On the distribution of the order over residue classes

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Abstract

For a fixed rational number $g \notin \{-1, 0, 1\}$ and integers a and d we consider the set $N_g(a, d)$ of primes p such that the order of g modulo p is congruent to $a(\text{mod } d)$. Under the Generalized Riemann Hypothesis (GRH), it can be shown that the set $N_g(a, d)$ has a natural density $\delta_g(a, d)$. Arithmetical properties of $\delta_g(a, d)$ are described and $\delta_g(a, d)$ is compared with $\delta(a, d)$: the average density of elements in a field of prime characteristic having order congruent to $a(\text{mod } d)$. It transpires that $\delta_g(a, d)$ has a strong tendency to be equal to $\delta(a, d)$, or at least to be close to it.

1 Introduction

Let $g \notin \{-1, 0, 1\}$ be a rational number. For a rational number u, let $\nu_p(u)$ denote the exponent of p in the canonical factorisation of u (throughout the letter p will be used to indicate prime numbers). If $\nu_p(g) = 0$, then there exists a smallest positive integer k such that $g^k \equiv 1(\text{mod } p)$. We put $\text{ord}_p(g) = k$. The number k is the *(residual) order* of $g(\text{mod } p)$. We let $N_g(a_1, d_1; a_2, d_2)$ be the set of primes p with $\nu_p(g) = 0$, $p \equiv a_1(\text{mod } d_1)$ and $\text{ord}_p(g) \equiv a_2(\text{mod } d_2)$. By $N_g(a_1, d_1; a_2, d_2)(x)$ we denote the number of primes $p \le x$ in $N_g(a_1, d_1; a_2, d_2)$. For convenience $N_g(0, 1; a, d)(x)$ is denoted as $N_g(a, d)(x)$. By GRH we denote the Generalized Riemann Hypothesis. By (a, b) and $[a, b]$ we denote the greatest common divisor, respectively lowest common multiple of a and b.

Theorem 1 [M-2]. (GRH). *The density $\delta_g(a_1, d_1; a_2, d_2)$ of the set of primes $N_g(a_1, d_1; a_2, d_2)$ exists. Moreover,*

$$N_g(a_1, d_1; a_2, d_2)(x) = \delta_g(a_1, d_1; a_2, d_2)\frac{x}{\log x} + O_{g,d}\left(\frac{x}{\log^{3/2} x}\right).$$

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Our primary interest is in $\delta_g(a,d)=\delta_g(0,1;a,d)$, but in studying this quantity it turns out to be fruitful to consider $N_g(a_1,d_1;a_2,d_2)(x)$. Theorem 3 for example is obtained from Theorem 4. By $K_{s,r}$ (with $r|s$) we denote the number field $\mathbb{Q}(\zeta_s,g^{1/r})$, where $\zeta_s=\exp(2\pi i/s)$. The density $\delta_g(a_1,d_1;a_2,d_2)$ can be expressed in terms of the degrees $[K_{s,r}:\mathbb{Q}]$ and certain intersection coefficients (with $(b,f)=1$),

$$c_g(b,f,v)=\begin{cases}1 & \text{if } \sigma_b|_{\mathbb{Q}(\zeta_f)\cap K_{v,v}}= \text{identity};\\ 0 & \text{otherwise,}\end{cases}$$

where σ_b is the automorphism of $\mathbb{Q}(\zeta_f)$ that sends ζ_f to ζ_f^b.

Theorem 2 [M-2]. (GRH). *We have*

$$\delta_g(a,d)=\sum_{\substack{t=1\\(1+ta,d)=1}}^{\infty}\sum_{\substack{n=1\\(n,d)|a}}^{\infty}\frac{\mu(n)c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt}:\mathbb{Q}]}. \tag{1}$$

In the proofs of various results below the determination of the intersection coefficients $c_g(b,f,v)$ plays an important rôle. Obviously $\mathbb{Q}(\zeta_f)\cap K_{v,v}$ is a subfield of the maximal abelian subfield, $K_{v,v}^{\rm ab}$, of $K_{v,v}$. It turns out, see [M-3], that $K_{v,v}^{\rm ab}$ is of the form $\mathbb{Q}(\zeta_v,\sqrt{\gamma})$ or of the form $\mathbb{Q}(\zeta_v,\zeta_{2v}\sqrt{\gamma})$ for some integer γ that can be explicitly given, where the latter case does not arise if $g>0$. Of course, the action of σ_b on $\mathbb{Q}(\zeta_f)\cap K_{v,v}=\mathbb{Q}(\zeta_f)\cap K_{v,v}^{\rm ab}$, which equals $\mathbb{Q}(\zeta_f)\cap\mathbb{Q}(\zeta_v,\sqrt{\gamma})$ or $\mathbb{Q}(\zeta_f)\cap\mathbb{Q}(\zeta_v,\zeta_{2v}\sqrt{\gamma})$, is easily determined.

The distribution of the order over congruence classes in case $d\nmid a$ seems to have been first studied by Chinen and Murata [CM] for $d=4$. In case $d|a$ the problem is much easier and unconditional results have been obtained, see [M-0, W-1, W-2]. In this case the density is always a rational number. Chinen and Murata restricted themselves to the case where g is positive and not a power of an integer. In their method $\delta_g(a,4)$ (for a is odd) is initially expressed as the sum of two fourfold sums. On making everything explicit, they obtained a long formula (distinguishing six cases) for $\delta_g(a,4)$ which was subsequently simplified by Zagier [Z]. The author expressed $\delta_g(a,4)$ as a single sum (see Theorem 4), which on evaluation gives a compact formula for $\delta_g(a,4)$ similar to Zagier's. In this formula an Euler product A_ψ appears. We put, for any Dirichlet character χ,

$$A_\chi=\prod_{\substack{p\\ \chi(p)\neq 0}}\left(1+\frac{[\chi(p)-1]p}{[p^2-\chi(p)](p-1)}\right).$$

The constants A_χ turn out to be rather basic in this setting (cf. Theorem 11). A table of numerical values of A_χ, with χ a Dirichlet character of modulus ≤ 12, is given in [M-0]. We let $\mathcal{G}$ be the set of rational integers that can not be written as $-g_0^h$ or g_0^h with $h>1$ an integer and g_0 a rational number. Note that almost all integers g are elements of $\mathcal{G}$.

Theorem 3 [M-1]. (GRH). *Let $D(g)$ denote the discriminant of the field $\mathbb{Q}(\sqrt{g})$. Let $g\in\mathcal{G}$. Then $\delta_g(\pm1,4)=1/6$ unless $D(g)$ is divisible by 8 and has no prime* *divisor congruent to* $1 (\mathrm{mod}\ 4)$*, in which case we have*

$$\delta_g(\pm 1,4)=\begin{cases}\frac{1}{6}\mp \mathrm{sgn}(g)\frac{A_\psi}{8}\prod_{p|\frac{D(g)}{8}}\frac{2p}{p^3-p^2-p-1} & \text{if } D(g)\neq \pm 8;\\ \frac{7}{48}\mp \mathrm{sgn}(g)\frac{A_\psi}{8} & \text{if } D(g)=\pm 8\end{cases},$$

where ψ *denotes the non-trivial character mod* 4.

`Remark 1`. We have, on invoking Theorem 12, $A_\psi = 0.643650679662525\cdots$.
`Remark 2`. Let $\epsilon_1(n)=1$ if $8|n$ and 0 otherwise. Then we can write, using e.g., Theorem 2 of Moree [M-0], for $2\nmid a$:

$$\delta_g(a,4)=\frac{\delta_g(1,2)}{2}+\epsilon_1(D(g))\mathrm{sgn}(g)A_\psi\frac{(-1)^{\frac{a+1}{2}}}{8}\prod_{p|\frac{D(g)}{8}}\frac{(1-\psi(p))p}{p^3-p^2-p-1}.$$

An explicit expression for $\delta_g(a,q^s)$ (in terms of A_χ's) with q a prime and $g\in\mathcal{G}$ is obtained in [M-2], but is omitted here for reasons of space.

Theorem 3 can be obtained from the following result (with $s=2$):

Theorem 4 [M-1]. (GRH). *We have, for a odd and* $s\geq 1$,

$$N_g(1,2^s;a,4)(x)=\frac{\delta_g(1,2^s;1,2)}{2}\frac{x}{\log x}+O_g(\frac{x}{\log^{3/2}x}),$$

and $N_g(3,4;a,4)(x)=\#\{p\leq x: p\equiv 3(\mathrm{mod}\ 4), (\frac{g}{p})=1\}/2$

$$+(-1)^{\frac{a-1}{2}}\frac{\Delta_g}{4}\frac{x}{\log x}+O_g(\frac{x}{\log^{3/2}x}),$$

where

$$\Delta_g=\sum_{\substack{\sqrt{-2}\in K_{2v,2v}\\ 2\nmid v}}\frac{h_\psi(v)}{[K_{2v,2v}:\mathbb{Q}]}-\sum_{\substack{\sqrt{2}\in K_{2v,2v}\\ 2\nmid v}}\frac{h_\psi(v)}{[K_{2v,2v}:\mathbb{Q}]},$$

with h_ψ *the Dirichlet convolution of* ψ *and the Möbius function, i.e.,* $h_\psi(n)=\sum_{d|n}\psi(d)\mu(n/d)$.

Remarkably, despite its arithmetic complexity $\delta_g(3,4;a,4)$ satisfies some easy properties.

Theorem 5 [M-1]. (GRH). *Write* $g=\pm g_0^h$*, where* g_0 *is positive and not an exact power of a rational number.*
1) *If* $g>0$ *and* h *is even, then* $\delta_g(2,3;1,3)\leq\delta_g(2,3;2,3)$*, otherwise* $\delta_g(2,3;1,3)\geq\delta_g(2,3;2,3)$*. We have equality iff* $\mathbb{Q}(\sqrt{g_0})=\mathbb{Q}(\sqrt{3})$ *and* $\nu_2(h)\in\{0,2\}$*. The same result holds with* $\delta_g(2,3;*,3)$ *replaced by* $\delta_g(*,3)$.
2) *If* $\delta_g(3,4;3,4)\neq\delta_g(3,4;1,4)$*, then* $\mathrm{sgn}(\delta_g(3,4;3,4)-\delta_g(3,4;1,4))=\mathrm{sgn}(g)$.
3) *If* $g\in\mathcal{G}$ *and* $2\nmid a$*, then* $\delta_g(3,4;a,4)+\delta_{-g}(3,4;a,4)=1/4$.

Let $\delta(p;a,d)$ denote the density of elements in $\mathbb{F}_p^*$ having order congruent to $a(\mathrm{mod}\ d)$. It is not so difficult to show that the average density $\delta(a,d)$ of elements of order congruent to $a(\mathrm{mod}\ d)$ in a field of prime characteristic exists. I.e., we have $\lim_{x\to\infty}\sum_{p\leq x}\delta(p;a,d)/\pi(x)=\delta(a,d)$, where $\pi(x)$ denotes the number of primes $p\leq x$. The quantity $\delta(a,d)$ can be studied by fairly elementary methods, but nevertheless turns out to exhibit behaviour similar to $\delta_g(a,d)$. An interpretation of $\delta(a,d)$ is that it is the g-average of $\delta_g(a,d)$:

Theorem 6 [M-3]. (GRH). *We have*

$$\frac{1}{2x}\sum_{|g|\le x}\delta_g(a,d)=\delta(a,d)+O(\frac{1}{\sqrt{x}}).$$

In some sense, if one takes out the Galois theory and degree aspects of formula (1), one obtains $\delta(a,d)$. More precisely, if one sets $c_g(1+ta,dt,nt)=1$ and $[K_{[d,n]t,nt}:\mathbb{Q}]=\varphi([d,n]t)nt$ (this is the maximal degree possible), then it can be shown that one obtains $\delta(a,d)$ out of $\delta_g(a,d)$ ([M-Av]). One has $\delta(\text{odd},4)=1/6$. It is not difficult to prove that for most integers g with $|g|\le x$ we have that $D(g)$ has a prime divisor congruent to $1(\text{mod }4)$. (Indeed, the size of the exceptional set is bounded above by $\ll_g x/\sqrt{\log x}$). Thus from Theorem 3 we infer that for almost all integers g with $|g|\le x$ we have, on GRH, $\delta_g(\text{odd},4)=\delta(\text{odd},4)=1/6$. If $\delta_g(a,4)\neq\delta(a,4)$, then the difference will be small in absolute value as is also obvious from Theorem 3. It turns out that these phenomena hold true in general. In case d equals a prime power it is still possible to write down an explicit formula for the density $\delta_g(a,d)$ from which the latter two properties can be similarly inferred [M-2]. For general d this seems to be difficult. Nevertheless, the following two results can be proved (where $k(d)=\prod_{p|d}p$ is the squarefree kernel of d):

Theorem 7 [M-3]. (GRH). *Let d be fixed. There are at most $O_d(x\log^{-1/\varphi(k_1(d))}x)$ integers g with $|g|\le x$ such that $\delta_g(a,d)\neq\delta(a,d)$ for some integer a. In particular,*

$$(\delta_g(0,d),\ldots,\delta_g(d-1,d))=(\delta(0,d),\ldots,\delta(d-1,d))$$

for almost all integers g, where

$$k_1(d)=\begin{cases}k(d) & \text{if } d \text{ is odd;}\\ 4k(d) & \text{otherwise,}\end{cases}\quad\text{and } k_2(d)=\begin{cases}k(d) & \text{if } d \text{ is odd;}\\ (4,d/2)k(d) & \text{otherwise.}\end{cases}$$

Theorem 8 [M-3]. (GRH). *Suppose that $g\in\mathcal{G}$. Set $D_1=|D(g)/(D(g),d)|$. Then*

$$\Big|\delta_g(a,d)-\delta(a,d)\Big|<\frac{3\cdot 2^{\omega(D_1)+2}}{\varphi(D_1)D_1},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n.

The following basic result reduces the study of $\delta_g(a,d)$ to that of $\delta_g(a,k_2(d))$.

Theorem 9 [M-3]. (GRH).
1) *If q is an odd prime dividing d_1, then $\delta_g(a,qd_1)=\delta_g(a,d_1)/q$.*
2) *If $8|d_1$, then $\delta_g(a,2d_1)=\delta_g(a,d_1)/2$.*
That is, we have $\delta_g(a,d)=\delta_g(a,k_2(d))k_2(d)/d$.

It is easy to see that $\delta(a,d)$ satisfies a similar and slightly stronger property: $\delta(a,d)=\delta(a,k(d))k(d)/d$.

In Theorem 4 it is seen that there is a difference in behaviour of $\text{ord}_p(g)$ when p is restricted to those primes with $p\equiv 1(\text{mod }4)$, respectively $p\equiv 3(\text{mod }4)$. A similar phenomenon (having a Galois theoretic explanation), is seen to hold in general.

Theorem 10 [M-3]. (GRH). *Suppose that* $(a,d)=(b,d)=1$.
1) *If* d *is odd, then* $\delta_g(1,k(d);a,d)=\delta_g(1,k(d);b,d)$.
2) *If* d *is even, then* $\delta_g(1,2k(d);a,d)=\delta_g(1,2k(d);b,d)$.

On the other hand, if $(a,d)\neq(b,d)$ then it seems that rarely $\delta_g(a,d)=\delta_g(b,d)$, cf. Theorem 3 with the first part of Theorem 4.

2 On the computation of $\delta_g(a,d)$

As in Theorem 3, in general $\delta_g(a,d)$ can be expressed in terms of linear combinations of the constants A_χ with coefficients coming from certain cyclotomic fields:

Theorem 11 [M-3]. (GRH). *Let* a *and* d *be arbitrary natural numbers. Then there exists an integer* $d_1|k_1(d)$ *such that*

$$\delta_g(a,d)=\sum_{\chi\in G_{d_1}} c_\chi A_\chi \text{ with } c_\chi\in\mathbb{Q}(\zeta_{o_\chi}),$$

where c_χ *can be explicitly computed,* G_{d_1} *denotes the group of Dirichlet characters modulo* d_1 *and* o_χ *the order of* χ *in* G_{d_1}.

The following result allows one to evaluate the constants A_χ easily with ten decimal digit precision and hence, by Theorem 11, the density $\delta_g(a,d)$.

Theorem 12 [M-Av]. *Let* $p_1(=2),p_2,\ldots$ *be the sequence of consecutive primes. Let* χ *be any Dirichlet character and* $n\geq 31$ *(hence* $p_n\geq 127$*). Then*

$$B_\chi=AL(2,\chi)L(3,\chi)L(4,\chi)S(n)R_1,$$

$$S(n)=\prod_{k=1}^{n}\left(1+\frac{\chi(p_k)}{p_k(p_k^2-p_k-1)}\right)\left(1-\frac{\chi(p_k)}{p_k^3}\right)\left(1-\frac{\chi(p_k)}{p_k^4}\right),$$

$$B_\chi=\prod_p\left(1+\frac{[\chi(p)-1]p}{[p^2-\chi(p)](p-1)}\right)=A_\chi\prod_{p|d}\left(1-\frac{1}{p(p-1)}\right),$$

$$A=\prod_p\left(1-\frac{1}{p(p-1)}\right)=0.3739558136\cdots,\text{ and }\frac{1}{1+p_{n+1}^{-3.85}}\leq|R_1|\leq 1+\frac{1}{p_{n+1}^{3.85}}.$$

The factor $AL(2,\chi)L(3,\chi)L(4,\chi)$ in the latter result is the beginning of an expansion of B_χ in terms of special values of L-series:

Theorem 13 [M-Av]. *One has*

$$B_\chi=A\frac{L(2,\chi)L(3,\chi)}{L(6,\chi^2)}\prod_{r=1}^{\infty}\prod_{k=3r+1}^{\infty}L(k,\chi^r)^{\lambda(k,r)},\text{ with }\lambda(k,r)\in\mathbb{Z}.$$

This formula can be used to approximate B_χ (and thus A_χ) with even higher numerical precision. The integers $\lambda(k,r)$ are related to so-called convoluted Fibonacci numbers, see [M-Fi], and exhibit certain monotonicity properties in both the k and r direction ([M-Fi]). These monotonicity properties are valid in a much more general setting, see [M-Wi].

3 On similar results for the index

The index, $[(\mathbb{Z}/p\mathbb{Z})^* : \langle g(\text{mod } p)\rangle]$, of the subgroup generated by $g(\text{mod } p)$ inside the multiplicative group of residues mod p, is denoted by $r_p(g)$ and called the *(residual) index* mod p of g. For this quantity similar questions can be asked with $\text{ord}_p(g)$ replaced by $r_p(g)$. The results under this replacement turn out to be rather similar, see [M-1, M-2, M-3, P], however, they are much easier to establish. A reason for this is that in the latter case intersection coefficients do not appear. It was in this context that the constants A_χ were introduced by Pappalardi [P].

4 On the proofs of the results

For reasons of space we can only give a small sample here. We sketch the proof of Theorem 2.

Sketch of proof of Theorem 2. On noting that $r_p(g)\text{ord}_p(g) = p - 1$ we obtain that $N_g(a,d)(x) = \sum_{t=1}^{\infty} V_g(a,d;t)(x)$, where

$$V_g(a,d;t)(x) := \#\{p \le x \ : \ r_p(g) = t,\ p \equiv 1 + ta(\text{mod } dt)\}.$$

In this infinite sum the terms with $t \ge \sqrt{\log x}$ are less easily individually computed, but since they are small they can be taken together to form an error term, which can be estimated by $O(x \log^{-3/2} x)$. If $(1+ta, d) > 1$, then there is at most one prime counted by $V_g(a,d;t)(x)$ and this prime has to divide d. In this way one obtains that

$$N_g(a,d)(x) = \sum_{t \le \sqrt{\log x},\ (1+ta,d)=1} V_g(a,d;t)(x) + O(\frac{x}{\log^{3/2} x}). \tag{2}$$

Note that $V_g(a,d;1)(x)$ counts the number of primes $p \equiv 1 + a(\text{mod } d)$ such that, moreover, g is a primitive root modulo p. This function, and indeed $V_g(a,d;t)(x)$, can be estimated by a variation of Hooley's classical argument [H]. However, we need to carry this out with a certain uniformity in t which forces us to keep track of the dependence on t of the various estimates. Furthermore, as will be explained shortly, the additional condition $p \equiv 1 + ta(\text{mod } dt)$ is responsible for bringing in the Galois theoretic intersection coefficients $c_g(1+ta, dt, nt)$. By inclusion and exclusion we find that

$$V_g(a,d;t)(x) = \sum_{n=1}^{\infty} \mu(n)\#\{p \le x : p \equiv 1 + ta(\text{mod } dt),\ nt | r_g(p)\}. \tag{3}$$

The counting functions in the latter sum can be estimated by an effective form of Chebotarev's density theorem, cf. Theorem 3 of [M-1] and the discussion immediately following that theorem. Namely, we are interested in those primes $p \equiv 1 + at(\text{mod } dt)$ that split completely in $K_{nt,nt} := \mathbb{Q}(\zeta_{nt}, g^{1/nt})$. These primes must have a Frobenius σ in $K_{[n,d]t,nt}$ with the property that $\sigma|_{\mathbb{Q}(\zeta_{dt})} = \sigma_{1+ta}$ and $\sigma|_{K_{nt,nt}}$ =id. If such a σ exists then certainly we must have $\sigma_{1+ta}|_{\mathbb{Q}(\zeta_{dt}) \cap K_{nt,nt}}$ =id, i.e. $c_g(1+ta, dt, nt) = 1$. Indeed, such a σ turns out to exist iff $c_g(1+ta, dt, nt) =$

1. On applying Chebotarev's density theorem one then finds, assuming the Riemann Hypothesis (RH) holds for the field $K_{[d,n]t,nt}$, that

$$\#\{p \le x : p \equiv 1 + ta (\text{mod } dt),\ nt | r_g(p)\} = \frac{c_g(1+ta, dt, nt)}{[K_{[d,n]t,nt}]}\mathrm{Li}(x) + O(\sqrt{x}\log x).$$

Again there is a problem with the tail in the series in (3), but again it can be reasonably estimated and one obtains that

$$V_g(a, d; t)(x) = \sum_{P(n) \le (\log x)/6} \mu(n)\#\{p \le x : p \equiv 1 + ta (\text{mod } dt),\ nt | r_g(p)\} + E(x),$$

where $P(n)$ denotes the greatest prime factor of n and $E(x)$ is the estimate for the tail. On combining the latter two displayed estimates, one then arrives at a usable estimate for $V_g(a, d; t)(x)$. On combining this with (2), the proof of Theorem 2 is then easily completed. $\square$

Working with the sharpest known unconditional version of Chebotarev's density theorem leads to an error term which is too weak for our purposes. Actually, as is clear from the above sketch it is not required to assume GRH. It suffices to assume RH for the number fields involved in the proof. So for Theorem 2 it suffices to require RH for the number fields $K_{[d,n]t,nt}$ with n squarefree, $(n, d)|a$ and $(1 + ta, d) = 1$.

5 Numerical experiments

The problem considered here allows for numerical experiments. We give here a small sample of data so obtained. In the cases studied, the numerics seemed to agree well with the theoretical predictions.

Table 1: Experimental and theoretical densities for $d = 5$

$\delta\backslash a$	0	1	2	3	4
$\delta(*, 5)$	$0.20833\underline{3}$	$0.23542\underline{1}$	$0.17799\underline{3}$	$0.23400\overline{3}$	$0.14424\overline{8}$
$\approx \delta_{-11}(*, 5)$	0.208347	0.235422	0.178007	0.233974	0.144250
$\delta_{-11}(*, 5)$	$\delta(0, 5)$	$\delta(1, 5)$	$\delta(2, 5)$	$\delta(3, 5)$	$\delta(4, 5)$
$\approx \delta_{-5}(*, 5)$	0.208348	0.264146	0.194858	0.233282	0.099365
$\delta_{-5}(*, 5)$	$\delta(0, 5)$	$0.26413\underline{5}$	$0.19486\overline{5}$	$0.23329\overline{4}$	$0.09937\underline{1}$
$\approx \delta_2(*, 5)$	0.208333	0.240673	0.178706	0.229270	0.143017
$\delta_2(*, 5)$	$\delta(0, 5)$	$0.24068\overline{1}$	$0.17869\overline{1}$	$0.22926\underline{4}$	$0.14302\underline{9}$
$\approx \delta_5(*, 5)$	0.208348	0.232581	0.292840	0.054488	0.211742
$\delta_5(*, 5)$	$\delta(0, 5)$	$0.23258\overline{5}$	$0.29284\overline{8}$	$0.05449\overline{3}$	$0.21173\overline{7}$

If an entry is in a row labelled $\approx \delta_g(*, 5)$ and in column a, then the number given equals $N_g(a, 5)(x)/\pi(x)$ rounded to 6 decimals with $x = 2038074743$ (and hence $\pi(x) = 10^8$). The theoretical values are given with 6 digit precision, with a bar over the last digit indicating that if the number is to be rounded off, it should be rounded upwards. The density $\delta(0, 5) = 5/24$ (unconditional result).

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