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Abstract

Let φ denote Euler's phi function. For a fixed odd prime q we investigate the first and second order terms of the asymptotic series expansion for the number of $n \leqslant x$ such that $q \nmid \varphi(n)$. Part of the analysis involves a careful study of the Euler-Kronecker constants for cyclotomic fields. In particular, we show that the Hardy-Littlewood conjecture about counts of prime k-tuples and a conjecture of Ihara about the distribution of these Euler-Kronecker constants cannot be both true.

1 Introduction

Let φ denote Euler's phi function. For a fixed odd prime q, let $\mathcal{E}_q(x) = |\{n \leqslant x : q \nmid \varphi(n)\}|$. Since $q \nmid \varphi(n)$ if and only if $q^2 \nmid n$ and $p \nmid n$ for all primes $p \equiv 1 \,(\mathrm{mod}\ q)$, it follows that $\mathcal{E}_q(x) = \mathcal{A}_q(x) + \mathcal{A}_q(x/q)$, where $\mathcal{A}_q(x)$ is the number of integers $n \leqslant x$ with $q \nmid n$ and $p \nmid n$ for all primes $p \equiv 1 \,(\mathrm{mod}\ q)$. The asymptotic formula for $\mathcal{A}_q(x)$ was found by Landau in 1909 ([25]; see also [26, §176–183]), and the asymptotic for $\mathcal{E}_q(x)$ follows immediately. We have

$$\mathcal{E}_q(x) \sim \frac{e_0(q)x}{(\log x)^{\frac{1}{q-1}}} \tag{1}$$

for some constant $e_0(q)$.

A standard application of the Selberg-Delange method (e.g., [6, Theorem B]) gives an asymptotic expansion

$$\mathcal{E}_q(x) = \frac{x}{(\log x)^{1/(q-1)}} \left(e_0(q) + \frac{e_1(q)}{\log x} + \cdots + \frac{e_k(q)}{\log^k x} + O_k\left(\frac{1}{\log^{k+1} x}\right)\right), \tag{2}$$

with $e_j(q)$ being certain constants depending on q. The basis of (2) is an analysis of the Dirichlet series generating function for n with $q \nmid \varphi(n)$, namely

$$h_q(s) = (1 + q^{-s}) \prod_{\substack{p \neq q \\ p \not\equiv 1 \pmod q}} (1 - p^{-s})^{-1} = (1 - q^{-2s})\zeta(s) \prod_{p \equiv 1 \pmod q} (1 - p^{-s}), \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function. Roughly speaking, the Selberg-Delange method provides an asymptotic expansion for $\sum_{n \leqslant x} a_n$ in decreasing powers of $\log x$ provided that

Mathematics Subject Classification (2000). 11N37, 11Y60 the Dirichlet series $\sum_{n=1}^{\infty} a_n n^s$ behaves like $\zeta(s)^z$ for some fixed complex number z. If a_n is multiplicative, this means that a_p has average value z over primes p. In our case, $z = \frac{q-2}{q-1}$ by the prime number theorem for arithmetic progressions. We observe that the primes $p \equiv 1 \pmod q$ are precisely those primes which split completely in $K(q) := \mathbb{Q}(e^{2\pi i/q})$ and thus $\zeta_{K(q)}(s)$, the Dedekind zeta function of $K(q)$, comes into play. We prove the following in Section 2.

Proposition 1. *Let q be an odd prime. Then*

$$h_q(s) = \frac{(1-q^{-2s})\zeta(s)}{\Big[C(q,s)(1-q^{-s})\zeta_{K(q)}(s)\Big]^{\frac{1}{q-1}}}, \tag{4}$$

where

$$C(q,s) = \prod_{\substack{p \neq q \\ f_p \geqslant 2}} \Big(1 - \frac{1}{p^{sf_p}}\Big)^{\frac{q-1}{f_p}}, \tag{5}$$

and $f_p = \operatorname{ord}_q p$ (the least positive f with $p^f \equiv 1 \pmod q$). Furthermore,

$$e_0(q) = \frac{1-q^{-2}}{\Gamma(\frac{q-2}{q-1})\Big(C(q)(1-\frac{1}{q})\alpha_{K(q)}\Big)^{\frac{1}{q-1}}}, \qquad \alpha_{K(q)} = \operatorname*{Res}_{s=1} \zeta_{K(q)}(s).$$

Spearman and Williams [46] expressed $e_0(q)$ in terms of the parameters of $K = K(q)$; namely,

$$e_0(q) = \frac{(q+1)(q-1)^{\frac{q-2}{q-1}}\Gamma(\frac{1}{q-1})\sin(\frac{\pi}{q-1})}{2^{\frac{q-3}{2(q-1)}} q^{\frac{3(q-2)}{2(q-1)}} \pi^{\frac{3}{2}} (h(K)R(K)C(q))^{\frac{1}{q-1}}}, \tag{6}$$

where $h(K)$ denotes the class number of K and $R(K)$ is its regulator. Spearman and Williams gave a rather involved description of $C(q)$, see Section 2.2. Making use of the Euler product for $\zeta_{K(q)}(s)$, we will show that actually $C(q) = C(q,1)$. We have, for example, $C(3) = \prod_{p\equiv 2 \pmod 3}(1-1/p^2)$ (this is Lemma 3.1 of [46]). Our argument also gives a very short proof of an estimate of a product from [46] (inequality (24) below). On using that $\Gamma(\frac{1}{q-1})\Gamma(\frac{q-2}{q-1}) = \frac{\pi}{\sin(\pi/(q-1))}$ and formula (16) for $\alpha_{K(q)}$ below, it is seen that the $e_0(q)$ as given in Proposition 1 matches the formula (6).

1.1 The second order term in (2)

One of the main topics of this paper is the problem of the behavior of the second order term $e_1(q)$ from (2). In particular, we study which of the following two approximations is asymptotically closer to $\mathcal{E}_q(x)$, the *naive approximation*

$$\mathcal{N}_q(x) = \frac{e_0(q)x}{\log^{1/(q-1)} x}$$

or the *Ramanujan type* estimate

$$\mathcal{R}_q(x) = e_0(q)\int_2^x \frac{dt}{\log^{1/(q-1)} t} = \frac{e_0(q)}{(\log x)^{\frac{1}{q-1}}}\left(1 + \frac{1}{(q-1)\log x} + O\left(\frac{1}{\log^2 x}\right)\right).$$

For every odd prime q this can be decided with a sufficiently good approximation of $e_1(q)/e_0(q)$.

Theorem 1. *We have*

$$\frac{e_1(q)}{e_0(q)} = \frac{1-\gamma}{q-1} + \begin{cases} O\left(\frac{\log^2 q}{q^{3/2}}\right) & \text{unconditionally with an effective constant,} \\ O_\varepsilon\left(\frac{1}{q^{2-\varepsilon}}\right) & \forall \varepsilon > 0, \text{ unconditionally with an ineffective constant} \\ O\left(\frac{\log^2 q}{q^2}\right) & \text{if there are no exceptional zeros for } q \\ O\left(\frac{(\log q)\log\log q}{q^2}\right) & \text{on ERH for } L\text{-functions modulo } q. \end{cases}$$

Here $\gamma = 0.57721566\ldots$ is Euler's constant, and in this paper, an exceptional zero is a real number $\beta > 1 - 1/(9.645908801 \log q)$ that is a zero of $L(s, \chi_q)$, with χ_q being the real, nonprincipal character modulo q.

Remark. McCurley [33, Theorem 1.1] showed that for each q, the region $\Re s \geqslant 1 - 1/(9.645908801 \log \max(q, q|\Im s|, 10))$ contains at most one zero of $\prod_{\chi \bmod q} L(s, \chi)$, and if the zero exists, it is real, simple and a zero of $L(s, \chi_q)$. Throughout this paper, by ERH we mean that all nontrivial zeros of the Dirichlet L-functions for characters modulo q lie on the critical line $\Re s = \frac{1}{2}$.

Theorem 2. *Let q be an odd prime. For $q \leqslant 67$ the Ramanujan type approximation is asymptotically better than the naive approximation for $\mathcal{E}_q(x)$, for all remaining primes the naive approximation is asymptotically better. That is, $e_1(q)/e_0(q) > \frac{1}{2(q-1)}$ precisely when $q \leqslant 67$.*

Theorem 1 reveals in fact that neither $\mathcal{N}_q(x)$ nor $\mathcal{R}_q(x)$ capture the second term of the expansion (2).

For comparison purposes, recall that Gauss's approximation $\mathrm{li}(x) = \int_2^x dt/\log t$ is a much better estimate of $\pi(x)$, the number of primes up to x, than is the naive approximation $x/\log x$. Similarly, if $B(x)$ denotes the counting function of integers $n \leqslant x$ that can be written as sum of two squares, Ramanujan in his first letter (16 Jan. 1913) to Hardy claimed that, for every $r \geqslant 1$,

$$B(x) = K \int_2^x \frac{dt}{\sqrt{\log t}} + O\left(\frac{x}{\log^r x}\right), \tag{7}$$

where K is a certain constant. Landau [24] had proved in 1908 that $B(x) \sim Kx/\sqrt{\log x}$: a weaker assertion. The issue of which of the two asymptotic formulas is a better approximation was settled by Shanks [45], who showed that (7) is false for every $r > 3/2$. Similarly, in an unpublished manuscript, possibly included with his final letter (12 Jan. 1920) to Hardy, Ramanujan discussed congruence properties of $\tau(n)$, the coefficient of q^n in $q\prod_{k=1}^{\infty}(1-q^k)^{24}$, and $p(n)$, the partition function (see [1] or [3]). For special primes q, Ramanujan claimed that "it can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k = C_q \int_2^n \frac{dx}{(\log x)^{\delta_q}} + O\left(\frac{n}{\log^r n}\right), \tag{8}$$

where r is an positive number". Although asymptotically correct (as shown by Rankin), the third author [36] showed that all claims of the form (7) are false for every $r > 1 + \delta_q$. The proof involves computing the *Euler-Kronecker constant* for the generating series $\sum_{q \nmid \tau(n)}^{\infty} n^{-s}$ with several decimals of accuracy.

Our argument for Theorems 1 and 2 proceed by first relating the $e_1(q)/e_0(q)$ to two additional quantities,

$$S(q) = \frac{1}{q-1}\frac{C'(q,1)}{C(q,1)} = \sum_{p\neq q, f_p\geqslant 2} \frac{\log p}{p^{f_p}-1} \tag{9}$$

and the *Euler-Kronecker constant*

$$\mathcal{EK}_{K(q)} = \lim_{s\to 1}\left(\frac{\zeta_{K(q)}(s)}{\alpha_{K(q)}} - \frac{1}{s-1}\right). \tag{10}$$

Proposition 2. *We have*

$$(q-1)\frac{e_1(q)}{e_0(q)} = 1-\gamma+\frac{(3-q)\log q}{(q-1)^2(q+1)} + S(q) + \frac{\mathcal{EK}_{K(q)}}{q-1}.$$

In Section 4, we prove the following upper estimates for $S(q)$:

Theorem 3.
(a) *We have $S(q) \leqslant 45/q$ for all q;*
(b) *Let $\epsilon > 0$ be fixed. The inequality $S(q) < \epsilon/q$ holds for $(1+o(1))\pi(x)$ primes $q \leqslant x$.*

The analysis used to prove Theorem 3 depends on estimates for linear forms in logarithms to deal with the summands with p and f_p both small.

1.2 Euler-Kronecker constants for cyclotomic fields

In Section 3 we study the distribution of $\mathcal{EK}_{K(q)}$ as q runs through the primes. In particular, we will give explicit estimates for these constants needed for proving Theorems 1 and 2.

In [21], Ihara remarks that it seems very likely that always $\mathcal{EK}_{K(q)} > 0$ (this was checked numerically for $q \leqslant 8000$ by Mahora Shimura, assuming ERH). Ihara observed that $\mathcal{EK}_K$ can be conspicuously negative and that this occurs when K has many primes having small norm. However, in the case of $K(q)$ the smallest norm is q and therefore is rather large as q increases.

Using a new, fast algorithm (requiring computation of $L(1,\chi)$ for all characters modulo q), we performed a search for small values of $\mathcal{EK}_{K(q)}$. The details of the algorithm and computation are described later in Section 3. One negative value was found, at $q = 964477901$. We discuss later in the subsection the reason why this q, and conjecturally infinitely many others, have negative Euler-Kronecker constants.

Theorem 4. *For $q = 964477901$, we have*

$$\mathcal{EK}_{K(q)} = -0.18237\ldots$$

In [21], Ihara also proved, under the assumption of ERH, the one-sided bound $\mathcal{EK}_{K(q)} \leqslant (2+o(1))\log q$. In [22], Ihara made the following stronger conjecture.

Conjecture I (Ihara). *For any $\epsilon > 0$, if q is sufficiently large then*

$$\left(\frac{1}{2}-\epsilon\right)\log q < \mathcal{EK}_{K(q)} < \left(\frac{3}{2}+\epsilon\right)\log q.$$

We will show, assuming the Hardy-Littlewood conjectures for counts of prime k-tuples, that the lower bound in Ihara's conjecture is false and, even more, that $\mathcal{EK}_{K(q)}$ is infinitely often negative. In 1904, Dickson [7] posed a wide generalization of the twin prime conjecture that is now known as the "prime k-tuples conjecture". It states that whenever a set of linear forms $a_i n + b_i$ ($1 \leqslant i \leqslant k$, $a_i \geqslant 1$, $b_i \in \mathbb{Z}$) have no fixed prime factor (there is no prime p that divides $\prod_i (a_i n + b_i)$ for all n), then for infinitely many n, all of the numbers $a_i n + b_i$ are prime. This expresses a kind of local-to-global principle for prime values of linear forms, but is has not been proven for any k-tuple of forms with $k \geqslant 2$. Later, Hardy and Littlewood [18] conjectured an asymptotic formula for the number of such n. There have been extensive numerical studies of prime k-tuples, especially in the case $a_1 = \cdots = a_k = 1$, providing evidence for these conjectures (e.g. [12], [13]).

In connection with $\mathcal{EK}_{K(q)}$, we need to understand special sets of forms. We say that a set $\{a_1, \ldots, a_k\}$ of positive integers is an *admissible set* if the collection of forms n and $a_i n + 1$ ($1 \leqslant i \leqslant k$) have no fixed prime factor. We need the following weak form of the Hardy-Littlewood conjecture:

Conjecture HL. *Suppose $\mathcal{A} = \{a_1, \ldots, a_k\}$ is an admissible set. The number of primes $n \leqslant x$ for which the numbers $a_i n + 1$ are all prime is $\gg_{\mathcal{A}} x(\log x)^{-k-1}$.*

Theorem 5. *Assume Conjecture HL. Then*

$$\liminf_{q \to \infty} \frac{\mathcal{EK}_{K(q)}}{\log q} = -\infty.$$

The basis of our theorem is the following formula for $\mathcal{EK}_{K(q)}$.

Proposition 3. *We have*

$$\begin{aligned}\mathcal{EK}_{K(q)} &= -\frac{\log q}{q-1} + \lim_{x \to \infty} \left[\log x - (q-1) \sum_{\substack{n \leqslant x \\ n \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n} \right] \\ &= -\frac{\log q}{q-1} - (q-1)S(q) + \lim_{x \to \infty} \left[\log x - (q-1) \sum_{\substack{p \leqslant x \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1} \right].\end{aligned}$$

It is expected that the primes $p \equiv 1 \,(\text{mod } q)$ behave very regularly for $p > q^{1+\varepsilon}$ (arbitrary fixed $\varepsilon > 0$). It is irregularities in the distribution of the $p \leqslant q^{1+\varepsilon}$ which provide the variation in the values of $\mathcal{EK}_{K(q)}$.

Put $a(1) = 0$ and inductively define $a(n)$ to be the smallest integer exceeding $a(n-1)$ such that, for every prime r, the set $\{a(i) (\text{mod } r) : 1 \leqslant i \leqslant n\}$ has at most $r - 1$ elements (using the Chinese remainder theorem it is easily seen that the sequence is infinite). Given the prime k-tuples conjecture an equivalent statement is that $a(n)$ is minimal such that there are infinitely many primes q with $q + a(i)$ prime for $1 \leqslant i \leqslant n$. We have $\{a(i)\}_{i=1}^{\infty} = \{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, \ldots\}$. This is sequence A135311 in the OEIS [41] and is called 'the greedy sequence of prime offsets'. Given the prime k-tuples conjecture another equivalent statement is that $a(n)$ is minimal such that $a(1) = 0$ and there are infinitely many primes q with $a(i)q + 1$ prime for $2 \leqslant i \leqslant n$, $n \geqslant 2$. Define i_0 to be the smallest integer satisfying

$$\sum_{i=2}^{i_0} \frac{1}{a(i)} > 2,$$

A computer calculation gives $i_0 = 2089$ and $a(i_0) = 18932$.

Proposition 4. *Suppose that the number of primes q such that $a(i)q+1$ is a prime for $2 \leqslant i \leqslant 2089$ is $\gg x/(\log x)^{2090}$. Then $\mathcal{EK}_{K(q)} < 0$ for $\gg x/(\log x)^{2090}$ primes $q \leqslant x$.*

We note here that when $q = 964477901$, then $aq+1$ is prime for

$$a \in \{2, 6, 8, 12, 18, 20, 26, 30, 36, 56, \ldots\}.$$

The strongest unconditional result about the distribution of primes in arithmetic progressions, the Bombieri-Vinogradov theorem, implies that the primes $p \equiv 1 \pmod q$ with $p > q^2(\log q)^A$ are well-distributed for most q. The Elliott-Halberstam conjecture [8] goes further: Let $\pi(x; q, 1)$ denote the number of primes $p \leqslant x$ such that $p \equiv 1 \pmod q$, and let $\mathrm{li}(x) = \int_2^x dt/\log t$. For convenience, write

$$E(q; x) = \pi(x; q, 1) - \frac{\mathrm{li}(x)}{\phi(q)}.$$

Conjecture EH (Elliott-Halberstam). *For every $\varepsilon > 0$ and $A > 0$,*

$$\sum_{q \leqslant x^{1-\varepsilon}} |E(q; x)| \ll_{A,\varepsilon} \frac{x}{(\log x)^A}.$$

Theorem 6. *(i) Assume Conjecture EH. For every $\varepsilon > 0$, the bounds*

$$1 - \varepsilon < \frac{\mathcal{EK}_{K(q)}}{\log q} < 1 + \varepsilon$$

hold for almost all primes q (that is, the number of exceptional $q \leqslant x$ is $o(\pi(x))$).

(ii) Assume Conjectures HL and EH. Then the set $\{\mathcal{EK}_{K(q)}/\log q : q \text{ prime}\}$ is dense in $(-\infty, 1]$.

If, as widely believed, $E(x; q)$ is small for *all* $q \leqslant x^{1-\varepsilon}$, we may make a stronger conclusion.

Conjecture 1. *The set of limit points of $\{\mathcal{EK}_{K(q)}/\log q : q \text{ prime}\}$ is $(-\infty, 1]$.*

To determine the maximal order of $-\mathcal{EK}_{K(q)}$, one needs to assume a version of Conjecture HL with the implied constant in the $\gg$-symbol explicitly depending on $\{a_1, \ldots, a_k\}$. The heuristic argument in [14, Proposition 5 and §9] suggests that perhaps

$$\liminf \frac{\mathcal{EK}_{K(q)}}{(\log q)(\log\log\log q)} = -1.$$

Our conditional results about $\mathcal{EK}_{K(q)}$ are proved using standard methods of analytic number theory, and are very similar to the conditional bounds given by Granville in [14] for the class number ratio $h_q^- := h(\mathbb{Q}(e^{2\pi i/q}))/h(\mathbb{Q}(\cos 2\pi/q))$. Kummer in 1851 conjectured that, as $q \to \infty$, one has

$$h_q^- \sim 2q\Big(\frac{q}{4\pi^2}\Big)^{(q-1)/4}.$$

This conjecture is the analog of the conjecture that $\mathcal{EK}_{K(q)} \sim \log q$ as $q \to \infty$. We will make use of several results from [14].

Our Theorem 5 is reminiscent of a theorem of Hensley and Richards [20], who showed the incompatibility of the prime k-tuples conjecture and a conjecture of Hardy and Littlewood about primes in short intervals.

Assuming ERH, Ihara [22] defined a function $c(q)$ as follows:

$$c(q) := \Big(\sum_{\rho} \frac{q^{\rho-1/2}}{\rho(1-\rho)}\Big)/\sum_{\rho} \frac{1}{\rho(1-\rho)} = \Big(\sum_{\rho} \frac{\cos(\tau \log q)}{\frac{1}{4}+\tau^2}\Big)/\sum_{\rho} \frac{1}{\frac{1}{4}+\tau^2},$$

where $\rho = 1/2 + i\tau$ runs over all non-trivial zeros of $\zeta_{K(q)}(s)$. We have $|c(q)| \leqslant 1$ and

$$\Big(\int_{\infty}^{\infty} \frac{\cos(t \log q)}{\frac{1}{4}+t^2} dt\Big)/\Big(\int_{-\infty}^{\infty} \frac{dt}{\frac{1}{4}+t^2}\Big) = \frac{1}{\sqrt{q}}.$$

Thus, assuming that the distribution of τ modulo $2\pi/\log q$ for small $|\tau|$ is rather uniform, we would maybe expect that $\sqrt{q}c(q)$ approximates 1 closely. Ihara [22, Proposition 3] showed that under ERH we have

$$\frac{\mathcal{EK}_{K(q)}}{\log q} = \frac{3}{2} + (\sqrt{q}c(q) - 1) + O\Big(\frac{1}{\log q}\Big). \tag{11}$$

However, assuming ERH and Conjecture HL, it follows from this and Theorem 5 that

$$\liminf_{q\to\infty} \sqrt{q}c(q) = -\infty.$$

Furthermore, assuming Conjecture EH, Theorem 6 (i) and (11) lead to the conjecture that the normal order of $\sqrt{q}c(q)$ is $1/2$.

Another connection between $\mathcal{EK}_{K(q)}$ and the nontrivial zeros ρ of $\zeta_{K(q)}(s)$ is given by [19]:

$$\sum_{\zeta_{K(q)}(\rho)=0} \frac{1}{\rho} = \mathcal{EK}_{K(q)} - (q-1)(\log 2 + \gamma) + \frac{1}{2}(q-2)\log q - \frac{(q-1)}{2}\log \pi. \tag{12}$$

Since, at least conjecturally, $\mathcal{EK}_{K(q)}$ has normal order $\log q$ this parameter seems to 'measure' a subtle effect in the distribution of the zeros.

Finally, we like to point out that this paper is a very much reworked version of an earlier preprint by the third author [38]. In it a proof of Theorem 2 on ERH is given. From the perspective of computational number theory, this proof is far easier and less computation intensive than the one that does not assume ERH given here.

2 Analytic Theory

2.1 Propositions 1 and 2

For a general number field K we have, for $\mathrm{Re}(s) > 1$, the *Dedekind zeta function*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}.$$

Here, $\mathfrak{a}$ runs over non-zero ideals in $\mathcal{O}_K$, the ring of integers of K, $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$ and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbb{C} - \{1\}$, and that at $s = 1$ it has a simple pole with residue α_K, where [16, Theorem 61]

$$\alpha_K = \frac{2^{r_1}(2\pi)^{r_2} h(K) R(K)}{w(K)\sqrt{|d(K)|}}, \tag{13}$$

where K has r_1 (resp. $2r_2$) real (resp. complex) embeddings, class number $h(K)$, regulator $R(K)$, $w(K)$ roots of unity, and discriminant $d(K)$. We also have the Laurent expansion

$$\zeta_K(s) = \frac{\alpha_K}{s-1} + \gamma_K + \gamma_1(K)(s-1) + \gamma_2(K)(s-1)^2 + \cdots . \tag{14}$$

The constants $\gamma_j(\mathbb{Q})$ and $\gamma_{\mathbb{Q}}$ are known as the Stieltjes constants. In particular, we have $\gamma_{\mathbb{Q}} = \gamma$. The constant $\mathcal{EK}_K = \gamma_K/\alpha_K$ is called the *Euler-Kronecker constant* in Ihara [21] and Tsfasman [48], the reason for this being that in the case when K is imaginary quadratic the well-known Kronecker limit formula expresses γ_K in terms of special values of the Dedekind η–function.

Put

$$\tilde{\zeta}_K(s) = s(1-s)\Big(\frac{\sqrt{|d(K)|}}{2^{r_2}\pi^{[K:\mathbb{Q}]/2}}\Big)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\zeta_K(s).$$

Then it is known that the functional equation $\tilde{\zeta}_K(s) = \tilde{\zeta}_K(1-s)$ holds. Since $\tilde{\zeta}_K(s)$ is entire of order 1, one has the following Hadamard product factorization:

$$\tilde{\zeta}_K(s) = \tilde{\zeta}_K(0)e^{\beta_K s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

with some complex number β_K. Hashimoto et al. [19] show that

$$-\beta_K = \sum_{\rho}\frac{1}{\rho} = \mathcal{EK}_K - (r_1+r_2)\log 2 + \frac{1}{2}\log|d(K)| - \frac{[K:\mathbb{Q}]}{2}(\gamma + \log\pi) + 1,$$

where the sum is over the zeros of $\zeta_K(s)$ in the critical strip. On specializing this to the case $K(q)$, equation (12) is obtained.

Prime ideals of small norm in the ring of integers of K have a decreasing effect on $\mathcal{EK}_K$ as the following result (see, e.g., [19]) shows:

$$\mathcal{EK}_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\leqslant x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big). \tag{15}$$

We recall some facts from the theory of cyclotomic fields needed for our proofs. For a nice introduction to cyclotomic fields, see [50]. The following result, see e.g. [40, Theorem 4.16], describes the splitting of primes in the ring of integers of a cyclotomic field.

Lemma 1. *(cyclotomic reciprocity law). Let $K = \mathbb{Q}(e^{2\pi i/m})$. If the prime p does not divide m and $f = \operatorname{ord}_m p$, then the principal ideal $p\mathcal{O}_K = \mathfrak{p}_1\cdots\mathfrak{p}_g$ with $g = \varphi(m)/f$, and all $\mathfrak{p}_i$'s distinct and of degree f.*

However, if p divides m, $m = p^a m_1$ with $p \nmid m_1$ and $f = \operatorname{ord}_{m_1} p$, then $p\mathcal{O}_k = (\mathfrak{p}_1\cdots\mathfrak{p}_g)^e$ with $e = \varphi(p^a)$, $g = \varphi(m_1)/f$, and all $\mathfrak{p}_i$'s distinct and of degree f.

In case $K = K(q)$, we have $r_1 = 0$, $2r_2 = q-1$, $w(K) = 2q$ (as K contains exactly $\{\pm 1, \pm\omega, \pm\omega^2, \ldots, \pm\omega^{q-1}\}$ as roots of unity, with $\omega = e^{2\pi i/(q-1)}$) and furthermore $D(K) = (-1)^{q(q-1)/2}q^{q-2}$, and thus we obtain from (13) that

$$\alpha_{K(q)} = \mathrm{Res}_{s=1}\zeta_K(s) = 2^{\frac{q-3}{2}}q^{-\frac{q}{2}}\pi^{\frac{q-1}{2}}h(K)R(K). \tag{16}$$

For cyclotomic fields K the Euler product for $\zeta_K(s)$ can be written down explicitly using the "cyclotomic reciprocity law". We find that

$$\begin{aligned}\zeta_{K(q)}(s) &= \left(1-\frac{1}{q^s}\right)^{-1}\prod_{p\neq q}\left(1-\frac{1}{p^{sf_p}}\right)^{\frac{1-q}{f_p}} \\ &= \left(1-\frac{1}{q^s}\right)^{-1} C(q,s)^{-1}\prod_{p\equiv 1 (\text{mod } q)}\left(1-\frac{1}{p^s}\right)^{1-q}.\end{aligned} \tag{17}$$

It is also well-known (see, e.g., [16, Theorem 65]) that

$$\zeta_{K(q)}(s) = \zeta(s)\prod_{\chi\neq\chi_0} L(s,\chi), \tag{18}$$

where the product is over characters χ modulo q, with χ_0 being the principal character, and $L(s,\chi)$ the Dirichlet L-function associated with χ.

Proof of Proposition 1. First, (4) follows by combining (3) and (17). By (18),

$$h_q(s) = (1+q^{-s})\left(\zeta(s)(1-q^{-s})\right)^{\frac{q-2}{q-1}} C(q,s)^{-\frac{1}{q-1}}\prod_{\chi\neq\chi_0} L(s,\chi)^{-\frac{1}{q-1}}. \tag{19}$$

For $\chi\neq\chi_0$, $L(s,\chi)$ is analytic and nonzero at $s=1$. Hence, $f(s)=h_q(s)(s-1)^{(q-2)/(q-1)}s^{-1}$ is analytic in a neighborhood of $s=1$ and has a power series expansion there. Moreover, $\prod_{\chi\neq\chi_0} L(s,\chi)$ has no zeros in the region $\Re s \geqslant 1-a_q(\log(|\Im s|+2))^{-1}$ for some positive a_q. Therefore, $h_q(s)/s$ has an expansion around the point $s=1$ of the form

$$\frac{h_q(s)}{s} = \frac{1}{(s-1)^{(q-2)/(q-1)}}\Big(c_0(q)+c_1(q)(s-1)+\cdots+c_k(q)(s-1)^k+\cdots\Big),$$

To apply the Selberg-Delange method, we also need a mild growth condition on $h_q(s)\zeta(s)^{-\frac{q-2}{q-1}}$. The function $C(q,s)$ is analytic for $\Re s > \frac{1}{2}$, and uniformly bounded in the half-plane $\Re s \geqslant \frac{3}{4}$. For $\sigma \geqslant 1-\frac{a_q}{2\log(|t|+2)}$,

$$\Big|\prod_{\chi\neq\chi_0} L(\sigma+it)\Big|^{-1} \ll_q (\log(|t|+2))^{q-2}.$$

By [47, §II.5, Theorem 3], an asymptotic expansion (2) holds with the coefficients satisfying $e_j(q) = c_j(q)/\Gamma(\frac{q-2}{q-1}-j)$. □

Proof of Proposition 2. By Proposition 1 and the functional equation $\Gamma(z)=(z-1)\Gamma(z-1)$, we have

$$\begin{aligned}\frac{e_1(q)}{e_0(q)} &= -\frac{1}{q-1}\frac{c_1(q)}{c_0(q)} = -\frac{f'(1)}{(q-1)f(1)} \\ &= \frac{1}{q-1}\left(1-\lim_{s\to 1^+}\left(\frac{1-\frac{1}{q-1}}{s-1}+\frac{h_q'(s)}{h_q(s)}\right)\right).\end{aligned}$$

By the Laurent expansion $\zeta(s)=(s-1)^{-1}+\gamma+O(|s-1|)$, we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1}+\gamma+O(|s-1|) \qquad (|s-1|\leqslant 1). \tag{20}$$

Hence, by logarithmic differentiation of (19),

$$\lim_{s\to 1^+}\frac{1-\frac{1}{q-1}}{s-1}+\frac{h_q'(s)}{h_q(s)}=-\frac{\log q}{q+1}+\frac{(q-2)\log q}{(q-1)^2}+\frac{q-2}{q-1}\gamma-S(q)-\frac{1}{q-1}\sum_{\chi\neq\chi_0}\frac{L'(1,\chi)}{L(1,\chi)}.$$

By (14), (20) and logarithmic differentiation of (18), we have

$$\mathcal{EK}_{K(q)}=\gamma+\sum_{\chi\neq\chi_0}\frac{L'(1,\chi)}{L(1,\chi)}. \tag{21}$$

On combining the various formulas the proof is completed. □

2.2 The constant $C(q)$

Spearman and Williams put, for a generator χ_q of the group of characters modulo q,

$$C(q)=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^r}\Big(1-\frac{1}{p^{(q-1)/(r,q-1)}}\Big)^{(r,q-1)}. \tag{22}$$

From this definition it is not a priori clear that it does not depend on the choice of χ_g. However, Spearman and Williams show that it indeed does not.

Proposition 5. *We have $C(q)=C(q,1)$.*

Proof. We claim that if $\chi_g(p)=\omega^r$, then $f_p=(q-1)/(r,q-1)$. We have $1=\chi_g(p^{f_p})=\omega^{rf_p}$. It follows that $(q-1)|rf_p$ and thus $q_r=(q-1)/(r,q-1)$ must be a divisor of f_p. On the other hand, since $\chi_g(a)=1$ if and only if $a=1$, it follows from $\omega^{rq_r}=\chi_g(p^{q_r})=1$ and $q_r|f_p$, that $f_p=q_r$. Thus, we can rewrite (22) as

$$C(q)=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^r}\Big(1-\frac{1}{p^{f_p}}\Big)^{\frac{q-1}{f_p}}. \tag{23}$$

Note that $p\neq q$ and $f_p\geqslant 2$ iff $\chi_g(p)=\omega^r$ for some $1\leqslant r\leqslant q-2$. This observation in combination with the absolute convergence of the double product (23), then shows that $C(q)=C(q,1)$. □

Remark. Proposition 5 says that $1/C(q)$ is the contribution at $s=1$ of the primes $p\neq q$, $p\not\equiv 1(\text{mod } q)$ to the Euler product (17) of $K(q)$.

2.3 On Mertens' theorem for arithmetic progressions

A crucial ingredient in the paper of Spearman and Williams is the asymptotic estimate [46, Proposition 6.3] that

$$\prod_{\substack{p\leqslant x\\ p\equiv 1(\text{mod } q)}}\Big(1-\frac{1}{p}\Big)=\Big(\frac{qe^{-\gamma}}{(q-1)\alpha_{K(q)}C(q)\log x}\Big)^{\frac{1}{q-1}}\Big(1+O_q\Big(\frac{1}{\log x}\Big)\Big). \tag{24}$$

An alternative, much shorter proof of the estimate (24) can be obtained on invoking Mertens' theorem for algebraic number fields.

Lemma 2. *Let α_K denote the residue of $\zeta_K(s)$ at $s = 1$. Then,*

$$\prod_{N\mathfrak{p}\leqslant x}\Big(1-\frac{1}{N\mathfrak{p}}\Big)=\frac{e^{-\gamma}}{\alpha_K\log x}\Big(1+O_K\Big(\frac{1}{\log x}\Big)\Big),$$

where the product is over the prime ideals $\mathfrak{p}$ in $\mathcal{O}_K$ having norm $\leqslant x$.

Proof. Similar to that of the usual Mertens' theorem (see e.g. Rosen [42] or Lebacque [31]). □

Proof of estimate (24). We invoke Lemma 2 with $K = K(q)$ and work out the product over the prime ideals more explicitly using the cyclotomic reciprocity law, Lemma 1. One finds, for $x \geqslant q$, that it equals

$$\Big(1-\frac{1}{q}\Big)\prod_{\substack{p\leqslant x\\ p\equiv 1(\text{mod } q)}}\Big(1-\frac{1}{p}\Big)^{q-1}\prod_{\substack{p^{f_p}\leqslant x,\ p\neq q\\ p\not\equiv 1(\text{mod } q)}}\Big(1-\frac{1}{p^{f_p}}\Big)^{\frac{q-1}{f_p}}=$$

$$\Big(1+O_q\Big(\frac{1}{\sqrt{x}}\Big)\Big)\Big(1-\frac{1}{q}\Big)C(q)\prod_{\substack{p\leqslant x\\ p\equiv 1(\text{mod } q)}}\Big(1-\frac{1}{p}\Big)^{q-1},$$

where we used that for $k \geqslant 2$,

$$\prod_{p^k>x}(1-p^{-k})^{-1}=1+O(\sum_{n^k>x}n^{-k})=1+O(x^{1/k-1}).$$

Thus, on invoking Lemma 2, we deduce (24). □

For recent work on this theme, the reader is referred to the papers by Languasco and Zaccagnini [27, 28, 29, 30].

3 Estimates for the Euler-Kronecker constants $\mathcal{EK}_{K(q)}$

3.1 Unconditional bounds for $\mathcal{EK}_{K(q)}$

Proof of Proposition 3. Apply (21), the orthogonality of characters, and the relation (e.g. [26, §55] or [35, §6.2, Exercise 4])

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\log x-\gamma+o(1)\qquad(x\to\infty)$$

to obtain the first claimed bound. The sum on n equals

$$\sum_{\substack{p\leqslant x\\ p\equiv 1 \pmod q}}\frac{\log p}{p-1}-A(x)+B(x),$$

where

$$A(x)=\sum_{\substack{p\leqslant x,p^a>x\\ p\equiv 1 \pmod q}}\frac{\log p}{p^a},\qquad B(x)=\sum_{\substack{p^a\leqslant x\\ p^a\equiv 1 \pmod q\\ p\not\equiv 1 \pmod q}}\frac{\log p}{p^a}.$$

Clearly, $\lim_{x\to\infty} B(x) = S(q)$. The last estimate we need is $\lim_{x\to\infty} A(x) = 0$, which is proved as follows:

$$A(x) \leqslant \sum_{a=2}^{\infty} \sum_{n>x^{1/a}} \frac{\log n}{n^a} \ll \sum_{a=2}^{\infty} \frac{\log x}{a^2 x^{1-1/a}} \ll \frac{\log x}{\sqrt{x}}. \qquad \square$$

Remark 1. Alternatively one can prove Proposition 3 on making the limit formula (15) explicit for $K(q)$ using Lemma 1.

Remark 2. Proposition 3 can be used to approximate, nonrigorously, the value of $\mathcal{EK}_{K(q)}$. For example, when $q = 964477901$, the right side in Proposition 3 stays very close to -0.18 for $10^6 \leqslant x/q \leqslant 10^7$; see Theorem 4.

Proposition 6. *If $y \geqslant 10q$ and $q \geqslant 11$, then*

$$\sum_{\substack{p\leqslant y \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} \leqslant \frac{2\log y + 2(\log q)\log\log(y/q)}{q-1}.$$

Proof. By the Montgomery-Vaughan sharpening of the Brun-Titchmarsh inequality [34], we have

$$\pi(y;q,1) \leqslant \frac{2y}{(q-1)\log(y/q)},$$

and hence, by partial summation,

$$\begin{aligned}
\sum_{\substack{p\leqslant y \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} &= \frac{\pi(y;q,1)\log y}{y-1} + \int_{2q}^{y} \frac{\pi(t;q,1)}{(t-1)^2}\left(\log t - \frac{t-1}{t}\right) dt \\
&\leqslant \frac{2}{q-1}\left(\frac{y\log y}{(y-1)\log(y/q)} + \int_{2q}^{y} \frac{t}{(t-1)^2} + \frac{t\log q - (t-1)}{(t-1)^2\log(t/q)} dt\right) \\
&\leqslant \frac{2}{q-1}\left(\frac{y}{y-1}\left(1 + \frac{\log q}{\log 10}\right) + \int_{2q}^{y} \frac{1}{t} + \frac{2}{(t-1)^2} + \frac{\log q}{t\log(t/q)}\, dt\right) \\
&\leqslant \frac{2}{q-1}\left(1.01 + 0.44\log q + \log(\frac{y}{2q}) + \frac{2}{2q-1} + (\log q)(\log\log(\frac{y}{q}) - \log\log 2)\right) \\
&\leqslant \frac{2\log y + 2(\log q)\log\log(y/q)}{q-1}. \qquad \square
\end{aligned}$$

Proposition 7. *Uniformly for $z \geqslant 2$, $\delta > 0$ and $0 < \varepsilon \leqslant 1$, the number of primes $q \leqslant z$ for which*

$$\sum_{\substack{p\leqslant q^{1+\varepsilon} \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} \geqslant \delta \frac{\log q}{q}$$

is $O(\varepsilon\pi(z)/\delta)$.

Proof. By sieve methods (e.g. [17, Theorem 5.7]), for an even $k \geqslant 2$, the number of prime $q \leqslant z$ with $kq+1$ prime is $O(\frac{k}{\phi(k)}\frac{z}{\log^2 z})$ uniformly in k. Thus, the number of primes q in question is

$$\leqslant \sum_{q\leqslant z} \frac{q}{\delta \log q} \sum_{\substack{p\leqslant q^{1+\varepsilon} \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} \leqslant \frac{1}{\delta} \sum_{\substack{k\leqslant z^{\varepsilon} \\ 2|k}} \sum_{\substack{k^{1/\varepsilon}\leqslant q\leqslant z \\ kq+1 \text{ prime}}} \frac{\log(kq+1)}{k\log q}$$

$$\ll \frac{z}{\delta \log^2 z} \sum_{k\leqslant z^{\varepsilon}} \frac{1}{\phi(k)} \ll \frac{\varepsilon}{\delta} \frac{z}{\log z},$$

where we used the well-known estimate $\sum_{n\leqslant x} \varphi(n)^{-1} = O(\log x)$. □

Lemma 3. *Let $q \geqslant 10000$ be prime and let χ be the quadratic character modulo q. If $L(\beta_0, \chi) = 0$, then*

$$\beta_0 \geqslant 1 - \frac{3.125 \min(2\pi, \frac{1}{2}\log q)}{\sqrt{q}\log^2 q}.$$

Proof. By Dirichlet's class number formula [5, §6, (15) and (16)],

$$L(1,\chi) = \begin{cases} \frac{\pi h(-q)}{\sqrt{q}} & q \equiv 3 \pmod 4 \\ \\ \frac{h(q)\log u}{\sqrt{q}} & q \equiv 1 \pmod 4), \end{cases}$$

where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, and u is the smallest unit in $\mathbb{Q}(\sqrt{d})$ satisfying $u > 1$. Since $u > \sqrt{q}$ and $h(-s) \geqslant 2$ for $s > 163$, we obtain for $q > 163$ that $L(1,\chi) \geqslant \min(2\pi, \frac{1}{2}\log q)q^{-1/2}$. Assume $\beta_0 \geqslant 1 - 0.2q^{-1/2}$, else there is nothing to prove. Let $V(t) = \sum_{n\leqslant t} \chi(n)$. By the Pólya-Vinogradov inequality ([5, §23, (2)] or [35, §9.4]), for $t > u > 0$,

$$|V(t) - V(u)| < \frac{2}{\sqrt{q}} \sum_{a=1}^{(q-1)/2} \frac{1}{\sin(\pi a/q)} \leqslant \frac{2}{\sqrt{q}} \int_{1/2}^{q/2} \frac{dt}{\sin(\pi t/q)}$$

$$= \frac{2\sqrt{q}}{\pi} \log\cot\left(\frac{\pi}{4q}\right) < \frac{2}{\pi}\sqrt{q}\log(4q/\pi).$$

Hence, for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $y \geqslant 100$,

$$|L'(\sigma,\chi)| \leqslant y^{1-\sigma} \sum_{n\leqslant y} \frac{\log n}{n} + \int_y^{\infty} |V(t) - V(y)| \frac{\sigma \log t - 1}{t^{1+\sigma}}\, dt$$

$$\leqslant y^{1-\sigma} \left(\frac{\log^2 y}{2} + \frac{2}{\pi}\sqrt{q}\log(\frac{4q}{\pi})\frac{\log y}{y} \right).$$

Taking $y = q^{0.67}$ gives

$$|L'(\sigma,\chi)| \leqslant q^{0.67(1-\sigma)}(0.316\log^2 q) \leqslant 0.32\log^2 q.$$

The mean value theorem implies $(1-\beta_0)(0.32\log^2 q) \geqslant L(1,\chi)$ and the lemma follows. □

3.2 Numerical calculation of $\mathcal{EK}_{K(q)}$

The identity (21) is useful for numerically calculating $\mathcal{EK}_{K(q)}$ for small q. For example, cf. [37],

$$\mathcal{EK}_{K(3)} = \gamma + \frac{L'(1,\chi_3)}{L(1,\chi_3)} = 0.9454972808716807032397499941581890 73\ldots,$$

where χ_3 stands for the only non-principal character modulo 3. For larger q we use the following formulas. First,

$$L(1,\chi) = -\frac{1}{q}\sum_{r=1}^{q-1}\chi(r)\psi\left(\frac{r}{q}\right), \qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{25}$$

We also use

$$-L'(1,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n} = (\log q)L(1,\chi) + \frac{1}{q}\sum_{r=1}^{q-1}\chi(r)T\left(\frac{r}{q}\right), \tag{26}$$

where

$$T(y) = \sum_{m=0}^{\infty}\left(\frac{\log(m+y)}{m+y} - \frac{\log(m+1)}{m+1}\right).$$

Here, the term $(m+1)^{-1}\log(m+1)$ is a convergence factor, included so that the terms in the sum on m are $O(m^{-2}\log m)$. The advantage of using (25) and (26) is that for each q, there are only $q-1$ values of ψ and $q-1$ sums $T(r/q)$ to compute. With these values in hand, there are, however, still $\gg q^2$ operations (additions, subtractions, multiplications, divisions) needed using a naive algorithm to compute all of the numbers $L(1,\chi)$ and $L'(1,\chi)$. A significant speed-up is achieved by observing that the vector of sums on r on the right sides of (25) and (26) are discrete Fourier transform coefficients. Specifically, let g be a primitive root of q, χ_1 the character with $\chi_1(g) = e^{2\pi i/(q-1)}$ and for $1 \leqslant k \leqslant q-1$, let r_k be the integer in $[1, q-1]$ satisfying $g^k \equiv r_k \pmod{q}$. The characters modulo q are $\chi_0, \chi_1, \chi_1^2, \ldots, \chi_1^{q-2}$ and for $\chi = \chi_1^j$, the sum in (25) is $\sum_{k=1}^{q-1} e^{2\pi ijk/(q-1)}\psi(r_k/q)$ and the sum on r in (26) is $\sum_{k=1}^{q-1} e^{2\pi ijk/(q-1)}T(r_k/q)$. Fast Fourier Transform (FFT) algorithms may be used to recover $L(1,\chi)$ and $L'(1,\chi)$ from the vectors $(\psi(r_1/q), \ldots, \psi(r_{q-1}/q))$ and $(T(r_1/q), \ldots, T(r_{q-1}/q))$, respectively, with $O(q\log q)$ operations.

A program to compute the numbers $L(1,\chi)$ and $L'(1,\chi)$ was written in the C language, making use of the FFT library `fftw` [10]. Running on a Dell Inspiron 530 desktop computer with Ubuntu Linux, 2GB RAM and a 2.0 GHz processor, the program computed $\mathcal{EK}_{K(q)}$ for all prime $q \leqslant 25000$ in 87 seconds. All computations were performed using high precision arithmetic (80-bit "long double precision" floating point numbers). In order to handle very large q (larger than about 5×10^7) a machine with more memory was required. A suitably modified version of the program was run on a large cluster computer, with 256GB RAM, 48 core AMD Opteron 6176 SE processors (4 sockets, 12 cores/socket), operating system Ubuntu Linux 10.04.3 LTS x86_64. The computation of $\mathcal{EK}_{K(q)}$ for $q = 964477901$ took 64 minutes of CPU time on this system. This gave Theorem 4.

Lemma 4. *For $q \leqslant 30000$, we have $0.315\log q \leqslant \mathcal{EK}_{K(q)} \leqslant 1.627\log q$.*

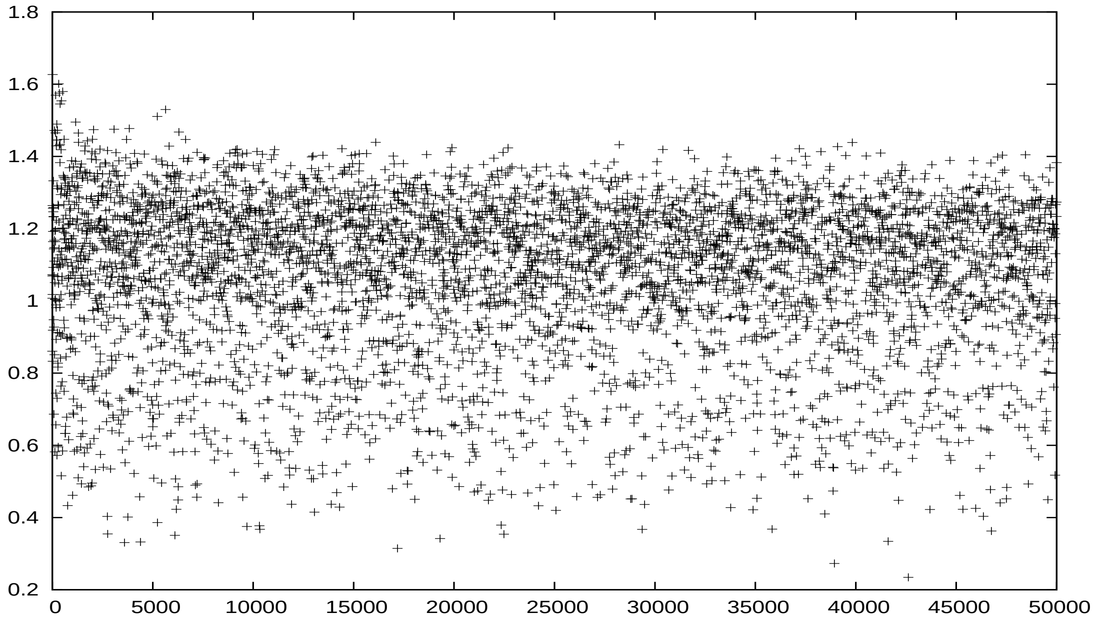

Figure 1: $\mathcal{EK}_{K(q)}/\log q$ for $q \leqslant 50000$

The largest value of $\mathcal{EK}_{K(q)}/\log q$ among $q \leqslant 30000$ is $\mathcal{EK}_{K(19)}/\log 19 = 1.626\ldots$ and the smallest is $\mathcal{EK}_{K(17183)}/\log 17183 = 0.315\ldots$. Lemma 4 suffices for the application to Theorem 2.

In the next subsection, we will discuss more about the likely distribution of the Euler-Kronecker constants. Figure 1 displays a scatter plot of $\mathcal{EK}_{K(q)}/\log q$ for the primes $q \leqslant 50000$.

3.3 Conditional bounds for $\mathcal{EK}_{K(q)}$

Lemma 5. *(i) For all $C > 0$ and for all except $O(\pi(u)/(\log u)^C)$ primes $q \leqslant u$,*

$$\mathcal{EK}_{K(q)} = 2\log q - q \sum_{\substack{p \leqslant q^2 \\ p \equiv 1 \pmod q}} \frac{\log p}{p-1} + O_C(\log\log q).$$

(ii) Assuming ERH, the above inequality holds for all prime q (the implied constant in the $O_C(\log\log q)$ term being absolute in this case).

(iii) Assume Conjecture EH and fix $C > 0$ and $\varepsilon > 0$. For all except $O(\pi(u)/(\log u)^C)$ primes $q \leqslant u$,

$$\mathcal{EK}_{K(q)} = (1+\varepsilon)\log q - q \sum_{\substack{p \leqslant q^{1+\varepsilon} \\ p \equiv 1 \pmod q}} \frac{\log p}{p-1} + O_C(\log\log q).$$

Proof. Part (i) is a straightforward combination of Proposition 3 and the Bombieri-Vinogradov theorem [5, §28] (cf. Proposition 2 of [14]). The latter states that for all $A > 0$ there is a B so that

$$\sum_{q \leqslant \sqrt{x}/\log^B x} |E(x;q)| \ll \frac{x}{(\log x)^A}.$$

For any $x \geqslant z > q$, partial summation implies

$$\sum_{\substack{y\leqslant p\leqslant x\\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} = \frac{\log\left(\frac{x-1}{y-1}\right)}{q-1} + G_q(x,z), \tag{27}$$

where

$$G_q(x,z) = \left[\frac{E(x;t)\log t}{t-1}\right]_y^x + \int_y^x \left(\frac{\log t}{(t-1)^2} - \frac{1}{t^2-t}\right) E(t;q)\,dt.$$

Let B be the constant corresponding to $A = C+3$, let z be large and put $y = z^2(\log z)^{2B+1}$. For any $t \geqslant y$, $2z \leqslant \sqrt{t}(\log t)^{-B}$ and so

$$S(t;z) := \sum_{z<q\leqslant 2z} |E(t;q)| \ll \frac{t}{(\log t)^{C+3}}.$$

We obtain

$$\sum_{z<q\leqslant 2z} \sup_{x>y} |G_q(y,x)| \ll \sup_{t\geqslant y} \frac{S(t;z)\log t}{t} + \int_y^\infty \frac{S(t;x)\log t}{t^2}\,dt \ll \frac{1}{(\log z)^{C+1}}.$$

Thus, the summand on the left is $\geqslant 1/(2z)$ for $O(z(\log z)^{-C-1})$ primes $q \in (z, 2z]$. Summing over dyadic intervals, we find that $\sup_{x>y} |G_q(y,x)| \geqslant 1/q$ for $O(\pi(u)/\log^C u)$ primes $q \leqslant u$. For the other (non-exceptional) q, from Proposition 3 and Theorem 3 we obtain

$$\mathcal{EK}_{K(q)} = 2\log(y-1) + O(1) - (q-1) \sum_{\substack{p\leqslant y\\ p\equiv 1 \pmod q}} \frac{\log p}{p-1},$$

where $y \asymp q^2(\log q)^{2B+1}$. Finally, the Brun-Titchmarsh inequality and partial summation gives

$$\sum_{\substack{q^2<p\leqslant y\\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} \ll \frac{\log(y/q^2)}{q} \ll_C \frac{\log\log q}{q}.$$

This proves (i). To obtain (ii), insert into (27) the bound $E(t;q) \ll \sqrt{t}\log q$ valid under ERH (apply partial summation to [5, §20, (14)]), take $y = q^2(\log q)^{C+10}$ and argue as in part (i). To prove (iii), substitute Conjecture EH for the Bombieri-Vinogradov Theorem and take $y = z^{1+\varepsilon}$ in the above argument. □

Remark. Part (ii) of Lemma 5 may also be deduced from a general bound for $\mathcal{EK}_K$ due to Ihara [21, Proposition 2].

Lemma 6. *For any $M > 0$, there is an admissble set $\{a_1, \dots, a_k\}$ with $\sum_i 1/a_i > M$.*

Remark. According to Granville [14], Lemma 6 was conjectured by Erdős in 1988 and a proof is given in [14, Theorem 3]. We show below that Lemma 6 is actually a simple corollary of Erdős' 1961 paper [9].

Proof. Let $p_1 = 3$ and, recursively for each $k \geqslant 2$, let p_k be the smallest prime for which $p_k \not\equiv 1 \pmod{p_j}$ for all $j < k$. Thus $p_2 = 5$, $p_3 = 17$, $p_4 = 23$, etc. Erdős in [9], answering a question of S. Golomb, proved that $\sum_{k=1}^{\infty} 1/p_k$ diverges. For a given M, let J be so large that if $\mathcal{B} = \{2(p_j + 1) : 1 \leqslant j \leqslant J\}$, then $\sum_{b\in\mathcal{B}} 1/b > M$. We now deduce that $\mathcal{B}$ is admissible. Let $F(n) = n\prod_{b\in\mathcal{B}}(bn + 1)$. Observe that by construction, if r is prime and $r = p_j$ for some j, then none of the elements of $\mathcal{B}$ are congruent to $2 \pmod r$. Hence, if $4n \equiv -1 \pmod r$, then $r \nmid F(n)$. If r is a prime and $r \neq p_j$ for every j, then none of the elements of $\mathcal{B}$ are congruent to $1 \pmod r$. Consequently, if $2n \equiv -1 \pmod r$, then $r \nmid F(n)$. □

Proof of Theorem 5 and Proposition 4. Let $M \geqslant 0$ be arbitrary. Using Lemma 6, there is an admissible set $\{a_1, \ldots, a_k\}$ so that $\sum_i 1/a_i > M + 2$. By Lemma 5 (i), for all but $O(u/\log^{k+2} u)$ primes $q \leqslant u$,

$$\mathcal{EK}_{K(q)} = 2\log q + O_M(\log\log q) - q \sum_{\substack{p\leqslant q^2 \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1}.$$

Assuming Conjecture HL, there are $\gg u/\log^{k+1} u$ primes $q \leqslant u$ for which $a_i q + 1$ is prime for $1 \leqslant i \leqslant k$. For such primes $q > a_k + 1$,

$$q \sum_{\substack{p\leqslant q^2 \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} \geqslant \sum_{i=1}^{k} \frac{\log q}{a_i} > (M+2)\log q.$$

Theorem 5 follows.

Proposition 4 follows by taking $M = 0$ in the above argument and noting that we may take an admissible set with $k = 2089$. □

Proof of Theorem 6. Fix $\eta > 0$. Assuming Conjecture EH and using Lemma 5 (iii), we see that for all but $O(\pi(u)/\log^C u)$ primes $q \leqslant u$,

$$\mathcal{EK}_{K(q)} = (1+\eta^2)\log q + O_C(\log\log q) - q \sum_{\substack{p\leqslant q^{1+\eta^2} \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1}. \tag{28}$$

On the other hand, by Lemma 7 (with $\delta = \eta/2$ and $\varepsilon = \eta^2$), for all but $O(\eta\pi(u))$ primes $p \leqslant u$, the above sum on p is $\leqslant (\eta \log q)/q$. Hence, taking $C = 1$, for all but $O(\eta\pi(u))$ primes $p \leqslant u$, $(1-\eta)\log q \leqslant \mathcal{EK}_{K(q)} \leqslant (1+\eta)\log q$ for large enough q. As η is arbitrary, part (i) follows.

To show part (ii) concerning limit points of $\mathcal{EK}_{K(q)}/\log q$, start with (28) and let $\varepsilon = \eta^2$. Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be an admissible set and let $m(\mathcal{A}) = \sum_i 1/a_i$. Assuming Conjecture HL, there are $\gg u/\log^{k+1} u$ primes $q \leqslant u$ such that $a_i q + 1$ is prime for $1 \leqslant i \leqslant k$. By sieve methods [17, Theorem 5.7], the number of primes $q \leqslant u$ for which $a_i q + 1$ is prime $(1 \leqslant i \leqslant k)$ and $bq + 1$ is also prime is $O(\frac{b}{\phi(b)} u/\log^{k+2} u)$, where the implied constant depends on $\mathcal{A}$. Summing over even $b \leqslant q^{\varepsilon}$, $b \in \mathcal{A}$, we find that there are $O(\varepsilon u/\log^{k+1} u)$ primes $q \leqslant u$ with $bq + 1$ prime for some $b \leqslant q^{\varepsilon}$, $b \notin \mathcal{A}$. If ε is small enough, depending on $\mathcal{A}$, then there are $\gg u/\log^{k+1}$ primes $q \leqslant u$ for which $qa_i + 1$ is prime $(1 \leqslant i \leqslant k)$ and

$qb+1$ is composite for all $b \leqslant q^\varepsilon$ such that $b \notin \mathcal{A}$. For such q, (28) with $C = k+2$ implies that

$$\mathcal{EK}_{K(q)} = (1+\varepsilon - m(\mathcal{A}))\log q + O_k(\log\log q).$$

As ε is arbitrary, we see that $1 - m(\mathcal{A})$ is a limit point of $\{\mathcal{EK}_{K(q)}/\log q \,:\, q \text{ prime}\}$. Finally, it follows immediately from Lemma 6 that $\{m(\mathcal{A}) : \mathcal{A} \text{ admissible}\}$ is dense in $[0,\infty)$. Indeed, given any $x > 0$ and $\delta > 0$, there is an admissible set of integers $> 1/\delta$ with $m(\mathcal{A}) > x$. As any subset of an admissible set is also admissible, there is a subset $\mathcal{A}'$ of $\mathcal{A}$ with $|m(\mathcal{A}') - x| < \delta$. □

4 Upper bounds for $S(q)$

We will give explicit upper bounds in Theorem 3 for $S(q)$, making use of explicit estimates for prime numbers from [43]. Note that $f_p \geqslant 2$ implies that $q|(p^{f_p}-1)/(p-1)$, that is,

$$\frac{p^{f_p}-1}{p-1} = qn_p, \quad n_p \geqslant 1. \tag{29}$$

Lemma 7. *For $x \geqslant 2$,*

$$\log x - 0.605 \leqslant \sum_{p\leqslant x}\frac{\log p}{p-1} \leqslant \begin{cases}\log x - 0.142 & (x \geqslant 9)\\ \log x - \frac{1}{2} & (x \geqslant 467.4).\end{cases}$$

Also,

$$\sum_{p\geqslant x}\frac{\log p}{p^3-1} \leqslant \frac{0.6}{x^2} \quad (x > 2).$$

Proof. For the first estimate, we note that

$$\sum_{p\leqslant x}\frac{\log p}{p-1} = \sum_{p\leqslant x}\frac{\log p}{p} + \sum_{p\leqslant x}\frac{\log p}{p(p-1)}.$$

The latter sum can be easily bounded by 0.756. The first estimate then is derived on invoking [43, Theorems 6, 21] to deal with $x \geqslant 1000$ and direct numerical calculation for smaller x. For $x \geqslant 7481$ one has $0.98x \leqslant \sum_{p\leqslant x}\log p \leqslant 1.01624x$, as was shown by Rosser and Schoenfeld [43, Theorems 9 and 10]. From this one easily infers that for $x \geqslant 7481$

$$\sum_{p>x}\frac{\log p}{p^3-1} \leqslant \frac{x}{x^3-1}\Big(-0.98 + 1.01624\left(\frac{3}{2}\right)\frac{x^3}{(x^3-1)}\Big).$$

For $k = 2$, the right side is $\leqslant 1.0525x^{-1}$ and for $k = 2$, the right side is $\leqslant 0.545x^{-2}$. For $x < 7481$, we explicitly calculate the sum using

$$\sum_{p>x}\frac{\log p}{p^3-1} = -\frac{\zeta'(3)}{\zeta(3)} - \sum_{p\leqslant x}\frac{\log p}{p^3-1}.$$

□

4.1 A simple upper bound

Lemma 8. *Let q be a prime with $q \geqslant 5$. We have*

$$S(q) \leqslant \frac{\log q + 1}{2q}.$$

Proof. First, if $f_p = 2$, then $p = 2kq - 1$ for a positive integer k. As $p \geqslant 13$, we have $p^2 - 1 \geqslant 6(p+1)^2/7$. Thus,

$$\begin{aligned}\sum_{p \equiv -1 \pmod q} \frac{\log p}{p^2 - 1} &\leqslant \frac{7}{6} \sum_{k=1}^{\infty} \frac{\log(2kq)}{4k^2 q^2} \\ &= \frac{(7/6)\left(\zeta(2)\log(2q) - \zeta'(2)\right)}{4q^2} \leqslant \frac{0.48 \log q + 0.61}{q^2}.\end{aligned}$$

Next, suppose $p > q$ and $f_p \geqslant 3$. Combining the latter estimate and Lemma 7, we conclude that

$$\sum_{p > q, f_p \geqslant 2} \frac{\log p}{p^{f_p} - 1} \leqslant \frac{0.48 \log q + 1.21}{q^2}. \tag{30}$$

Now suppose $p < q$ (so that $f_p \geqslant 3$). If $q \geqslant 83$, by Lemma 7

$$S'(q) = \sum_{p < q} \frac{\log p}{p^{f_p} - 1} \leqslant \frac{1}{q} \sum_{p < \sqrt{q}} \frac{\log p}{p - 1} + \sum_{p > \sqrt{q}} \frac{\log p}{p^3 - 1} \leqslant \frac{0.5 \log q + 0.458}{q}.$$

On combining this estimate with (30) yields the claimed bound for $q \geqslant 83$. For $5 \leqslant q < 83$, direct calculation shows that $S'(q) \leqslant \frac{\log q - 0.5}{2q}$ and the claimed bound on $S(q)$ follows from (30). □

Lemma 8 is strong enough in order to prove Theorem 2. However, with a refined analysis, we can obtain a sharper inequality when q is large.

4.2 Refined upper bound

Note that in case q is a Mersenne prime we have

$$S(q) \geqslant \frac{\log 2}{2^{f_2} - 1} = \frac{\log 2}{q}.$$

Actually, the only q we have been able to find for which $S(q) > (\log 2)/q$ are the Mersenne primes. It thus is conceivable that if q is not a Mersenne prime, then always $S(q) < (\log 2)/q$. For a given $\epsilon > 0$ it also appeared to us that the primes q for which $S(q) > \epsilon/q$ have density zero. In what follows, we prove that this is the case. In general $S(q)$ is relatively large if q almost equals a number of the form $p^r - 1$ with p small. For example, if $2q = 3^r - 1$ for some r (e.g. when $r = 3, 7, 13, 71$), then $S(q) > (\log 3)/(2q)$. The above remarks show that the upper bound in the first part of Theorem 3, except for the constant, is likely optimal.

Proof of Theorem 3. We prove both (a) and (b) simultaneously. If $5 \leqslant q \leqslant 10^{30}$, Lemma 8 gives $S(q) < 35.1/q$ and (a) follows. Now suppose $q > 10^{30}$. We first consider three ranges for p:

(i) $p > q$,

(ii) $p < q$ and $f_p \leqslant F = \lceil \frac{\log q}{3 \log\log q} \rceil$,

(iii) $f_p \geqslant F + 1$ and $p > \log^4 q$.

Inequality (30) gives a good bound for the contribution of the primes in the range (i) to $S(q)$. Note that given $f \geqslant 3$, there are at most $f-1$ primes $p < q$ with $f_p = f$. By (29), $q \leqslant 2p^{f-1}$, hence the contribution to $S(q)$ from a given f is

$$\leqslant \frac{(f-1)\log[(q/2)^{1/(f-1)}]}{(q/2)^{f/(f-1)} - 1} \leqslant 2.83 \frac{\log q}{q^{1+\frac{1}{f-1}}}.$$

If $f \leqslant F$, then $q^{\frac{1}{f-1}} \geqslant \log^3 q$ and the contribution to $S(q)$ from such f is

$$\leqslant \frac{2.83(F-2)}{q \log^2 q} \leqslant \frac{2.83}{3q(\log q)\log\log q}. \tag{31}$$

For p counted in the range (iii), $p^{f_p - 1} \geqslant p^F \geqslant q^{4/3}$. By Lemma 7, the contribution to $S(q)$ from such p is

$$\leqslant \frac{1}{q^{4/3}} \sum_{\log^4 q < p < q} \frac{\log p}{p-1} \leqslant \frac{\log q}{q^{4/3}}. \tag{32}$$

By (30), (31) and (32), the contribution to $S(q)$ from p in ranges (i)–(iii) is

$$O\left(\frac{1}{q \log q}\right) \qquad \text{and also} \qquad \leqslant \frac{1}{310q}. \tag{33}$$

The primes p not considered in ranges (i)–(iii) satisfy $p \leqslant \log^4 q$ and $f_p > F$. We now take a brief interlude to prove (b). The contribution to $S(q)$ from those p with $f_p \geqslant F' = \lceil \frac{2\log q}{\log 2} \rceil$ is $\leqslant 2\sum_p (\log p)p^{-F'} = O(q^{-2})$. As $f_p | (q-1)$, we have dealt with all ranges unless $q-1$ has a divisor in (F, F'). But this is rare; specifically, by Theorems 1 and 6 of [15], the number of $q \in (x, 2x]$ with such a divisor is $O(\pi(x)(\log\log\log x / \log\log x)^{-0.086})$. By (33), (b) follows.

Next, we continue proving (a), by considering further ranges:

(iv) $p \leqslant e^{41}$,

(v) $e^{41} < p \leqslant \log^4 q$ and $n_p \geqslant \min(p, f_p)$,

(vi) $e^{41} < p \leqslant \log^4 q$ and $n_p < \min(p, f_p)$.

Trivially, by Lemma 7, the contribution to $S(q)$ in case (iv) is

$$\leqslant \frac{1}{q} \sum_{p \leqslant e^{41}} \frac{\log p}{p-1} \leqslant \frac{40.5}{q}. \tag{34}$$

For ranges (v) and (vi), observe that $\log q \geqslant e^{41/4}$. Since $f_p \geqslant \frac{\log q}{\log p}$, the contribution to $S(q)$ in case (v) is

$$\begin{aligned} &\leqslant \frac{1}{q \log q} \sum_{e^{41} < p \leqslant \log^4 q} \frac{\log^2 p}{p-1} + \sum_{p > e^{41}} \frac{\log p}{qp(p-1)} \\ &\leqslant \frac{4\log\log q(4\log\log q - 40.895)}{q \log q} + \frac{10^{-10}}{q} \leqslant \frac{1}{416q}. \end{aligned} \tag{35}$$

Here we used again Lemma 7, together with the fact that the maximum of $x(x-b)e^{-x/4}$ occurs at $x=(b+8+\sqrt{b^2+64})/2$ (here $x=4\log\log q$).

Now consider range (vi). We will show that f_p is prime. Indeed, assume that f_p is composite. Then

$$\frac{p^{f_p}-1}{p-1}=\prod_{\substack{d|f_p\\d>1}}\Phi_d(p),$$

where $\Phi_d(X)\in\mathbb{Z}[X]$ is the dth cyclotomic polynomial. There exists some divisor $d_0>1$ of f_p such that $q\mid\Phi_{d_0}(p)$ (in fact $d_0=f_p$, but this is not needed for the proof). Hence,

$$n_p\geqslant\prod_{\substack{d|f_p\\d\neq 1,d_0}}\Phi_d(p).$$

Since f_p is not prime, the number f_p has at least three divisors. Let $d_1>1$ be any divisor of f_p different from d_0. Then

$$n_p\geqslant\Phi_{d_1}(p)>(p-1)^{\phi(d_1)}\geqslant p-1,$$

so $n_p\geqslant p$, a contradiction. Hence, $f=f_p$ is a prime factor of $q-1$. By Fermat's Little Theorem, $p^f\equiv p(\bmod f)$. Further, if $p\equiv 1(\bmod f)$, then $(p^f-1)/(p-1)$ is a multiple of f. Otherwise, $p-1$ is invertible modulo f, and since $p^f-1\equiv p-1(\bmod f)$, we get that $(p^f-1)/(p-1)$ is congruent to 1 modulo f. Hence,

$$qn_p=\frac{p^f-1}{p-1}\equiv 0,1\pmod f,$$

and since $q\equiv 1(\bmod f)$, we conclude that $n_p\equiv 0,1(\bmod f)$. But $n_p<f$, hence $n_p=1$ and

$$\frac{p^f-1}{p-1}=q. \tag{36}$$

On writing the left hand side as $\sum_{j=0}^{f-1}p^j$, we that in particular, $p|(q-1)$. Since $q-1$ has at most $\frac{\log q}{\log\log q}$ prime factors $>\log q$, the contribution to $S(q)$ from $p\in(\log q,\log^4 q]$ is

$$\leqslant\frac{\log q}{q\log\log q}\cdot\frac{4\log\log q}{\log q-1}\leqslant\frac{4.004}{q}. \tag{37}$$

Let $\mathcal{P}$ be the set of primes satisfying (36) which are in the interval $(e^{41},\log q]$. We cover the interval in dyadic intervals of the form $\mathcal{I}_k=[2^k,2^{k+1})$ with $2^k\leqslant\log q$, and we look at $\mathcal{P}_k=\mathcal{P}\cap\mathcal{I}_k$. We will show below that $\mathcal{P}_k$ has at most one element, and hence

$$\frac{1}{q}\sum_{p\in\mathcal{P}}\frac{\log p}{p}\leqslant\frac{1}{q}\sum_{k\geqslant 59}\frac{k\log 2}{2^k-1}\leqslant\frac{1}{10^{15}q}.$$

Combined with (33), (34), (35) and (37), this proves the theorem.

Now assume that $\mathcal{P}_k$ has at least two elements for some k, so that $k\geqslant 59$. Let $p_1<p_2$ be any two elements in $\mathcal{P}_k$ with

$$q=\frac{p_1^{f_1}-1}{p_1-1}=\frac{p_2^{f_2}-1}{p_2-1}.$$

Since the function $f \mapsto (p^f - 1)/(p-1)$ is increasing for all fixed p, it follows that $f_1 > f_2$. Now

$$(p_2 - 1)p_1^{f_1} - (p_1 - 1)p_2^{f_2} = p_2 - p_1. \tag{38}$$

Thus,

$$\left| \frac{(p_1 - 1)}{(p_2 - 1)} p_2^{f_2} p_1^{-f_1} - 1 \right| = \frac{p_2 - p_1}{(p_2 - 1)p_1^{f_1}} < \frac{1}{p_1^{f_1}} \leqslant \frac{1}{2^{kf_1}}. \tag{39}$$

On the left, we use a lower bound for a linear form in three logarithms. Note that since $p_2 > p_1$ this expression is not zero. Now all three rational numbers $(p_1 - 1)/(p_2 - 1)$, p_1 and p_2 have height $< 2^{k+1}$. Thus, Matveev's bound from [32] (see also Theorem 9.4 in [4]) tells us at once that

$$\begin{aligned} \log \left| \frac{(p_1 - 1)}{(p_2 - 1)} p_2^{f_2} p_1^{-f_1} - 1 \right| &> -1.4 \times 30^6 \times 3^{4.5} (1 + \log(4f_1)) \left(\log(2^{k+1}) \right)^3 \\ &> -4.77 \times 10^{10} (k+1)^3 (1 + \log(4f_1)). \end{aligned} \tag{40}$$

Thus, comparing bounds (39) and (40), we get that

$$k f_1 \log 2 < 4.77 \times 10^{10} (k+1)^3 (1 + \log(4f_1)).$$

Since $k \geqslant 59$,

$$f_1 < \frac{4.77 \times 10^{10}}{\log 2} \left(\frac{k+1}{k} \right) (k+1)^2 (1 + \log(4f_1)) < 7 \times 10^{10} (k+1)^2 \log(4f_1).$$

Here, we used the fact that $\log(4f_1) \geqslant \log(4F) > 37$, so $1 + \log(4f_1) < \frac{38}{37} \log(4f_1)$. This gives

$$4f_1 < 2.876 \times 10^{11} (k+1)^2 \log(4f_1).$$

For $A > 10^{12}$, the inequality $x < A \log x$ implies that $x < \frac{9}{8} A \log A$ and hence

$$f_1 < 8.1 \times 10^{10} (k+1)^2 \left(26.4 + 2 \log(k+1) \right).$$

Since $f_1 \log p_1 > \log q$, $\log p_1 < (k+1) \log 2$ and $2^k \leqslant \log q$, we have

$$2^k \leqslant \log q < (\log 2) \times 10^{11} (k+1)^3 (26.6 + 2 \log(k+1)).$$

This implies $k \leqslant 58$, a contradiction. □

5 Proof of theorems 1 and 2

Let

$$E_q(t) = \Psi(t; q, 1) - \frac{t}{q-1}, \text{ where } \Psi(t; q, 1) = \sum_{\substack{n \leqslant x \\ n \equiv 1 \pmod q}} \Lambda(n).$$

Let $R = 9.645908801$. We say that β_0 is an *exceptional zero* for a prime q if $\beta_0 \geqslant 1 - 1/(R \log q)$ and $L(\beta_0, \chi) = 0$, where χ is the quadratic character modulo q. Let $B(q) = 1$ if β_0 exists, and $B(q) = 0$ otherwise.

Lemma 9. *Suppose $q \geqslant 10000$ is prime. Then, for $x \geqslant e^{R\log^2 q}$,*

$$|E_q(x)| \leqslant \frac{1.012x^{\beta_0}}{q}B(q) + \frac{8}{9}x\sqrt{\frac{\log x}{R}}\exp\left\{-\sqrt{\frac{\log x}{R}}\right\}.$$

The proof of Lemma 9 comes from estimates in McCurley [33], and will be given later in Section 6.

Proof of Theorem 1. Propositions 2 and 3 imply that

$$(q-1)\frac{e_1(q)}{e_0(q)} = 1-\gamma-\frac{2\log q}{q^2-1} - S(q) + \lim_{x\to\infty}\left[\frac{\log x}{q-1} - \sum_{\substack{n\leqslant x\\ n\equiv 1 \pmod n}}\frac{\Lambda(n)}{n}\right]. \quad (41)$$

By partial summation, for any $y > 2q$ we have

$$\lim_{x\to\infty}\Big(\sum_{\substack{y<n\leqslant x\\ n\equiv 1 \pmod q}}\frac{\Lambda(n)}{n} - \frac{\log(x/y)}{q-1}\Big) = -\frac{E_q(y)}{y} + \int_y^\infty \frac{E_q(t)}{t^2}\,dt. \quad (42)$$

By Lemma 9,

$$\Big|\int_y^\infty \frac{E_q(t)}{t^2}\,dt - \frac{E_q(y)}{y}\Big| \leqslant B(q)\frac{1.012(2-\beta_0)y^{\beta_0-1}}{(1-\beta_0)q} + \frac{8}{9}\left(\frac{2RW^2+(4R+1)W+4R}{e^W}\right), \quad (43)$$

where $W = \sqrt{\frac{\log y}{R}}$.

Taking $y = \exp(4R\log^2 q)$ (so that $W = 2\log q$), we obtain

$$\left|(q-1)\frac{e_1(q)}{e_0(q)} - (1-\gamma)\right| \ll \frac{\log y}{q} + \frac{B(q)}{1-\beta_0} + \sum_{\substack{n\leqslant y\\ n\equiv 1 \pmod q}}\frac{\Lambda(n)}{n}.$$

By Proposition 6 and Theorem 3, the above sum on n is

$$\leqslant S(q) + \frac{2\log y + 2(\log q)\log\log(y/q)}{q-1} \ll \frac{\log^2 q}{q}.$$

The first three parts of Theorem 1 now follow: for the first part, use Lemma 3; for the second part use Siegel's theorem [5, §21] which states that for every $\varepsilon > 0$, $\beta_0 \geqslant 1 - C(\varepsilon)q^{-\varepsilon}$ for an (ineffective) constant $C(\varepsilon)$; for the third part, we assume β_0 doesn't exist.

Finally, on ERH we have $E_q(t) \ll t^{1/2}\log^2 t$, uniformly in $q \leqslant t$ [5, §20, (14)]. Hence, if $y \geqslant q$ then

$$\Big|\int_y^\infty \frac{E_q(t)}{t^2}\,dt - \frac{E_q(y)}{y}\Big| \ll \frac{\log^2 y}{y^{1/2}}.$$

Taking $y = q^3$ in the above argument yields $\mathcal{EK}_{K(q)} = O((\log q)(\log\log q))$ and hence the final estimate in Theorem 1. □

Remarks. The estimate $\mathcal{EK}_{K(q)} = O((\log q)\log\log q)$, valid under ERH, was proved independently by Badzyan [2]. Note that a third way to establish it is by using [21, Proposition 2]. Unconditionally, Ihara et al. [23] have shown that $\mathcal{EK}_{K(q)} \ll_\varepsilon q^\epsilon$ (implicit in the third estimate in Theorem 1). In a more recent paper [39], Kumar Murty proved that $|\mathcal{EK}_{K(q)}|$ is $O(\log q)$ on average:

$$\sum_{Q/2<q\leqslant Q} |\mathcal{EK}_{K(q)}| \ll (\pi(Q)-\pi(Q/2))\log Q.$$

Proof of Theorem 2. By (41)–(43) (ignoring the summands in (41) with $n \leqslant y$), together with the exceptional zero estimate in Lemma 3, we have for $q \geqslant 10000$ the estimate

$$(q-1)\frac{e_1(q)}{e_0(q)} \leqslant 1-\gamma+\frac{\log y}{q-1}+1.015\frac{y^{-D/(q^{1/2}\log^2 q)}\log^2 q}{Dq^{1/2}}+\frac{8}{9}\left(\frac{2RW^2+(4R+1)W+4R}{e^W}\right),$$

where $D = 3.125\max(2\pi, \frac{1}{2}\log q)$. When $q \geqslant 30000$, we take $y = e^{1.44R\log^2 q}$, so that $W = 1.2\log q$ and $D \geqslant 16.1$. A short calculation reveals that $e_1(q)/e_0(q) < \frac{1}{2}$.

For $q < 30000$ we use the results of explicit calculation of $\mathcal{EK}_{K(q)}$ (e.g., Table 1 and Lemma 4). □

6 Proof of Lemma 9

In [33], McCurley gives estimates for $E_q(x)$ under the assumption that the exceptional zero β_0 doesn't exist. It is simple to modify the arguments to handle the case when β_0 does exist. Define

$$L=\log q, \quad X=\sqrt{\frac{\log x}{R}}, \quad x=e^{\lambda RL^2}, \quad \lambda=(1+\alpha)^2, \quad H=q^\alpha.$$

In particular,

$$X=(1+\alpha)L=\log(qH). \tag{44}$$

Also, since $q \geqslant 10000$, we have $x \geqslant 10^{355}$. We take $\eta = \frac{1}{2}$ in [33, Theorem 2.1], which gives

$$\left|N(T,\chi)-\frac{T}{\pi}\log\left(\frac{qT}{2\pi e}\right)\right| \leqslant C_1\log(qT)+C_2,$$

where $C_1 = 0.9185$, $c_2 = 5.512$ and $N(T,\chi)$ is the number of zeros of $L(s,\chi)$ with imaginary part in $[-T,T]$ and real part in $(0,1)$. Lemma 3.5 of [33] concerns bounds for $\sum_{\chi\neq\chi_0}|b(\chi)|$ (where $b(\chi)$ is the constant term in the Laurent expansion of $\frac{L'}{L}(s,\chi)$ about $s=0$) and it is assumed that β_0 doesn't exist. However, by [33, (3.16)], the existence of β_0 contributes an extra amount $\leqslant \frac{1}{14}q^{1/2}\log^2 q$ to the sum. The estimate in this lemma is thus increased by an amount $\leqslant 0.06$ if β_0 exists.

We apply [33, Theorem 3.6] with $m=2$ and $\delta = 2/H \leqslant 0.0002$. In the notation of this theorem,

$$A_2(\delta)=\delta^{-2}\left(1+2(1+\delta)^3+(1+2\delta)^3\right) \leqslant 4.003\delta^{-2}. \tag{45}$$

Denote by $\rho = \beta + i\gamma$ a generic zero of a non-principal L-function with $0<\beta<1$. Then we have

$$\frac{q-1}{x}|E_q(x)| < (1+\delta)\sum_{\chi\neq\chi_0}\sum_{\rho:|\gamma|\leqslant H}\frac{x^{\beta-1}}{|\rho|}+\frac{4.003}{\delta^2}\sum_{\chi\neq\chi_0}\sum_{\rho:|\gamma|>H}\frac{x^{\beta-1}}{|\rho(\rho+1)(\rho+2)|}+\delta+\varepsilon_1, \tag{46}$$

q	$S(q)$	$qS(q)$	$\mathcal{EK}_{K(q)}$	$\mathcal{EK}_{K(q)}/\log q$	$(q-1)\dfrac{e_1(q)}{e_0(q)}$
3	0.351646	1.054940	0.945497	0.860628	1.247179
5	0.077777	0.388887	1.720624	1.069083	0.897187
7	0.122829	0.859805	2.087594	1.072811	0.866519
11	0.009100	0.100103	2.415425	1.007310	0.657441
13	0.046201	0.600623	2.610757	1.017859	0.673826
17	0.004437	0.075432	3.581976	1.264280	0.642487
19	0.011009	0.209173	4.790409	1.626934	0.692657
23	0.000829	0.019080	2.611289	0.832815	0.536910
29	0.000347	0.010088	3.093731	0.918758	0.529900
31	0.036585	1.134139	4.314442	1.256394	0.599845
37	0.000929	0.034387	4.304938	1.192200	0.540802
41	0.000449	0.018445	3.971521	1.069461	0.520422
43	0.000218	0.009397	4.378627	1.164157	0.525317
47	0.000129	0.006083	4.799394	1.246548	0.525580
53	0.000214	0.011346	4.337736	1.092548	0.505056
59	0.000065	0.003863	5.433516	1.332548	0.515399
61	0.001438	0.087727	5.071085	1.233578	0.507672
67	0.000268	0.018017	5.292139	1.258626	0.502328
71	0.000612	0.043471	5.255258	1.232853	0.497650
73	0.001374	0.100374	4.066949	0.947905	0.479861
79	0.000496	0.039250	4.998276	1.143914	0.486679
83	0.000073	0.006119	3.033136	0.686409	0.459221
89	0.000349	0.031120	4.164090	0.927696	0.469899
97	0.000171	0.016587	4.891240	1.069191	0.473429
101	0.000012	0.001283	5.297012	1.147751	0.475323
103	0.000032	0.003301	5.144339	1.109954	0.472822
107	0.000030	0.003234	5.458274	1.168087	0.473907
109	0.000025	0.002756	6.906638	1.472207	0.486372
113	0.000024	0.002809	4.021730	0.850729	0.458353
127	0.005911	0.750763	5.088599	1.050454	0.468785
131	0.000029	0.003827	2.836826	0.581889	0.444355
137	0.000034	0.004791	4.937000	1.003459	0.458862
139	0.000079	0.011060	5.889168	1.193474	0.465287
149	0.000008	0.001234	5.983424	1.195741	0.462998

Table 1: Approximate values of $S(q)$, $\mathcal{EK}_{K(q)}$ and $e_1(q)/e_0(q)$.

where, using the modified Lemma 3.5 of [33],

$$\varepsilon_1 < \frac{q}{x}\left(\frac{\log q \log x}{\log 2} + \frac{q\log q}{4} + 15\log^2 q + 56\log q + 12\right) < 10^{-300}Xe^{-X}. \tag{47}$$

To estimate the sums over ρ, let

$$R(T) = C_1\log(qT) + C_2, \qquad \phi_n(t) = t^{-n-1}\exp\left\{-\frac{\log x}{R\log(qt)}\right\}.$$

By [33, Lemma 3.7], for each $\chi \neq \chi_0$,

$$\sum_{\substack{\rho:|\gamma|\leqslant H\\ \rho\neq\beta_0}} \frac{x^{\beta-1}}{|\rho|} < \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \tag{48}$$

where, by (44),

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2\sqrt{x}}\left(\frac{\lambda L^2}{\pi} + \frac{2+\alpha}{\pi}L + \frac{R(H)}{H} + 2R(1) + C_1\right) + \frac{qL+\alpha L^2}{x} < 10^{-100}Xe^{-X},\\
\varepsilon_3 &= \phi_0(H)R(H) = \frac{C_1X+C_2}{H}e^{-X} < 0.00016Xe^{-X},\\
\varepsilon_4 &= \frac{1}{2}\int_1^H \phi_0(t)\log\left(\frac{qt}{2\pi}\right)dt < \frac{1}{2}\int_1^H \phi_0(t)\log(qt)\,dt\\
&= \frac{\log^2 x}{2R^2}\int_{(1+\alpha)L}^{(1+\alpha)^2L}\frac{e^{-u}}{u^3}\,du < \frac{\log^2 x}{2R^2(1+\alpha)^3L^3}\int_{(1+\alpha)L}^{\infty} e^{-u}\,du = \frac{Xe^{-X}}{2}.
\end{aligned}$$

Therefore,

$$\varepsilon_2 + \varepsilon_3 + \varepsilon_4 < 0.5002Xe^{-X}. \tag{49}$$

For each $\chi \neq \chi_0$, [33, Lemma 3.8] implies that

$$\sum_{\rho:|\gamma|>H} \frac{x^{\beta-1}}{|\rho(\rho+1)(\rho+2)|} < \varepsilon_5 + \varepsilon_6 + \varepsilon_7, \tag{50}$$

where

$$\begin{aligned}
\varepsilon_5 &= \frac{1}{2H^2\sqrt{x}}\left(\frac{H}{2\pi}(1+\alpha)L + 2R(H) + \frac{C_1}{3}\right) + \frac{4L}{xH^2} < 10^{-100}\frac{Xe^{-X}}{H^2},\\
\varepsilon_7 &= R(H)\phi_2(H) = \frac{C_1X+C_2}{H^3}e^{-X} < 0.00016\frac{Xe^{-X}}{H^2},\\
\varepsilon_6 &= \frac{1}{2}\int_H^\infty C_1\phi_3(t) + \phi_2(t)\log\left(\frac{qt}{2\pi}\right)dt < \frac{1}{2}\int_H^\infty \phi_2(t)\log(qt)\,dt\\
&= \frac{q^2\lambda L^2}{4}\int_{\sqrt{2}}^\infty ue^{-\frac{X}{\sqrt{2}}(u+\frac{1}{u})}\,du = \frac{q^2\lambda L^2}{2\pi}K_2(2\sqrt{2}X,\sqrt{2}),
\end{aligned}$$

where K_2 is the incomplete Bessel function. By [44, Lemmas 4 and 5],

$$K_2(z,x) \leqslant \left(x+\frac{2}{z}\right)\left(\frac{x^2}{z(x^2-1)}\right)e^{-\frac{z}{2}(x+1/x)} \qquad (x>1, z>0),$$

hence

$$\varepsilon_6 \leqslant \frac{q^2}{2\pi}\left(X+\frac{1}{2}\right)e^{-3X} = \frac{X}{2\pi}\left(1+\frac{1}{2X}\right)\frac{e^{-X}}{H^2} \leqslant \frac{0.1678Xe^{-X}}{H^2}.$$

Therefore,

$$\varepsilon_5+\varepsilon_6+\varepsilon_7 < \frac{0.168Xe^{-X}}{H^2}. \tag{51}$$

By (45),

$$\frac{\delta}{q-1} \leqslant \frac{2.0003}{qH} = 2.0003e^{-X} < \frac{2.0003}{L}Xe^{-X}.$$

Combining this with estimates (46), (47), (48), (49), (50) and (51), we conclude that

$$\begin{aligned}|E_q(x)| &< B(q)\frac{(1+\delta)x^{\beta_0}}{(q-1)\beta_0} + Xe^{-X}x\left[(1+\delta)(0.5002)+10^{-300}+0.168\frac{A_2(\delta)}{H^2}\right]+\frac{\delta x}{q-1}\\ &< B(q)\frac{1.012x^{\beta_0}}{q}+\frac{8}{9}xXe^{-X}. \quad \square\end{aligned}$$

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