Traces and dimensions in braided categories

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Abstract

With any even Hecke symmetry R (that is a Hecke type solution of the Yang-Baxter equation) we associate a quasitensor category. We formulate conditions on R under which the category constructed in this way turns out to be a natural generalization of the category of $sl(n)$-modules. The categorical dimensions for the objects of the category are defined and explicitly calculated. Our main result states that these dimensions are equal to those of $U_q(sl(n))$-modules and can be expressed in terms of the Schur functions. An important property of the approach developed here is its universal character — it does not depend on a specific form of the Hecke symmetry. In particular, it is applicable for the family of even nonquasiclassical Hecke symmetries constructed in [G].

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1 Introduction

The main purpose of this paper is to introduce some braided categories arising from "nonquasiclassical" Hecke symmetries constructed in [G] and to suggest a way of defining and computing dimensions of objects of these categories. Let us precise from the very beginning that by braided categories we mean monoidal tensor or quasitensor ones whose objects are vector spaces with the usual tensor product and whose natural associativity isomorphisms are identical (for the terminology, cf. for example [CP]). Thus, the structure of such a category is principally determined by natural commutativity isomorphisms

$$R_{U,V} : U \otimes V \to V \otimes U, \qquad U, V \in \mathrm{Ob}\,(\mathcal{C}). \tag{1.1}$$

The term "natural" means here that for any two morphisms $f:\ U \to U'$ and $g:\ V \to V'$ one has:

$$(g \otimes f) \circ R_{U,V} = R_{U',V'} \circ (f \otimes g). \tag{1.2}$$

Here $\mathcal{C}$ stands for a category in question and $\mathrm{Ob}\,(\mathcal{C})$ stands for the set of its objects. These isomorphisms are assumed to satisfy the so-called Yang-Baxter equation. They play the role of the usual flip which transposes the factors. We call the isomorphisms (1.1) YB operator, braiding, or quantum R-matrix.

Initially, categories equipped with such morphisms were introduced by S. MacLane [McL1]. However, he only considered involutary YB operators ($R^2 = \mathrm{id}$) called in what follows "symmetries". A new wave of interest in braided categories (but with noninvolutary braidings) arose in connection with integrable system theory. More precisely, such categories appeared as those of modules of Drinfeld-Jimbo quantum groups (QG) $U_q(\mathbf{g})$ playing an important role in this theory (we will denote this category $U_q(\mathbf{g})-\mathrm{Mod}$). Besides, the QG have found many other interesting applications, in particular, in noncommutative geometry.

The notion of quantum (or q-) dimension of objects of the category is one of the basic notions related to QG which turns out to be very useful in noncommutative geometry. In accordance with the usual tradition the dimension of a given object $V \in \mathrm{Ob}\,(\mathcal{C})$ is the value of the trace operator

$$\mathrm{tr}\,:\mathrm{End}\,(V) \to \mathbb{K} \tag{1.3}$$

on the identity element $\mathrm{id} \in \mathrm{End}\,(V)$. Hereafter $\mathbb{K}$ stands for the basic field (always $\mathbb{C}$ or $\mathbb{R}$).

Let us note that we identify $\mathbb{K}$ and the $\mathrm{End}\,(I)$ where I is the identity object in the sense of [CP]. Moreover, we use the notation $\mathrm{End}\,(V) = \mathrm{Hom}\,(V, V)$ for the objects of the category which sometimes are called internal (endo)morphisms (cf. [DM]). By contrast, the morphisms of the category in question will be called "categorical morphisms". For example, categorical morphisms of the category $\mathbf{g}-\mathrm{Mod}$ of modules over a Lie algebra $\mathbf{g}$ are elements of $\mathrm{Hom}\,(U, V)$, $U, V \in \mathrm{Ob}\,(\mathbf{g}-\mathrm{Mod})$, commuting with the action of $\mathbf{g}$.

The problem we want to discuss now is a categorical definition of trace (1.3). What is a reasonable generalization of the basic property of the classical trace

$$\mathrm{tr}\,[X,\, Y] = 0, \qquad \forall\, X, Y \in \mathrm{End}\,(V) \tag{1.4}$$

where $[X,\, Y]$ is the commutator of two endomorphisms[1]?

If the braiding in a given braided category is involutary and invertible by column in the sense of formula (2.47) then there exists a natural generalization of the above commutator such that relation (1.4) is still valid for this commutator and an appropriate trace. For example it is so for a super-commutator and super-trace defined in a super-category. Other examples can be found in [G].

Let us note that the definition of the trace (1.3) in a category equipped with a symmetry makes use of this symmetry and of the identification $\mathrm{End}\,(V) = V \otimes V^*$ where V^* is an object dual to V (all categories in question are assumed to be rigid, i.e. for any object of the category its dual is also an object). Up to our knowledge for the first time such a trace has been introduced in [DP].

Once such a trace is defined one can introduce *a categorical dimension* $\dim(V)$ of an object V by setting $\dim(V) = \mathrm{tr}\,(\mathrm{id})$. It is easy to see that

$$\dim(U \oplus V) = \dim(U) + \dim(V),\;\; \dim(U \otimes V) = \dim(U)\dim(V), \tag{1.5}$$

i.e. the dimension can be considered as an additive and multiplicative (a-m) functional on the objects of the category.

However, a direct application of the approach of [DP] to the category $U_q(\mathbf{g})-\mathrm{Mod}$ gives "dimension" which is not an a-m functional. This is the reason why one has to introduce a correction in the definition of "dimensions" turning them into an a-m functional. The correction is connected to the so-called ribbon element in the corresponding QG. The image of this element is somewhere called "twist" (cf. [T]). The categories possessing a braiding and a twist are called ribbon. The trace in such categories is defined via some combination of the ribbon element and another element u due to Drinfeld (cf. [CP]) and this leads to the dimension which is an a-m functional.

Some generalization of this construction of the trace has been suggested in [BW]. Suggested in that paper is a way to define the notion of dimension in some categories without any braiding.

[1] Another form of this property is $\mathrm{tr}\,(g\,X\,g^{-1}) = \mathrm{tr}\,X$ for any invertible $g \in \mathrm{End}\,(V)$. Sometimes this property is attributed to the quantum trace (cf. [T]). However, in this case one means by g an invertible categorical morphism (see above). Of course, this property fails if g is an arbitrary (but still invertible) element of $\mathrm{End}\,(V)$.

Such a category is introduced as that of H-modules where H is a Hopf algebra. Instead of a universal quantum R-matrix (i.e. the element giving rise to a braiding) the authors of [BW] use another element allowing to identify any object V with its second dual V^{**}.

In the present paper we restrict ourselves to categories equipped with braidings and suggest another way of introducing traces and dimensions for them.

Before discussing this way let us describe the categories in question. A Yang-Baxter operator

$$R : V^{\otimes 2} \to V^{\otimes 2} \tag{1.6}$$

will be called *a Hecke symmetry* if in addition to the Yang-Baxter equation

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23} \tag{1.7}$$

it also satisfies the relation

$$(q\,\mathrm{id}\, - R)(q^{-1}\,\mathrm{id}\, + R) = 0, \quad q \in \mathbb{K}. \tag{1.8}$$

Here the standard tensor notations have been used for the equations and operators in tensor product of spaces. Equation (1.7) is written in $V^{\otimes 3}$ and one assumes

$$R_{12} = R \otimes \mathrm{id}, \quad R_{23} = \mathrm{id} \otimes R,$$

where id is the identity operator on V.

The nonzero parameter q is taken to be generic, which means it is not a root of unity: $\forall k \in \mathbb{N},\ k \neq 1,\ q^k \neq 1$. As a consequence none of the so-called *q-numbers* k_q is equal to zero

$$k_q \equiv \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \quad \forall k \in \mathbb{N}. \tag{1.9}$$

Let us assign to such a Hecke symmetry "symmetric" $\Lambda_+(V)$ and "skew-symmetric" $\Lambda_-(V)$ algebras by

$$\Lambda_+(V) = T(V)/\{\mathrm{Im}\,(q\,\mathrm{id}\, - R)\}, \quad \Lambda_-(V) = T(V)/\{\mathrm{Im}\,(q^{-1}\mathrm{id}\, + R)\}. \tag{1.10}$$

Here $T(V)$ stands for the free tensor algebra of the space V and $\{I\}$ denotes the ideal generated by a subset I in a given algebra. Let $\Lambda^l_\pm(V)$ be a homogeneous component of the algebra $\Lambda_\pm(V)$ of the degree l. We call a Hecke symmetry *even* if there exists an integer p such that the component $\Lambda^l_-(V)$ is trivial at $l > p$ and it is one-dimensional at $l = p$. The integer p will be called a rank of V and be denoted $\mathrm{rk}\,(V)$.

In the latter 80's one of the authors (D.G.) constructed examples of Hecke type braidings which differ drastically from those related to the QG (see [G] and references therein). Namely, it was shown that there exist a lot of Hecke symmetries R such that $n = \dim(V) > p = \mathrm{rk}\,(V)$. Here by $\dim(V)$ we mean the usual dimension of the space V. Note that these Hecke symmetries are not deformations of the usual flip and we call them nonquasiclassical. To the contrary, the Hecke symmetry coming from the QG $U_q(sl(n))$ is a deformation of the flip and we call it quasiclassical.

Then the following problem naturally arises. Given an even Hecke symmetry, is it possible to associate with it a category looking like $U_q(\mathbf{g})-\mathrm{Mod}$ and how to define traces and corresponding (categorical) dimensions of its objects? We would like to point out that in what follows we use the term "dimension" in the categorical sense. The use of classical dimensions will be explicitly specified.

In this paper we introduce a braided category generated by a vector space V equipped with a Hecke symmetry satisfying the additional condition (2.33). This category will be shown to be a natural generalization of the category $U_q(sl(n)) - \mathrm{Mod}$. We call it the Schur-Weyl (SW) category and denote as SW(V). Note that for involutary YB operators a similar construction was presented in [GM].

Any object of such a category is a direct sum of simple ones and in this sense the category SW(V) is spanned by simple (basic) objects. These basic objects are labelled by partitions (or what is the same by Young diagrams)

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_{p-1}), \quad \lambda_{i+1} \le \lambda_i, \tag{1.11}$$

λ_i being nonnegative integers. In the sequel we will use the notation $\lambda \vdash k$ for each λ being a partition of the integer k that is $\sum_i \lambda_i = k$. The number of nonzero components of a partition λ is called its *height* and will be denoted $\ell(\lambda)$. For an object corresponding to a diagram λ we introduce the notation V_λ.

Let us stress that the tensor product of two basic objects is a direct sum of basic ones. This naturally leads us to the notion of Grothendieck ring of the category which turns out to be the same as for the categories of $sl(n)$- (or $U_q(sl(n))$)-modules. The only difference is that the role of the classical dimension $n = \dim(V)$ is played by the rank $p = \mathrm{rk}\,(V)$. Thus, we can consider the dimensions of objects of the category as a functional on the Grothendieck ring. Since it is an a-m functional we come to the problem of describing the whole family of a-m functionals.

An exhaustive answer to this question is given in Proposition 16 stating that such a functional f is determined by its values on objects $V_{(1^k)}$, $k = 1, 2, ..., p-1$, and its value on a basic object V_λ is equal to

$$f(V_\lambda) = s_\lambda(\alpha_1, \alpha_2, ..., \alpha_p)$$

where s_λ is the Schur function (polynomial) in p variables corresponding to the partition λ. The numbers $(-\alpha_i)$ are the roots of the polynomial

$$\phi(t) = t^p + f_1 t^{p-1} + \ldots + f_{p-1} t + 1, \qquad f_k = f(V_{(1^k)}).$$

Note, that the classical (usual) dimension is also an a-m functional. Therefore, the above result allows one to calculate the classical dimension of spaces V_λ provided that the Poincaré series $P_-(t)$ (see Remark 20) is known (for involutary symmetries such a calculation has been done in [GM]). Remark, that if a Hecke symmetry $R(q)$ (1.6) is a deformation of an involutary one $R(1)$ then the usual dimension of corresponding spaces V_λ will be the same in both cases. Indeed, being an integer number such a dimension is stable under deformation, while the categorical dimension is a function in q which becomes integer at $q = 1$.

Having classified all a-m functionals one can put the question: which a-m functionals come from SW categories? More precisely, for which a-m functional on a given Grothendieck ring (with a fixed p) there exists a SW category such that the categorical dimension of its objects coincides with this functional?

Now we are going back to the problem of defining trace on objects $\mathrm{End}\,(V)$. Our definition of trace is rather simple and natural. As a trace we will take a properly normalized (nontrivial) morphism (1.3) in the category. Note that for any object $V \in \mathrm{Ob}\,(\mathrm{SW(V)})$ the trivial component in the product $V \otimes V^*$ is unique. So, morphism (1.3) on a given V (being nothing but a multiple of the projection on the trivial component) is uniquely defined up to a (nontrivial) factor and the problem of defining the trace reduces to finding a "proper" normalization of traces on separate objects (thus, this normalization is a collective phenomenon). Our "proper" normalization is fixed by the requirement that the corresponding dimension should be an a-m functional.

Having defined the trace and corresponding dimension in a SW category we calculate the categorical dimension $\dim_R(V_\lambda)$ of basic objects (it suffices for finding dimensions of all objects). The result is rather surprising. The dimension is given by the formula

$$\dim_q(V_\lambda) = s_\lambda(q^{p-1}, q^{p-3}, ..., q^{-(p-1)}) \tag{1.12}$$

which coincides with the well-known Weyl result for the classical dimensions at $q = 1$. Let us note that in the case of the category $U_q(sl(n))-$Mod formula (1.12) is equivalent to that obtained in [Go]. Thus, the quantum dimension does not depend on which case (quasiclassical or not) we are dealing with. So, we get just the formula for dimensions in the category $U_q(sl(n))$ − Mod with $n = p$.

However, behind this nondetailed information about the objects (their dimensions) different structures are hidden. For example, traces (1.3) in different categories are defined via some matrix C entering the formula for the trace on the basic space V and extended in a proper way onto the whole category (the extension of C matrix to the space V_λ will be denoted C_λ). It is worth mentioning that the matrices C_λ are essentially different in quasiclassical and nonquasiclassical cases while the categorical dimensions of objects which are nothing but the properly normalized usual traces of C_λ depend only on λ and $p = \mathrm{rk}\,(V)$ via formula (1.12).

The paper is organized as follows. In Section 2 we construct all the necessary elements of Schur-Weyl category: the class of objects, categorical morphisms, the quasitensor and rigid structure. In Section 3 we describe the full set of a-m functionals on the objects of SW(V), define the trace in End(U), $\forall\, U \in$ Ob(SW(V)), and calculate $\dim_q U$ as the value of trace on identity morphism in End(U). There we give the proof of our main result (1.12).

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2 Schur-Weyl category generated by a Hecke symmetry

In this section we describe the construction of Schur-Weyl category SW(V) (see Introduction) generated by a finite dimensional vector space V equipped with a Hecke symmetry. We supply the category with attributes of that $U_q(sl(n))$-Mod. In particular, all categories SW(V) are $\mathbb{K}$-linear rigid quasitensor category (for precise definitions see [CP, McL2]). As was shortly outlined in Introduction, the basic objects of SW(V) are some linear subspaces V_λ of the tensor algebra $T(V)$, while the morphisms will be specific linear mappings of the objects (see below). Consequently, the $\mathbb{K}$-linear structure of such a category is obvious: as the direct sum of objects we will take the usual direct sum of linear spaces and the trivial (zero) vector space will play the role of null object of the category. So we should only define the structure of monoidal category and fix braiding isomorphisms converting the monoidal structure into the quasitensor (or braided) one. Besides, to have a rigid category, we must ensure that the dual space V^* of an arbitrary $V \in$ Ob(SW(V)) is also an object of the category. Now we pass to the explicit construction of the mentioned components of the category SW(V).

2.1 *Class of objects*

Let us fix some finite dimensional vector space V, $\dim V = n$, and choose a basis $\{e_i\}$ in this space. Consider a linear operator R (1.6) whose action on the basis elements reads:

$$R(e_i \otimes e_j) = e_r \otimes e_s\, R_{ij}^{rs},$$

where the summation over the repeated indices is understood. In what follows we assume R to be a Hecke symmetry with a generic $q \in \mathbb{K}$.

Besides, we will suppose R to be an even symmetry of a rank $p \le n = \dim V$. This means that the Poincaré series $P_-(t)$ related to the "skew-symmetric" algebra $\wedge_-(V)$ (1.10) is a monic polynomial of p-th degree, that is a polynomial in t whose leading coefficient is equal to 1.

Throughout the paper we will use the compact notation for the relations including matrices and vector spaces. In such notations the indices will stand for the number of space rather than for a particular matrix or vector component. A basis vector e_i in the k-th matrix space is denoted as $e_{\langle k|}$, while for the basis vector of dual space we will write $e^{|k\rangle}$. The same principle is applied to components of arbitrary tensors.

For example, the above relation will look like[2]

$$R(e_{\langle 1|}\otimes e_{\langle 2|}) := e_{\langle 1|}\otimes e_{\langle 2|}\, R^{|12\rangle}_{\langle 12|} := e_{\langle 1|}\otimes e_{\langle 2|}\, R_{12}.$$

In this notation the summation over the repeated indices is represented by an expression which contains the same upper and lower case indices *with properly oriented brackets* as shown in the examples below:

$$e_{\langle 1|}\otimes e^{|1\rangle} := \sum_i e_i\otimes e^i, \quad e_{\langle 1|}\cdot T_1 := e_{\langle 1|}\, T^{|1\rangle}_{\langle 1|} := \sum_j e_j T^j_i, \quad M_1 R_{12} N_2 := \sum_{a,b} M^{i_1}_{\ a} R^{a i_2}_{\ j_1 b} N^b_{j_2}$$

but

$$e^{|1\rangle}\otimes e_{\langle 1|} := e^i\otimes e_j, \quad e_{\langle 1|}\otimes e^{|2\rangle} := e_i\otimes e^j, \quad T_1\cdot e_{\langle 1|} := T^i_j\, e_k$$

and so on.

Now we use the fact, that in each homogeneous component $V^{\otimes m}$ of tensor algebra $T(V)$ one can realize the so called *local representation* of the Hecke algebra H_m [DJ] via a given Hecke symmetry[3].

The Hecke algebra H_m is generated by the unit element id_H and $m-1$ generators σ_k subject to the following relations:

$$\left.\begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j|\ge 2 \\ (\sigma_i - q\,\mathrm{id}_H)(\sigma_i + q^{-1}\,\mathrm{id}_H) = 0 \end{array}\right\}\ i = 1,2,\dots m-1.$$

The local representation of H_m in $V^{\otimes m}$ is of the form:

$$\sigma_i \to \rho_R(\sigma_i) \equiv R_{i i+1} = \mathrm{id}_{i-1}\otimes R\otimes \mathrm{id}_{m-i-1} \in \mathrm{End}(V^{\otimes m}). \tag{2.1}$$

In the further construction of SW(V) category the central role belongs to the fact that the Hecke algebra H_m for a generic q is semisimple and can be decomposed into a direct sum of ideals. Moreover, since the Hecke algebra H_m at a generic q is isomorphic to the group algebra $\mathbb{K}[S_m]$ of the m-th order permutation group S_m, its primitive idempotents generating the ideals in mentioned decomposition can be put into one-to-one correspondence with the set of all standard Young tableaux connected with each possible partition λ of the integer m. Speaking more explicitly, given the algebra H_m and a partition $\lambda\vdash m$, one can construct some polynomials Y^λ_{ii} in generators $\{\sigma_i\}$ which turn out to be the primitive idempotents of Hecke algebra H_m (for detailed description of such a construction see review [OP]):

$$\mathrm{id}_H = \sum_{\lambda\vdash m}\sum_{i=1}^{d_\lambda} Y^\lambda_{ii}, \tag{2.2}$$

$$Y^\lambda_{ii}\, Y^\mu_{jj} = \delta_{ij}\delta^{\lambda\mu}\, Y^\lambda_{ii}. \tag{2.3}$$

In the above formulas the index i runs from 1 to $d_\lambda \equiv \dim\lambda$, that is to the number of all standard Young tableaux, corresponding to the Young diagram of the partition $\lambda\vdash m$. Recall,

[2]The symbol := means "by definition".

[3]For the review on Hecke algebra the reader is referred to the recent work [OP]

that a Young tableau is called *standard* if it is filled with successive integers from 1 to m in such a way that the integers increase from left to right in each row and from top to bottom in each column. The standard Young tableaux (and hence the primitive idempotents Y_{ii}^{λ}) may be lexicographically ordered in many ways and for definiteness we fix an order in which the first idempotent Y_{11}^{λ} corresponds to the Young tableau filled by integers consequently increasing by 1 when going down in each column. Here is an example for the Young tableau corresponding to the partition $\lambda = (3, 2^2, 1)$:

$$Y_{11}^{\lambda} \longleftrightarrow \begin{array}{|c|c|c|} \hline 1 & 5 & 8 \\ \hline 2 & 6 \\ \cline{1-2} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \tag{2.4}$$

A primitive idempotent corresponding to the partition $\lambda = (1^m)$ will be called the antisymmetrizer and be denoted as $A^{(m)}$. There is one important circumstance about the images of $A^{(m)}$ in $\mathrm{End}(V^{\otimes m})$ with respect to representation (2.1). Since the YB operator R is taken to be an even Hecke symmetry of rank p, the image of the p-th order antisymmetrizer

$$\mathcal{A}^{(p)} := \rho_R(A^{(p)}) : V^{\otimes p} \to V^{\otimes p}$$

is one dimensional. So, the action of this projector on an arbitrary basis element of $V^{\otimes p}$ can be presented in the form:

$$\mathcal{A}^{(p)} e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|} = e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|}\, v^{|1\ldots p\rangle} u_{\langle 1\ldots p|}, \tag{2.5}$$

where components of the tensors u and v belong to the field $\mathbb{K}$. Due to (2.3) the normalization of u and v is fixed to be $u_{\langle 1\ldots p|} v^{|1\ldots p\rangle} = 1$. The images of antisymmetrizers $A^{(m)}$ with $m > p$ are trivial:

$$\mathrm{Im}\, \rho_R(A^{(m)}) = 0 \quad \forall m \geq p+1. \tag{2.6}$$

In accordance with general theory each primitive idempotent Y_{ii}^{λ} generates a left or right ideal of the Hecke algebra H_m by means of left or right multiplications on all possible elements of H_m. Equivalently, one can consider the left or right *regular module* over H_m which as a vector space is H_m itself and left or right action consists in the left or right multiplication by Hecke algebra elements. In these terms the left or right ideals generated by primitive idempotents will be irreducible left or right H_m submodules. Let us denote left and right submodules generated by Y_{ii}^{λ} as $M_{\lambda}^{l}(i)$ and $M_{\lambda}^{r}(i)$ respectively. As a direct consequence of (2.2) one can write the following decomposition:

$$H_m = \bigoplus_{\lambda \vdash m} \bigoplus_{i=1}^{d_\lambda} M_{\lambda}^{r}(i), \tag{2.7}$$

and a similar decomposition holds for left submodules $M_{\lambda}^{l}(i)$.

Consider in more detail the structure of the component of (2.7) corresponding to a fixed partition $\lambda \vdash m$. Let us denote

$$\bar{M}_{\lambda} \equiv \bigoplus_{i=1}^{d_\lambda} M_{\lambda}^{r}(i) \equiv \bigoplus_{i=1}^{d_\lambda} M_{\lambda}^{l}(i). \tag{2.8}$$

The set $\bar{M}_{\lambda}$ is a two sided submodule in H_m. Let us dwell upon the question of linear basis in $\bar{M}_{\lambda}$. As is known from the theory of Hecke algebra, $\bar{M}_{\lambda}$ is isomorphic to the algebra $Mat_{d_\lambda}(\mathbb{K})$ of $d_\lambda \times d_\lambda$ matrices. Therefore in $\bar{M}_{\lambda}$ one can find the set of d_λ^2 linear independent quantities Y_{ij}^{λ}, which are some polynomials in generators σ_i of H_m obtained from the primitive idempotents

Y_{ii}^λ. These quantities form a linear basis in $\bar{M}_\lambda$ and obey the algebra of matrix units[4] E_{ij} which constitute the linear basis in $Mat_{d_\lambda}(\mathbb{K})$:

$$Y_{ij}^\lambda\, Y_{kl}^\lambda = \delta_{jk}\, Y_{il}^\lambda, \quad i,j,k,l = 1\dots d_\lambda. \tag{2.9}$$

Note, that for each fixed i the elements Y_{ij}^λ $1 \le j \le d_\lambda$ form a linear basis in the irreducible submodule $M_\lambda^r(i) \subset \bar{M}_\lambda$ generated by the primitive idempotent Y_{ii}^λ. And on the other hand, for each fixed j the elements Y_{ij}^λ $1 \le i \le d_\lambda$ form a linear basis in $M_\lambda^l(j)$ generated by Y_{jj}^λ. Therefore we have $\dim M_\lambda^r(i) = \dim M_\lambda^l(i) = d_\lambda$.

Otherwise stated, if we arrange the elements Y_{ij}^λ into a rectangular $d_\lambda \times d_\lambda$ matrix, then its rows will represent the linear basises of d_λ right submodules $M_\lambda^r(i)$, while the columns of the matrix will represent the linear basises of left submodules $M_\lambda^l(i)$. The diagonal entries of the matrix are the primitive idempotents Y_{ii}^λ.

An important property of Y_{ii}^λ consists in the following fact. For any two idempotents Y_{ii}^λ and Y_{jj}^λ corresponding to the same partition λ there exists an invertible element of H_m which transforms one of these idempotents into the other one. This means that the submodules $M_\lambda^r(i)$ (or $M_\lambda^l(i)$) with different i are isomorphic: they can be transformed into each other by *left* (respectively *right*) multiplication on some invertible element of Hecke algebra H_m. Note, that the whole $\bar{M}_\lambda$, being a two-sided submodule in H_m, is invariant with respect to such a multiplication.

In what follows we will not distinguish the isomorphic submodules corresponding to a partition λ and will consider them up to an isomorphism. Then each submodule $M_\lambda^r(i) \subset \bar{M}_\lambda$ generated by Y_{ii}^λ can be treated as the image of an irreducible H_m module M_λ^r (labelled only by the partition $\lambda \vdash m$) w.r.t. the following H_m module monomorphism:

$$M_\lambda^r \to M_\lambda^r(i) \subset \bar{M}_\lambda \subset H_m, \quad \lambda \vdash m. \tag{2.10}$$

In this sense the submodule $\bar{M}_\lambda$ is isomorphic (as the H_m module) to a direct sum of d_λ copies of the module M_λ^r:

$$\bar{M}_\lambda \equiv \bigoplus_{i=1}^{d_\lambda} M_\lambda^r(i) \cong (M_\lambda^r)^{\oplus d_\lambda}.$$

This formula reflects the well known fact that the regular representation of a finite dimensional algebra decomposes into a direct sum of irreducible modules and the multiplicity of each module is equal to its dimension.

Remark 1 *It is worth mentioning that the module M_λ^r can be mapped into $\bar{M}_\lambda$ in many different ways and submodules $M_\lambda^r(i)$ (generated by a fixed choice of primitive idempotents $\{Y_{ii}^\lambda\}$) are only particular cases of all possible monomorphisms. Indeed, let us take an arbitrary set of orthonormal $d_\lambda \times d_\lambda$ projectors $P^{(i)}$ of the rank 1:*

$$P^{(i)} \cdot P^{(j)} = \delta^{ij}\, P^{(i)}, \quad \sum_{i=1}^{d_\lambda} P^{(i)} = \mathrm{id}\,.$$

Then, as is evident from above relations and (2.9), the quantities $X_{ii}^\lambda = \sum_{r,s} P_{rs}^{(i)} Y_{rs}^\lambda$ are also a set of primitive idempotents in H_m leading to another decomposition of $\bar{M}_\lambda$ into a direct sum of right submodules which will represent other possible monomorphisms of M_λ^r into $\bar{M}_\lambda$.

[4]Matrix unit E_{ij} $1 \le i,j \le m$ of the matrix algebra $Mat_m(\mathbb{K})$ is the $m \times m$ matrix with the only nonzero entry at the intersection of the i-th row and the j-th column which is equal to 1.

Now with each right[5] submodule $M^r_\lambda(i) \subset H_m$ we can associate a space $V_\lambda(i) \subset V^{\otimes m}$ in the following way:

$$V_\lambda(i) = \operatorname{Im} \rho_R(M^r_\lambda(i)). \tag{2.11}$$

Besides, we will deal with a space $\bar{V}_\lambda$ which is the image of $\bar{M}_\lambda$ (2.8):

$$\bar{V}_\lambda = \operatorname{Im} \rho_R(\bar{M}_\lambda) \subset V^{\otimes m}. \tag{2.12}$$

The spaces $V_\lambda(i)$ with different i (and all other $V_\lambda(X) \in \bar{V}_\lambda$ which are ρ_R-images of other possible monomorphisms[6] of M^r_λ into $\bar{M}_\lambda$) are isomorphic as vector spaces and we will not distinguish them. Instead, we will deal with a space V_λ which (like M^r_λ) gives rise to a class of isomorphic embeddings $V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes m}$ and each $V_\lambda(i)$ (or any other $V_\lambda(X)$ as well) is just a particular representative of this class of isomorphic spaces.

Remark 2 *Formula (2.11) has a nontrivial meaning for $k \geq 1$, where at $k = 1$ the only space V_λ is the space V itself. For the future convenience we extend the formula to the case $k = 0$. Namely, we will take by definition $V_{\lambda \vdash 0} = V_0 \equiv \mathbb{K}$.*

Now we can define the class of objects of the SW(V) category generated by a finite dimensional space V over a field $\mathbb{K}$ equipped with a Hecke symmetry R (1.6)–(1.8).

Definition 3 *To each fixed nonnegative integer $k \in \mathbb{Z}_+$ (for $k = 0$ see remark 2 above) and each possible partition $\lambda \vdash k$ we put into correspondence a space V_λ isomorphic to any $V_\lambda(i)$ in (2.11). The spaces V_λ, $\lambda \vdash k$, $k \in \mathbb{Z}_+$ are the basic objects of the category SW(V). The whole class of objects of the category is formed by direct sums of a finite number of basic objects V_λ. Thus, the spaces $V_\lambda(i)$ can be treated as the space V_λ equipped with an embedding $\{V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes k}\}$.*

Remark 4 *The objects V_λ should be simple objects of our category. However, some objects V_λ of the category will be identified with each other (in particular, $V_{(1^p)}$ and V_0). So, finally simple objects of our category will be V_λ modulo the mentioned identification. Up to this identification we will sometimes use the notation $[V_\lambda]_k$ for the family of embeddings $\{V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes k}\}$.*

Let us turn now to the definition of morphisms of our category.

2.2 *Morphisms of the first kind and the structure of quasitensor category*

As a morphism we will take some linear mappings among the objects, which will be described explicitly below.

Definition 5 *The morphisms of the first kind are defined as follows :*

i) The set $Mor(V_\lambda, V_\lambda)$ for any basic object V_λ contains only multiples of the identical morphism. That is for any morphism $f: V_\lambda \to V_\lambda$ we have by definition $f = a\,\mathrm{id}$ for some $a \in \mathbb{K}$.

ii) For arbitrary objects of SW(V) category the morphisms of the first kind are represented by a set of linear mappings:

$$\forall\, k \in \mathbb{N},\ \forall\, \tau \in H_k, \quad \phi^k(\tau) = \rho_R(\tau): \quad V^{\otimes k} \to V^{\otimes k}. \tag{2.13}$$

It can be easily shown, that being restricted to the set of subspaces $[V_\lambda]_k$ any mapping ϕ^k sends each $V_\lambda(i) \in [V_\lambda]_k$ to an isomorphic space or to zero space[7]. This means that nontrivial

[5]The choice of *right* submodules is made since we prefer to use the *left* action of $T \in \mathrm{End}(V)$ on V. Since an arbitrary element of $M^r_\lambda(i)$ has the form $Y^\lambda_{ii} f(\sigma_1 \ldots \sigma_m)$ (f being a polynomial in the generators of H_m), then with such a choice the projector $\rho_R(Y^\lambda_{ii})$ will be the last element acting on $V^{\otimes m}$ in formula (2.11).

[6]See Remark 1

[7]For example, $\rho_R(\sigma_i + q^{-1}\,\mathrm{id}_H)$ $1 \leq i \leq k-1$ sends to zero the image of the antisymmetrizer $A^{(k)}$.

mappings ϕ^k at most change the embedding of V_λ into $V^{\otimes k}$ and therefore are multiples of identical morphism for the basic objects. Among the morphisms ϕ^k there is no one which would send V_λ to V_μ with $\lambda \neq \mu$. A different kind of morphisms is considered in subsection 2.3.

A similar situation takes place for the category of finite dimensional representations of a simple group G (or a simple algebra $\mathfrak{g}$). Indeed, if one has a linear mapping of an irreducible $\mathfrak{g}$–module into itself such that it commutes with the action of $\mathfrak{g}$ then by the Schur lemma such a mapping equals $a\,\mathrm{id}$ for some $a \in \mathbb{K}$.

The mentioned similarity becomes more obvious if we supply the objects of SW(V) category with the comodule structure over a quantum bialgebra $\mathcal{T}$ generated by the well known RTT relation. Namely, consider a $\mathbb{K}$-algebra generated by n^2 elements $T^i_{\,j}$ subject to the relations [FRT]:

$$R_{12}T_1T_2 = T_1T_2R_{12}. \tag{2.14}$$

The bialgebra structure is introduced by a coproduct Δ and a counit ε:

$$\Delta(T^i_{\,j}) = T^i_{\,k} \otimes T^k_{\,j} \qquad \varepsilon(T^i_{\,j}) = \delta^i_{\,j}.$$

Define the right comodule structure $\delta_r:\ V \to V \otimes \mathcal{T}$ on the space V as follows:

$$\delta_r(v) = e_k \otimes T^k_{\,i} v^i, \quad \forall\, v = v^i e_i \in V.$$

Such a coaction is extended to $V^{\otimes k}$ in the obvious way:

$$\delta_r : V^{\otimes k} \to V^{\otimes k} \otimes \mathcal{T} \quad e_{\langle 1|} \otimes \ldots \otimes e_{\langle k|} \to e_{\langle 1|} \otimes \ldots \otimes e_{\langle k|} \otimes T_1 \ldots T_k \tag{2.15}$$

and all the comodule properties are easily verified. Then each $V_\lambda(i)$ $\lambda \vdash k$ turns out to be an invariant subcomodule in $V^{\otimes k}$ since the coaction δ_r commute with the action of $\rho_R(M^r_\lambda(i))$ on $V^{\otimes k}$:

$$\forall k,\ \forall \lambda \vdash k, \quad \rho_R(M^r_\lambda)\, T_1 T_1 \ldots T_k = T_1 T_1 \ldots T_k\, \rho_R(M^r_\lambda).$$

This equation is a direct consequence of defining relations (2.14) and explicit formula for primitive idempotents of H_k which expresses each Y^λ_{ii} as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$.

Under certain conditions on matrix R the bialgebra (2.14) can be factorized to a Hopf algebra (see below). In the case of the Hecke symmetry related to the QG $U_q(sl(n))$ this Hopf algebra is in fact the algebra $\mathrm{Fun}_q(SL(n))$ — the quantized function algebra on the group $SL(n)$. The dual Hopf algebra (quantum universal enveloping algebra $U_q(sl(n))$) canonically acts on each V_λ, therefore we obtain a category of finite dimensional $U_q(sl(n))$–modules. In this case our mappings ϕ^k are just those, which commute with the action of $U_q(sl(n))$.

Our next step consists in constructing a monoidal structure, which allows us to "multiply" the objects of the category. This means that we want to define a covariant functor

$$\otimes :\ \mathrm{Ob(SW)} \times \mathrm{Ob(SW)} \to \mathrm{Ob(SW)}$$

with some associativity morphisms (for detail see [CP, McL2]). As such a functor we take the usual tensor product of linear spaces with associativity morphisms to be identical. So, we have to prove that the tensor product of any two objects of our category is also an object, that is it can be decomposed into a direct sum of basic objects. Evidently, one only needs to verify this property for the tensor product of basic objects V_λ.

Proposition 6 *For given $\lambda \vdash n$ and $\mu \vdash m$ the tensor product of two basic objects V_λ and V_μ can be expanded into a direct sum of basic objects V_ν, $\nu \vdash (n+m)$:*

$$V_\lambda \otimes V_\mu = c_{\lambda\mu}^{\ \nu}\, V_\nu$$

the coefficients $c_{\lambda\mu}^{\;\nu}$ being the Littlewood-Richardson ones entering the formula for product of Schur symmetric functions s_λ in p variables[8].

Proof. As is clear from the definition of basic objects V_λ of our category the structure of their tensor product is controlled by that of modules M_λ^r (2.10) and the assertion of the proposition can be reformulated in terms of these modules.

To do so let us consider an embedding of $V_\lambda \otimes V_\mu$ into $V^{\otimes(n+m)}$ of the form

$$V_\lambda \otimes V_\mu \to V_\lambda(i) \otimes V_\mu(j) \subset V^{\otimes(n+m)}$$

for some fixed i and j. As was defined in (2.11) above, the space $V_\lambda(i)$ is the image of the right H_n submodule $M_\lambda^r(i)$ under representation ρ_R (2.1).

Consider an embedding $\mathcal{J}:\ H_n \times H_m \hookrightarrow H_{n+m}$ which on generators looks as follows:

$$\begin{array}{c} \forall \sigma'_i \in H_n :\ \ \mathcal{J}(\sigma'_i) = \sigma_i \in H_{n+m} \quad 1 \le i \le n-1 \\ \forall \sigma''_j \in H_m :\ \ \mathcal{J}(\sigma''_j) = \sigma_{n+j} \in H_{n+m} \quad 1 \le j \le m-1 \end{array} \tag{2.16}$$

By construction $\mathcal{J}(H_n)$ and $\mathcal{J}(H_m)$ form two mutually commuting Hecke subalgebras in H_{m+n}.

Let $\lambda \vdash n$, $\mu \vdash m$ and $M'_\lambda(i)$, $M''_\mu(j)$ be two right submodules in H_n and H_m generated by the primitive idempotents $Y_{ii}^\lambda(\sigma')$ and $Y_{jj}^\mu(\sigma'')$ correspondingly. The image of the tensor product $M'_\lambda(i) \otimes M''_\mu(j)$ under (2.16) is a right $H_n \times H_m$ module $M_\lambda^r(i) \otimes M_\mu^r(j) \subset H_{n+m}$ and obviously

$$V_\lambda(i) \otimes V_\mu(j) = \mathrm{Im}\,\rho_R(M_\lambda^r(i) \otimes M_\mu^r(j)). \tag{2.17}$$

The right H_{n+m} module induced from $M_\lambda^r(i) \otimes M_\mu^r(j)$ is reducible and can be decomposed into a direct sum of irreducible right H_{n+m} submodules $M_\nu^r(k)$. This decomposition is the image (under monomorphism $M_\lambda^r \to M_\lambda^r(i)$) of the following relation between irreducible modules M_λ^r and M_μ^r:

$$M_\lambda^r \otimes M_\mu^r = c_{\lambda\mu}^{\;\nu}\, M_\nu^r \tag{2.18}$$

where the coefficients $c_{\lambda\mu}^{\;\nu}$ are equal to the multiplicity of irreducible H_{n+m} characters χ^ν in the character induced from $\chi^\lambda \times \chi^\mu$. Due to (2.17) the coefficients $c_{\lambda\mu}^{\;\nu}$ will also determine the expansion of tensor product of two basic objects V_λ and V_μ.

Now we use the fact that at generic q the Hecke algebra H_m is isomorphic to the group algebra $\mathbb{K}(S_m)$ of the m-th order permutation group S_m for all $m \ge 1$. Therefore, the coefficients $c_{\lambda\mu}^{\;\nu}$ defining the multiplicity of irreducible module M_ν^r in the tensor product $M_\lambda^r \otimes M_\mu^r$ are the same for Hecke algebra and for the algebra $\mathbb{K}(S_m)$. To complete the proof, note that as is well known from the representation theory of symmetric group (see for example [M]) the corresponding multiplicities coincide with the Littlewood-Richardson coefficients in the product of symmetric Schur functions s_λ. ∎

Remark 7 *The explicit calculation of the decomposition of $M_\lambda^r(i) \otimes M_\mu^r(j)$ into a direct sum of H_{n+m} submodules gives*

$$M_\lambda^r(i) \otimes M_\mu^r(j) = \alpha_{\lambda\mu}^{\;\nu}(k(i,j))\, M_\nu^r(k(i,j)), \quad \alpha_{\lambda\mu}^{\;\nu}(k(i,j)) \in \mathbb{Z}_+.$$

But the isomorphic submodules $M_\nu^r(k)$ with different k are images of the same module M_ν^r equipped with different monomorphisms (depending on i and j) $M_\nu^r \to \bar{M}_\nu \subset H_{n+m}$. The sum $\sum_k \alpha_{\lambda\mu}^{\;\nu}(k)$ does not depend on i and j and is equal to $c_{\lambda\mu}^{\;\nu}$ which defines the structure of the tensor product (2.18).

Now we should convert the monoidal category SW(V) into a quasitensor one (see [CP]). This means, that we need to define a set of *natural commutativity (or braiding) isomorphisms*

[8]The detailed description of Schur functions and related topics can be found in [M].

(1.1) for any pair $U, V \in \mathrm{Ob}(\mathrm{SW(V)})$. To be compatible with monoidal structure the braiding isomorphisms must satisfy the following property for any triple $U, V, W \in \mathrm{Ob}(\mathrm{SW(V)})$:

$$R_{U,V\otimes W} = (\mathrm{id}_V \otimes R_{U,W}) \circ (R_{U,V} \otimes \mathrm{id}_W) \quad R_{V\otimes W,U} = (R_{V,U} \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes R_{W,U}). \tag{2.19}$$

Let us recall that the associativity isomorphisms are taken to be identical.

Besides, considering two possible ways of transformation of $U \otimes V \otimes W$ into $W \otimes V \otimes U$ one comes to the following condition:

$$(R_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes R_{U,W}) \circ (R_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes R_{U,V}) \circ (R_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes R_{V,W}). \tag{2.20}$$

Let us begin with the simplest object — the space V. As a braiding isomorphism for $V \otimes V$ we take the Hecke symmetry R. Requirement (2.20) on $V^{\otimes 3}$ transforms into the Yang-Baxter equation (1.7) on R which is satisfied by the definition of R. The decomposition property (2.19) will be used as *a definition* of braiding isomorphisms for an arbitrary tensor power of the space V. That is we take:

$$R_{V,V^{\otimes k}} = R_k R_{k-1} \dots R_1 \quad R_{V^{\otimes k},V} = R_1 R_2 \dots R_k. \tag{2.21}$$

In the above formula a shorthand notation $R_i \equiv R_{ii+1}$ is used. Introducing one more notation for the chain of R matrices

$$R_{i\to j} \equiv \begin{cases} R_i R_{i+1} \dots R_j & \text{if } j = i+n \geq i \\ R_i R_{i-1} \dots R_j & \text{if } j = i-n \leq i \end{cases}$$

we can write a compact form for the braiding isomorphism in the general case:

$$\begin{aligned} R_{V^{\otimes n},V^{\otimes m}} &= R_{m\to(m+n-1)} R_{(m-1)\to(m+n-2)} \dots R_{1\to n} \\ &\equiv R_{m\to 1} R_{(m+1)\to 2} \dots R_{(n+m-1)\to n}. \end{aligned} \tag{2.22}$$

The two parts of this formula correspond to two possible ways of passing from $V^{\otimes n} \otimes V^{\otimes m}$ to $V^{\otimes m} \otimes V^{\otimes n}$. Note, that all mappings (2.22) are morphisms of the first kind.

As for the braiding isomorphisms for $\mathbb{K} \otimes V$ we take it to be the usual flip:

$$\mathbb{K} \otimes V = V \otimes \mathbb{K} = V. \tag{2.23}$$

By definition of the objects of the category SW(V) any tensor power $V^{\otimes m}$ can be decomposed into a direct sum of basic objects V_λ, $\lambda \vdash m$. Therefore by making use of formula (2.22) one can define the braiding isomorphism for the tensor product of two arbitrary basic objects $V_\lambda \otimes V_\mu$, (and, therefore, for the tensor product of any couple of objects) if we manage to prove that isomorphism (2.22) does not "destroy" the structure of embeddings of $V_\lambda \otimes V_\mu$ into $V^{\otimes n} \otimes V^{\otimes m}$. If it is the case then we can take the restriction of (2.22) onto $\bar{V}_\lambda \otimes \bar{V}_\mu$ as the braiding morphism for the tensor product of V_λ and V_μ.

That is we have to show the following. Consider two arbitrary embeddings $\mathcal{J}_i : V_\lambda \hookrightarrow V^{\otimes n}$ and $\mathcal{J}_j : V_\mu \hookrightarrow V^{\otimes m}$. Let $\mathcal{J}_i(V_\lambda) = V_\lambda(i)$ and $\mathcal{J}_j(V_\mu) = V_\mu(j)$. Let us recall that $V_\lambda(i)$ is defined by (2.11). We want to show that under isomorphism (2.22) one gets:

$$V_\lambda(i) \otimes V_\mu(j) \longrightarrow R_{\mu\lambda}(V_\mu(j) \otimes V_\lambda(i))$$

where $R_{\lambda\mu}$ is an invertible operator from $\mathrm{End}(\bar{V}_\mu \otimes \bar{V}_\lambda)$ (the space $\bar{V}_\lambda$ is defined in (2.12)) which *does not* depend on i and j.

Proposition 8 *For two given partitions $\lambda \vdash n$, $\mu \vdash m$ and two arbitrary integers $1 \le i \le d_\lambda$ and $1 \le j \le d_\mu$ consider the corresponding spaces $V_\lambda(i)$ and $V_\mu(j)$ as defined in (2.11). Then under isomorphism (2.22) one has*

$$V_\lambda(i) \otimes V_\mu(j) \longrightarrow R_{\mu\lambda}(V_\mu(j) \otimes V_\lambda(i)) \tag{2.24}$$

where the operator $R_{\mu\lambda}$ is defined on $\bar{V}_\lambda \otimes \bar{V}_\mu$ by means of the following formula:

$$R_{\mu\lambda} = \rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda) \cdot R_{V^{\otimes n},V^{\otimes m}} \equiv R_{V^{\otimes n},V^{\otimes m}} \cdot \rho_R(\mathbb{Y}^\lambda) \otimes \rho_R(\mathbb{Y}^\mu). \tag{2.25}$$

Here $R_{V^{\otimes n},V^{\otimes m}}$ is defined by (2.22) and

$$\mathbb{Y}^\lambda \equiv \sum_{i=1}^{d_\lambda} Y_{ii}^\lambda$$

is a central idempotent of the corresponding Hecke algebra.

Proof. The proof is based on the embedding of $H_n \times H_m$ into H_{n+m} introduced in Proposition 6. In accordance with (2.16) and (2.17) we can write

$$V_\lambda(i) \otimes V_\mu(j) = \mathrm{Im}\{\rho_R(Y_{ii}^\lambda) \otimes \rho_R(Y_{jj}^\mu)\}.$$

Considered on the arbitrary basis element of the space $V^{\otimes n} \otimes V^{\otimes m}$ the right hand side of the above formula reads:

$$e_{\langle 1|} \otimes \ldots \otimes e_{\langle n+m|} \to e_{\langle 1|} \otimes \ldots \otimes e_{\langle n+m|} \,\{\mathcal{Y}_{ii}^\lambda \otimes \mathcal{Y}_{jj}^\mu\}^{|1\ldots n+m\rangle}_{\langle 1\ldots n+m|} \tag{2.26}$$

where the matrices $\mathcal{Y}_{ii}^\lambda = \rho_R(Y_{ii}^\lambda)$ and $\mathcal{Y}_{jj}^\mu = \rho_R(Y_{jj}^\mu)$ are some polynomials in $R_1 \ldots R_{n-1}$ and $R_{n+1} \ldots R_{n+m-1}$ respectively:

$$\mathcal{Y}_{ii}^\lambda = \mathcal{Y}_{ii}^\lambda(R_1, \ldots, R_{n-1}), \quad \mathcal{Y}_{jj}^\mu = \mathcal{Y}_{jj}^\mu(R_{n+1}, \ldots, R_{n+m-1}).$$

Applying the braiding isomorphism (2.22) to $V^{\otimes n} \otimes V^{\otimes m}$ leads to the following result for formula (2.26) :

$$e_{\langle 1|} \otimes \ldots \otimes e_{\langle n+m|} \to e_{\langle 1|} \otimes \ldots \otimes e_{\langle n+m|} \,\{R_{V^{\otimes n},V^{\otimes m}} \cdot \mathcal{Y}_{ii}^\lambda \otimes \mathcal{Y}_{jj}^\mu\}$$

where the symbol $\cdot$ stands for the matrix multiplication.

Now one should take into account the following relations which are direct consequence of (2.22) and Yang-Baxter equation (1.7):

$$R_{V^{\otimes n},V^{\otimes m}} \cdot R_i = R_{i+m} \cdot R_{V^{\otimes n},V^{\otimes m}} \quad 1 \le i \le n-1$$

$$R_{V^{\otimes n},V^{\otimes m}} \cdot R_j = R_{j-n} \cdot R_{V^{\otimes n},V^{\otimes m}} \quad n+1 \le j \le n+m-1.$$

Using these relations one gets:

$$R_{V^{\otimes n},V^{\otimes m}} \cdot \mathcal{Y}_{ii}^\lambda(R_1 \ldots R_{n-1}) \otimes \mathcal{Y}_{jj}^\mu(R_{n+1} \ldots R_{n+m-1}) =$$
$$\mathcal{Y}_{jj}^\mu(R_1 \ldots R_{m-1}) \otimes \mathcal{Y}_{ii}^\lambda(R_{m+1} \ldots R_{m+n-1}) \cdot R_{V^{\otimes n},V^{\otimes m}}.$$

This formula proves (2.24). In order to find the form of $R_{\mu\lambda}$ we observe that $\mathbb{Y}^\lambda = \sum_i Y_{ii}^\lambda$ are *central* elements of H_n ($\lambda \vdash n$) and besides $Y_{ii}^\lambda \equiv Y_{ii}^\lambda \mathbb{Y}^\lambda \; \forall i, j$. Therefore

$$\mathcal{Y}_{jj}^\mu \otimes \mathcal{Y}_{ii}^\lambda \cdot R_{V^{\otimes n},V^{\otimes m}} \equiv \mathcal{Y}_{jj}^\mu \otimes \mathcal{Y}_{ii}^\lambda \cdot (\rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda) \cdot R_{V^{\otimes n},V^{\otimes m}}).$$

where $\rho_R(\mathbb{Y}^\lambda)$ is the projector onto $\bar{V}_\lambda \subset V^{\otimes(n+m)}$. Thus, we come to form (2.25) of $R_{\mu\lambda}$. ■

So, the operator $R_{\mu\lambda} = \rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda) R_{V^{\otimes n},V^{\otimes m}}$ does not depend on a concrete embedding of $V_\lambda \otimes V_\mu \in \mathrm{Ob(SW(V))}$ into $V^n \otimes V^m$ (i.e., it does not depend on indices i, j) and represents a braiding isomorphism for $V_\lambda \otimes V_\mu$.

2.3 *Morphisms of the second kind and reduction procedure*

As was mentioned above (see (2.6)) the image of the antisymmetrizer A^m with $m > p$ is identical zero in $V^{\otimes m}$. It can be shown that the same is true for any $\mathrm{Im}\,\rho_R(M^r_\lambda)$ in case if the height $\ell(\lambda) > p$. This means that the basic objects of our category V_λ are labelled by partitions with restricted height: $V_\lambda \not\equiv 0 \Leftrightarrow \ell(\lambda) \le p$.

However, there is another consequence of (2.5) – (2.6) which concerns the objects labelled by partitions with the height equal to p. Consider a partition $\lambda \vdash (p+m)$ for some nonnegative integer m such that $\ell(\lambda) = p$. Let μ denote a partition of m which is obtained from λ by striking the first column out of the Young diagram corresponding to λ. The parts of λ and μ are connected by the relation $\mu_i = \lambda_i - 1$, $\forall \lambda_i \neq 0$. As immediately follows from Proposition 6 and from the above remark about the maximal height of λ the following decomposition takes place:

$$V_\lambda \cong V_{(1^p)} \otimes V_\mu \tag{2.27}$$

where $V_{(1^p)}$ stands for the one dimensional space labelled by one-column diagram with p boxes: $\lambda = (1^p)$.

Isomorphisms (2.27) allow us to define the so called *reduction procedure* and to introduce morphisms of the second kind in our category, namely, those identifying V_λ and V_μ. For this purpose consider a mapping ψ sending the one-dimensional space $V_{(1^p)}$ into the field $\mathbb{K}$. Evidently, one only needs to fix the action of the mapping ψ on a basis vector of $V_{(1^p)} \cong \mathrm{Im}\,\rho_R(A^{(p)})$. In the general form such a mapping looks like

$$\psi(e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|}\, v^{|1\ldots p\rangle}) = 1 \in \mathbb{K}. \tag{2.28}$$

The choice of 1 in the above formula does not restrict generality of our consideration.

Let us find conditions under which the mapping ψ can be treated as a morphism in our category. Taking into account (1.2) and the fact that the identity operator is a morphism of the category we conclude that the following diagrams must be commutative:

$$\begin{array}{ccc} V_{(1^p)} \otimes V_\mu & \stackrel{\psi\otimes \mathrm{id}}{\longrightarrow} & \mathbb{K}\otimes V_\mu \\ {\scriptstyle R_{V_{(1^p)},V_\mu}}\downarrow & & \| \\ V_\mu \otimes V_{(1^p)} & \stackrel{\mathrm{id}\otimes\psi}{\longrightarrow} & V_\mu \otimes \mathbb{K} \end{array} \qquad \begin{array}{ccc} V_\mu \otimes V_{(1^p)} & \stackrel{\mathrm{id}\otimes\psi}{\longrightarrow} & V_\mu\otimes \mathbb{K} \\ {\scriptstyle R_{V_\mu,V_{(1^p)}}}\downarrow & & \| \\ V_{(1^p)} \otimes V_\mu & \stackrel{\psi\otimes\mathrm{id}}{\longrightarrow} & \mathbb{K} \otimes V_\mu \end{array}$$

Let us consider a particular case of these diagrams by putting $V_\mu = V$ (if they are commutative in this case the same will be true for any μ). Then by passing to the basis we have

$$V_{(1^p)} \otimes V: \quad e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|} \otimes e_{\langle p+1|}\, v^{|1\ldots p\rangle} \stackrel{\psi\otimes\mathrm{id}}{\longrightarrow} 1 \otimes e_{\langle p+1|} = e_{\langle p+1|} \otimes 1. \tag{2.29}$$

According to another way in the left diagram we should first apply the braiding morphism $R_{V_{(1^p)},V}$. Thus, we have

$$\begin{aligned} R_{V_{(1^p)},V}: \quad e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|} \otimes e_{\langle p+1|}\, v^{|1\ldots p\rangle} \quad & \longrightarrow \\ & e_{\langle 1|} \otimes \ldots \otimes e_{\langle p|} \otimes e_{\langle p+1|}\, R_1 \ldots R_p\, v^{|1\ldots p\rangle}. \end{aligned}$$

In order to simplify this expression we need the following useful relations (see [HIOPT]):

$$\begin{aligned} R_1 \ldots R_p\, \mathcal{A}^{(p)} &= (-1)^{p-1}\, q p_q\, \mathcal{A}^{(2,p+1)} \mathcal{A}^{(p)} \\ R_p \ldots R_1\, \mathcal{A}^{(2,p+1)} &= (-1)^{p-1}\, q p_q\, \mathcal{A}^{(p)} \mathcal{A}^{(2,p+1)} \\ \mathcal{A}^{(p)} \mathcal{A}^{(2,p+1)} \mathcal{A}^{(p)} &= p_q^{-2} \mathcal{A}^{(p)} \otimes \mathrm{id}_{p+1}. \end{aligned} \tag{2.30}$$

In these formulas the antisymmetrizer $\mathcal{A}^{(2,p+1)}$ has the same form as $\mathcal{A}^{(p)}$ but it depends on $R_2, \dots, R_p$ instead of $R_1, \dots, R_{p-1}$.

Now in virtue of (2.30) we obtain:

$$R_1 \dots R_p \, v^{|1\dots p\rangle} = v^{|2\dots p+1\rangle} \, N^{|1\rangle}_{\langle p+1|} \tag{2.31}$$

where the matrix N is defined as follows:

$$N^i_j \equiv (-1)^{p-1} q p_q \, u_{a_2\dots a_p j} v^{i a_2 \dots a_p} \quad \text{or} \quad N^{|1\rangle}_{\langle 1'|} = (-1)^{p-1} q p_q \, u_{\langle 2\dots p 1'|} v^{|12\dots p\rangle}. \tag{2.32}$$

Therefore the composition $(\mathrm{id} \otimes \psi) \circ R_{V_{(1^p)},V}$ gives the following result:

$$e_{\langle 1|} \otimes \dots \otimes e_{\langle p|} \otimes e_{\langle p+1|} \, v^{|1\dots p\rangle} \to e_{\langle 1|} \otimes \dots \otimes e_{\langle p|} \otimes e_{\langle p+1|} \, v^{|2\dots p+1\rangle} \, N^{|1\rangle}_{\langle p+1|} \to e_{\langle 1|} N^{|1\rangle}_{\langle p+1|} \otimes 1$$

which obviously differs from (2.29). These results can be made compatible iff N is a scalar matrix (i.e., it is proportional to the $n \times n$ unit matrix I). If it is so, then by multiplying the braiding R by an appropriate factor we can achieve the commutativity of the left diagram above. Indeed, if we replace the braiding R by aR, $a \in \mathbb{K}$, the factor a^p appears in the right hand side of (2.31). Choosing properly the factor a we can obtain $a^p N = I$.

Applying the same procedure for the space $V \otimes V_{(1^p)}$ (the right diagram above) we find:

$$e_{\langle 1|} \otimes e_{\langle 2|} \otimes \dots \otimes e_{\langle p+1|} \, v^{|2\dots p+1\rangle} \longrightarrow 1 \otimes e_{\langle p+1|} M^{|p+1\rangle}_{\langle 1|}$$

where[9]

$$M^i_j \equiv (-1)^{p-1} q p_q \, u_{j a_2 \dots a_p} v^{a_2 \dots a_p i} \quad \text{or} \quad M^{|1'\rangle}_{\langle 1|} \equiv (-1)^{p-1} q p_q \, u_{\langle 12\dots p|} v^{|2\dots p 1'\rangle}. \tag{2.32$'$}$$

For the same reason we assume the matrix M to be scalar. Then the right diagram can be also made commutative by a proper rescaling of R. However if we want simultaneously obtain the unit matrix instead of the matrices N and M we have to impose one more condition: $M = N$.

Therefore from now on we will suppose the Hecke symmetry R to satisfy the relation

$$M = N = aI, \quad a \in \mathbb{K}^\times. \tag{2.33}$$

Thus, the mapping ψ becomes a morphism of the category SW(V) after a proper renormalization of the braiding morphisms.

This normalization factor is easy to find. Indeed, using relations (2.30) (valid for *any* Hecke symmetry of the rank p) we find

$$M \cdot N = q^2 I. \tag{2.34}$$

So, if relation (2.33) is satisfied then by virtue of (2.34) we have $a = \pm q$ and hence

$$N = M = \pm q I. \tag{2.35}$$

Therefore to convert ψ into a morphism we should pass from the initial Hecke symmetry R to the braiding

$$\bar{R} = (\pm q)^{-\frac{1}{p}} R. \tag{2.36}$$

On the higher tensor powers this renormalization gives rise to a renormalized braiding as well

$$\bar{R}_{V^{\otimes n}, V^{\otimes m}} = (\pm q)^{-\frac{mn}{p}} R_{V^{\otimes n}, V^{\otimes m}}. \tag{2.37}$$

[9]Note that the matrices N and M differ by a factor from those considered in [G].

The mapping ψ (as well as its inverse ψ^{-1}) will be called a morphism of the second kind.

The sense of restriction (2.33) becomes more clear if objects of our category are supplied with a comodule structure over bialgebra (2.14). Consider shortly the consequences of this structure.

Let us define *a quantum determinant* $\det_q T$ of bialgebra (2.14) as follows [FRT]:

$$\det_q T \equiv u_{\langle 12\ldots p|} T_1 T_2 \ldots T_p v^{|12\ldots p\rangle}. \tag{2.38}$$

The commutation relations of $\det_q T$ with generators of (2.14) read (cf. [G]):

$$\det_q T \cdot T = (N^{-1} T N) \cdot \det_q T \quad \Leftrightarrow \quad \det_q T \cdot (NT) = (TN) \cdot \det_q T \tag{2.39}$$

or due to (2.34)

$$\det_q T \cdot T = (MTM^{-1}) \cdot \det_q T \quad \Leftrightarrow \quad \det_q T \cdot (TM) = (MT) \cdot \det_q T.$$

If we want the mapping ψ (2.28) to commute with the comodule structure we have to introduce one more condition:

$$\det_q T = 1.$$

But such a relation is compatible with the algebraic structure only if $\det_q T$ is a central element of algebra (2.14). By virtue of (2.39) this means that the matrix N (and hence M) must be scalar.

Remark 9 *However, it is not clear whether there exist Hecke symmetries such that the matrices N and M are scalar but not equal to each other. (Conjecturally, the condition "N and M are scalar" leads to "N and M are equal".) Nevertheless, the family of even Hecke symmetries satisfying (2.33) is sufficiently large. An explicit construction of such a type Hecke symmetries will be given by one of the authors (R.L) in a separate paper.*

Let us also remark that for the quasiclassical Hecke symmetries appearing from the universal R matrix of QG $U_q(sl(n))$ condition (2.33) holds true. The quantum determinant $\det_q T$ *and the unit element* 1 *generate the center of (2.14) [FRT]. But in general (even in the quasiclassical case)* $\det_q T$ *is not central.*

2.4 *The structure of rigid category*

To convert a (quasi)tensor category $\mathcal{C}$ into a *rigid* one we must specify the following data [CP]:

a. A mapping $*:\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ which for any $U \in \mathrm{Ob}(\mathcal{C})$ put into correspondence its left dual U^*

b. For any pair of dual objects U and U^* there exist evaluation $\mathbf{ev}_U$ and coevaluation $\boldsymbol{\pi}_U$ morphisms of $\mathcal{C}$

$$\boldsymbol{\pi}_U :\ \mathbb{K} \longrightarrow U \otimes U^*, \qquad \mathbf{ev}_U :\ U^* \otimes U \longrightarrow \mathbb{K}$$

such that the following diagrams are commutative:

$$\begin{array}{ccc} U & \xrightarrow{\boldsymbol{\pi}_U \otimes \mathrm{id}} & (U \otimes U^*) \otimes U \\ \mathrm{id}\downarrow & & \downarrow \mathrm{id} \\ U & \xleftarrow{\mathrm{id} \otimes \mathbf{ev}_U} & U \otimes (U^* \otimes U) \end{array} \qquad \begin{array}{ccc} U^* & \xrightarrow{\mathrm{id} \otimes \boldsymbol{\pi}_U} & U^* \otimes (U \otimes U^*) \\ \mathrm{id}\downarrow & & \downarrow \mathrm{id} \\ U^* & \xleftarrow{\mathbf{ev}_U \otimes \mathrm{id}} & (U^* \otimes U) \otimes U^* \end{array} \tag{2.40}$$

Let us begin with the case $U = V$. By our construction of the morphism ψ it is evident that the dual space V^* is nothing but $\Lambda^{p-1}(V)$. So, in order to satisfy (2.40) we should only fix a convenient basis converting $\Lambda^{p-1}(V)$ into a left dual space.

Definition 10 *The space $\Lambda^{p-1}(V)$ equipped with the basis*

$$e^i = e_{a_2} \otimes \ldots \otimes e_{a_p} v^{ia_2 \ldots a_p} \quad \text{or} \quad e^{|1\rangle} = e_{\langle 2|} \otimes \ldots \otimes e_{\langle p|} v^{|12\ldots p\rangle}. \tag{2.41}$$

will be called a left dual space to the space V. The morphisms $\mathbf{ev}_V$ and $\boldsymbol{\pi}_V$ are defined as follows (on basis vectors):

$$\begin{aligned} \mathbf{ev}_V: &\quad e^i \otimes e_j \to \delta^i_j 1 \\ \boldsymbol{\pi}_V \ : &\quad 1 \longrightarrow e_i \otimes e^i. \end{aligned} \tag{2.42}$$

In order to justify this definition we have to show the following.

Proposition 11 *Mappings (2.42) are morphisms of the SW(V) category and they satisfy property (2.40).*

Proof. The fact that $\boldsymbol{\pi}_V$ is a morphism is evident since

$$\boldsymbol{\pi}_V = \psi^{-1}.$$

To show that $\mathbf{ev}_V$ is a morphism let us consider the mapping

$$\varphi = (-1)^{p-1} p_q \psi \circ \mathcal{A}^p : \quad V^{\otimes p} \to \mathbb{K}.$$

It is a morphism by construction. In a basis form we get

$$\varphi : \ e_{i_1} \otimes \ldots \otimes e_{i_p} \to (-1)^{p-1} p_q u_{i_1 \ldots i_p}.$$

Applying this morphism to the element

$$e^i \otimes e_j = v^{ia_2 \ldots a_p} e_{a_2} \otimes \ldots \otimes e_{a_p} \otimes e_j$$

we get

$$\varphi(e^i \otimes e_i) = (-1)^{p-1} p_q v^{ia_2 \ldots a_p} u_{a_2 \ldots a_p j} = q^{-1} N^i_j = \pm \delta^i_j.$$

Thus, the mapping $\mathbf{ev}_V$ is nothing but the morphism $\pm\varphi$ restricted on $\Lambda^{p-1}(V) \otimes V$.

Diagrams (2.40) are obviously commutative. ■

Let us define now left duals to other simple objects of our category. First, introduce a dual object to $V^{\otimes m}$ by putting $(V^{\otimes m})^* = (V^*)^{\otimes m}$.

Proposition 12 *The space $(V^*)^{\otimes m}$ is dual to the space $(V^{\otimes m})$ being equipped with the mappings*

$$\mathbf{ev}_U : V^{\otimes m} \otimes V^{*\otimes m} \longrightarrow \mathbb{K}, \qquad \boldsymbol{\pi}_U : \mathbb{K} \longrightarrow V^{*\otimes m} \otimes V^{\otimes m}$$

which are defined to be:

$$\begin{aligned} \mathbf{ev}_U : &\quad e^{|m\rangle} \otimes \ldots \otimes e^{|1\rangle} \otimes e_{\langle 1|} \otimes \ldots e_{\langle m|} \to \delta^{|1\rangle}_{\langle 1|} \ldots \delta^{|m\rangle}_{\langle m|} \\ \boldsymbol{\pi}_U \ : &\quad 1_{\mathbb{K}} \to e_{\langle 1|} \otimes \ldots \otimes e_{\langle m|} \otimes e^{|m\rangle} \otimes \ldots \otimes e^{|1\rangle} \end{aligned} . \tag{2.43}$$

Proof is obvious.

Now we are ready to define the dual V^*_λ and corresponding morphisms $\boldsymbol{\pi}_{V_\lambda}$ and $\mathbf{ev}_{V_\lambda}$ for arbitrary basic object V_λ of the category SW(V).

Fix a basic object V_λ, $\lambda \vdash m$, and consider its particular embedding into $V^{\otimes m}$ in the form (2.11):

$$V_\lambda(i) = \operatorname{Im} \rho_R(Y^\lambda_{ii}).$$

This space is spanned by the following vectors

$$e_{\langle 1|}\otimes\ldots\otimes e_{\langle m|}\ \cdot(\mathcal{Y}^{\lambda}_{ii})^{|1\ldots m\rangle}_{\ \langle 1\ldots m|}$$

where $\mathcal{Y}^{\lambda}_{ii}$ is a matrix of Y^{λ}_{ii} in the representation ρ_R.

Let us define a space $V^*_\lambda(i)\subset V^{*\otimes m}$ as a linear span of the form:

$$V^*_\lambda(i)=\{w^*:\ w^*=w_{\langle 1\ldots m|}\,(\mathcal{Y}^{\lambda}_{ii})\cdot e^{|m\rangle}\otimes\ldots\otimes e^{|1\rangle}\}\tag{2.44}$$

where $w_{\langle 1\ldots m|}$ is an arbitrary tensor with components from $\mathbb{K}$. It is important, that the order of spaces in this formula is reversed comparing with the preceding relation while the matrix $\mathcal{Y}^{\lambda}_{ii}$ is the same.

Finally, we have the following.

Proposition 13 *The space V^*_λ being equipped with the morphisms $\mathbf{ev}_{V_\lambda}$ and $\boldsymbol{\pi}_{V_\lambda}$ of the form*

$$\begin{aligned}\mathbf{ev}_{V_\lambda}:&\quad V^*_\lambda\otimes V_\lambda\longrightarrow\mathcal{Y}^{\lambda}_{ii}\\ \boldsymbol{\pi}_{V_\lambda}\ :&\quad 1_{\mathbb{K}}\longrightarrow e_{\langle 1|}\otimes\ldots e_{\langle m|}\otimes\mathcal{Y}^{\lambda}_{ii}\cdot e^{|m\rangle}\otimes\ldots\otimes e^{|1\rangle}.\end{aligned}\tag{2.45}$$

becomes left dual to the space V_λ (provided that V_λ is realized as $V_\lambda(i)$).

Let us note that in fact the space V^*_λ is nothing but

$$V_\mu,\ \ \mu=(\lambda_1,\lambda_1-\lambda_{p-1},\lambda_1-\lambda_{p-2},...,\lambda_1-\lambda_2)$$

equipped with an embedding in the space $V^{*\otimes m}$. This shows that we do not need to construct left dual to the space V^*_λ because it is just the object V_λ.

Nevertheless, the problem of an explicit pairing

$$V_\lambda\otimes V^*_\lambda\to\mathbb{K}$$

converting V^*_λ into the right dual to V_λ (and V_λ in the left dual to V^*_λ) is very important. In order to describe this pairing in a particular case $V_\lambda=V$ let us calculate the braiding of V and $\Lambda^{p-1}(V)$ expressed via the basis $\{e^i\}$. Note, that we use the renormalized braiding $\bar R$ (2.36).

Proposition 14 *We have*

$$\bar R_{V,V^*}:\quad e_i\otimes e^j\longrightarrow e^r\otimes e_s\,\bar Q^{js}_{\ ir}\tag{2.46}$$

where in the given basis $\{e_i\}$ the operator $\bar Q$ satisfies the relation:

$$\bar R^{ia}_{\ jb}\,\bar Q^{bk}_{\ al}=\delta^i_{\,l}\delta^k_{\,j}\quad\Leftrightarrow\quad\bar Q^{ia}_{\ jb}\,\bar R^{bk}_{\ al}=\delta^i_{\,l}\delta^k_{\,j}.\tag{2.47}$$

Note, that if such $\bar Q$ exists then the corresponding braiding $\bar R$ is usually called *invertible by column.*

Proof. According to definition (2.41) for e^i and (2.37) for the renormalized braiding we get (omitting the obvious signs of tensor product):

$$e_i\otimes e^j=e_i\,e_{a_2}\ldots e_{a_p}\,v^{ja_2\ldots a_p}\stackrel{\bar R_{V,V^{\otimes(p-1)}}}{\longrightarrow}e_{s_1}\ldots e_{s_p}(\pm q)^{\frac{1-p}{p}}(R_{p-1}\ldots R_1)^{s_1\ldots s_p}_{\ ia_2\ldots a_p}v^{ja_2\ldots a_p}.$$

Now from (2.30) it is easy to find

$$(\pm q)^{\frac{1-p}{p}}(R_{p-1}\ldots R_1)^{s_1\ldots s_p}_{\ ia_2\ldots a_p}v^{ja_2\ldots a_p}=\bar Q^{js_p}_{\ ir}\,v^{rs_1\ldots s_{p-1}}.$$

Substituting this into the previous relation and applying again the definition (2.41) of the dual basis we find result (2.46). ■

Now we will introduce a pairing between V and V^* by putting

$$\bar{\mathbf{ev}}_V=\mathbf{ev}_V\circ R_{V,V^*}:\quad V\otimes V^*\to\mathbb{K}.$$

It is this operator which plays the central role in the next section where a categorical trace in the space $\mathrm{End}\,(V)$ will be introduced.

3 Additive-multiplicive functional and dimension

In this section we find all possible additive and multiplicative (a-m) functionals on objects of Schur-Weyl category SW(V) and suggest a definition of a trace in $\mathrm{End}(V)$ as morphism (1.3) in the category such that the corresponding dimension is an a-m functional. As was pointed out in the Introduction, the requirement that the trace should be a morphism in general is not a trivial one. In particular, this requirement gives rise to the notion of *quantum trace* in the category of finite dimensional modules over a special class of quasitriangular Hopf algebras — the so called ribbon Hopf algebras. In our approach the morphism property of trace can be obtained without any additional Hopf structure.

Let us recall that a mapping $f:\ \mathrm{Ob(SW(V))}\to\mathbb{K}$ will be called *an additive-multiplicative functional* (a-m functional) if it possesses the following property:

$$\forall\, U,V\in \mathrm{Ob(SW(V))}:\quad f(U\oplus V)=f(U)+f(V),\quad f(U\otimes V)=f(U)f(V). \tag{3.1}$$

Remark 15 *Introducing the Grothendieck ring of the category SW(V) and making use of the fact that our category is semisimple (i.e. any its object can be decomposed into a direct sum of simple ones) we conclude that each a-m functional on the category is in fact defined on the Grothendieck ring.*

The fact that we are working with a Hecke symmetry of the rank p leads to an important consequence for the set of a-m functionals. Namely we will show that the value of any a-m functional f on a basic object V_λ (and, hence, on an arbitrary object of SW(V)) is completely determined by $(p-1)$ numbers[10] $f(V_{(1^k)})$ $k=1,\dots,p-1$.

Proposition 16 *Given an a-m functional f, construct a p-th order polynomial in a formal variable t of the form:*

$$\phi(t)=t^p+f_1t^{p-1}\ldots+f_{p-1}t+1,\quad f_k\equiv f(V_{(1^k)}),\ \ k=1,\dots,p-1. \tag{3.2}$$

Let the numbers $(-\alpha_i)\in\mathbb{K}$ be the roots of $\phi(t)$, that is $\phi(-\alpha_i)=0$ $i=1,\dots,p$. Then for any basic object V_λ of SW(V) the following relation holds:

$$f(V_\lambda)=s_\lambda(\alpha_1,\alpha_2,\dots,\alpha_p) \tag{3.3}$$

where $s_\lambda(x_1,\dots,x_p)$ is a symmetric Schur function in p variables[11] *corresponding to the partition λ.*

Proof. First of all let us prove that the numbers $f_k=f(V_{(1^k)})$ do satisfy (3.3). In accordance with the definition of α_i we can write

$$\phi(t)=(t+\alpha_1)(t+\alpha_2)\ldots(t+\alpha_p).$$

Therefore, as immediately follows from this relation, the coefficient f_k is the k-th elementary symmetric function e_k in variables α_i [M]:

$$f_k=\sum_{i_1<i_2<\ldots<i_k}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}=e_k(\alpha_1,\dots,\alpha_k).$$

But since $s_{(1^k)}=e_k$ we conclude that the assertion of the proposition is valid for f_k.

The fact that the quantities $s_\lambda(\alpha_1,\alpha_2,\dots,\alpha_p)$ can be considered as values of an a-m functional follows immediately from the properties of Schur functions and from Proposition 6. In

[10] Due to (2.23) and reduction morphism ψ (2.28) we have to put for any a-m functional $f(\mathbb{K})=f(V_{(1^p)})=1$.

[11] Let us recall, that due to (2.23) all the partitions λ labelling the objects of our category has the height not greater than p therefore the right hand side of equation (3.3) is correctly defined for any object of SW(V).

order to show the relation (3.3) we should only check that once the quantities $f(V_\lambda)$, $\lambda=(1^k)$, are given then all quantities $f(V_\mu)$ for all other partitions are uniquely defined. This can be shown by induction in couples (m,k) where m is the number of columns in λ and k is the number of boxes in the last column. Namely, using the implication

$$f(V_\lambda\otimes V_\mu)=f(V_\lambda)f(V_\mu)=c_{\lambda\mu}^{\ \nu}f(V_\nu)\quad\Rightarrow\quad s_\lambda s_\mu=c_{\lambda\mu}^{\ \nu}f(V_\nu)$$

we can find the quantity $f(V_\lambda)$ for λ corresponding to a given couple (m,k) provided that all $f(V_\mu)$ where μ corresponds to the couples (l,r) such that $l<m$ or $l=m,\ r<k$ are already known. ■

In what follows we will use this proposition for a proper definition of the categorical trace. It will be defined as such a morphism (1.3) that the corresponding dimension becomes an a-m functional. By virtue of the above proposition we should only find the quantities (3.3) for $\lambda=(1^k)$.

Describe now this construction in detail. Begin with the basic object V. Since the space V is finite dimensional, we can identify $\mathrm{End}(V)$ and $V\otimes V^*$ in the usual way:

$$\forall\mathcal{F}\in\mathrm{End}(V)\quad\longleftrightarrow\quad e_i\otimes F^i_j e^j\in V\otimes V^*$$

F^i_j being a matrix of $\mathcal{F}$ in the basis $\{e_i\}$. Define the mapping $\mathrm{tr}_V:\ \mathrm{End}(V)\to\mathbb{K}$ as the following composition of morphisms:

$$\mathrm{tr}_V=\alpha\,\mathbf{ev}_V\circ\bar R_{V,V^*}. \tag{3.4}$$

Note, that tr_V is defined by the above relation up to an arbitrary nonzero factor α which will be specified later from the requirement that such a trace would be an a-m functional. In the fixed basis $\{e_i\}$ of V one can write this mapping in an explicit form. Taking $\bar R_{V,V^*}$ from Proposition 14 we find:

$$\forall\mathcal{F}\in\mathrm{End}(V):\quad e_i\to e_jF^j_i\quad\Rightarrow\quad\mathrm{tr}_V(\mathcal{F})=\alpha\,\mathrm{Tr}(F\cdot C) \tag{3.5}$$

where the symbol Tr means the usual matrix trace and the $n\times n$ matrix C is defined to be

$$C^i_j:=\sum_{a=1}^n Q^{ia}_{\ ja},\qquad Q\equiv(\pm q)^{-\frac{1}{p}}\bar Q.$$

Here the matrix $\bar Q$ is taken from (2.47) and the factor $(\pm q)^{\frac{1}{p}}$ is included into α.

The matrix C has the following useful properties [GPS]:

$$R_{12}C_1C_2=C_1C_2R_{12} \tag{3.6}$$

$$C^{|1\rangle}_{\langle 1'|}=\frac{p_q}{q^p}\,u_{\langle 1'23\dots p|}v^{|123\dots p\rangle}\quad\Rightarrow\quad\mathrm{Tr}\,C=\frac{p_q}{q^p}. \tag{3.7}$$

$$\mathrm{Tr}_{(2)}R_{12}C_2=\mathrm{id}_{(1)}. \tag{3.8}$$

The trace $\mathrm{tr}_{V^{\otimes k}}$ in $\mathrm{End}(V^{\otimes k})$ is defined (also up to an arbitrary factor $\beta_{(k)}$) as the mapping $V^{\otimes k}\otimes V^{*\otimes k}\to\mathbb{K}$:

$$\mathrm{tr}_{V^{\otimes k}}=\beta_{(k)}\,\mathrm{tr}^{(1)}_V\circ\mathrm{tr}^{(2)}_V\dots\circ\mathrm{tr}^{(k)}_V \tag{3.9}$$

where $\mathrm{tr}^{(i)}_V:\ V^{\otimes i}\otimes V^{*\otimes i}\to V^{\otimes(i-1)}\otimes V^{*\otimes(i-1)}$ reads

$$\mathrm{tr}^{(i)}_V=\mathrm{id}^{\otimes(i-1)}\otimes\alpha^{-1}\,\mathrm{tr}_V\otimes\mathrm{id}^{\otimes(i-1)}.$$

For any basic object V_λ $\lambda\vdash m$ one can find the explicit form of the trace in $\mathrm{End}(V_\lambda)\cong\mathrm{End}(V_\lambda(i))$ using definitions (2.11), (2.44) and property (3.6):

$$\forall\mathcal{F}\in\mathrm{End}(V_\lambda(i)):\quad\mathrm{tr}_{V_\lambda}(\mathcal{F})=\beta_{(m)}\,\mathrm{Tr}_{(1\dots m)}(F\cdot C_\lambda) \tag{3.10}$$

where $\mathrm{Tr}_{(1\dots m)}$ is the the usual trace in the matrix space $\mathrm{Mat}_{n\times n}^{\otimes m}(\mathbb{K})$ and the matrix C_λ is of the form:

$$C_\lambda = \mathcal{Y}_{ii}^\lambda(R_1\dots R_{m-1})C_1C_2\dots C_m. \tag{3.11}$$

Remark 17 *At the first sight the trace tr_{V_λ} depends on a concrete embedding $V_\lambda \to V_\lambda(i)$. But in fact our definition is invariant w.r.t. different embeddings of V_λ into $V^{\otimes m}$. Indeed, as was already noticed in the previous section, the primitive idempotents Y_{ii}^λ of the Hecke algebra H_m can be transformed into any other idempotent Y_{jj}^λ (with the same λ) with the help of an invertible element $X(i,j|\lambda)$ from H_m. The image $\mathcal{X} = \rho_R(X(i,j|\lambda))$ of the element $X(i,j|\lambda)$ under the representation ρ_R (2.1) represents a first kind morphism of SW(V) and according to (3.6) the string of matrices $C_1\dots C_m$ in (3.11) commute with $\mathcal{X}$. Therefore trace (3.10) is actually independent of the index i labelling the concrete embedding $V_\lambda \to V_\lambda(i)$.*

Find now the conditions on arbitrary factors α and $\beta_{(k)}$ which would guarantee the dimensions defined via the above trace to be an a-m functional.

Let us note that although the dimension defined by the usual trace is an a-m functional such a trace is not a morphism of the category if the braiding differs from the flip[12]. This is the reason why we should modify the usual trace by means of the matrix C (and all its extensions C_λ) multiplied by a proper factor α (3.5).

The fact that the matrix C extends to $\mathrm{End}(V^{\otimes m})$ in a "group like" way

$$C \;\to\; C^{\otimes m}$$

ensures the conservation of the additive and multiplicative structure of the functional (3.9) restricted on the identity operator (what is just categorical dimension) iff we put

$$\beta_{(k)} = \alpha^k. \tag{3.12}$$

So, if we want the trace tr_{V_λ} to be a morphism and giving rise to an a-m functional we have the only free parameter α at our disposal. This parameter (a normalization of the trace) can be found from the following condition $\dim_q V_{(1^p)} = 1$ (see footnote 10). The following lemma plays the key role in finding such a normalization.

Lemma 18 *For any Hecke symmetry of the rank p the following relation holds:*

$$\mathrm{Tr}_{(1\dots k)}(\mathcal{A}^{(k)}C_1\dots C_k) = q^{-pk}\frac{p_q!}{k_q!(p-k)_q!}, \quad 1\le k\le p \tag{3.13}$$

where the matrix $\mathcal{A}^{(k)}$ is the image of k-th order antisymmetrizer $A^{(k)}$ under the representation ρ_R (2.1).

Proof. For an arbitrary $m\in\mathbb{Z}$ introduce the auxiliary notation

$$\mathcal{R}_i(m) := R_i - \frac{q^m}{m_q}$$

where m_q is defined in (1.9). With this notation the matrix $\mathcal{A}^{(m)}(R_1,\dots,R_{m-1})$ can be presented in the form:

$$\mathcal{A}^{(m)} = \frac{(-1)^{m-1}}{m_q}\mathcal{A}^{(m-1)}\mathcal{R}_{m-1}(m-1)\mathcal{R}_{m-2}(m-2)\dots\mathcal{R}_1(1), \quad \mathcal{A}^{(1)} := \mathrm{id}.$$

[12] For example, if our category SW(V) is supplied with the comodule structure over the *RTT* Hopf algebra (2.14) then the usual trace is not compatible with the comodule structure.

For the sake of shortness we introduce the notation:

$$\mathrm{Tr}_{q\,(1\ldots k)}\, X := \mathrm{Tr}_{(1\ldots k)}\,(XC_1\ldots C_k).$$

Now we prove (3.13) by induction. For $k=1$ we get with the help of (3.7)

$$\mathrm{Tr}\, C = q^{-p} p_q$$

which gives the base of induction. Suppose that the assertion of the lemma is valid up to some integer $k<p$. Then using the simple relations

$$\begin{aligned}&\mathrm{Tr}_{q\,(k+1)}\mathcal{R}_k(k) = -q^{-p}\,\frac{(p-k)_q}{k_q}\,\mathrm{id}_{(k)}\\ &\mathcal{A}^{(k)}\mathcal{R}_{k-1}(k-1)\ldots\mathcal{R}_1(1) = (-1)^{k-1}k_q\,\mathcal{A}^{(k)}.\end{aligned}$$

one can complete the induction

$$\begin{aligned}\mathrm{Tr}_{q\,(1\ldots k+1)}\,\mathcal{A}^{(k+1)} &= \frac{(-1)^k}{(k+1)_q}\mathrm{Tr}_{q\,(1\ldots k+1)}\left(\mathcal{A}^{(k)}\mathcal{R}_k(k)\ldots\mathcal{R}_1(1)\right)\\ &= \frac{(-1)^k}{(k+1)_q}\mathrm{Tr}_{q\,(1\ldots k)}\left(\mathcal{A}^{(k)}\left[\mathrm{Tr}_{q\,(k+1)}\mathcal{R}_k(k)\right]\mathcal{R}_{k-1}(k-1)\ldots\mathcal{R}_1(1)\right)\\ &= (-1)^{k+1}\,q^{-p}\,\frac{(p-k)_q}{k_q(k+1)_q}\,\mathrm{Tr}_{q\,(1\ldots k)}\left(\mathcal{A}^{(k)}\mathcal{R}_{k-1}(k-1)\ldots\mathcal{R}_1(1)\right)\\ &= q^{-p}\frac{(p-k)_q}{(k+1)_q}\,\mathrm{Tr}_{q\,(1\ldots k)}\mathcal{A}^{(k)} = \frac{q^{-p(k+1)}\,p_q!}{(k+1)_q!(p-k-1)_q!}. \qquad\blacksquare\end{aligned}$$

This lemma implies that if we take $\alpha=q^p$ (which is equivalent to simple renormalization $C\to q^pC$ in all formulas for the trace) then we get $\dim_q(V_{(1^p)}=1$ and the categorical dimension related to the trace becomes an a-m functional well defined on objects of our category and hence on the corresponding Grothendieck ring.

Let us summarize the above consideration in the following proposition.

Proposition 19 *For any basic object V_λ, $\lambda\vdash m$, of the category SW(V) define the trace* tr_{V_λ} *as a morphism* $\mathrm{End}(V_\lambda)\to\mathbb{K}$ *of the form:*

$$\forall\,\mathcal{F}\in\mathrm{End}(V_\lambda):\quad \mathrm{tr}_{V_\lambda}(\mathcal{F}) = q^{pm}\,\mathrm{Tr}_{(1\ldots m)}(F\cdot C_\lambda).$$

Then such a trace is a morphism in the category SW(V) and the corresponding categorical dimension is an a-m functional. The dimension of any basic object V_λ is as follows:

$$\dim_q V_\lambda := \mathrm{tr}_{V_\lambda}(\mathrm{id}) = s_\lambda(q^{p-1},q^{p-3},\ldots,q^{1-p}) \tag{3.14}$$

s_λ being the Schur symmetric function (polynomial) in p variables.

Proof. To prove (3.14) we use Proposition 16 and Lemma 18. By virtue of the trace definition one gets from (3.13):

$$\dim_q V_{(1^k)} = q^{pk}\,\mathrm{Tr}_{q\,(1\ldots k)}\mathcal{A}^{(k)} = \frac{p_q!}{k_q!(p-k)_q!}\equiv f_k.$$

Therefore due to Proposition 16 we should find the roots of the polynomial $\phi(t)$ (3.2) with the above f_k. But as is well known the generating function for such f_k (which are q-binomial coefficients) is as follows

$$E_q(t) := \prod_{k=0}^{p-1}(q^{2k+1-p}+t),\quad f_{p-k} = \left[\frac{d^k}{dt^k}\,E_q(t)\right]_{t=0}\quad\Rightarrow\quad \phi(t)\equiv E_q(t).$$

Therefore the roots of the polynomial $\phi(t)$ with the coefficients $f_k = \dim_q V_{(1^k)}$ are the numbers $(-q^{2k+1-p})$, $k = 0, 1, \ldots, p-1$ and result (3.14) is now follows from Proposition 16. ■

Remark 20 *We want to complete the paper with the following observation concerning the Koszul complexes considered in [G]. As was shown there, the Poincaré series* $P_{\pm}(t)$

$$P_{\pm}(t) := \sum_l \dim \Lambda^l_{\pm} t^l$$

of "symmetric" and "skewsymmetric" algebras $\Lambda_{\pm}(V)$ *(see (1.10)) satisfy the relation*

$$P_+(t)\, P(-t) = 1. \tag{3.15}$$

We would like to point out that if we replace the usual dimensions in formula (3.15) by categorical ones relation (3.15) will be still valid. Moreover, the same is true if we replace the dimensions by the values of any a-m functional. It is not surprising since this fact reflects the well known relation between elementary and complete symmetric functions. And any a-m functional is nothing but a specialization of symmetric functions.

Thus, without constructing any deformed Koszul complex we can obtain some numerical characteristics of quantum objects merely replacing the usual dimensions by their q-analogs (if we disregard the property of the quantum differential to be a morphism of the category).

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