end{cases}$$

Here $I_i(t)$ *are defined in Section 6.*

Lemma 7.3. *Let* $\gamma = 2^c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $t = \alpha 2^a$. *Then*

$$A(\gamma, L_i(t)) = 2^{\min(0,a+c) \dim L_i(t)}$$

for $1 \le i \le 3$.

Lemma 7.4. *Let* $\gamma = 2^c \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ *and* $t = \alpha 2^a$. *Then*

$$A(\gamma, L_i(t)) = (-2)^{\min(0,a+c) \dim L_i(t)}$$

for $1 \le i \le 3$.

Proof. We verify Lemma 7.4 and leave Lemma 7.3 to the reader. Set $\gamma_2 = 2^a$. For $z = \begin{pmatrix} x \\ y \end{pmatrix} \in L^2$, one has

$$\mathrm{tr}\gamma q(z) = \gamma_2 \left((x,x) + (x,y) + (y,y)\right)$$

and thus

$$A(\gamma, L) = \int_{L^2} \psi\left(\gamma_2 \left((x,x) + (x,y) + (y,y)\right)\right) dxdy. \tag{7.2}$$

When $L = L_1(t) = \mathbb{Z}_2$, one has $(x,y) = txy$, and

$$A(\gamma, L_1(t)) = \int_{\mathbb{Z}_2^2} \psi(t\gamma_2(x^2 + xy + y^2))dxdy = I_3(t\gamma_2).$$

When $L = L_2(t) = \mathbb{Z}_2^2$, $(x,y) = t(x_1y_2 + x_2y_1)$ and

$$A(\gamma, L_2(t)) = \int_{\mathbb{Z}_2^4} \psi(t\gamma_2(2x_1x_2 + x_1y_2 + x_2y_1 + 2y_1y_2)dx_1dx_2dy_1dy_2.$$

So

$$\begin{aligned} A(\gamma, L_2(t)) &= \int_{\mathbb{Z}_2^2} \mathrm{char}(\mathbb{Z}_2)(t\gamma_2(2x_2 + y_2))\ \mathrm{char}(\mathbb{Z}_2)(t\gamma_2(2y_2 + x_2))dx_2\,dy_2 \\ &= \int_{\mathbb{Z}_2^2} \mathrm{char}(\mathbb{Z}_2)(t\gamma_2 y_2)\ \mathrm{char}(\mathbb{Z}_2)(t\gamma_2 x_2)dx_2\,dy_2 \\ &= 2^{2\min(1,a+c)}. \end{aligned}$$

Finally, when $L = L_3(t)$, we rewrite (7.2) as

$$A(\gamma, L_3(t)) = \int_{\mathbb{Z}_2^2} \psi(\frac{3}{4}\gamma_2(y,y))dy \int_{\mathbb{Z}_2^2} \psi(\gamma_2(x + \frac{1}{2}y, x + \frac{1}{2}y)dx.$$

Substituting $x' = x + \frac{1}{2}y$, and breaking the integral into 4 parts, one has

$$\begin{aligned} A(\gamma, L_3(t)) = &\sum_{a,b\in\frac{1}{2}\mathbb{Z}/\mathbb{Z}} \int_{(2a+2\mathbb{Z}_2)\times(2b+2\mathbb{Z}_2)} \psi(\frac{3}{4}\gamma_2(y,y))dy \int_{(a+\mathbb{Z}_2)\times(b+\mathbb{Z}_2)} \psi(\gamma_2(x',x'))dx' \\ &= \frac{1}{4}\sum_{a,b} I_3(6t\gamma_2; a, b) I_3(2t\gamma_2, a, b). \end{aligned}$$

Now applying Lemma 6.2, one obtains

$$\begin{aligned} A(\gamma, L_3(t)) &= \frac{1}{4} I_3(2t\gamma_2)^2 + \frac{3}{4}\ \mathrm{char}(2\mathbb{Z}_2)(2t\gamma_2) \\ &= \begin{cases} \frac{1}{4}+\frac{3}{4}=1 & \text{if } t\gamma_2 \in \mathbb{Z}_2, \\ \frac{1}{4} I_3(2t\gamma_2)^2 & \text{if } t\gamma_2 \notin \mathbb{Z}_2 \end{cases} \\ &= (-2)^{2\min(0,a+c)}. \end{aligned}$$

Proof of Lemma 1.2. The case $\gamma \in \mathcal{A}_r(-c)$ with $r \geq 0$ follows from Lemmas 7.1 and 7.2. We check the case $r = -1$ and omit the case $r = -2$ here. Indeed, Lemmas 7.1 and 7.3 imply

$$\begin{aligned} A(\gamma, L) &= \prod_{1\le i\le H} A(\gamma, L_1(\epsilon_i 2^{l_i}) \prod_{1\le j\le M} A(\gamma, L_2(2^{m_j})) \prod_{1\le k\le N} A(\gamma, L_3(2^{n_k}) \\ &= 2^{\sum_i \min(0, l_i - c)} 2^{2\sum_j \min(0, m_j - c)} 2^{2\sum_k \min(0, n_k - c)} \\ &= 2^{g_1(c)}. \end{aligned}$$

8. The integral $B(T, \gamma)$.

In this section, we compute the integral

$$B(T,\gamma) = \int_{\mathrm{GL}_2(\mathbb{Z}_2)} \psi(-\ \mathrm{tr}(k.T\gamma))dk = \int_{\mathrm{GL}_2(\mathbb{Z}_2)} \psi(-Tk.\gamma)dk. \tag{8.1}$$

Here $T, \gamma \in \mathrm{Sym}_2(\mathbb{Q}_2)$ are symmetric matrices over $\mathbb{Q}_2$ of order 2, and we denote $k.T = kT^tk$ for the action of $\mathrm{GL}_2(\mathbb{Q}_2)$ on $\mathrm{Sym}_2(\mathbb{Q}_2)$. Here dk is the restriction of the standard Haar measure on $M_2(\mathbb{Z}_2)$ to $\mathrm{GL}_2(\mathbb{Z}_2)$. Clearly $B(T, \gamma)$ are $\mathrm{GL}_2(\mathbb{Z}_2)$-invariant in both variables in the sense

$$B(k.T, \gamma) = B(T, k.\gamma) = B(T, \gamma) \tag{8.2}$$

for every $k \in \mathrm{GL}_2(\mathbb{Z}_2)$. It is also clear that

$$B(T, \gamma) = B(\gamma, T) \tag{8.3}$$

is symmetric on T and γ.

Let

(8.4)

$$K = K_1(2) = \{k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_2) : c \equiv 0 \bmod 2, a \equiv d \equiv 1 \bmod 2\}.$$

Then

$$\mathrm{GL}_2(\mathbb{Z}_2) = \cup K g_i, \quad g_0 = 1, g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{8.5}$$

Define

$$B_1(T,\gamma) = \int_K \psi(-Tk.\gamma)dk, \tag{8.6}$$

and

$$\begin{aligned} J(t,\gamma) &= \int_{\mathbb{Z}_2^*}\int_{\mathbb{Z}_2} \psi(t(x_1,x_2)\gamma^t(x_1,x_2))dx_1dx_2 \\ &= \int_{\mathbb{Z}_2^*}\int_{\mathbb{Z}_2} \psi(t(\gamma_1x_1^2 + 2\gamma_2x_1x_2 + \gamma_3x_2^2))dx_1dx_2. \end{aligned} \tag{8.7}$$

Lemma 8.1. *Let the notation be as above. Then*

$$B(T,\gamma) = B_1(T,\gamma) + B_1(T,g_1.\gamma) + B_1(T,g_2.\gamma), \tag{8.8}$$

Lemma 8.2. *Let* $T = \ diag(t_1,t_3)$, *and* $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$. *Then*

$$B_1(T,\gamma) = \frac{1}{2}J(-t_1,\gamma)J(-t_3, \begin{pmatrix} \gamma_3 & 2\gamma_2 \\ 2\gamma_2 & 4\gamma_1 \end{pmatrix}).$$

Proof. It follows from the formula

$$\mathrm{tr}Tk.\gamma = t_1(\gamma_1k_1^2 + 2\gamma_2k_1k_2 + \gamma_3k_2^2) + t_3(\gamma_3k_4^2 + 4\gamma_2k_4k_3 + 4\gamma_1k_3^2)$$

for $k = \begin{pmatrix} k_1 & k_2 \\ 2k_3 & k_4 \end{pmatrix} \in K = K_1(2)$.

Lemma 8.3. *Let the notation be as in Lemma 8.2. Then*

$$J(t,\gamma) = \frac{1}{2}\begin{cases} char(\mathbb{Z}_2)(\frac{8t\det\gamma}{\gamma_3})I_1(2t\gamma_3)\psi(\frac{t\det\gamma}{\gamma_3}) & if\ \frac{2\gamma_2}{\gamma_3} \in 2\mathbb{Z}_2, \\ char(\mathbb{Z}_2^2)(8t\gamma_1, 2t\gamma_3)\psi(t\gamma_1) & if\ \frac{2\gamma_2}{\gamma_3} \in \mathbb{Z}_2^*, \\ char(\mathbb{Z}_2^2)(8t\gamma_1, 2t\gamma_2)\psi(t\gamma_1) & if\ \frac{2\gamma_2}{\gamma_3} \notin \mathbb{Z}_2 \end{cases}$$

Proof. Indeed, one has by definition

$$\begin{aligned} J(t,\gamma) &= \int_{\mathbb{Z}_2^*} \psi(t\gamma_1k_1^2)dk_1 \int_{\mathbb{Z}_2} \psi(t\gamma_3\left(k_2^2 + \frac{2k_1\gamma_2}{\gamma_3}k_2\right)dk_2 \\ &= \int_{\mathbb{Z}_2^*} \psi(t\gamma_1k_1^2)I_1(t\gamma_3, \frac{2k_1\gamma_2}{\gamma_3})dk_1. \end{aligned}$$

When $\frac{2\gamma_2}{\gamma_3} \in 2\mathbb{Z}_2$, one has by Lemma 6.1

$$\begin{aligned} J(t,\gamma) &= I_1(2t\gamma_3)\int_{1+2\mathbb{Z}_2}\psi(\frac{t\det\gamma}{\gamma_3}k_1^2)dk_1 \\ &= \frac{1}{2}I_1(2t\gamma_3)\psi(\frac{t\det\gamma}{\gamma_3})\int_{\mathbb{Z}_2}\psi(\frac{4t\det\gamma}{\gamma_3}(k_1^2+k_1))dk_1 \\ &= \frac{1}{2}I_1(2t\gamma_3)\psi(\frac{t\det\gamma}{\gamma_3})\ \mathrm{char}(\mathbb{Z}_2)(\frac{8t\det\gamma}{\gamma_3}) \end{aligned}$$

as desired. When $\frac{2\gamma_2}{\gamma_3} \in \mathbb{Z}_2^*$, one has by Lemma 8.1 (notice that $k_1 \in \mathbb{Z}_2^*$)

$$\begin{aligned} J(t,\gamma) &= \mathrm{char}(\mathbb{Z}_2)(2t\gamma_3)\int_{1+2\mathbb{Z}_2}\psi(t\gamma_1k_1^2)dk_1 \\ &= \frac{1}{2}\ \mathrm{char}(\mathbb{Z}_2)(2t\gamma_3)\psi(t\gamma_1)\int_{\mathbb{Z}_2}\psi(4t\gamma_1(k_1^2+k_1))dk_1 \\ &= \frac{1}{2}\ \mathrm{char}(\mathbb{Z}_2)(2t\gamma_3)\psi(t\gamma_1)\ \mathrm{char}(\mathbb{Z}_2)(8t\gamma_1) \end{aligned}$$

as desired. The case $\frac{2\gamma_2}{\gamma_3} \notin \mathbb{Z}_2$ is the same.

Now we have our first main formula of this section.

Proposition 8.4. *Assume that $T = \mathit{diag}(t_1, t_3)$ with $t_3/t_1 \in \mathbb{Z}_2$. Then*

(1) When $\frac{\gamma_2}{\gamma_3} \in \mathbb{Z}_2$

$$8B_1(T,\gamma) = \begin{cases} \mathit{char}(\mathbb{Z}_2^2)(\frac{8t_1\det\gamma}{\gamma_3}, \frac{8t_3\det\gamma}{\gamma_1})I_1(-2t_1\gamma_3)I_1(-8t_3\gamma_1) \\ \qquad \cdot\psi(-\frac{t_1\det\gamma}{\gamma_3} - \frac{t_3\det\gamma}{\gamma_1}) \\ \mathit{char}(\mathbb{Z}_2^3)(\frac{8t_1\det\gamma}{\gamma_3}, 8t_3\gamma_3, 8t_3\gamma_1)I_1(-2t_1\gamma_3)\psi(-\frac{t_1\det\gamma}{\gamma_3} - t_3\gamma_3) \\ \mathit{char}(\mathbb{Z}_2^3)(\frac{8t_1\det\gamma}{\gamma_3}, 8t_3\gamma_3, 4t_3\gamma_2)I_1(-2t_1\gamma_3)\psi(-\frac{t_1\det\gamma}{\gamma_3} - t_3\gamma_3) \end{cases}$$

according to $\frac{\gamma_2}{\gamma_1} \in 2\mathbb{Z}_2, \mathbb{Z}_2^*$ or $\notin \mathbb{Z}_2$.

(2) When $\frac{2\gamma_2}{\gamma_3} \in \mathbb{Z}_2^*$, one has then

$$8B_1(T,\gamma) = \begin{cases} \mathit{char}(\mathbb{Z}_2^3)(8t_1\gamma_1, 2t_1\gamma_3, \frac{8t_3\det\gamma}{\gamma_1})I_1(-8t_3\gamma_1)\psi(-t_1\gamma_1 - \frac{t_3\det\gamma}{\gamma_1}) \\ \mathit{char}(\mathbb{Z}_2^2)(8t_1\gamma_1, 2t_1\gamma_3)\psi(-t_1\gamma_1 - t_3\gamma_3), \\ \mathit{char}(\mathbb{Z}_2^3)(8t_1\gamma_1, 2t_1\gamma_3, 4t_3\gamma_2)\psi(-t_1\gamma_1 - t_3\gamma_3), \end{cases}$$

according to $\frac{\gamma_2}{\gamma_1} \in 2\mathbb{Z}_2, \mathbb{Z}_2^*$ or $\notin \mathbb{Z}_2$.

(3) When $\frac{2\gamma_2}{\gamma_3} \notin \mathbb{Z}_2$*, one has*

$$8B_1(T,\gamma) = \begin{cases} char(\mathbb{Z}_2^3)(8t_1\gamma_1, 2t_1\gamma_2, \frac{8t_3\det\gamma}{\gamma_1})I_1(-8t_3\gamma_1)\psi(-t_1\gamma_1 - \frac{t_3\det\gamma}{\gamma_1}), \\ char(\mathbb{Z}_2^3)(8t_1\gamma_1, 2t_1\gamma_2, 8t_3\gamma_3)\psi(-t_1\gamma_1 - t_3\gamma_3), \\ char(\mathbb{Z}_2^3)(8t_1\gamma_1, 2t_1\gamma_2, 8t_3\gamma_3)\psi(-t_1\gamma_1 - t_3\gamma_3), \end{cases}$$

according to $\frac{\gamma_2}{\gamma_1} \in 2\mathbb{Z}_2, \mathbb{Z}_2^*$ *or* $\notin \mathbb{Z}_2$*.*

Corollary 8.5. *Let* $T = diag(t_1, t_3)$ *with* $t_3/t_1 \in \mathbb{Z}_2$*, and* $\gamma = diag(\gamma_1, \gamma_3)$ *with* $\gamma_3/\gamma_1 \in \mathbb{Z}_2$*. Let* $g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *as in (8.5).*

(1) One has

$$B_1(T,\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2)(8t_1\gamma_1)I_1(-2t_1\gamma_3)I_1(-8t_3\gamma_1)\psi(-t_1\gamma_1 - t_3\gamma_3),$$

(2) One has

$$B_1(T, g_1.\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2^2)(8t_1\gamma_3, 8t_3\gamma_1)I_1(-2t_1\gamma_1)I_1(-8t_3\gamma_3)\psi(-t_1\gamma_3 - t_3\gamma_1).$$

(3) When $\gamma_3/\gamma_1 \in 2\mathbb{Z}_2$*, one has*

$$B_1(T, g_2.\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2^2)(8t_1\gamma_3, 8t_3\gamma_1)I_1(-2t_1(\gamma_1+\gamma_3))\psi(-\frac{t_1\gamma_1\gamma_3}{\gamma_1+\gamma_3} - t_3(\gamma_1+\gamma_3)).$$

In particular, if $c_1 \le c_3 - 3$*, then*

$$B_1(T, g_2.\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2^2)(8t_1\gamma_3, 8t_3\gamma_1)I_1(-2t_1\gamma_1)\psi(-t_1\gamma_3 - t_3\gamma_1).$$

(4) When $\gamma_3/\gamma_1 = u \in \mathbb{Z}_2^*$*, one has*

$$B_1(T, g_2.\gamma) = \begin{cases} \frac{1}{8}\ char(\mathbb{Z}_2)(4t_1\gamma_1)\psi(-t_1\gamma_1 - 2t_3\gamma_1) & \text{if } u \equiv 1 \bmod 4, \\ \frac{1}{8}\ char(\mathbb{Z}_2)(2t_1\gamma_1)\psi(-t_1\gamma_1) & \text{if } u \equiv -1 \bmod 4. \end{cases}$$

Proof. We verify (3) and (4), and leave (1) and (2) to the reader. Write

$$g_2.\gamma = \begin{pmatrix} \tilde\gamma_1 & \tilde\gamma_2 \\ \tilde\gamma_2 & \tilde\gamma_3 \end{pmatrix}$$

with $\tilde\gamma_1 = \tilde\gamma_2 = \gamma_3$ and $\tilde\gamma_3 = \gamma_1 + \gamma_3$.

When $c_1 < c_3$, one has $\frac{2\tilde{\gamma}_2}{\tilde{\gamma}_3} \in 2\mathbb{Z}_2$, and $\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1} = 1 \in \mathbb{Z}_2^*$. So Proposition 8.4 gives

$$\begin{aligned} 8B_1(T, g_2.\gamma) = & \ \mathrm{char}(\mathbb{Z}_2^3)(\frac{8t_1\gamma_1\gamma_3}{\tilde{\gamma}_3}, 8t_3\tilde{\gamma}_3, 8t_3\tilde{\gamma}_1) I_1(-t_1\tilde{\gamma}_3)\psi\left(-\frac{t_1\gamma_1\gamma_3}{\tilde{\gamma}_3} - t_3\tilde{\gamma}_3\right) \\ = & \ \mathrm{char}(\mathbb{Z}_2^2)(8t_1\gamma_3, 8t_3\gamma_1) I_1(-t_1(\gamma_1+\gamma_3))\psi\left(-\frac{t_1\gamma_1\gamma_3}{\gamma_1+\gamma_3} - t_3\tilde{\gamma}_3\right). \end{aligned}$$

When $\gamma_3 = \gamma_1 u$ with $u \equiv 1 \bmod 4$, one has $\tilde{\gamma}_3 = \gamma_1(1+u)$ and $\frac{2\tilde{\gamma}_2}{\tilde{\gamma}_3} = \frac{2}{1+u} \in \mathbb{Z}_2^*$. One still has $\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1} = 1 \in \mathbb{Z}_2^*$, and one has then by Proposition 8.4

$$\begin{aligned} 8B_1(T, g_2.\gamma) = & \ \mathrm{char}(\mathbb{Z}_2^3)(8t_1\gamma_3, 2t_1\gamma_1(1+u), 8t_3\gamma_1(1+u))\psi(-t_1\gamma_3 - t_3(t_1+\gamma_3)) \\ = & \ \mathrm{char}(4t_1\gamma_1)\psi(-t_1\gamma_1 - 2t_3\gamma_1). \end{aligned}$$

Here we have used the fact $t_i\gamma_3 \equiv t_i\gamma_1 \bmod 4t_1\gamma_1$.

When $\gamma_3 = \gamma_1 u$ with $u \equiv -1 \bmod 4$, one has $\tilde{\gamma}_3 = \gamma_1(1+u)$ and $\frac{2\tilde{\gamma}_2}{\tilde{\gamma}_3} = \frac{2}{1+u} \notin \mathbb{Z}_2$. One still has $\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1} = 1 \in \mathbb{Z}_2^*$, and one has then by Proposition 8.4

$$\begin{aligned} 8B_1(T, g_2.\gamma) = & \ \mathrm{char}(\mathbb{Z}_2^3)(8t_1\gamma_3, 2t_1\gamma_3, 8t_3\gamma_1(1+u))\psi(-t_1\gamma_3 - t_3(t_1+\gamma_3)) \\ = & \ \mathrm{char}(\mathbb{Z}_2)(2t_1\gamma_1)\psi(t_1\gamma_1). \end{aligned}$$

Proof of Lemma 3.1: In our new notation, $t_1 = \alpha_1 2^a$, $t_3 = \alpha_2 2^a$, $\gamma_1 = \beta_1 2^{-c}$, and $\gamma_3 = \beta_2 2^{r-c}$.

We first look at the case $r \geq 1$. We have by Lemma 6.1

$$B(T, \gamma) = B(\gamma, T) = B_1(\gamma, T) + B_2(\gamma, T) + B_3(\gamma, T)$$

where $B_2(\gamma, T) = B_1(\gamma, g_1.T)$, and $B_3(\gamma, T) = B_1(\gamma, g_2.T)$. Applying Corollary 8.5, one has

$$\begin{aligned} 8B_1(\gamma, T) = & \ \mathrm{char}(\mathbb{Z}_2)(2^{a+3-c}) I_1(-2^{a+1-c}\alpha_2\beta_1)\psi(-2^{a-c}\alpha_1\beta_1 - 2^{a-c+r}\alpha_2\beta_2) \\ = & \begin{cases} 0 & \text{if } c > a+3, \\ I_1(-\frac{\alpha_2\beta_1}{4})\psi(-\frac{\alpha_1\beta_1 + 2^r\alpha_2\beta_2}{8}) & \text{if } c = a+3, \\ I_1(-\frac{\alpha_2\beta_1}{2})\psi(-\frac{\alpha_1\beta_1 + 2^r\alpha_2\beta_2}{4}) & \text{if } c = a+2, \\ 0 & \text{if } c = a+1, \\ 1 & \text{if } c \leq a, \end{cases} \end{aligned}$$

$$\begin{aligned} 8B_2(\gamma, T) = & \ \mathrm{char}(\mathbb{Z}_2)(2^{a+3-c}) I_1(-2^{a+1-c}\alpha_1\beta_1)\psi(-2^{a-c}\alpha_2\beta_1 - 2^{a-c+r}\alpha_1\beta_2) \\ = & \begin{cases} 0 & \text{if } c > a+3, \\ I_1(-\frac{\alpha_1\beta_1}{4})\psi(-\frac{\alpha_2\beta_1 + 2^r\alpha_1\beta_2}{8}) & \text{if } c = a+3, \\ I_1(-\frac{\alpha_1\beta_1}{2})\psi(-\frac{\alpha_2\beta_1 + 2^r\alpha_1\beta_2}{4}) & \text{if } c = a+2, \\ 0 & \text{if } c = a+1, \\ 1 & \text{if } c \leq a, \end{cases} \end{aligned}$$

and

$$8B_3(\gamma, T) = \begin{cases} \mathrm{char}(\mathbb{Z}_2)(2^{a+2-c})\psi(-2^{a-c}\alpha_1\beta_1 - 2^{a+1-c+r}\alpha_1\beta_2) & \text{if } \alpha_2 \equiv \alpha_1 \bmod 4, \\ \mathrm{char}(\mathbb{Z}_2)(2^{a+2-1})\psi(2^{a-c}\alpha_1\beta_1) & \text{if } \alpha_2 \equiv -\alpha_1 \bmod 4 \end{cases}$$

When $c = a+3$, one has then $I_1(\frac{b}{4}) = \frac{1}{2}\psi(\frac{b}{8})$ for any $b \in \mathbb{Z}_2^*$ and thus

$$\begin{aligned} 8B(T, \gamma) &= \frac{1}{2}\psi(-\frac{\alpha_2\beta_1}{8})\psi(-\frac{\alpha_1\beta_1 + 2^r\alpha_2\beta_2}{8}) + \frac{1}{2}\psi(-\frac{\alpha_1\beta_1}{8})\psi(-\frac{\alpha_2\beta_1 + 2^r\alpha_1\beta_2}{8}) \\ &= \frac{1}{2}\psi(-\frac{\alpha\beta_1}{8})\left(\psi(-2^{r-3}\alpha_1\beta_2) + \psi(-2^{r-3}\alpha_2\beta_2)\right). \end{aligned}$$

When $c = a+2$, $I_1(\frac{b}{2}) = \frac{1}{2}(1 + \psi(\frac{b}{4}))$ for every $b \in \mathbb{Z}_2^*$. So

$$\begin{aligned} 8(B_1 + B_2) &= \frac{1}{2}\left(1 + \psi(-\frac{\alpha_2\beta_1}{4})\right)\psi(-\frac{\alpha_1\beta_1 + 2^r\alpha_2\beta_2}{4}) \\ &\quad + \frac{1}{2}\left(1 + \psi(-\frac{\alpha_1\beta_1}{4})\right)\psi(-\frac{\alpha_2\beta_1 + 2^r\alpha_1\beta_2}{4}) \\ &= \psi(2^{r-2})\left(\psi(-\frac{\alpha\beta_1}{4}) + \frac{1}{2}\psi(-\frac{\alpha_1\beta_1}{4}) + \frac{1}{2}\psi(-\frac{\alpha_2\beta_1}{4})\right). \end{aligned}$$

Here we have used the fact $\psi(2^{r-2}b) = \psi(2^{r-2})$ for $r \geq 1$. If $\alpha_2 \equiv \alpha_1 \bmod 4$, then

$$\begin{aligned} 8B(T, \gamma) &= \psi(2^{r-2})\left(\psi(-\frac{\alpha\beta_1}{4}) + \frac{1}{2}\psi(-\frac{\alpha_1\beta_1}{4}) + \frac{1}{2}\psi(-\frac{\alpha_2\beta_1}{4})\right) + \psi(-\frac{\alpha_1\beta_1}{4}) \\ &= \begin{cases} 1 & \text{if } r = 1, \\ -1 + 2\psi(-\frac{\alpha_1\beta_1}{4}) & \text{if } r \geq 2. \end{cases} \\ &= -\psi(2^{r-2}) + (\psi(2^{r-2}) + 1)\psi(-\frac{\alpha_1\beta_1}{4}) \end{aligned}$$

If $\alpha_2 \equiv -\alpha_1 \bmod 4$, then

$$8B(T, \gamma) = 8(B_1 + B_2) = \psi(2^{r-2}).$$

Notice that, in both cases, one has

$$8B(T, \gamma) = \begin{cases} \psi(-\frac{\alpha\beta_1}{4}) + \psi(-\frac{\alpha_1\beta_1}{4}) + \psi(-\frac{\alpha_2\beta_1}{4}) & \text{if } r \geq 2, \\ (-1)^{\frac{\alpha_1 - \alpha_2}{2}} & \text{if } r = 1. \end{cases}$$

The case $c \leq a+1$ is obvious. This proves (1) – (3) of Lemma 3.1. For the case $r = 0$, we use

$$B(T, \gamma) = B_1(T, \gamma) + B_2(T, \gamma) + B_3(T, \gamma).$$

By Corollary 8.5, one has

$$\begin{aligned} 8B_1(T,\gamma) = & \ \mathrm{char}(\mathbb{Z}_2)(2^{a+3-c}) I_1(-2^{a+1-c}\alpha_1\beta_2) \\ & \cdot I_1(-2^{a+3-c}\alpha_2\beta_1)\psi(-2^{a-c}(\alpha_1\beta_1+\alpha_2\beta_2)) \\ = & \begin{cases} 0 & \text{if } c > a+2, \\ I_1(-\frac{\alpha_1\beta_2}{2})\psi(-\frac{\alpha_1\beta_1+\alpha_2\beta_2}{4}) & \text{if } c = a+2, \\ 0 & \text{if } c = a+1, \\ 1 & \text{if } c \le a, \end{cases} \end{aligned}$$

$$\begin{aligned} 8B_2(T,\gamma) = & \ \mathrm{char}(\mathbb{Z}_2)(2^{a+3-c}) I_1(-2^{a+1-c}\alpha_1\beta_1) \\ & \cdot I_1(-2^{a+3-c}\alpha_2\beta_2)\psi(-2^{a-c}(\alpha_1\beta_2+\alpha_2\beta_1)) \\ = & \begin{cases} 0 & \text{if } c > a+2, \\ I_1(-\frac{\alpha_1\beta_1}{2})\psi(-\frac{\alpha_1\beta_2+\alpha_2\beta_1}{4}) & \text{if } c = a+2, \\ 0 & \text{if } c = a+1, \\ 1 & \text{if } c \le a, \end{cases} \end{aligned}$$

and

$$\begin{aligned} 8B_3(T,\gamma) &= \begin{cases} cha(\mathbb{Z}_2)(2^{a+2-c})\psi(-2^{a-c}\alpha_1\beta_1 - 2^{a+1-c}\alpha_2\beta_1) & \text{if } \beta_2 \equiv \beta_1 \bmod 4 \\ cha(\mathbb{Z}_2)(2^{a+1-c})\psi(-2^{a-c}\alpha_1\beta_1) & \text{if } \beta_2 \equiv -\beta_1 \bmod 4 \end{cases} \\ &= \begin{cases} 0 & \text{if } c > a+2, \\ -\psi(-\frac{\alpha_1\beta_1}{4}) & \text{if } c = a+2, \beta_2 \equiv \beta_1 \bmod 4, \\ 0 & \text{if } c = a+2, \beta_2 \equiv -\beta_1 \bmod 4, \\ -1 & \text{if } c = a+1, \\ 1 & \text{if } c \le a. \end{cases} \end{aligned}$$

Since $I_1(\frac{b}{2}) = \frac{1}{2}(1+\psi(\frac{b}{2}))$ for $b \in \mathbb{Z}_2^*$, one has when $c = a+2$ and $\beta_2 \equiv \beta_1 \bmod 4$

$$\begin{aligned} 8B(T,\gamma) &= 2I_1(-\frac{\alpha_1\beta_1}{2})\psi(-\frac{\alpha\beta_1}{)} - \psi(-\frac{\alpha_1\beta_1}{4}) \\ &= \psi(-\frac{\alpha\beta_1}{4}) - \psi(-\frac{\alpha_2\beta_1}{4}) - \psi(-\frac{\alpha_1\beta_1}{4}) \end{aligned}$$

as claimed. When $c = a+2$ and $\beta_2 \equiv -\beta_1 \bmod 4$, one has

$$\begin{aligned} 8B(T,\gamma) &= \frac{1}{2}\left(1+\psi(\frac{\alpha_1\beta_1}{4}\right)\psi(\frac{(\alpha_2-\alpha_1)\beta_1}{4}) + \frac{1}{2}\left(1+\psi(-\frac{\alpha_1\beta_1}{4}\right)\psi(\frac{(\alpha_1-\alpha_2)\beta_1}{4}) \\ &= \psi(\frac{(\alpha_2-\alpha_1)\beta_1}{4}) \end{aligned}$$

as claimed.

Proof of Lemma 4.1: The proof of Lemma 4.1 is again a routine calculation from Corollary 8.5 and is left to the reader. Proposition 8.4 also gives

Corollary 8.6. *Let* $T = \operatorname{diag}(t_1, t_3)$ *with* $t_3/t_1 \in \mathbb{Z}_2$. *Let* $g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *as in (8.5).*

(1) *When* $\gamma = \gamma_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *one has*

$$B_1(T, \gamma) = B_1(T, g_1.\gamma) = \frac{1}{8}\ \mathit{char}(\mathbb{Z}_2)(2t_1\gamma_2),$$

and

$$B_1(T, g_2.\gamma) = \frac{1}{8}\ \mathit{char}(\mathbb{Z}_2)(4t_1\gamma_2)\psi(-2t_3\gamma_2).$$

(2) *When* $\gamma = \gamma_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, *one has*

$$B_1(T, \gamma) = B_1(T, g_1.\gamma) = \frac{1}{8}\ \mathit{char}(\mathbb{Z}_2)(4t_1\gamma_2)\psi(-2(t_1 + t_3)\gamma_2),$$

and

$$B_1(T, g_2.\gamma) = \frac{1}{8}\ \mathit{char}(\mathbb{Z}_2)(4t_1\gamma_2)\psi(-2(t_1 + 3t_3)\gamma_2).$$

Proposition 8.7. *Let* g_i *be as in* (8.5) *with* $i = 0, 1, 2$. *Let* $T = t\begin{pmatrix} d_1 & 1 \\ 1 & d_1 \end{pmatrix}$, *and* $\gamma = \gamma_2 \begin{pmatrix} d_2 & 1 \\ 1 & d_2 \end{pmatrix}$, *with* $d_j = 0, 2$. *Then one has*

$$B_1(T, g_i.\gamma) = \frac{1}{8}\ \mathit{char}(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2).$$

Proof. First assume $T = t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We will prove a slightly stronger result. For a general $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$ and $k = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, one has

$$\mathrm{tr} Tk.\gamma = 2t(\gamma_1 k_3 + \gamma_2 k_4)k_1 + 2t(\gamma_2 k_3 + \gamma_3 k_4)k_2.$$

So

$$\begin{aligned} B_1(-T, \gamma) &= \int_{2\mathbb{Z}_2 \times \mathbb{Z}_2^*} dk_3 dk_4 \int_{1+2\mathbb{Z}_2} \psi(2t(\gamma_1 k_3 + \gamma_2 k_4)k_1) dk_1 \\ &\quad \cdot \int_{\mathbb{Z}_2} \psi(2t(\gamma_2 k_3 + \gamma_3 k_4)k_2) dk_2 \\ &= \frac{1}{2} \int_{2\mathbb{Z}_2 \times \mathbb{Z}_2^*} \psi(2t\gamma_1 k_3 + 2t\gamma_2 k_4)\ \mathrm{char}(\mathbb{Z}_2)(4t(\gamma_1 k_3 + \gamma_2 k_4)) \\ &\quad \cdot\ \mathrm{char}(\mathbb{Z}_2)(2t(\gamma_2 k_3 + \gamma_3 k_4)) dk_3 dk_4. \end{aligned}$$

If $\gamma_1/\gamma_2 \in \mathbb{Z}_2$, then $k_3 \in 2\mathbb{Z}_2$ and $k_4 \in \mathbb{Z}_2^*$ imply

$$\mathrm{char}(\mathbb{Z}_2)(4t(\gamma_1 k_3 + \gamma_2 k_4)) = \mathrm{char}(\mathbb{Z}_2)(4t\gamma_2).$$

Moreover, when $4t\gamma_2 \in \mathbb{Z}_2$, one has

$$\mathrm{char}(\mathbb{Z}_2)(2t(\gamma_2 k_2 + \gamma_3 k_3)) = \mathrm{char}(\mathbb{Z}_2)(2t\gamma_3).$$

So

$$\begin{aligned} B_1(T,\gamma) &= \frac{1}{2}\ \mathrm{char}(\mathbb{Z}_2^2)(4t\gamma_2, 2t\gamma_3) \int_{2\mathbb{Z}_2\times\mathbb{Z}_2^*} \psi(2t\gamma_1 k_3 + 2t\gamma_2 k_4) dk_3\, dk_4 \\ &= \frac{1}{8}\ \mathrm{char}(\mathbb{Z}_2^2)(4t\gamma_2, 2t\gamma_3)\psi(2t\gamma_2) \end{aligned}$$

So we have proved

Lemma 8.8. *Let* $T = t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*, and* $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$ *with* $\gamma_1/\gamma_2 \in \mathbb{Z}_2$*. Then*

$$B_1(T,\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2^2)(4t\gamma_2, 2t\gamma_3)\psi(2t\gamma_2).$$

Now the case $T = t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ follows from this lemma immediately. The case $T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ follows from the following lemma easily.

Lemma 8.9. *Let* $T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$*, and* $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$ *with* $\gamma_1/\gamma_2, \gamma_3/\gamma_2 \in 2\mathbb{Z}_2$*. Then*

$$B_1(T,\gamma) = \frac{1}{8}\ char(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2).$$

Proof. One checks that

$$\mathrm{tr} Tk.\gamma = 2t\gamma_1\left(k_1^2 + (k_3 + \frac{2\gamma_2}{\gamma_1}k_2 + \frac{\gamma_2}{\gamma_1}k_4)k_1\right) + Q$$

with

$$Q = 2\gamma_3 k_4^2 + 4\gamma_2 k_3 k_4 + 2\gamma_3 k_2 k_4 + 2\gamma_2 k_2 k_3 + 2\gamma_1 k_3^2 + 2\gamma_3 k_2^2.$$

Now Lemma 1.1 implies

$$\begin{aligned}
&\int_{\mathbb{Z}_2^*} \psi(2t\gamma_1 \left(k_1^2 + (k_3 + \frac{2\gamma_2}{\gamma_1}k_2 + \frac{\gamma_2}{\gamma_1}k_4)k_1\right))dk_1 \\
&= \operatorname{char}(\mathbb{Z}_2)(2t\gamma_1 \left(k_3 + \frac{2\gamma_2}{\gamma_1}k_2 + \frac{\gamma_2}{\gamma_1}k_4\right)) - \frac{1}{2}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_1 \left(k_3 + \frac{2\gamma_2}{\gamma_1}k_2 + \frac{\gamma_2}{\gamma_1}k_4\right)) \\
&= \operatorname{char}(\mathbb{Z}_2)(2t\gamma_2) - \frac{1}{2}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_2) \\
&= \frac{1}{2}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2).
\end{aligned}$$

On the other hand, when $4t\gamma_2 \in \mathbb{Z}_2$, $k_3 \in 2\mathbb{Z}_2$, one has $2t\gamma_3 \in Z$ and thus $Q \in Z$. So

$$\begin{aligned}
B_1(-T,\gamma) &= \int \psi(Q)dk_2dk_3dk_4 \int_{\mathbb{Z}_2^*} \psi(2t\gamma_1 \left(k_1^2 + (k_3 + \frac{2\gamma_2}{\gamma_1}k_2 + \frac{\gamma_2}{\gamma_1}k_4)k_1\right))dk_1 \\
&= \frac{1}{2}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2)\int \psi(Q)dk_2dk_3dk_4 \\
&= \frac{1}{2}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2)\int_{\mathbb{Z}_2\times 2\mathbb{Z}_2\times\mathbb{Z}_2^*} dk_2dk_3dk_4 \\
&= \frac{1}{8}\operatorname{char}(\mathbb{Z}_2)(4t\gamma_2)\psi(2t\gamma_2).
\end{aligned}$$

Proof of Lemma 2.1 Now Lemma 2.1 follows from Corollary 8.6 and Proposition 8.7. We check the case $T \in \mathcal{A}_{-1}(a)$ with and leave the other case to the reader. When $r < 0$, Proposition 8.7 asserts

$$\begin{aligned}
B(T,\gamma) &= B_1(T,\gamma) + B_1(T, g_1.\gamma) + B_1(T, g_2.\gamma) \\
&= \frac{3}{8}\operatorname{char}(\mathbb{Z}_2)(2^{a+2-c})\psi(2^{a+1-c}) \\
&= \frac{3}{8}f_2(a,c).
\end{aligned}$$

When $r \geq 0$, Corollary 8.6 implies

$$\begin{aligned}
B(T,\gamma) &= B(\gamma, T) \\
&= B_1(\gamma, T) + B_1(\gamma, g_1.T) + B_1(\gamma, g_2.T) \\
&= \frac{1}{4}\operatorname{char}(\mathbb{Z}_2)(2^{a+1-c}) + \frac{1}{8}\operatorname{char}(\mathbb{Z}_2)(2^{a+2-c})\psi(2^{a+1-c+r}) \\
&= \begin{cases} \frac{1}{8}f_1(a,c)f_2(a,c) & \text{if } r = 0, \\ \frac{1}{8}f_1(a,c) & \text{if } r > 0. \end{cases}
\end{aligned}$$

9. The constant $C(\gamma)$.

In this section, we compute the constant $C(\gamma)$ in (1.5) when $n = 2$ and prove Lemma 1.4. We start with some general calculation related with $C(\gamma)$. Recall that $C(\gamma)$ is defined via

$$C(\gamma) \int_{\mathrm{GL}_n(\mathbb{Z}_2)} \phi(k.\gamma) dk = \int_{O_\gamma} \phi(t)\, dt \tag{9.1}$$

for every locally constant function ϕ on $\mathcal{O}_\gamma$. For a positive integer $m > 0$, Let $O_{m,\gamma}$ be the $(1+2^m M_n(\mathbb{Z}_2))$-orbit of γ, and let $O_m(\gamma) = (1+2^m M_n(\mathbb{Z}_2)) \cap O(\gamma)$ be the stabilizer of γ in $1 + 2^m M_n(\mathbb{Z}_2)$. Let ϕ be the characteristic function of $O_{m,\gamma}$, then (9.1) gives

$$C(\gamma)\mathrm{vol}((1 + 2^m M_n(\mathbb{Z}_2))O(\gamma)) = \mathrm{vol}(O_{m,\gamma}).$$

So we have

$$C(\gamma) = \frac{2^{mn^2}\mathrm{vol}(O_{m,\gamma})}{[O(\gamma) : O_m(\gamma)]}. \tag{9.2}$$

Let

$$\gamma = \ \mathrm{diag}(\epsilon_1 2^{l_1}, \cdots , \epsilon_H 2^{l_H}, 2^{m_1}A, \cdots , 2^{m_M}A, 2^{n_1}B, \cdots , 2^{n_N}B),$$

with $\epsilon_i \in \mathbb{Z}_2^*$, $l_i, m_j, n_k \in \mathbb{Z}_2$, and $H + 2M + 2N = n$, and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We reorder l_i, m_j, m_j, n_k, n_k into an increasing sequence $l'_1 \le l'_2 \le \cdots \le l'_n$. We assume that γ is invertible, i.e, $l'_n < \infty$ (Notice that these numbers are not the same as those in Section 1).

Lemma 9.1. *Let the notation be as above. Then one has when $m >> 0$*

$$vol(O_{m,\gamma}) = 2^{-\frac{mn(n+1)}{2} - n} \prod_{1 \le i \le n} 2^{-(n-i+1)l'_i}.$$

Proof. A typical element in $O_{m,\gamma}$ is

$$\alpha = (\alpha_{ij}) = (1 + 2^m k).\gamma = \gamma + 2^m(k\gamma + \gamma^t k) + 2^{2m}k.\gamma$$

with $k=(k_{ij}) \in M_n(\mathbb{Z}_2)$. We choose $m>0$ so that $m+l_1' \ge l_n'+4$. Then we have

$$\alpha_{ij}-\gamma_{ij} \equiv 2^m \sum_{a=1}^{n} k_{ia}(\gamma_{ja}+k_{ja}\gamma_{ia}) \bmod 2^{m+l_n'+4}.$$

From this, one can easily check that $\alpha_{ij}-\gamma_{ij}$ runs over the set $2^{m+\delta_{ij}+l_{ij}} M_n(\mathbb{Z}_2)$, where δ_{ij} is the usual Kronecker symbol, and l_{ij} are given as follows.

$$l_{ii} = \begin{cases} l_i & \text{if } 1 \le i \le H, \\ m_j & \text{if } 1 \le j = [\frac{i-H+1}{2}] \le M, \\ n_k & \text{if } 1 \le k = [\frac{i-H+1}{2}] - M \le N, \end{cases}$$

and $l_{ij} = \min(l_{ii}, l_{jj})$. So

$$vol(O_{m,\gamma}) = 2^{-\frac{mn(n+1)}{2}-n} \prod_{1 \le i \le j \le n} 2^{-l_{ij}}.$$

Clearly, l_i' appears as l_{ij} with multiplicity $(n-i+1)$, and so

$$\prod_{1 \le i \le j \le n} 2^{-l_{ij}} = \prod_{1 \le i \le n} 2^{-(n+1-i)l_i'}.$$

Next we compute $[O(\gamma) : O_m(\gamma)]$ when γ is of rank 2. There are three cases.

Lemma 9.2. *Let $\gamma = diag(\gamma_1, \gamma_3)$ with $\gamma_3/\gamma_1 \in \mathbb{Z}_2$. Let $r = ord_2(\gamma_3/\gamma_1)$. Then*

$$[O(\gamma) : O_m(\gamma)] = \begin{cases} 2^{m+2} & \text{if } c \ge 3, \\ 2^{m+1} & \text{if } 1 \le r \le 2, \\ 2^{m+1} & \text{if } r=0, \gamma_3/\gamma_1 \equiv 1 \mod 4, \\ 2^m & \text{if } r=0, \gamma_3/\gamma_1 \equiv -1 \mod 4. \end{cases}$$

Proof. Let $\Delta = -\gamma_3/\gamma_1$. It is easy to check that $[O(\gamma) : SO(\gamma)] = 2$ with

$$SO(\gamma) = \{k \in O(\gamma) : \det k = 1\} = \{\begin{pmatrix} k_1 & k_2 \\ \Delta k_2 & k_1 \end{pmatrix} : k_1^2 - \Delta k_2^2 = 1, k_1, k_2 \in \mathbb{Z}_2\}.$$

Let $K = \mathbb{Q}_2(\sqrt{\Delta})$ be the quadratic ètale extension of $\mathbb{Q}_2$ (i.e., $K = \mathbb{Q}_2 \times \mathbb{Q}_2$ is allowed). Let $G = K^1$ be the norm one subgroup of K^* in the nonsplit case and $G = \{(x, x^{-1}) : x \in \mathbb{Z}_2^*\}$ in the split case. Write $\Delta = 2^{2a}\Delta_0$ with $\operatorname{ord}_2\Delta_0 = 0$ or 1. As in the Appendix, we set

$$G_n := \{g = a + b\delta_0 \in G : b \in 2^n\mathbb{Z}_2, a-1 \in 2^n\Delta_0\mathbb{Z}_2\}.$$

Here $\delta_0 = \sqrt{\Delta_0}$ in the nonsplit case and $(\sqrt{\Delta_0}, -\sqrt{\Delta_0})$ in the split case. Let $\delta = 2^a\delta_0 = \sqrt{\Delta}$. Then we have a group isomorphism

$$SO(\gamma) \longrightarrow \{\pm 1\}G_a, \quad k \mapsto g = k_1 + \delta k_2,$$

under which $O_m(\gamma)$ maps onto G_{a+m}. When $r = \operatorname{ord}_2(\gamma_3/\gamma_1) \geq 3$, either $a \geq 2$ or $a = 1$ and $\operatorname{ord}_2\Delta_0 = 1$. In either case, one has $-1 \notin G_a$ and Propositions A.8 in the Appendix A gives

$$[SO(\gamma) : O_m(\gamma)] = 2[G_a : G_{a+m}] = 2^{m+1}.$$

When $1 \leq r \leq 2$, $a = 1$ with $\operatorname{ord}_2\Delta_0 = 0$ or $a = 1$ with $\operatorname{ord}_2\delta_0 = 1$. In either case, $-1 \in G_a$, and

$$[SO(\gamma) : O_m(\gamma)] = [G_a : G_{a+m}] = 2^m.$$

Finally, when $r = 0$, one has $a = 0$ and $\Delta_0 = \Delta = -\gamma_3/\gamma_1 \in \mathbb{Z}_2^*$. So $-1 \in G_0$. Proposition A.8 in the Appendix A implies

$$[SO(\gamma) : O_m(\gamma)] = [G_a : G_{a+m}] = \begin{cases} 2^m & \text{if } \Delta_0 \equiv -1 \bmod 4, \\ 2^{m-1} & \text{if } \Delta_0 \equiv 1 \bmod 4. \end{cases}$$

This proves the Lemma.

Lemma 9.3. *When* $\gamma = 2^c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq 0$, *one has*

$$[O(\gamma) : O_m(\gamma)] = 2^m.$$

Proof. Let $SO(\gamma)$ be the special orthogonal group associated to γ as in the proof of Lemma 9.2. One has again $[O(\gamma) : SO(\gamma)] = 2$. It is easy to check that

$$\mathbb{Z}_2^* \longrightarrow SO(\gamma), \quad \alpha \mapsto \operatorname{diag}(\alpha, \alpha^{-1})$$

is a group isomorphism, which maps $1 + 2^m\mathbb{Z}_2$ onto $\mathcal{O}_m(\gamma)$. From this, it is easy to see that

$$[SO(\gamma) : O_m(\gamma)] = 2^{m-1}.$$

Lemma 9.4. *Let* $\gamma = 2^c \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. *Then*

$$[O(\gamma) : O_m(\gamma)] = 3 \cdot 2^m$$

for $m \ge 1$.

Proof. Again we have $[O(\gamma); SO(\gamma)] = 2$. Notice that for $k \in \mathrm{SL}_2(\mathbb{Z}_2)$, $k.\gamma = \gamma$ if and only if

$$\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} k_4 & -k_3 \\ -k_2 & k_1 \end{pmatrix},$$

that is,

$$k_1^2 + k_1 k_2 + k_2^2 = 1, \quad k_3 = -k_2, \quad k_4 = k_1 + k_2. \tag{9.3}$$

Let $K = \mathbb{Q}_2(\sqrt{-3})$, and $G = K^1$ is the subgroup of norm one elements in K^*. Then

$$SO(\gamma) \longrightarrow G, \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \mapsto k_1 + \frac{-1+\sqrt{-3}}{2} k_2 = (k_1 - \frac{1}{2}k_2) + \frac{1}{2}k_2\sqrt{-3}$$

is a group isomorphism, under which $O_m(\gamma)$ maps onto G_m. Indeed, one can verify using (9.3) that if $\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in SO_m(\gamma)$, then $k_2 \in 2^{m+1}\mathbb{Z}_2$. Now Proposition A.8 in the Appendix A implies

$$[SO(\gamma) : O_m(\gamma)] = [G : G_m] = 3 \cdot 2^{m-1}$$

for $m \ge 2$.

Proof of Lemma 1.4 Now Lemma 1.4 follows from (9.2) and Lemmas 9.1-9.4 immediately.

Appendix A: Structure of $U(1)$ over a 2-adic field.

In this appendix, we study the structure of $U(1)$ over a 2-adic field, which is needed in Section 9.

Let F be a finite field extension of $\mathbb{Q}_2$ with ring of integers $\mathcal{O}$ and a uniformizer π. Let $K = F(\delta)$ be a quadratic field extension of F with $\Delta = \delta^2 \in F^*$. We assume that $\nu(\Delta) = 0$ or 1, where ν is the 2-adic valuation on F such that $v(\pi) = 1$. Let

$$r = \nu(2), \quad t = \max\{n : \Delta - x^2 \in \pi^n \mathcal{O} \text{ for some } x \in O^*\}. \tag{A.1}$$

By Hensel's lemma, one can show $t \le 2r$ (otherwise $\Delta \in F^{*2}$). Let

$$G = K^1 = \{g \in K : \, g\bar{g} = 1\} \tag{A.2}$$

be the group of norm one elements in K. Here $\bar{g} = a - b\delta$ if $g = a + b\delta$. We denote for any integer n

$$G_n = \{g = a + b\delta \in G : b \in \pi^n \mathcal{O}, \text{ and } a - 1 \in \pi^n \Delta \mathcal{O}\}. \tag{A.3}$$

Clearly, G_n is a subgroup of G when $n \ge 0$, we will later prove that G_n is always a group.

The next two simple lemmas will be fundamental for our calculation.

Lemma A.1. *Let the notation be as above. Then the map*

$$g : F \bigcup \{\infty\} \longrightarrow G, \quad x \mapsto g(x) = \frac{\delta + x}{\delta - x} = a(x) + b(x)\delta \tag{A.4}$$

is a bijection of sets with $g(x)^{-1} = g(-x)$. Here

$$a(x) - 1 = \frac{2x^2}{\Delta - x^2}, \quad b(x) = \frac{2x}{\Delta - x^2}, \quad \text{and} \quad a(x) + 1 = \frac{2\Delta}{\Delta - x^2}, \tag{A.5}$$

Moreover, if we write

$$g(x_1)g(x_2)^{-1} = g(x_1)g(-x_2) = g(x_3), \tag{A.6}$$

then

$$a(x_3) - 1 = \frac{2\Delta(x_1 - x_2)^2}{(\Delta - x_1^2)(\Delta - x_2^2)}, \quad b(x_3) = \frac{2(x_1 - x_2)(\Delta - x_1 x_2)}{(\Delta - x_1^2)(\Delta - x_2^2)}. \tag{A.7}$$

Lemma A.2. *Let $x \mapsto g(x) = a(x) + b(x)delta$ the bijection defined in Lemma A.1.*

(1) When $n \ge 1$, one has

$$x \in \pi^n \mathcal{O}^* \Leftrightarrow \begin{cases} a(x) - 1 \in \pi^{r+2n} \Delta^{-1} \mathcal{O}^*, \\ b(x) \in \pi^{r+n} \Delta^{-1} \mathcal{O}^*, \\ a(x) + 1 \in \pi^r \Delta \mathcal{O}^*.. \end{cases}$$

(2) When $n \ge 1$, one has

$$x \in \pi^{-n} \mathcal{O}^* \Leftrightarrow \begin{cases} a(x) - 1 \in \pi^r \Delta^{-1} \mathcal{O}^*, \\ b(x) \in \pi^{r+n} \mathcal{O}^*, \\ a(x) + 1 \in \pi^{r+2n} \Delta \mathcal{O}^*. \end{cases}$$

(3) *When* $\nu(\Delta) = 1$*, one has*

$$x \in \mathcal{O}^* \Leftrightarrow \begin{cases} a(x) - 1 \in \pi^r \mathcal{O}^* \\ b(x) \in \pi^r \mathcal{O}^*, \\ a(x) + 1 \in \pi^{r+1}\mathcal{O}^*. \end{cases}$$

(4) *When* $\nu(\Delta) = 0$*, one has, for* $x \in \mathcal{O}^*$ *and* $0 \le c \le t$*,*

$$\Delta - x^2 \in \pi^c \mathcal{O}^* \Leftrightarrow \begin{cases} a(x) \pm 1 \in \pi^{r-c}\mathcal{O}^*, \\ b(x) \in \pi^{r-c}\mathcal{O}^*. \end{cases}$$

Here t *is the integer defined in* $(A.1)$

Proof. The "$\Rightarrow$" parts follow from Lemma A.1 directly. Since the conditions on either side are mutually exclusive, the "$\Leftarrow$" parts follow from the "$\Rightarrow$" parts.

The following two propositions follow from Lemma A.2 directly. I learnt Proposition A.3 first from J.S. Li's private note ([Li]).

Proposition A.3 (Li). *Assume first that* $\nu(\Delta) = 1$*. Then* $G = G_{r-1}$ *and for* $n \ge r$

$$G_n = \{g(x) :\ x \in \pi^{n-r+1}\mathcal{O}\} = \{g = a + b\delta : b \in \pi^n \mathcal{O}, a - 1 \in \pi^{2n-r+1}\mathcal{O}\}.$$

Moreover, one has

(1) $[G : G_r] = 2$.

(2) $[G_n : G_{n+1}] = q$ *for* $n \ge r$.

Proposition A.4. *Assume* $\Delta \in \mathcal{O}^*$ *and* $n > r$*. Then*

$$G_n = g(\pi^{n-r}\mathcal{O}) = \{g = a + b\delta :\ b \in \pi^n\mathcal{O}, a - 1 \in \pi^{2n-r}\mathcal{O}\} \tag{1}$$

is a subgroup of G*, and*

$$[G_n : G_{n+1}] = q. \tag{2}$$

From now on, we assume $\Delta \in \mathcal{O}^*$ and $n \longrightarrow r$. Recall that t is the largest integer such that

$$\Delta = x_0^2 \bmod \pi^t$$

has a solution $x_0 \in \mathcal{O}^*$. We fix such an x_0. Depending on whether t is an even or odd number, we write in this appendix $t = 2c$ or $t = 2c + 1$. Recall that $t \le 2r$.

Lemma A.5. *Assume that $\Delta \in \mathcal{O}^*$. Then $G = G_{r-t}$, and G_n is a subgroup of G for every integer n. Moreover,*

(1) For $0 \le \alpha \le c$, one has

$$G_{r-2\alpha} = \{g(x) : \nu(x - x_0) \le \alpha\}.$$

Moreover, $G_{r-2\alpha-1} = G_{r-2\alpha}$ if $\alpha < c$.

(2) For $0 < \alpha \le c$ and $2\alpha < t$, one has

$$G_{r-2\alpha} - G_{r-2\alpha+1} = \{g(x) : x - x_0 \in \pi^\alpha \mathcal{O}^*\}.$$

(3) When $t = 2c$, one has

$$G - G_{r-2c+1} = \{g(x) : x - x_0 \in \pi^c \mathcal{O}\}.$$

(4) When $t = 2c + 1$, one has

$$G - G_{r-2c} = \{g(x) : x - x_0 \in \pi^{c+1} \mathcal{O}\}.$$

Proof. By Lemma A.2, one has

$$g(x) \in \begin{cases} G_{r+1} & \text{if } x \in \pi\mathcal{O}, \\ G_r & \text{if } x \notin \mathcal{O}, \\ G_{r-\beta} - G_{r-\beta-1} & \text{if } x \in \mathcal{O}^* \text{ and } \nu(\Delta - x^2) = \beta \ge 0. \end{cases}$$

In particular, $g(x) \in G_r$ for all $x \notin \mathcal{O}^*$.

For any $x \in \mathcal{O}^*$, write

$$x = x_0 + \pi^\alpha y \tag{A.9}$$

with $\alpha \ge 0$ and $y \in \mathcal{O}^*$. If $\alpha > c$, then

$$x^2 \equiv x_0^2 \equiv \Delta \bmod \pi^t.$$

By the definition of t, one has then $\nu(\Delta - x^2) = t$ and thus $g(x) \in G_{r-t}$. If $\alpha \le c$ and $2\alpha < t$, then

$$x^2 - \Delta = \pi^{2\alpha}\left(\frac{x_0^2 - \Delta}{\pi^{2\alpha}} + \frac{2}{\pi^\alpha} x_0 y + y^2\right) \in \pi^{2\alpha}\mathcal{O}^*.$$

So Lemma A.2 implies that

$$g(x) \in G_{r-2\alpha} - G_{r-2\alpha+1}, \tag{A.10}$$

whenever $\alpha \le c$ and $2\alpha < t$. This proves

$$G = G_{r-t}, \quad G_{r-2\alpha} = G_{r-2\alpha-1}, 0 \le \alpha < c$$

and claim (2). Notice that $\nu(x - x_0) \le 0 \le \alpha$ when $x \notin \mathcal{O}^*$. This, together with $(A.10)$, implies claim (1). Claim (3) is the same as Claim (2) except for the fact that $g(x) \in G$ even when $\nu(x - x_0) > c$. Claim (4) follows from Claim (1). Now we check G_n is a group. We may assume $n = r - 2\alpha$ with $0 \le \alpha \le c$. By induction, we may assume $\nu(x_1 - x_0) = \alpha$, i.e, $g(x_1) \in G_{r-2\alpha} - G_{r-2\alpha+1}$, and $g(x_2) \in G_{r-2\alpha}$. Finally, when $0 \le \nu(x_2 - x_0) = \beta \le \alpha$, one has $\Delta - x_2^2 \in \pi^{2\beta}\mathcal{O}^*$,

$$x_1 \equiv x_2 \mod \pi^\beta, \text{ and } \Delta - x_1x_2 \equiv \Delta - x_2^2 \equiv 0 \mod \pi^\beta.$$

Write $g(x_1)g(x_2)^{-1} = g(x_3)$ as in Lemma $A.1$, then

$$a(x_3) - 1 = \frac{2\Delta(x_1 - x_2)^2}{(\Delta - x_1^2)(\Delta - x_2^2)} \in \pi^{r+2\beta-2\alpha-2\beta}\mathcal{O} = \pi^{r-2\alpha}\mathcal{O},$$

and

$$b(x_3) = \frac{2(x_1 - x_2)(\Delta - x_1x_2)}{(\Delta - x_1^2)(\Delta - x_2^2)} \in \pi^{r+2\beta-2\alpha-2\beta}\mathcal{O} =\in \pi^{r-2\alpha}\mathcal{O}.$$

So $g(x_3) \in G_{r-2\alpha}$. When $x_2 \notin \mathcal{O}$, one has $-g(x_2) \in G_{r+1}$, and then

$$-g(x_1)g(x_2)^{-1} = -g(x_3) \in G_{r-2\alpha}$$

by what has just been proved. In summary, $G_{r-2\alpha}$ is a subgroup of G.

Proposition A.6. *Assume that $\Delta \in \mathcal{O}^*$, and let $t = 2c$ or $2c+1$ as before. Then*

(1) $[G_r : G_{r+1}] = 2(q-1)$.

(2) *When $0 < \alpha \le c$ and $2\alpha < t$, one has*

$$[G_{r-2\alpha} : G_{r-2\alpha+1}] = [G_{r-2\alpha} : G_{r-2\alpha+2}] = q.$$

(3) *When $t = 2c$, one has $[G : G_{r-2c+1}] = [G : G_{r-2c+2}] = q+1$.*

(4) *When $t = 2c+1$, one has $[G : G_{r-2c}] = 2$.*

Proof. By Lemma A.5, one has

$$G_r = \{g(x) : \nu(x - x_0) \le 0\} = \{g(x) : x \notin \mathcal{O} \text{ or } x - x_0 \in \mathcal{O}^*\}.$$

Let $H = \{g(x) : x - x_0 \in \mathcal{O}^*\}$, then $G_r = H \cup (-H)$, and H is a subgroup of G. For $x_1, x_2 \in x_0 + \mathcal{O}^*$, one has

$$\Delta - x_1^2 \in \mathcal{O}^*, \Delta - x_2^2 \in \mathcal{O}^*.$$

Write $g(x_1)g(x_2)^{-1} = g(x_3)$ as in Lemma A.1, one has then that

$$a(x_3) - 1 = \frac{2\Delta(x_1 - x_2)^2}{(\Delta - x_1^2)(\Delta - x_2^2)} \in \pi^{r+1}\mathcal{O}$$

if and only if $x_1 \equiv x_2 \mod \pi$. One also has

$$b(x_3) = \frac{2(x_1 - x_2)(\Delta - x_1 x_2)}{(\Delta - x_1^2)(\Delta - x_2^2)} \in \pi^{r+1}$$

when $x_1 \equiv x_2 \mod \pi$. So $g(x_1) \in g(x_2)G_{r+1}$ if and only if $x_1 \equiv x_2 \mod \pi$, and thus $x \mapsto g(x)$ induces a bijection between $\{x \in \mathcal{O}/\pi, x \neq x_0\}$ and H/G_{r+1}. So $[H : G_{r+1}] = (q-1)$ and $[G_r : G_{r+1}] = 2(q-1)$. This proves (1). When $0 < \alpha \le c$ and $2\alpha < t$, similar argument shows that $a \mapsto g(x_0 + \pi^\alpha a)$ induces a bijection between $(\mathcal{O}/\pi)^*$ and $(G_{r-2r} - G_{r-2\alpha+1})/G_{r-2\alpha+1}$. So

$$[G_{r-2\alpha} : G_{r-2\alpha+1}] = 1 + \#(\mathcal{O}/\pi)^* = q.$$

This proves (2). Claim (3) is the same as Claim (2) except for the fact that $a = 0$ is allowed (i.e., $g(x_0) \in G - G_{r-2c+1}$). For claim (4), if $g(x) \in G - G_{r-2c}$, one has $x - x_0 \in \pi^{c+1}$ by Lemma A.5. So Formula (A.7) implies that $g(x)g(x_0)^{-1} \in G_{r-2c}$, and thus $[G : G_{r-2c}] = 2$.

In Section 9, we also need the split case $\Delta \in \mathbb{F}^{*2}$. In this case, $K = F \oplus F$ is a quadratic etale extension of F, and $\delta = (\sqrt{\Delta}, -\sqrt{\Delta}) \in \mathbb{Z}_2^*$ for a fixed square-root $\sqrt{\Delta}$ of Δ in F. We identify $K^1 = \{(x, x^{-1}) : x \in F^*\} \cong F^*$. We take in this case

$$G = \{(a + b\sqrt{\Delta}, a - b\sqrt{\Delta}) \in K^1 : a + b\sqrt{\Delta} \in \mathcal{O}^*\} \cong \mathcal{O}^*,$$

and

$$G_n = \{(a + b\sqrt{\Delta}, a - b\sqrt{\Delta}) \in G : b \in \pi^n\mathcal{O}, a - 1 \in \pi^n\mathcal{O}\} \cong 1 + \pi^n\mathcal{O}$$

as in $(A.3)$. Then we have

Proposition A.7. *In the split case, one has*

$$[G : G_1] = q - 1, \quad \textit{and} \quad [G_n : G_{n+1}] = q, \ \textit{for } n \ge 1.$$

The following special case will be used in Section 9.

Proposition A.8. *Let $F = \mathbb{Q}_2$, and $K = F(\sqrt{\Delta})$ with $ord_2 \Delta = 0$ or 1.*

(1) When $K/\mathbb{Q}_2$ is a ramified field extension, one has $G = G_0$, and $[G_n : G_{n+1}] = 2$ for every $n \geq 0$.

(2) When $K/\mathbb{Q}_2$ is an inert or split extension, one has

$$[G_n : G_{n+1}] = \begin{cases} 2 & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

and

$$[G : G_0] = \begin{cases} 3 & \text{if inert,} \\ 1 & \text{if split.} \end{cases}$$

Appendix B. The Gross-Keating invariants.

This appendix is essentially due to Steve Kudla, who is very generous to allow it be included here.

Gross and Keating define invariants of a $\mathbb{Z}_p$–lattice L with a quadratic form Q as follows. Let

$$(x, y) = Q(x + y) - Q(x) - Q(y)$$

be the associated bilinear form. Then $Q(x) = \frac{1}{2}(x, x)$. For any $\mathbb{Z}_p$–basis $\psi = (\psi_1, \psi_2, \dots, \psi_n)$ of L, write

$$Q(\sum_i x_i \psi_i) = \sum_{i \leq j} q_{ij}\, x_i x_j = x T^t x. \tag{B.1}$$

Here the $q_{ij} \in \mathbb{Z}_p$, $x = (x_1, \cdots, x_n) \in \mathbb{Z}_p^n$, and $T = (t_{ij}) = (\frac{1}{2}(\psi_i, \psi_j))$ is a symmetric matrix over $\frac{1}{2}\mathbb{Z}_p$ with $t_{ii} \in \mathbb{Z}_p$. We have

$$q_{ij} = \begin{cases} 2t_{ij} & \text{if } i < j \\ t_{ii} & \text{if } i = j, \end{cases}$$

for the q_{ij} defined by (1). Note that $t_{ij} \in \frac{1}{2}\mathbb{Z}_p$. Let

(B.2)
$$S(\psi) = \{\ y = (y_1, \dots, y_n) \in \mathbb{Z}^n \mid \quad y_1 \leq y_2 \leq \cdots \leq y_n \quad \text{and } \frac{1}{2}(y_i + y_j) \leq o(q_{ij}), \quad \forall i \leq j\ \}.$$

Here $o(q_{ij}) = \operatorname{ord}_p(q_{ij})$. Let

$$S = \cup_\psi S(\psi).$$

Finally, define

$$
\begin{aligned}
a_1 &= \max_{y \in S}\{y_1\} \\
a_2 &= \max_{y=(a_1,y_2,\dots)\in S}\{y_2\} \\
&\dots \\
a_n &= \max_{y=(a_1,\dots,a_{n-1},y_n)\in S}\{y_n\}
\end{aligned}
\tag{B.3}
$$

If $(a_1, \dots, a_n) \in S(\psi)$ we will say that ψ is *optimal*; such a basis exists, by the definition. It should be pointed out however that it happens sometimes $a_i < o(Q(\psi_i)$ even when ψ is optimal.

Actually, Gross and Keating consider the case $n = 3$. In that case, they further define a sign $\epsilon = \epsilon(L)$—the Gross-Keating constant—when $a_1 \equiv a_2 \bmod (2)$ and $a_2 < a_3$ as follows. For ψ optimal, let $\epsilon = +1$ if the span of ψ_1 and ψ_2 is isotropic and -1 if it is anisotropic. We want to have an explicit expression for these invariants when a Jordan decomposition of the lattice L is given. When $p \neq 2$, it is a fact that if ψ is a basis with

$$T(\psi) = \operatorname{diag}(\epsilon_1 p^{\alpha_1}, \dots, \epsilon_n p^{\alpha_n}), \tag{B.4}$$

where $\alpha_1 \leq \cdots \leq \alpha_n$ and the ϵ_i's are units, then

$$(a_1, \dots, a_n) = (\alpha_1, \dots, \alpha_n). \tag{B.5}$$

So, the interesting case is when $p = 2$.

Following Kitaoka ([Ki]), for a lattice L, let

$$n(L) = \text{ ideal in } \mathbb{Z}_p \text{ generated by } \{\ Q(x) \mid x \in L\ \},$$

and

$$s(L) = \frac{1}{2}(L, L). \tag{B.6}$$

Then

$$o(n(L)) = \min\{\ o(q_{ij})\ \}, \tag{B.7}$$

and

$$o(s(L)) = \min\{\ o(t_{ij})\ \}. \tag{B.8}$$

In particular, for $n > 1$,

$$2s(L) \subset n(L) \subset s(L). \tag{B.9}$$

Lemma B.1. *For any L, $a_1 = o(n(L))$.*

Proof. Let $\alpha = o(n(L))$. Then, for any basis ψ, $(\alpha, \dots, \alpha) \in S(\psi)$, and hence $\mathfrak{a} \le a_1$. On the other hand, if ψ is an optimal basis, then for all $i \le j$,

$$a_1 \le \frac{1}{2}(a_i + a_j) \le o(q_{ij}),$$

so that $a_1 \le \alpha$.

For $r = 1, \dots, n$, let $\wedge^r(L)$ be the rth exterior product of the lattice L. It is the $\mathbb{Z}_p$ submodule spanned by the vectors

$$\psi_{\mathbf{i}} = \psi_{i_1} \wedge \dots \psi_{i_r}$$

where $\mathbf{i} = (i_1, \dots, i_r)$ with $i_1 < i_2 \dots < i_r$. Define a symmetric bilinear form B_r on $\wedge^r(L)$ by

$$B_r(\psi_{\mathbf{i}}, \psi_{\mathbf{j}}) := \det\big(B(\psi_{i_a}, \psi_{j_b})\big) = \det(T(\psi)_{\mathbf{ij}}) = \text{ det. of the } \mathbf{ij}\text{th minor of } T(\psi),$$

where a indexes the row and b indexes the column. Note that, for $r = 1$, this agrees with Kitaoka's notation. The values of B_r can lie in $2^{-r}\mathbb{Z}_p$. In any case, the values

$$\rho_r(L) := \min_{\mathbf{i},\mathbf{j}}(o(B_r(\psi_{\mathbf{i}}, \psi_{\mathbf{j}}))) = s(\wedge^r(L))$$

depend only on L, and $\rho_1(L) = o(S(L))$. Also, $\rho_n(L) = \det(T(\psi))$.

Lemma B.2. *(i) For any n, $a_1 + a_2 \le \rho_2(L) + 2$.*
(ii) For $n = 3$,

$$a_1 + a_2 + a_3 \le o(\det(T)) + 2.$$

Proof. Suppose that ψ is an optimal basis, so that, entrywise:

$$\begin{pmatrix} a_1 & \frac{1}{2}(a_1 + a_2) - 1 & \dots \\ & a_2 & \dots \\ & \dots & \dots \end{pmatrix} \le \begin{pmatrix} o(t_{11}) & o(t_{12}) & \dots \\ & o(t_{22}) & \dots \\ & \dots & \dots \end{pmatrix}$$

Now consider the quantities $o(\det(T(\psi)_{\mathbf{ij}})$. For example, if $r = 2$, then, for a principal minor, we have

$$o(t_{ii}t_{jj} - t_{ij}^2) \ge \min\{o(t_{ii}) + o(t_{jj}), 2o(t_{ij})\}.$$

But

$$a_i + a_j \le o(t_{ii}) + o(t_{jj}), \qquad \text{and} \qquad a_i + a_j - 2 \le 2o(t_{ij}).$$

so that

$$a_i + a_j \leq o(t_{ii}t_{jj} - t_{ij}^2) + 2,$$

For a nonprincipal minor,

$$o(t_{i_1j_1}t_{i_2j_2} - t_{i_2j_1}t_{i_1j_2}) \geq \min\{o(t_{i_1j_1}) + o(t_{i_2j_2}), o(t_{i_2j_1}) + o(t_{i_1j_2})\},$$

$$o(t_{i_1j_1}) + o(t_{i_2j_2}) \geq \frac{1}{2}(a_{i_1} + a_{j_1} + a_{i_2} + a_{j_2}) - 2,$$

and

$$o(t_{i_1j_2}) + o(t_{i_1j_2}) \geq \frac{1}{2}(a_{i_1} + a_{j_1} + a_{i_2} + a_{j_2}) - 2.$$

Thus,

$$a_1 + a_2 \leq \rho_2(L) + 2.$$

Note, for example, that when $n = 2$, the extreme case $a_1 + a_2 = o(\det(T)) + 2$ does occur.

For $r = 3$, we have, say for $n = 3$,

$$\begin{aligned} o(\det(T)) \geq \min\{&o(t_{11}) + o(t_{22}) + o(t_{33}), o(t_{12}) + o(t_{23}) + o(t_{13}) + 1, \\ &o(t_{11}) + 2o(t_{23}), o(t_{22}) + 2o(t_{13}), o(t_{33}) + 2o(t_{12}) \ \} \\ \geq{}& a_1 + a_2 + a_2 - 2. \end{aligned}$$

Proposition B.3. *Assume $n = 2$. Let L be an $\mathbb{Z}_2$-integral quadratic lattice of rank 2 and let T be the matrix of L with respect to a basis $e = (e_1, e_2)$ of L.*

(1) When $T = diag(\epsilon_1 p^\alpha, \epsilon_2 p^\beta)$, where $0 \leq \alpha \leq \beta$, one has $n(L) = s(L)$ and

$$(a_1, a_2) = \begin{cases} (\alpha, \beta) & \text{if } \alpha \not\equiv \beta \text{ mod } 2, \\ (\alpha, \beta + 1) & \text{if } \alpha \equiv \beta \text{ mod } 2 \text{ and } \epsilon_1 \equiv \epsilon_2 \text{ mod } 4, \\ (\alpha, \beta + 2) & \text{if } \alpha \equiv \beta \text{ mod } 2 \text{ and } \epsilon_1 \equiv -\epsilon_2 \text{ mod } 4. \end{cases}$$

Note that the basis ψ which diagonalizes Q is not *optimal in the last two cases.*

(2) When

$$T(\psi) = 2^\alpha \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \qquad a = 0, 2,$$

one has

$$(a_1, a_2) = (\alpha + 1, \alpha + 1).$$

Proof. (1) In this case, clearly $n(L) = s(L) = 2^\alpha$. So $a_1 = \alpha$. By Lemma B.2, one has then $a_1 + a_2 \leq \alpha + \beta + 2$ and so $a_2 \leq \beta + 2$. Since $(\alpha, \beta) \in S(e)$, one has $a_2 \geq \beta$.

First assume that $\beta \not\equiv \alpha \bmod 2$. If $a_2 > \beta$, let $\psi = e(\alpha_{ij})$ be an optimal basis, then

$$\beta < a_2 \leq o(Q(\psi_2)) = \epsilon_1 2^\alpha \alpha_{12}^2 + \epsilon_2 2^\beta \alpha_{22}^2$$

implies that $\alpha_{12}, \alpha_{22} \in 2\mathbb{Z}_2$, and thus $(\alpha_{ij}) \notin \mathrm{GL}_2(\mathbb{Z}_2)$, a contradiction. So $a_2 = \beta$.

Now assume $\beta \equiv \alpha \bmod 2$. If $\epsilon_1 + \epsilon_2 \equiv 0 \bmod 4$, choose

$$(\psi_1, \psi_2) = (e_1, e_2) \begin{pmatrix} 1 & 2^{\frac{\beta-\alpha}{2}} \\ 0 & 1 \end{pmatrix}.$$

Then $Q(\psi_2) = 2^\beta(\epsilon_1 + \epsilon_2) \in 2^{\beta+2}\mathbb{Z}_2$, and $q_{12} = (\psi_1, \psi_2) = 2^{\beta+1}\epsilon_1)$. So $(\alpha, \beta + 2) \in S(\psi)$, and $(a_1, a_2) = (\alpha, \beta + 2)$ as claimed. On the other hand, if $\epsilon_1 \equiv \epsilon_2 \bmod 4$, i.e., $\epsilon_1 + \epsilon_2 \equiv 2 \bmod 4$, the same basis ψ gives $(\alpha, \beta + 1) \in S(\psi)$. Furthermore, for any optimal basis $\psi = e(\alpha_{ij})$ with $\mathrm{GL}_2(\mathbb{Z}_2)$, at least one of α_{12} or α_{22} is a unit, one has then

$$a_2 \leq o(Q(\psi_2)) = o(\epsilon_1 2^\alpha \alpha_{12}^2 + \epsilon_2 2^\beta \alpha_{22}^2) \leq \beta + 1.$$

So $(a_1, a_2) = (\alpha, \beta)$.

(2) In the second case, clearly $n(L) = 2^{\alpha+1} = 2s(L)$. So $a_1 = \alpha + 1$ by Lemma B.1. Lemma B.2 implies then $a_2 = \beta + 1$ too.

Next, we consider the case $n = 3$.

Proposition B.4. *Let*

$$T = \ diag(\epsilon_1 2^\alpha, 2^\beta \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}) \ \textit{with } a = 0, 2.$$

Then

$$(a_1, a_2, a_3) = \begin{cases} (\alpha, \beta+1, \beta+1) & \textit{if } \alpha \leq \beta + 1, \\ (\beta+1, \beta+1, \alpha) & \textit{if } \alpha > \beta + 1. \end{cases}$$

Proof. First assume that $\alpha \leq \beta + 1$. Then $n(L) = s(L) = 2^\alpha$, and thus $a_1 = \alpha$ by Lemma B.1. Notice that $(\alpha, \beta + 1 \beta + 1) \in S(e)$, where e is a basis of L with respect to which the matrix of L is T. So Lemma B.2 implies $(a_1, a_2, a_3) = (\alpha, \beta + 1, \beta + 1)$. Now assume that $\alpha > \beta + 1$, then $n(L) = 2^{\beta+1} = 2s(L)$ and $a_1 = \beta + 1$. This implies $a_2 \geq \beta + 1$. Lemma B.2 implies then $(a_1, a_2, a_3) = (\beta + 1, \beta + 1, \alpha)$.

Proposition B.5. *$T =$ diag$(\epsilon_1 2^{\beta_1}, \epsilon_2 2^{\beta_2}, \epsilon_3 2^{\beta_3})$ with $\beta_1 \le \beta_2 \le \beta_3$.*

(1) When $\beta_2 \not\equiv \beta_1$ mod 2, one has $(a_1, a_2, a_3) = (\beta_1, \beta_2, \beta_3 + 2)$.

(2) When $\beta_2 \equiv \beta_1$ mod 2 and $\beta_3 \le \beta_2 + 1$, one has $(a_1, a_2, a_3) = (\beta_1, \beta_2 + 1, \beta_3 + 1)$.

(3) When $\beta_2 \equiv \beta_1$ mod 2, and $\beta_3 \ge \beta_2 + 2$, one has

$$(a_1, a_2, a_3) = \begin{cases} (\beta_1, \beta_2 + 1, \beta_3 + 1) & \text{if } \epsilon_1 \equiv \epsilon_2 \text{ mod } 4, \\ (\beta_1, \beta_2 + 2, \beta_3) & \text{if } \epsilon_1 \equiv -\epsilon_2 \text{ mod } 4. \end{cases}$$

For clarity, we divide the proof into several lemmas.

Lemma B.6. *When $\beta_2 \not\equiv \beta_1$ mod 2, one has $(a_1, a_2, a_3) = (\beta_1, \beta_2, \beta_3 + 2)$.*

Proof. We first check that $(\beta_1, \beta_2, \beta_3 + 2) \in S(\psi)$ for some basis ψ. When $\beta_3 - \beta_1$ is odd, set

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 0 & 2^{\frac{\beta_3 - \beta_1 + 1}{2}} \alpha_{13} \\ 0 & 1 & 2^{\frac{\beta_3 - \beta_2}{2}} \beta_{123} \\ 0 & 0 & 1 \end{pmatrix}$$

Then $q_{33} = Q(\psi_3) = 2^{\beta_3}(2\epsilon_1 \alpha_{13}^2 + \epsilon_2 \alpha_{23}^2 + \epsilon_3)$. Clearly, there are $\alpha_{13}, \alpha_{23} \in \mathbb{Z}_2$ such that $o(q_{33}) \ge \beta_3 + 2$. For this choice, it is easy to see that $(\beta_1, \beta_2, \beta_3 + 2) \in S(\psi)$. When $\beta_3 - \beta_1$ is even, $\beta_3 - \beta_2$ has to be odd. The same argument with the role of β_1 and β_2 switched works. Now, by Lemma B.2, it suffices to prove $a_2 = \beta_2$. Clearly, $a_2 \ge \beta_2$. If $a_2 > \beta_2$, let $\psi = (\psi_1, \psi_2, \psi_3)$ be an optimal basis with $\psi = e(\alpha_{ij})$ and $(\alpha_{ij}) \in \mathrm{GL}_2(\mathbb{Z}_2)$. However

$$\beta_1 < \beta_2 < a_j \le o(Q(\psi_j)) = o(\epsilon_1 2^{\beta_1} \alpha_{1j}^2 + \epsilon_2 2^{\beta_2} \alpha_{2j}^2 + \epsilon_3 2^{\beta_3} \alpha_{3j}^2)$$

for $j = 2, 3$ implies that $\alpha_{ij} \in 2\mathbb{Z}_2$ for $i = 1, 2$ and $j = 2, 3$. This implies $(\alpha_{ij}) \notin \mathrm{GL}_2(\mathbb{Z}_2)$.

Lemma B.7. *When $\beta_2 \equiv \beta_1$ mod 2 and $\beta_3 \le \beta_2 + 1$, one has $(a_1, a_2, a_3) = (\beta_1, \beta_2 + 1, \beta_3 + 1)$.*

Proof. When $\beta_3 = \beta_2$, take

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2 - \beta_1}{2}} & 2^{\frac{\beta_2 - \beta_1}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 1, \beta_2 + 1) \in S(\psi)$. By Lemma B.2, this has to be (a_1, a_2, a_3).

When $\beta_3 = \beta_2 + 1$ and $\epsilon_1 + \epsilon_2 \equiv 2 \bmod 4$, take

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2-\beta_1}{2}} & 2^{\frac{\beta_2-\beta_1}{2}} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 1, \beta_3 + 1) \in S(\psi)$. By Lemma B.2, this has to be (a_1, a_2, a_3).

Finally, when $\beta_3 = \beta_2 + 1$ and $\epsilon_1 + \epsilon_2 \equiv 0 \bmod 4$, take

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2-\beta_1}{2}} & 2^{\frac{\beta_2-\beta_1}{2}} \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 1, \beta_3 + 1) \in S(\psi)$. By Lemma B.2, this has to be (a_1, a_2, a_3).

Lemma B.8. *Assume $\beta_2 \equiv \beta_1$ mod 2, and $\beta_3 \geq \beta_2 + 2$. Then*

$$(a_1, a_2, a_3) = \begin{cases} (\beta_1, \beta_2 + 1, \beta_3 + 1) & \text{if } \epsilon_1 \equiv \epsilon_2 \bmod 4, \\ (\beta_1, \beta_2 + 2, \beta_3) & \text{if } \epsilon_1 \equiv -\epsilon_2 \bmod 4. \end{cases}$$

Proof. First assume that $\epsilon_1 \equiv -\epsilon_2 \bmod 4$, i.e., $\epsilon_1 + \epsilon_2 \equiv 0 \bmod 4$. Set

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2-\beta_1}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 2, \beta_3) \in S(\psi)$. We claim that $a_3 \geq \beta_3$. This, together with Lemma B.2, implies $(a_1, a_2, a_3) = (\beta_1, \beta_2 + 2, \beta_3)$.

To prove $a_3 \geq \beta_3$, we may assume that $\beta_2 < a_2 \leq \beta_3$. Let $\psi = e(\alpha_{ij})$ be an optimal basis. If $\alpha_{22} \in \mathbb{Z}_2^*$, then $(a_1, a_2, \beta_3) \in S(\{\psi_1, \psi_2, e_3\})$ and thus $a_3 \geq \beta_3$. If $\alpha_{23} \in \mathbb{Z}_2^*$, then $(a_1, a_2, \beta_3) \in S(\{\psi_1, \psi_3, e_3\})$ and thus $a_3 \geq \beta_3$. Indeed,

$$\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}(a_1 + a_3) \leq o((\psi_1, \psi_3)),$$

$$\frac{1}{2}(a_1 + \beta_3) \leq o((\psi_1, e_3)) = \beta_3 + o(\alpha_{31}),$$

and

$$\frac{1}{2}(a_2 + \beta_3) \leq \beta_3 \leq o((\psi_3, e_3)) = \beta_3 + o(\alpha_{33}).$$

Clearly one has also

$$a_1 \leq o(Q(\psi_1)), \quad a_2 \leq a_3 \leq o(Q(\psi_3)), \quad , \beta_3 = o(Q(e_3)).$$

Finally, if $\alpha_{22}, \alpha_{23} \in 2\mathbb{Z}_2$, then $\alpha_{12}, \alpha_{13} \in 2\mathbb{Z}_2$ due to the fact $o(Q(\psi_2)) \ge a_2 > \beta_1$ and $o(Q(\psi_3)) > \beta_1$. This implies that $(\alpha_{ij}) \notin \mathrm{GL}_2(\mathbb{Z}_2)$, a contradiction.

Now assume that $\epsilon_1 \equiv \epsilon_2 \mod 4$, i.e., $\epsilon_1 + \epsilon_2 \equiv 2 \mod 4$. Then one easily check $(\beta_1, \beta_2 + 1, \beta_3) \in S(\psi)$, where ψ is the basis just given. So $a_2 \ge \beta_2 + 1$. In fact, we can do better. If $\beta_3 \equiv \beta_1 \mod 2$, choose

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2-\beta_1}{2}} & 2^{\frac{\beta_3-\beta_1}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 1, \beta_3 + 1) \in S(\psi)$. If $\beta_3 \not\equiv \beta_1 \mod 2$, choose

$$(\psi_1, \psi_2, \psi_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 2^{\frac{\beta_2-\beta_1}{2}} & 2^{\frac{\beta_3-\beta_1-1}{2}} \\ 0 & 1 & 2^{\frac{\beta_3-\beta_2-1}{2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1, \beta_2 + 1, \beta_3 + 1) \in S(\psi)$. This, together with Lemma B.2, implies that $(a_1, a_2, a_3) = (\beta_1, \beta_2 + 1, \beta_3 + 1)$ as claimed if we can prove $a_2 = \beta_2 + 1$.

If $a_2 \ge \beta_2 + 2$, let $\psi = e(\alpha_{ij})$ be an optimal basis for it. Then

$$Q(\psi_2) = \epsilon_1 2^{\beta_1} \alpha_{12}^2 + \epsilon_2 2^{\beta_2} \alpha_{22}^2 + \epsilon_3 2^{\beta_3} \alpha_{32}^2 \in 2^{\beta_2+2}\mathbb{Z}_2,$$

and so $\alpha_{12} = 2^{\frac{\beta_2-\beta_1}{2}} \tilde{\alpha}_{12}$ with $\tilde{\alpha}_{12} \in \mathbb{Z}_2$, and

$$\epsilon_1 \tilde{\alpha}_{12}^2 + \epsilon_2 \alpha_{22}^2 + \epsilon_3 2^{\beta_3-\beta_2} \alpha_{32}^2 \equiv 0 \mod 4.$$

So $\epsilon_1 \tilde{\alpha}_{12}^2 + \epsilon_2 \alpha_{22}^2 \equiv 0 \mod 4$. Since $\epsilon_1 + \epsilon_2 \equiv 2 \mod 4$, this implies $\tilde{\alpha}_{12}, \alpha_{22} \in 2\mathbb{Z}_2$. For the same reason, $o(Q(\psi_3)) \ge a_3 \ge a_2 \ge \beta_2 + 2$ implies $\alpha_{13}\alpha_{23} \in 2\mathbb{Z}_2$. However, this implies $(\alpha_{ij}) \notin \mathrm{GL}_2(\mathbb{Z}_2)$, a contradiction. Therefore $a_2 = \beta_2 + 1$, which proves the proposition.

Appendix C: Correction and reformulation in non-dyadic case.

In [Ya], we deal with p-adic local density in the cases $n = 1$ and $n = 2 \ne p$. In this appendix, we correct some typos and reformulate some formulae. The notation $\alpha(X, T, S)$ there is the same as $W(X, T, L)$ in this paper, where S is the matrix of L.

C1. We assume that $n = 2$ and $p \ne 2$. [Ya, Propositions 8.6 and 8.7] is reformulated as Theorem 5.7. In fact, [Ya, Prop 8.7] has a minor problem when $a = 0$ which is corrected in the new version. A computer program for computing the local density when $p \ne 2$ using [Ya, Theorem 7.1] was written by Song Guo and is available in my web page

http://www.math.wisc.edu/∼ thyang/programs.

There are a couple of typos in the formulaes leading to [Ya, Theorem 7.1]—the main formula. Using the notation in [Ya], the correct ones are the following.

When $a = b$,

$$I_{2,2} = v(a+1)\delta^+(a+1)((\frac{-\alpha_1\alpha_2}{p}) - p^{-1}) \sum_{0<k\leq a} \delta^+(k)v(k)p^{a+d(a+1)+d(k)}X^{a+1+k}.$$

When $a < b$, $I_{2,3}$ missed a factor $X^{k_1+k_2}$ and should be

$$I_{2,3} = (1-p^{-1}) \sum_{0<k_2<a+1<k_1\leq b} v(k_1)v(k_2)\delta^+(k_2)g(k_1)p^{\frac{a+k_1}{2}+d(k_1)+d(k_2)}X^{k_1+k_2}.$$

Of course, the p-adic case with $p \neq 2$ can also be computed in this paper's notation. The relation with [Ya, Theorem 7.1] is as follows.

Proposition C1. *One has for $T =$ diag$(\alpha_1 p^a, \alpha_2 p^b) \in \mathcal{A}_t(a)$ with $t = b - a \geq 0$,*

$$W(X, T, L) = 1 + \sum_{0<c\leq b+1} \sum_{0\leq r<c} S_r(-c, T, L)X^{2c-r} + \sum_{0<c\leq b+1} S_{\geq c}(-c, T, L)X^c.$$

Here S_r and $S_{\geq c}$ are given as follows.

(1) When $a = b$, i.e., $t = 0$, one has

$$\sum_{0<c\leq a} S_0(-c, T, L)X^{2c} = (1-p^{-1})I_{2,8},$$
$$S_0(-a-1, T, L)X^{2a+2} = I_{2,6},$$
$$\sum_{0<r<c\leq a} S_r(-c, T, L)X^{2c-r} = (1-p^{-1})I_{2,1},$$
$$\sum_{0<r<c=a+1} S_r(-c, T, L)X^{2c-r} = (1-p^{-1})I_{2,2},$$
$$\sum_{0<c\leq a} S_{\geq c}(-c, T, L)X^c = I_{1,1},$$
$$S_{\geq a+1}(-a-1, T, L)X^{a+1} = I_{1,2}.$$

(2) When $a < b$, i.e., $t > 0$, one has

$$\sum_{0<c\le a} S_0(-c,T,L)X^{2c} = (1-p^{-1})I_{2,8},$$

$$S_0(-a-1,T,L)X^{2a+2} = (1-p^{-1})I_{2,7},$$

$$S_r(-c,T,L) = 0 \quad \text{if } r < c-a-1,$$

$$\sum_{0<r<c\le a} S_r(-c,T,L)X^{2c-r} = (1-p^{-1})I_{2,1},$$

$$\sum_{0<r<c=a+1} S_r(-c,T,L)X^{2c-r} = (1-p^{-1})I_{2,2},$$

$$\sum_{a+1<c\le b}\ \sum_{c-a\le r<c} S_r(-c,T,L)X^{2c-r} = (1-p^{-1})I_{2,3},$$

$$\sum_{a+1<c\le b}\ \sum_{0<r\le c-a-1} S_r(-c,T,L)X^{2c-r} = (1-p^{-1})I_{2,5},$$

$$\sum_{c-a\le r<c=b+1} S_r(-c,T,L)X^{2c-r} = (1-p^{-1})I_{2,4},$$

$$\sum_{0<r\le b-a} S_r(-b-1,T,L)X^{2b+2-r} = I_{2,6},$$

$$\sum_{0<c\le a} S_{\ge c}(-c,T,L)X^{c} = I_{1,1},$$

$$S_{\ge a+1}(-a-1,T,L)X^{c} = I_{1,2},$$

$$\sum_{a+1<c\le b} S_{\ge c}(-c,T,L)X^{c} = I_{1,3},$$

$$S_{\ge b+1}(-b-1,T,L)X^{b+1} = I_{1,4}.$$

C2. Now we assume $n = 1$ and p is any prime, including 2. In [KRY, Proposition 13.4], we reformulate [Ya, Proposition 8.3] uniformly in terms of local L-function. Here we reformulate it in terms of the Gross-Keating invariants uniformly. There are quite a few obvious typos in [Ya, Proposition 8.3]. We thank J.K. Yu for pointing them and other minor mistakes in [Ya] out.

Proposition C2. *Let $L^0 = \{X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{Z}_p)\}$ with quadratic form $q(X) = -\kappa \det X$, where $\kappa \in \mathbb{Z}_p^*$. For $t = \alpha p^a \in \mathbb{Q}_p^*$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}_{\ge 0}$. Let $(0, a_2)$ be the Gross-Keating invariant of diag$(-\kappa, t)$. Let $\mu_p = (\kappa\alpha, p)$.*

(1) When a_2 is odd, one has

$$\frac{W(X,t,L^0)}{1-p^{-2}X^2} = \sum_{0\le k\le \frac{a_2-1}{2}} (p^{-1}X^2)^k.$$

(2) When a_2 is even, one has

$$W(X,t,L^0) = (1-p^{-2}X^2)\sum_{0\le k\le \frac{a_2}{2}} (p^{-1}X^2)^k - \mu_p p^{-\frac{a_2}{2}-1}(X^{a_2+1}-\mu_p p^{-1}X^{a_2+2}).$$

It might be interesting to notice that $-\kappa$ is $\det L^0$. In complement to the split case, we also record the corresponding formula for the ramified case $L = L^{ra}$ (with the quadratic form twisted by $-\kappa$ as above.

When a_2 is odd, one has

$$W(X,t,L^{ra}) = \sum_{0\le k\le \frac{a_2+1}{2}} \delta_{p,i}(p^{-1}X^2)^k.$$

Here and in the next formula,

$$\delta_{p,i} = \begin{cases} 1 & \text{if } i=0, [\frac{a_2}{2}]+1, \\ 1-p & \text{if } 1\le i\le [\frac{a_2}{2}]. \end{cases}$$

When a_2 is even, one has

$$W(T,t,L^{ra}) = \sum_{0\le k\le \frac{a_2}{2}} \delta_{p,i}(p^{-1}X^2)^k - \mu_p p^{-\frac{a_2}{2}-1}X^{a_2+1}.$$

Bibliography

[GK] B. Gross and K. Keating, *On the intersection of modular correspondences*, Invent. Math. **112** (1993), 225–245.

[HS] Y. Hironaka and F. Sato, *Local densities of representations of quadratic forms over p-adic integers*, J. Number Theory **83** (2002), 106-136.

[Ka] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. **121** (1999), 415-452.

[Ki] Y. Kitaoka, *Arithmetic of quadratic forms.*, Cambridge Tracts in Mathematics **106**, 1993.

[Ku] S. S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. Math. **146** (1997), 545-646.

[KR1] S. Kudla and M. Rapoport, *Height pairings on Shimura curves and p-adic uniformization*, Invent. Math. **141** (2002), 153-223.

[KR2] S. Kudla and M. Rapoport, in preparation.

[KRY] S. Kudla, M. Rapoport, and T.H. Yang, *Derivatives of Eisenstein series and Faltings heights*, to appear in Compositio Math., pp88.

[Li] J.S. Li, *Private note.*

[Si] C.L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. Math. **36** (1935), 527-606.

[Wa] G. L. Watson, *The* 2*-adic density of a quadratic form*, Mathematika **23** (1976), 94-106.

[Ya] T.H. Yang, *An explicit formula for local densities of quadratic forms*, J. Number Theory **72** (1998), 309-356.

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