Hamiltonian circle actions and π_1 of the space of Hamiltonians

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Abstract

We study the fundamental group of the space of Hamiltonians of certain compact symplectic 6-manifolds admitting Hamiltonian circle actions. In fact, we prove that the existence of a Hamiltonian S^1 action provides and non-trivial element of the fundamental group of the space of Hamiltonians of such spaces.

1 Introduction

In the study of symplectic structures on manifolds, the topology of the Hamiltonian group plays a central role [4, 7, 2, 8] and, in particular, its fundamental group [10]. Let (M, Ω) be a closed symplectic manifold. Let $\mathrm{Ham}(M, \Omega)$ denote the group of Hamiltonian symplectomorphims endowed with the $\mathcal{C}^\infty$ topology, given as follows: a diffeomorphism ϕ belongs to $\mathrm{Ham}(M, \Omega)$ if

1. $\phi^*\Omega = \Omega$,
2. there exists an isotopy $\phi_t : M \to M$, $t \in [0, 1]$, with $\phi_0 = id_M$ and $\phi_1 = \phi$, and

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3. there exists a time dependent *Hamiltonian* function $H_t : M \to \mathbb{R}$ such that
$$\Omega(\cdot, \frac{\partial \phi_t}{\partial t}) = d(H_t \circ \phi_t).$$

Since M is closed there is a unique Hamiltonian function F_t with the property that $\int_M F_t \Omega^n = 0$ for every t. Such a function will be called a *normalized* Hamiltonian.

The group $\pi_1(\mathrm{Ham}(M, \Omega))$ has been calculated only in very few cases, such as Riemann surfaces [3] or 4-dimensional ruled surfaces [4, 1, 2]. In the 6-dimensional case practically no information is available. For instance, $\mathrm{Ham}(\mathbb{R}^{2n}, dxdy)$ is contractible for $n = 1, 2$, while nothing is known for $n = 3$.

The Hofer norm [7] is defined as follows: for $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega))$,

$$\nu(\gamma) := \inf_{F_t} \int_0^1 (\max_x F_t - \min_x F_t) dt,$$

where the infimum is taken over all normalized Hamiltonian functions F_t that generate ϕ_t, and the maximum and minimum are taken over all the points $x \in M$. It is known that

$$\nu(-\gamma) = \nu(\gamma) \quad \text{and} \quad \nu(\gamma + \gamma') \leq \nu(\gamma) + \nu(\gamma'),$$

where we have used additive notation, since $\pi_1(\mathrm{Ham}(M, \Omega))$ is an abelian group.

Let us define the *positive* and *negative* parts of ν separately,

$$\begin{aligned} \nu_+(\gamma) &= \inf_{F_t} \textstyle\int_0^1 \max_x F_t(x) dt, \\ \nu_-(\gamma) &= \inf_{F_t} \textstyle\int_0^1 - \min_x F_t(x) dt, \end{aligned}$$

where the infimum is taken over all the normalized Hamiltonian functions that generate a loop representing γ. One can verify that

$$\nu_+(\gamma) = \nu_-(-\gamma) \quad \text{and} \quad \nu(\gamma) \geq \nu_+(\gamma) + \nu_-(\gamma).$$

Note that that equality holds in all the known examples. The calculation of $\nu_+(\gamma)$ is non-trivial even in the case $M = S^2$ and γ is the class of the rotation with two fixed points (see [10]).

The main theorem of this note is the following.

Theorem 1. *Let $M = \mathbb{P}(E \oplus \mathbb{C})$ where E is a complex line bundle and $\mathbb{C}$ is the trivial line bundle over $\mathbb{CP}^2$, with first Chern class $c_1(E) = n\rho$, $n = 0, 1, 2, 3$, and ρ the generator of $H^2(\mathbb{CP}^2)$. Let M be endowed with the symplectic structure in the class of $c_1(TM) + t\rho$, where $t = 0$ for $n = 0, 1, 2$ and $0 < t \leq 1$ for $n = 3$. Then*

$$\begin{cases} \nu_-(k\gamma) \geq k \geq 0, & k \in \mathbb{Z} \text{ and } n > 0, \\ \nu_-(k\gamma) = 0, & \text{if } k \text{ is even and } n = 0, \\ \nu_-(k\gamma) = 1, & \text{if } k \text{ is odd and } n = 0. \end{cases}$$

where γ is the loop of Hamiltonians given by a (global) S^1-action on M that rotates the fibers $\mathbb{CP}^1$ of M over $\mathbb{CP}^2$. Then,

$$\lim_{k \to \infty} \frac{\nu_-(k\gamma)}{k} \geq 1,$$

and γ is an element of infinite order in $\pi_1(Ham(M, \Omega))$.

Remark. In fact, the only property of the action needed in the proof is that the fixed point set be the two canonical sections of $\mathbb{P}(E \oplus \mathbb{C})$.

This type of lower bound on the norm and the asymptotic growth of a loop are more general than those in [10] since the manifolds are not necessarily monotonic. Moreover, we see that the existence of a Hamiltonian circle action produces an element of infinite order in $\pi_1(\mathrm{Ham}(M, \Omega))$, which may also be true in more general situations.

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2 Preliminaries and Quantum Cohomology

The approach we shall use to study $\pi_1(\mathrm{Ham}(M, \Omega))$ is by means of *symplectic fibrations* over S^2 (cf. [12, 8]) and quantum cohomology.

2.1 Symplectic fibrations

Given a loop $\phi : S^1 \times M \to M$ of Hamiltonian diffeomorphisms, define

$$P_\phi = M \times D^2_+ \cup_\phi M \times D^2_-$$

by

$$\begin{array}{ccc} M \times \partial D^2_+ & \longrightarrow & M \times \partial D^2_- \\ \\ (x,t) & \mapsto & (\phi_t(x), -t), \end{array}$$

where we consider the two dimensional disc D^2_+ with its positive orientation. Thus, P_ϕ fibers over $S^2 = D^2_+ \cup D^2_-$ with fiber equal to M, where the orientation in S^2 is the one induced from D^2_+. We denote this fibration by $p : P_\phi \to S^2$. The assumption that ϕ is a Hamiltonian loop implies that p is a symplectic fibration, i.e. there exists a closed 2-form τ on P_ϕ that extends the symplectic form of the fibers. Homotopic loops produce isomorphic symplectic fibrations and, in particular, if $[\phi_t] = 0$ in $\pi_1(\mathrm{Ham}(M,\Omega))$, then $P_\phi \cong M \times S^2$. For a loop ϕ_t representing the class γ, we write P_γ or P_ϕ. We denote by $Vert$ the kernel of p_*.

There are two canonical classes attached to P_ϕ. One is the first Chern class of the vertical bundle $c_1(Vert)$ denoted by c_ϕ, which is the extension of $c_1(TM)$ to the total space P_ϕ of the fibration. The other class is an extension of $[\Omega]$, the class of the symplectic form on the fiber. Such an extension is possible due to the Hamiltonian nature of the loop [9]. In particular, we consider the unique class with the following properties,

$$u_\phi \mid_F = \Omega \quad \text{and} \quad u_\phi^{n+1} = 0,$$

where $2n = \mathrm{dim} M$. Such a class is called the *coupling class* of the fibration. In this case, we observe that $\Omega_\epsilon = u_\phi + \epsilon p^* \tau$ is a symplectic form on P_ϕ for ϵ big enough, where τ is a two form on S^2 with $\int_{S^2} \tau = 1$.

2.2 Quantum Homology

In order to find the bound for ν_-, we need to use the homomorphism defined by Seidel [12]

$$\Psi : \pi_1(\mathrm{Ham}(M,\Omega)) \longrightarrow QH_*^\times,$$

where $QH_*^\times$ denotes the group of units in the even part of the small Quantum homology of M with coefficients in a real Novikov ring. Namely, let Λ be the Novikov ring of the group $\mathcal{H} := H_2^S(M, \mathbb{R})/\sim$, where $H_2^S(M, \mathbb{R})$ is the image of $\pi_2(M)$ in $H_2(M, \mathbb{R})$, and $B \sim B'$ if $\Omega(B) = \Omega(B')$ and $c_1(TM) \cdot B = c_1(TM) \cdot B'$. Define the *small quantum homology ring* by

$$QH_*(M, \Lambda) = H_*(M, \mathbb{R}) \otimes \Lambda,$$

so that for any $a, b \in H_*(M)$ their product is defined by

$$a * b = \sum_{B \in \mathcal{H}} (a * b)_B \otimes e^{-B},$$

where $(a * b)_B \in H_*(M)$ is the class such that for any other $c \in H_*(M)$

$$(a * b)_B \cdot c = \Phi_B^M(a, b, c).$$

Here $\Phi_B^M(a, b, c)$ denotes the Gromov-Witten invariant that counts the number of pseudo-holomorphic curves representing the class B and intersecting generic representatives of $a, b, c \in H_*(M)$. We can also define a grading by

$$deg(a \otimes e^{-B}) = deg(a) - 2c_1(TM)B.$$

Roughly speaking, $\Psi(\gamma)$ counts the number of pseudo-holomorphic sections of $p : P_\phi \to S^2$. In general, there is no canonical section of p. Hence, we define a *generalized section* to be a homology class $\sigma_\phi \in H_2(P_\phi, \mathbb{R})$ such that $p_*(\sigma_\phi) = [S^2]$ and $c_\phi(\sigma_\phi) = 0 = u_\phi(\sigma_\phi)$ if Ω and $c_1(TM)$ are linearly independent. If they are linearly dependent, then we just require $c_\phi(\sigma_\phi) = 0$. Thus,

$$\Psi(\gamma) = \sum_{B \in \mathcal{H}} a_B \otimes e^B,$$

where $\sigma_\phi - B \in H_2(P_\phi, \mathbb{Z})$, and $a_B \in H_*(M)$ is the class such that for $c \in H_*(M)$

$$a_B \cdot_M c = \Phi_{\sigma_\phi - B}^{P_\phi}([M], [M], c). \tag{1}$$

Notice that the left hand side in equation (1) is the intersection product in M, while the right hand side is the Gromov-Witten invariant in P_ϕ.

Next, consider the following valuation map on $QH_*(M)$,

$$val : QH_*(M) \longrightarrow \mathbb{R}$$

$$val(\sum_{B \in \mathcal{H}} a_B \otimes e^B) = \sup\{\Omega(B) \mid a_B \neq 0\}.$$

The relevant property we will use is the following [13]. If $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega))$, then

$$\nu_-(\gamma) \geq val(\Psi(\gamma)).$$

3 Proof of the main theorem

Consider the manifold $M = \mathbb{P}(E \oplus \mathbb{C})$, where $E \to \mathbb{CP}^2$ is a complex line bundle with first Chern class $c_1(E) = n\rho$, $n = 0, 1, 2, 3$, and ρ is the generator of $H^2(\mathbb{CP}^2)$. Here, $\mathbb{C}$ denotes a trivial line bundle.

The manifold M fibers over $\mathbb{CP}^2$ with fiber equal to the sphere S^2, $\pi : M \to \mathbb{CP}^2$. Clearly, there are two sections, $Z_+ = \mathbb{P}(0 \oplus \mathbb{C})$ and $Z_- = \mathbb{P}(E \oplus 0)$, with normal bundles $E|_{Z_+}$ and $E^*|_{Z_-}$ respectively. Let $\phi_t : M \to M$ be a loop of Hamiltonians whose fixed point set is $Z_+ \cup Z_-$, and such that that $\pi \circ \phi_t = \pi$, i.e. ϕ_t descends to the identity in $\mathbb{CP}^2$. For instance, consider the loop of Hamiltonians that rotate the fiber $2\pi\theta$ degrees, $\theta \in [0, 1]$.

In this case, $P_\phi = M \times D^2_+ \cup_\phi M \times D^2_-$, fibers over $\mathbb{CP}^2 \times S^2$, $\Pi : M \to \mathbb{CP}^2 \times S^2$, and the fibration $p : P_\phi \to S^2$ is the composition of Π with the projection pr_{S^2} onto the second factor,

$$p : P_\phi \to \mathbb{CP}^2 \times S^2 \to S^2.$$

In fact, it is not hard to see that

$$P_\phi \cong \mathbb{P}((pr^*_{S^2} H^{-1} \otimes pr^*_{\mathbb{CP}^2} E) \oplus \mathbb{C}),$$

where H^{-1} is the complex line bundle over S^2 of degree -1.

Next, we give a description of c_ϕ, the first Chern class of the vertical tangent bundle to P_ϕ with respect to p, and of the coupling class u_ϕ. Notice that TP_ϕ splits as $Vert(p) \oplus p^*TS^2$ as well as $Vert(\Pi) \oplus \Pi^*T(\mathbb{CP}^2 \times S^2)$. Therefore $c_\phi = c_1(Vert(p)) = c_1(Vert(\Pi)) + c_1(\Pi^*T\mathbb{CP}^2)$.

We have the following form $\eta = c_1(\mathcal{O}_M(1))$ on M. It generates the cohomology of the fiber, $\eta^2 = c_1(E)\eta = n\rho\eta$, and $c_1(TM) = 3\rho + (2\eta - c_1(E)) = 2\eta + (3-n)\rho$. This class can be represented by a symplectic form [9], which we shall denote by Ω, for $n = 0, 1, 2$. In the case of $n = 3$, we consider the class $\eta + t\rho$, where $0 < t \leq 1$, and a representative shall also be denoted by Ω.

Similarly, in P_ϕ there is a form $\widehat{\eta}$ that generates the cohomology of the fiber (of Π), and such that, $\widehat{\eta}^2 = c_1(H^{-1} \otimes E)\widehat{\eta} = (n\rho - \tau)\widehat{\eta}$, and

$$c_1(Vert(\Pi)) = 2\widehat{\eta} - (n\rho - \tau).$$

Thus,

$$c_\phi = 2\widehat{\eta} - (n\rho - \tau) + 3\rho = 2\widehat{\eta} + (3-n)\rho + \tau.$$

Notice that $\widehat{\eta}$ is an extension of η to P_ϕ. Hence, the coupling form u_ϕ is equal to $2\widehat{\eta} + (n-3)\rho + \kappa\tau$, where κ is the constant for which $u_\phi^4 = 0$. That is

$$\kappa = \frac{-\int_{P_\phi}(2\widehat{\eta} + \delta\rho)^4}{\int_{P_\phi} 4(2\widehat{\eta} + \delta\rho)^3\tau},$$

where $\delta = 3-n$ for $n = 0,1,2$ and $\delta = t$ for $n = 3$. Notice that for $n = 1,2,3$, $\kappa < 0$.

Let σ_- denote the section over $S^2 \times \{x\}$ given by $\mathbb{P}(0 \oplus \mathbb{C}) \subset P_\phi|_{S^2\times\{x\}}$, $x \in \mathbb{CP}^2$. Observe that $c_\phi(\sigma_-) = -1$, for $n = 0,1,2,3$. For $n = 0,1,2$, $c_1(TM) = [\Omega]$ and we need to find a section σ_ϕ such that $c_\phi(\sigma_\phi) = 0$. The second homology group $H_2(M,\mathbb{Z})$ is generated by the fiber class F, and the class $\hat{L}_-$ represented by a line in Z_-. The desired σ_ϕ is then given by $\sigma_- + \frac{1}{2}F$. For $n = 3$, $c_\phi = 2\widehat{\eta} + \tau$, while $u_\phi = 2\widehat{\eta} + t\rho + \kappa\tau$, thus the desired section is $\sigma_\phi = \sigma_- + \frac{1}{2}F + \frac{-\kappa+1}{t}\hat{L}_-$.

Now, $\Psi(\gamma) = \sum_B a_B \otimes e^B$, where $a_b \cdot c = \Phi^{P_\phi}_{\sigma_\gamma - B}([M],[M],c)$. Instead, we shall calculate invariants of the form $\Phi^{P_\phi}_{\sigma_- - B}([M],[M],c)$ since $\sigma_- \in H_2(P_\phi,\mathbb{Z})$. If this invariant is different from 0 then

$$\dim \mathcal{M}_{0,3}(\sigma_- - B, P_\phi, J) + 6 + 6 + \dim(c) = 3(8),$$

that is,

$$-3 \le c_\phi(\sigma_- - B) \le 0. \tag{2}$$

Now $\sigma_\gamma - B$ has to be of the form $\sigma_- - \alpha\hat{L}_- - \beta F$, for some α, $\beta \in \mathbb{Z}$. The inequality (2) means that

$$-2 \le (n-3)\,\alpha - 2\beta \le 1 \tag{3}$$

where $\alpha \le 0$ to ensure that $\Pi_*(\sigma_- - B)$ is represented by a pseudo-holomorphic curve in $S^2 \times \mathbb{CP}^2$. Furthermore, given the symplectic structures on P_ϕ and M we also need

$$c_1(Vert(\Pi))(\sigma_- - \alpha\hat{L}_- - \beta F) \ge -c_1(H^{-1} \otimes E)(\Pi_*(\sigma_- - \alpha\hat{L}_- - \beta F)).$$

Equivalently,

$$\begin{cases} -1-2\beta \geq 1, & \text{if } \alpha < 0, \\ -1-2\beta \geq -1, & \text{if } \alpha = 0. \end{cases} \tag{4}$$

Therefore the only possibility is $\alpha = 0 = \beta$, so that

$$\dim(c) = -2c_\phi(\sigma_-) = 2,$$

and the invariants we need are the following.

Lemma 1.

$$\Phi^{P_\phi}_{\sigma_-}([M],[M],\hat{L}_-) = 0,$$
$$\Phi^{P_\phi}_{\sigma_-}([M],[M],F) = 1.$$

These invariants are computed as a direct application of Theorem 4.2 in [6] $\square$

Hence, for $n = 0,1,2$, $\Psi(\gamma) = a_{\frac{1}{2}F} \otimes e^{\frac{1}{2}F}$, where $a_{\frac{1}{2}F} = Z_+$, and $val(\Psi(\gamma)) = \Omega(\frac{1}{2}F) = 1$. Thus

$$\nu_-(\gamma) \geq 1.$$

For $n = 3$, $\Psi(\gamma) = a_{\frac{1}{2}F + \frac{1-\kappa}{t}\hat{L}_-} \otimes e^{\frac{1}{2}F + \frac{1-\kappa}{t}\hat{L}_-}$, where $a_{\frac{1}{2}F + \frac{1-\kappa}{t}\hat{L}_-} = Z_+$, and $val(\Psi(\gamma)) = \Omega(\frac{1}{2}F + \frac{1-\kappa}{t}\hat{L}_-) = 1 + 1 - \kappa \geq 1$. Thus

$$\nu_-(\gamma) \geq 1,$$

which proves Theorem 1 for $k = 1$.

Recall that $c_1(M) = 2\eta + (3-n)\rho$. For $n = 0,1,2$, $c_1(M)$ has a representative given by a symplectic form Ω. For $n = 3$, we take $\Omega = 2\eta + t\rho$, where $0 < t \leq 1$.

3.1 Case $n = 0$

In this case $Z_+ = Z_- = [pt \times \mathbb{C}P^2]$. Now, $Z * Z = \sum a_B \otimes e^{-B}$, so that for any other $c \in H_2(M,\mathbb{Z})$

$$a_B \cdot c = \Phi^M_B(Z,Z,c),$$

which is different form 0 only if

$$2c_1(M) \cdot B + \dim c = 4.$$

The only possibilities are $B=0$, and $B=F$,

$$\begin{aligned}
\Phi_0^M(Z,Z,c) &= Z\cdot Z\cdot c = 0,\\
\Phi_F^M(Z,Z,pt) &= 1.
\end{aligned}$$

Thus,

$$a_F = [M] \quad \text{and} \quad Z * Z = [M]\otimes e^{-F},$$

so that

$$Z\otimes e^{F/2} * Z\otimes e^{F/2} = [M],$$

and

$$val(\Psi(2\gamma)) = 0.$$

3.2 Case $n=1$

The (small) quantum cohomology of M is given by (cf. [11])

$$QH^*(M,\Lambda) = \mathbb{R}\{\eta,\rho,e^F,e^{\hat{L}_-}\}/\mathcal{R},$$

where the set of relations is

$$\mathcal{R} = \{\rho^3 = (\eta-\rho)e^{-\hat{L}_-},\ \eta^2 = \eta\rho + e^{-F}\}.$$

By Poincaré duality and the correspondence between the intersection product in homology and the exterior product in cohomology, we can use the above description to find information about the quantum homology of M. In particular, $Z_+ = \mathrm{PD}(\eta)$, so that,

$$(Z_+\otimes e^{F/2})^{*2} = \hat{L}_+\otimes e^F + [M],$$
$$(Z_+\otimes e^{F/2})^{*3} = [pt]\otimes e^{3F/2} + (Z_+ + \pi^{-1}(L))\otimes e^{F/2}.$$

By using an induction argument, it is not hard to see that $val(Z_+^{*k}) = 0$ and so,

$$val(\Psi(k\gamma)) = \Omega(kF/2) = k.$$

3.3 Case $n=2$

The (small) quantum cohomology of M is given by (cf. [5, 14])

$$QH^*(M,\Lambda) = \mathbb{R}\{\eta,\rho,e^F,e^{\hat{L}_-}\}/\mathcal{R},$$

where the set of relations is

$$\mathcal{R} = \{\rho^3 = (\rho^2 - 2\eta\rho) \otimes e^{-\hat{L}_-} + (8\rho - 4\eta) \otimes e^{-2\hat{L}_-} + e^{-\hat{L}_- - F},\ \eta^2 = 2\eta\rho + e^{-F}\}.$$

This time we have, for example,

$$(Z_+ \otimes e^{F/2})^{*2} = 2\hat{L}_+ \otimes e^F + [M],$$

$$(Z_+ \otimes e^{F/2})^{*3} = 4[pt]e^{3F/2} + (Z_+ + 2\pi^{-1}(L)) \otimes e^{F/2} - 4[M] \otimes e^{-\hat{L}_- + F/2}.$$

Once again, it is not hard to see by induction that $val(Z_+^{*k}) = 0$ and so

$$val(\Psi(k\gamma)) = \Omega(kF/2) = k.$$

3.4 Case $n = 3$

The (small) quantum cohomology of M is of the form

$$QH^*(M, \Lambda) = \mathbb{R}\{\eta, \rho, e^F, e^{\hat{L}_-}\}/\mathcal{R},$$

where the set of relations is

$$\mathcal{R} = \{\rho^3 = \sum_{\beta \geq 1} a_{F+\beta\hat{L}_-} \otimes e^{-F-\beta\hat{L}_-},\ \eta^2 = 3\eta\rho + e^{-F} + \sum_{\beta \geq 1} b_{F+\beta\hat{L}_-} e^{-F-\beta\hat{L}_-}\},$$

$a_{F+\beta\hat{L}_-} \in H^2(M, \mathbb{Z})$ and $b_{F+\beta\hat{L}_-} \in H^0(M, \mathbb{Z})$. For example,

$$\begin{aligned} (Z_+ \otimes e^{F/2+(1-\kappa)\hat{L}_-/t})^{*2} &= 3\hat{L}_+ \otimes e^{F+2(1-\kappa)\hat{L}_-/t} + [M]e^{2(1-\kappa)\hat{L}_-/t} \\ &+ \sum_{\beta \geq 1} a_{F+\beta\hat{L}_-} \otimes e^{-\beta\hat{L}_- + 2(1-\kappa)\hat{L}_-/t}. \end{aligned}$$

By induction, $val(Z_+^{*k}) = 0$ and

$$val(\Psi(k\gamma)) = \Omega(kF/2 + k(1-\kappa)\hat{L}_-/t) = k(2-\kappa) \geq k.$$

This concludes the proof of Theorem 1. $\square$

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