ON THE LOOP INEQUALITY FOR EUCLIDEAN BUILDINGS

Jacek Świątkowski

Instytut Matematyczny
Uniwersytet Wrockawski
Pl. Grunwaldzki 2/4
50-384 Wrockaw
POLAND

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI/95-127

ON THE LOOP INEQUALITY FOR EUCLIDEAN BUILDINGS

JACEK ŚWIĄTKOWSKI

The geometric study of euclidean buildings from the point of view presented in this paper, is motivated by the idea of extending the results in analysis on discrete groups to groups which act on buildings. The interesting examples of such groups, acting simply transitively on sets of vertices of euclidean buildings of type $\tilde{A}_n$, were discovered recently (see [Cartwright]). Since the Cayley graphs of those groups, with respect to natural sets of generators, coincide with 1–skeletons of buildings, the geometry of those 1–skeletons strongly influences analytical properties of the corresponding groups. The results of this paper will be applied in above spirit in the forthcoming joint paper with Alain Valette.

I would like to express my thanks to Alain Valette, for hospitality during my visit in Universite de Neuchatel, and for involving me into the subject. I also acknowledge the hospitality of Max–Planck–Institut für Mathematik in Bonn, where the essential part of this paper was written.

§0. Formulation of main results.

0.1. In this paper we deal with buildings Δ satisfying the following two properties:

(0.1.1) Δ is *euclidean*, i.e. its apartments are euclidean Coxeter complexes;

(0.1.2) Δ is *uniformly thick*, i.e. there is $q \in \mathbf{N}$ (called *thickness* of Δ), $q \geq 2$, such that each face of codim 1 in Δ is contained in exactly $q+1$ chambers.

The main reference book for buildings and other notions related to them, their properties, as well as the notation used in the paper, is [Tits].

0.2. Denote by $\Delta^{(i)}$ the i–skeleton of Δ. A *polygonal curve* in $\Delta^{(1)}$ is a sequence $(v_0, v_1, ..., v_n)$ of vertices $v_i \in \Delta^{(0)}$, such that (v_i, v_{i+1}) is an edge in $\Delta^{(1)}$ for $i = 0, 1, ..., n-1$. Number n is called the *lenght* of a polygonal curve. For $v_1, v_2 \in \Delta^{(0)}$ define $d_\Delta(v_1, v_2)$ to be the minimal lenght of a polygonal curve in $\Delta^{(1)}$ joining v_1 with v_2. Then d_Δ is a metric on $\Delta^{(0)}$.

0.3. A *ball* of nadius $N \in \mathbf{N}$ and center $v \in \Delta^{(0)}$ is the set

$$B_N(v) := \{w \in \Delta^{(0)} : d_\Delta(v, w) \leq N\}.$$

Similarly, we define a *sphere* in Δ by

$$S_N(v) := \{w \in \Delta^{(0)} :\ d_\Delta(v,w) = N\}.$$

A *norm* of the ball $B_N(v)$ is the number

$$\|B_N(v)\|_r := \sum_{w \in B_N(v)} (1 + d_\Delta(v,w))^r,$$

where $r \in \mathbf{N}$.

0.4. Fix $v_0 \in \Delta^{(0)}$ and numbers $N, k \in \mathbf{N}$. An *N–loop* of length k in (Δ, v_0) is a sequence $(v_0, v_1, ..., v_k)$ of vertices of $\Delta^{(0)}$ such that $d(v_i, v_{i+1}) \le N$ for $i = 0, 1, ..., k-1$, and $v_k = v_0$.

0.5. Theorem. Let Δ be a euclidean, uniformly thick building, and v_0 its vertex. Then there exist constants $C > 0$ and $r \in \mathbf{N}$ such that

$$\#\{N\text{–loops of length } 2k \text{ in } (\Delta, v_0)\} \le [C \cdot N^r \cdot \#\, B_N(v_0)]^k\ .$$

0.6. Theorem. Let Δ be a uniformly thick euclidean building of type $\widetilde{A}_n$, and v_0 its vertex. Then there exist constants $C' > 0$ and $r' \in \mathbf{N}$, such that

(a) $\#\{\text{strict } N\text{–loops of length } 2k \text{ in } (\Delta, v_0)\} \le [C' \cdot N^{r'} \cdot \#\, S_N(v_0)]^k$, where the N–loop $(v_0, ..., v_k)$ is *strict* if $d_\Delta(v_i, v_{i+1}) = N$ for $i = 0, 1, ..., k-1$.

(b) $\#\{N\text{–loops of length } 2k \text{ in } (\Delta, v_0)\} \le [C' \cdot \|B_N(v_0)\|_r']^k$.

Building of type $\widetilde{A}_n$ is a one which appartments are Coxeter complexes of type $\widetilde{A}_n$. See [Tits], [Brown] or [Bourbaki] for more details.

0.7. Remark. Define an *N–path* of length k in (Δ, v_0) by omitting the condition $v_k = v_0$ in definition 0.4. Then it follows from lemma 1.13.(b) that

$$(\frac{1}{V} \cdot \#\, B_N(v_0))^{2k} \le \#\{N\text{-paths of length } 2k \text{ in } (\Delta, v_0)\} \le (V \cdot \#\, B_N(v_0))^{2k}\ .$$

Since in our situation of thick euclidean building the number $\#\, B_N(v_0)$ grows exponentially with N, we can read the theorem in the following rough way:

> The number of N–loops in Δ is "not much bigger" than the square root of the number of N–paths in Δ, of the same length.

§1. Properties of buildings.

Definitions and natations.

1.1. Let Δ be a building and $cham\,\Delta$ the set of all its chambers. A *gallery* in Δ is a sequence $\gamma = (C_0, \sigma_0, C_1, \sigma_1, ..., C_{n-1}, \sigma_{n-1}, C_n)$ such that $C_i \in cham\,\Delta$ for $i = 0, 1, , ..., n,$

and σ_i is a codim 1 face in both C_i and C_{i+1} for $i = 0, 1, ..., n-1$. The number n is called the *length* of gallery γ. We say that γ *starts* at C_0 and *ends* with C_n, or that γ is a gallery *between* C_0 and C_n. We say that γ *stammers* at C_i, if $C_i = C_{i+1}$.

We shall write shortly $\gamma = (C_0, C_1, ..., C_n)$ if it doesn't lead to a confusion.

1.2. Let $C, D \in cham\,\Delta$. Put $dist_\Delta(C, D)$ to be the minimal length of gallery in Δ between C and D. Then $dist_\Delta$ is a metric on the set $cham\,\Delta$.

A *minimal* gallery between C and D is one (in general non unique) with length equal to $\mathrm{dist}_\Delta(C, D)$.

1.3. For $v, w \in \Delta^{(0)}$ put

$$\varepsilon_\Delta(v, w) = \min\{N : \forall C \in cham\,\Delta \cap st(v)\ \exists D \in cham\,\Delta \cap st(w) : dist_\Delta(C, D) \leq N\},$$

where $st(v) = st_\Delta(v) := \{C \in cham\,\Delta : v \in C\}$ is the *star* of vertex v.

Note that ε_Δ is not symmetric in general, but it satisfies the triangle inequality

$$\varepsilon_\Delta(u, w) \leq \varepsilon_\Delta(u, v) + \varepsilon_\Delta(v, w)\ . \tag{1.3.1}$$

1.4. Fix an apartment Σ in Δ, and a chamber C of Σ. Define the function $\lambda = \lambda_C : Cham\,\Delta \to \mathbf{N} \cup \{0\}$ by $\lambda(D) := dist_\Delta(C, D)$. Define then a *folding map* $\varphi = \varphi_{\Sigma,C} : \Delta \to \Sigma$ to be the unique chamber map such that:

$$\varphi|_\Sigma = id_\Sigma\ ; \tag{1.4.1}$$

$$\lambda(D) = \lambda(\varphi(D)) \text{ for any } D \in cham\,\Delta\ . \tag{1.4.2}$$

For the existence and uniqueness of the folding map defined as above, see [Tits] theorem 3.3, page 42, where it is called retraction.

Statement of main results of the paragraph.

From now on Δ will always denote a euclidean, uniformly thick building, as defined in 0.1. We formulate five propositions which will be used in next paragraph in the proofs of Theorems 0.5 and 0.6. We give the proofs of Propositions 1.5 – 1.7 in the last part of this Section, while the proofs of propositions 1.8 and 1.9 are left to §3 and §4.

1.5. Proposition. There is a constant $V \in \mathbf{N}$ such that for any $v_0, v, w \in \Delta^{(0)}$, with $d_\Delta(v, w) \leq N$ we have $q^{\varepsilon_\Delta(v,w)} \leq V \cdot (\# B_N(v_0))$.

1.6. Proposition. Under notation of 1.4, let $v \in \Delta^{(0)}$ and γ_1, γ_2 be two minimal galleries in Σ between chamber C_0 such that $\varphi(v) \in C_0$, and chamber C_n. Let Γ_i, for $i = 1, 2$, denotes the family of all galleries η in Δ such that $\varphi(\eta) = \gamma_i$ and η starts with a chamber containing v. Denote by $E(\eta)$ the ending chamber of the gallery η. Then

$$\{E(\eta) : \eta \in \Gamma_1\} = \{E(\eta) : \eta \in \Gamma_2\}\ .$$

1.7. Proposition. Under notation of 1.4, let $v \in \Delta^{(0)}$ and γ be a gallery in Σ which starts at a chamber C_0 such that $\varphi(v) \in C_0$, ends with a chamber C_n, and stammers L times. Let Γ be a family of all galeries η in Δ such that $\varphi(\eta) = \gamma$ and η starts with a chamber containing v. Then

$$\#\Gamma \leq A \cdot q^{[n+\lambda(C_n)-\lambda(C_0)+L]/2} ,$$

where $A > 0$ is a constant depending only on Δ.

1.8. Proposition. Under notation of 1.4, there is a natural number M depending only on Δ, such that the critical locus of the map $\varphi|_{\Sigma'} : \Sigma' \to \Sigma$, for any apartment Σ' in Δ, is contained in not more than M hyperplanes of Σ'.

(*Critical locus* is the set of these codim 1 faces σ of Σ', for which both chambers in Σ' containing σ are mapped by φ on to the same chamber of Σ.)

1.9. Proposition. For any uniformly thick euclidean building Δ of type $\widetilde{A}_n$, there are constants $C > 0$ and $d \in \mathbf{N}$, such that for any $v \in \Delta^{(0)}$ and any natural N

$$\# B_N(v) \leq C \cdot (1+N)^d \cdot \# S_N(v) .$$

This last Proposition will be used in the proof of Theorem 0.6.

Proofs of propositions 1.5 – 1.7.

1.10. Lemma. Let Δ, Σ, φ and λ be as in 1.4.
(a) If D_1, D_2 are adjacent chambers in Σ, then $\lambda(D_1) - \lambda(D_2) = \pm 1$.
(b) Let σ be a codim 1 face in Δ, and let D_0 be one of $q+1$ chambers containing σ, for which the value $\lambda(D_0)$ is minimal. Then $\lambda(D) = \lambda(D_0) + 1$ for all other q chambers containing σ. In particuliar, $\varphi(D)$ is the chamber in Σ adjacent to $\varphi(D_0)$ by $\varphi(\sigma)$, for all other q chambers containing σ.

Proof of (a): Note that $\lambda(D)$ is the number of hyperplanes in Σ that separate chamber C from D. If chambers D_1, D_2 are adjacent, the unique hyperplane containing their common face separates exactly one of them from C.

Proof of (b): If Σ' is an apartment containing C and D_0, denote by E the chamber in Σ' adjacent to D_0 by face σ. Then $\lambda(E) = \lambda(D_0) + 1$. Suppose that $\lambda(D) = \lambda(D_0)$ for some other chamber D in Δ containing σ. Then, there exists a minimal gallery from E to C passing throngh D. Since $C, E \in \Sigma'$, any minimal gallery joining them must be contained in Σ'. But $D \notin \Sigma'$, contradiction.

The last sentence in statement of (b) is a consequence of fact that the folding map φ preserves values of function λ.

1.11. Lemma. Let Δ, Σ, C and φ be as in 1.4, v be a vertex of C and w a vertex of Σ. Then

(a) for any $w' \in \varphi^{-1}(w) \quad d_\Delta(v,w') = d_\Sigma(v,w)$;
(b) $\#\varphi^{-1}(w) = q^k$, where $k = \min\{\lambda(D) : D$ is a chamber of Σ adjacent to vertex $w\}$;
(c) $q^{\varepsilon_\Sigma(v,w)} \leq \#\varphi^{-1}(w)$.

Proof of (a): Since we can push any polygonal curve in $\Delta^{(1)}$ by φ to $\Sigma^{(1)}$, we have $d_\Sigma(v,w) \leq d_\Delta(v,w')$.

Conversely, let Σ' be an apartment in Δ containing C and w'; then by the property of the folding map, $\varphi|_{\Sigma'} : \Sigma' \to \Sigma$ is an isomorphism (compare [Tits] 3.3, page 42). Thus we can lift any polygonal curve from $\Sigma^{(1)}$ to $\Sigma'^{(1)}$, getting $d_\Delta(v,w') \leq d_\Sigma(v,w)$.

Proof of (b): Put k as in the statement of part (b). Let $\gamma = (C_0, C_1, ..., C_k)$, with $C_0 = C$, be a minimal gallery joining chamber C with a chamber C_k adjacent to w, for which the value of function λ is smallest among chambers of $st(w)$. Then for any $w' \in \varphi^{-1}(w)$ there is a gallery $\gamma' = (C'_0, ..., C'_k)$ with $C'_0 = C, w' \in C'_k$, lifted from γ by φ, i.e. such that $\varphi(\gamma') = \gamma$, or more precisely $\varphi(C'_i) = C_i$ for $i = 0, 1, ..., k$. For distinct vertices of $\varphi^{-1}(w)$ the corresponding galleries are distinct, and vice versa. Since, by lemma 1.10.(b), there are exactly q^k of such lifted galleries, part (b) of the Lemma follows.

Proof of (c): Part (c) is a direct consequence of part (b) and the fact that $\varepsilon_\Sigma(v,w) \geq k$.

1.12. Recall that fixing a chamber C of Δ, there is a unique chamber map $\rho_C : \Delta \to C$ such that $\rho_C|_C = id_C$. It is called the *contraction* of Δ onto C (see [Tits] 3.8, p. 44–45). Two vertices $v, w \in \Delta^{(0)}$ are said to *have the same type*, if $\rho_C(v) = \rho_C(w)$ (this does not depend on the choice of C).

1.13. Lemma:
(a) The number $\# B_N(v)$ depends only on the type of v.
(b) There is a constant $V \in \mathbf{N}$ depending only on the building Δ such that for any $v, v' \in \Delta^{(0)}$ and any $N \in \mathbf{N}$

$$\# B_N(v) \leq V \cdot \# B_N(v') \ .$$

Proof of (a): Let $type(v) = type(v')$, Σ and Σ' be any apartments in Δ containing v and v'. Then, there is an isomorphism $\kappa : \Sigma \to \Sigma'$ such that $\kappa(v) = v'$. Let C be a chamber of Σ containing v, and take $C' = \kappa(C)$. Consider functions $\lambda, \lambda' : cham\,\Delta \to \mathbf{N} \cup \{0\}$ defined with respect to C and C' respectively, and folding maps $\varphi : \varphi_{\Sigma,C} : \Delta \to \Sigma, \ \ \varphi' = \varphi'_{\Sigma',C'} : \Delta \to \Sigma'$ defined as in 1.4. For a vertex $w \in \Delta^{(0)}$ define $\lambda(w) := \min\{\lambda(D) : D \in st(w) \cap cham\,\Delta\}$, and similarly $\lambda'(w)$. Then if $w' = \kappa(w)$, we get $\lambda(w) = \lambda'(w')$, and thus it follows from Lemma 1.11 (b) that $\#\varphi^{-1}(w) = \#(\varphi')^{-1}(w')$.

Denote by $B_N^\Sigma(v)$, $B_N^{\Sigma'}(v')$ the balls of radius N with respect to metrics d_Σ, $d_{\Sigma'}$ in $\Sigma^{(0)}$ and $\Sigma'^{(0)}$ respectively (compare 0.2). Then, by lemma 1.11.(a), $B_N(v) = \varphi^{-1}(B_N^\Sigma(v))$, $B_N(v^1) = (\varphi')^{-1}(B_N^{\Sigma'}(v^1))$, and the lemma follows by noting that $\kappa(B_N^\Sigma(v)) = B_N^{\Sigma'}(v')$.

Proof of (b): Note that all types of vertices in Δ are represented by vertices of any chamber. Thus, there exists $r \in \mathbf{N}$ such that for any $v, w \in \Delta^{(0)}$ there is $v' \in \Delta^{(0)}$ such that $type(w') = type(w)$ and $d_\Delta(v,w') \leq r$.

Put $A = \max\{\#(st(v) \cap cham\,\Delta) : v \in \Delta^{(0)}\}$, and note that

$$\# B_N(v) \leq \# B_{N+r}(w') \leq A^r \cdot \# B_N(w') = A^r \cdot \# B_N(w) \ ,$$

where the last equality follows from part (a). Now, the proof of part (b) is completed by putting $V = A^r$.

1.14. Proof of Proposition 1.5.

Let C be a chamber containing v, Σ an a apartment containing C, $\varphi = \varphi_{\Sigma,C} : \Delta \to \Sigma$ the folding map as defined in 1.4, and $w_0 = \varphi(w)$. Then, by 1.11.(c)

$$q^{\varepsilon_\Delta(v,w)} = q^{\varepsilon_\Sigma(v,w_0)} \leq \# \varphi^{-1}(w_0)$$

and since by 1.11.(a) $\varphi^{-1}(w_0) \subset B_N(v)$, we use 1.13.(b) to get

$$q^{\varepsilon_\Delta(v,w)} \leq \# B_N(v) \leq V \cdot \# B_N(v_0) \ .$$

1.15. Proof of Proposition 1.6.

The proposition follows from the fact, that if $\varphi : \Sigma \to \Sigma$ is a type–preserving chamber endomorphism of a Coxeter complex Σ, then the set $X = \{D \in \mathrm{Cham}\,\Sigma : \varphi(D) = D\}$ is convex in Σ, i.e. for any chambers $D_1, D_2 \in X$ all chambers of any minimal gallery in Σ between D_1 and D_2 are in X (see [Tits], Remark 2.20, p. 26).

Indeed, let $\eta_1 \in \Gamma_1$ be a gallery between chambers $\widetilde{C}_0$ and $E(\eta_1)$, and let Σ' be an apartment containing both $\widetilde{C}_0$ and $E(\eta_1)$, and hence the whole gallery η_1. There is a unique isomorphism $\psi : \Sigma \to \Sigma'$ such that $\psi(\gamma_1) = \eta_1$. Then $\varphi \circ \psi = id$ on a convex hull of $\{C_0, C_n\}$, and thus $\varphi \circ \psi(\gamma_2) = \gamma_2$. This implies that $\psi(\gamma_2)$ is in Γ_2, with $E(\psi(\gamma_2)) = E(\eta_1)$, and the Proposition follows.

1.16. Observe that the lemma 1.10.(b) implies the following

Corollary. Let $C_1, C_2 \in cham\,\Sigma$ (not necessarily distinct), and σ be their common codim 1 face. Fix $C_0 \in cham\,\Delta$ with $\varphi(C_0) = C_1$, and denote by σ_0 the face of C_0 such that $\varphi(\sigma_0) = \sigma$. Then the set

$$Y = \{D \in cham\,\Delta : \varphi(D) = C_2, \ \sigma_0 \in D\}$$

satisfies

$$\#Y \leq q \qquad \text{if} \qquad \lambda(C_2) \geq \lambda(C_1) \ ; \tag{1.16.1}$$

$$\#Y = 1 \qquad \text{if} \qquad \lambda(C_2) < \lambda(C_1) \ . \tag{1.16.2}$$

Note that the only case in which (1.16.1) is not the equality, happens when $C_1 = C_2$, and $\lambda(C_1) < \lambda(D_0)$, where D_0 is the only chamber in Σ adjacent to C_1 by σ.

1.17. Proof of Proposition 1.7.

Put $A = \max\{\#(st(v) \cap cham\,\Delta) : v \in \Delta^{(0)}\}$, and denote by $S(\eta)$ the starting chamber of a gallery η. It is easy to observe that $\#\{S(\eta) : \eta \in \Gamma\} \leq A$.

If $\gamma = (C_0, C_1, ..., C_n)$, denote

$$a = a(\gamma) := \#\{i : \lambda(C_i) < \lambda(C_{i+1})\} ;$$

$$b = b(\gamma) := \#\{i : \lambda(C_i) > \lambda(C_{i+1})\} ;$$

$$c = c(\gamma) := \#\{i : \lambda(C_i) = \lambda(C_{i+1})\} .$$

Then

$$a + b + c = n,\ c = l,\ a - b = \lambda(C_n) - \lambda(C_0) . \tag{1.17.1}$$

By Corollary 1.16 we have $\#\Gamma \leq A \cdot q^{a+c}$, and since from (1.17.1) we have $a + c = [n + \lambda(C_n) - \lambda(C_0) = l]/2$, the Proposition follows.

§2. Types of loops and the proofs of Theorems 0.5 and 0.6.

2.1. Let Σ, C and φ be as in 1.4. Then, a *type* of an N–loop $(v_0, v_1, ..., v_k = v_0)$ in (Δ, v_0) (with respect to folding map φ) is a sequence $(\varphi(v_0), \varphi(v_1), ..., \varphi(v_n))$ of vertices of Σ.

Lemma: The number of all types of N–loops of length $2k$ in (Δ, v_0) is bounded by $(C \cdot N^{\dim \Sigma})^{2k}$, for some constant C depending only on Δ.

Proof: The lemma follows from the fact that there is a constant C such that

$$\#\{w \in \Sigma^{(0)} : d_\Sigma(v, w) \leq N\} \leq C \cdot N^{\dim \Sigma} .$$

This estimate can be easily obtained, e.g. by considerations involving volumes of equal disjoint euclidean balls around the vertices of Σ.

2.2. Lemma. Let Σ be a euclidean Coxeter complex, and let $A = \max\{\operatorname{diam} st_\Sigma(v) : v \in \Sigma^{(0)}\}$, where the diameter is taken with respect to the metric $dist_\Sigma$. Then for any vertices v, w in $\Sigma^{(0)}$ we have

$$\varepsilon_\Sigma(v, w) \leq A \cdot d_\Sigma(v, w) .$$

Proof: Observe that if $d_\Sigma(v_1, v_2) = 1$, then there is $D \in st_\Sigma(v_1) \cap st_\Sigma(v_2)$, which implies that for any $C \in st_\Sigma(v_1)$ it holds $dist_\Sigma(C, D) \leq A$ and thus $\varepsilon_\Sigma(v_1, v_2) \leq A$. The lemma follows by induction, with use of the triangle inequality (1.3.1) for ε_Σ.

2.3. Define an *N–jump* in Δ to be any pair (v, w) of vertices of Δ satisfying $d_\Delta(v, w) \leq N$. Note that any pair (v_i, v_{i+1}) for $i = 0, 1, ..., k - 1$, where $(v_0, ..., v_k)$ is an N–loop, is an N–jump.

Let (v, w) be an N–jump. Then, by Lemma 2.2, for any $C \in cham\,\Delta$ such that $v \in C$, there exists a minimal gallery $\gamma = (C = C_0, C_1, ..., C_m)$ in Δ, with $w \in C_m$ and the length

$m \leq A \cdot N$. Chosen such a gallery γ, define a *type of N–jump* (v, w) (with respect to the folding map φ) to be the sequence

$$(\varphi(v), \varphi(C_0), \varphi(C_{i_1}), \tau_1, \varphi(C_{i_2}), \tau_2, ..., \varphi(C_{i_l}), \tau_l, \varphi(C_m), \varphi(w))$$

with $i_j < i_{j+1}$, and such that $\varphi(C_{i_j})$ for $i = 1, 2, ..., l$ are all the chambers at which the gallery $\varphi(\gamma)$ in Σ stammers, i.e. those for which $\varphi(C_{i_j}) = \varphi(C_{i_j+1})$; furthermore $\tau_j = \varphi(C_{i_j} \cap C_{i_j+1})$ for $j = 1, 2, ..., l$.

Note that a type of N–jump is a sequence of simplices in Σ. The N–jump may have many different types (with respect to fixed folding map φ) depending on the choice of minimal gallery γ.

We will call the vertices $\varphi(v)$ and $\varphi(w)$ the *starting* and *ending* points of a type respectively.

2.4. Lemma. The number of all types of N–jumps for which the starting and ending points in Σ are fixed is bounded by $C \cdot N^{\dim \Sigma \cdot (M+1)}$, for some constants C, M depending only on Δ.

Proof: Observe that, by proposition 1.8, the number l appearing in the definition of type of N–jump above, satisfies $l \leq M$. Moreover, by Lemma 2.2 we get $\mathrm{dist}_\Sigma(\varphi(C_{i_j}), \varphi(C_{i_{j+1}})) \leq A \cdot N$ for $j = 0, 1, ..., l$, where we put $i_0 = 0$ and $i_{l+1} = m$. Then the lemma follows from the fact that there is a constant $\alpha > 0$ such that for any $C \in cham\,\Sigma$

$$\#\{D \in cham\,\Sigma : \mathrm{dist}_\Sigma(C, D) \leq A \cdot N\} \leq \alpha \cdot N^{\dim \Sigma_n} .$$

This last estimate is clearly related to the one used in the proof of Lemma 2.1, and it can be derived in the same way by passing to the dual of the complex Σ.

2.5. Proof of Theorem 0.5.

Let C, Σ, φ and λ be as in 1.4, and let $v_0 \in C$ be a vertex.

Step 1. We shall first estimate the number of all N–jumps (v, w) of the given type

$$T = (u_0, E_0, E_1, \tau_1, E_2, \tau_2, ..., E_l, \tau_l, E_{l+1}, u_1)$$

and with fixed starting point v.

For $i = 0, 1, ..., l$, fix a minimal gallery ξ_i in Σ joining E_i with E_{i+1}. Denote by ξ a gallery in Σ composed out of galleries ξ_i by putting them one after another in a sequence, separating the last chamber of ξ_i and the first chamber of ξ_{i+1} (both being the same chamber E_{i+1}) by the face τ_{i+1}.

To each N–jump of the given type T, there corresponds a minimal gallery γ in Δ, with respect to which the N–jump has type T. Then the gallery $\varphi(\gamma)$ can by cut into pieces γ_i for $i = 0, 1, ..., l$ such that γ_i is a minimal gallery between E_i and E_{i+1} in Σ. Applying Proposition 1.6 consecutively to pairs of minimal galleries (γ_i, η_i), for $i = 0, 1, ..., l$, we get that for any N–jump of type T, the corresponding gallery γ can be chosen in such a way that $\varphi(\gamma) = \xi$. We shall estimate the number of such galleries.

Since we assume that the starting vertex v of N–jumps is fixed, we can apply directly Proposition 1.7 to get a bound by

$$A \cdot q^{[\text{length}\,(\gamma)+\lambda(E_{l+1})-\lambda(E_0)+l]/2} ,$$

but since $\lambda(E_{l+1}) \leq \lambda(u_1)+A$, $\lambda(E_0) \geq \lambda(u_0)-A$, and $l \leq M$ (where A, M are the constants of Lemma 2.2 and Proposition 1.8 respectively), we get the estimate by

$$A \cdot q^{A+M/2} \cdot q^{\text{length}\,(\gamma)/2} \cdot q^{[\lambda(u_1)-\lambda(u_0)]/2} .$$

Finaly, for any N–jump (v, w) of type T we have length$(\gamma) \leq \varepsilon_\Delta(v, w)$, and thus by Proposition 1.5

$$q^{\text{length}\,(\gamma)} \leq q^{\varepsilon_\Delta(v,w)} \leq V \cdot \# B_N(v_0) .$$

This gives us the estimate

$$\#\{N\text{–jumps of type } T \text{ starting at } v\} \leq C \cdot q^{[\lambda(u_1)-\lambda(u_0)]/2} \cdot [\# B_N(v_0)]^{1/2} .$$

Step 2. By Step 1 and Lemma 2.4, for any $u_0, u_1 \in \Sigma^{(0)}$ with $d_\Sigma(u_0, u_1) \leq N$, and for any $v \in \varphi^{-1}(u_0)$ we obtain the following ineqality:

$$\#\{w \in \varphi^{-1}(u_1) : \, (v, w) \text{ is an } N\text{–jump in } \Delta\} \leq C \cdot [\# B_N(v_0)]^{1/2} \cdot N^{\dim \Sigma(M+1)} \cdot q^{[\lambda(u_1)-\lambda(u_0)]/2} .$$

Step 3. Let $u = (u_0, u_1, ..., u_{2k})$ be a type of N–loop in (Δ, v_0). Then, by estimate of Step 2 we get

$$\#\{N\text{–loops of type } u\} \leq [C \cdot [\# B_N(v_0)]^{1/2} \cdot N^{\dim \Sigma(M+1)}]^{2k} = [C^2 \cdot [\# B_N(v_0)] \cdot N^{2 \dim \Sigma(M+1)}]^k$$

since

$$q^{[\lambda(u_1)-\lambda(u_0)]/2} \cdot q^{[\lambda(u_2)-\lambda(u_1)]/2} \cdot \ldots \cdot q^{[\lambda(u_{2k})-\lambda(u_{2k-1})]/2} = q^{[\lambda(u_{2k})-\lambda(u_0)]/2} = q^0 = 1$$

becouse $u_{2k} = u_0$.

Step 4. The theorem follows by applying the estimate of lemma 2.1 for the number of different types of loops.

2.6. Proof of Theorem 0.6.

Note that using Proposition 1.9 we get the inequality

$$N^r \cdot \# B_N(v_0) \leq N^r \cdot C \cdot (1+N)^d \cdot \# S_N(v_0) \leq C_1 \cdot N^{r'} \cdot \# S_N(v_0). \tag{2.6.1}$$

Furthermore, since $N^{r'} \cdot \# S_N(v_0) \leq (1+N)^{r'} \cdot \# S_N(v_0) \leq ||B_N(v_0)||_{r'}$, we get

$$N^r \cdot \# B_N(v_0) \leq C_1 \cdot ||B_N(v_0)||_{r'}. \tag{2.6.2}$$

In view of Theorem 0.5, part (a) of Theorem 0.6 follows easily from (2.6.1), and part (b) from (2.6.2).

§3. Proof of Proposition 1.8.

3.1. Denote by $\mathcal{D}$ the subcomplex of appartment Σ' consisting of all faces of critical locus of the folding map φ, i.e. of those codimension 1 faces σ of Σ', for which both chambers of Σ' adjacent to σ are mapped by φ onto the same chamber. Topological components of the complement $\Sigma' \setminus \mathcal{D}$ will be called *regularity components* of Σ' with respect to φ.

3.2. Lemma.
(a) Regularity components are convex.
(b) Each regularity component contains a cone sector of Σ'.

See 3.7 below, or [Brown] VI.6, page 164, for the definition of cone sector (which is called a sector in [Brown]).

3.3. Proof of Lemma 3.2 (a): It is well known that for chamber subcomplexes of Coxeter complexes the (affine) convexity condition is equivalent to the combinatorial convexity condition expressed in the obvious manner in terms of minimal galleries (compare [Brown], Exercise 2, page 15). Therefore, to prove Lemma 3.2 (a), it is enough to prove that no minimal gallery between two chambers C_1, C_2 of the same regularity component, passes through the critical locus. (We say that a chamber belongs to regularity component, if its interior is contained in this component.)

Since each regularity component of Σ' is embedded isometrically into Σ by the folding map φ, we get that

$$dist_{\Sigma'}(C_1, C_2) = dist_{\Sigma}(\varphi(C_1), \varphi(C_2)) \tag{3.3.1}$$

for any chambers C_1, C_2 of the same regularity component in Σ'. If a minimal gallery crossed the critical locus, then its image by φ would stammer, and hence be not minimal. But then there would be a shorter gallery between $\varphi(C_1)$ and $\varphi(C_2)$, giving a contradiction to (3.3.1). Thus the part (a) of the Lemma follows.

3.4. Proposition. Let C_0, C_1 be two chambers of a euclidean Coxeter complex Σ. Denote by $Ray(C_0, C_1)$ the subcomplex of Σ, consisting of all chambers D, for which there is a minimal gallery from C_0 to D passing through C_1. Then $Ray(C_0, C_1)$ contains a cone sector of Σ.

We will first apply Proposition 3.4 to prove Lemma 3.2 (b), and skip the proof of it until 3.7–3.9.

3.5. Proof of Lemma 3.2 (b): Consider a chamber C of some regularity component $\mathcal{R}$ of Σ', and denote $C_1 := \varphi(C)$ to be the image of chamber C by the folding map φ.

Claim. The image $\varphi(\mathcal{R})$ of the regularity component $\mathcal{R}$ contains all chambers of $Ray(C_0, C_1)$, where C_0 is the chamber with respect to which the folding map φ is defined via the function λ_{C_0}, as in 1.4.

Proof of Claim: Consider a cell D of $Ray(C_0, C_1)$, and a minimal gallery γ from C to D, which is a part of some minimal gallery from C_0 to D (it exists by the definition of $Ray(C_0, C_1)$). Then the values of the function $\lambda = \lambda_{C_0}$ strictly increase along the gallery γ. If $\gamma = (E_0, \sigma_0, E_1, \ldots, \sigma_{k-1}, E_k)$, where $E_0 = C$ and $E_k = D$, then according to Lemma 1.10 (b), to each chamber F of $\varphi^{-1}(E_i)$ there correspond q chambers of $\varphi^{-1}(E_{i+1})$ adjacent to F. But this means that the codimension 1 face τ of F lying in $\varphi^{-1}(\sigma_i)$ cannot belong to the critical locus, since there is no other chamber adjacent to τ in $\varphi^{-1}(E_i)$. This shows that all the gallery γ is contained in the image $\varphi(\mathcal{R})$, and the claim follows.

In view of the above Claim, part (b) of Lemma 3.2 follows from Proposition 3.4.

3.6. Proof of Proposition 1.8. Denote by N_1 the maximal number of non–parallel reflection hyperplanes in a euclidean Coxeter complex. This number is well defined for a building Δ, since appartments in Δ are isomorphic Coxeter complexes.

Since, by Lemma 3.2 (a), each regularity component is convex, it is equal to the intersection of several open halfspaces. The number of those (essential) halfspaces is bounded by $2N_1$, since at most two of the hyperplanes bounding these halfspaces can be parallel to each other. (In fact, since each regularity component contains a cone sector, no two bounding hyperplanes can be parallel, and thus their number is not bigger than N_1.)

On the other hand, if two cone sectors in a Coxeter complex Σ are disjoint, they determine distinct chambers in the spherical Coxeter complex Σ_∞ at infinity (see [Brown] VI.9, Lemma 4, page 176). Thus, in view of Lemma 3.2 (b), the number of regularity components of appartment Σ' is bounded by the number N_2 of chambers of the spherical Coxeter complex Σ'_∞ at infinity. But this number N_2 is also a well defined number for building Δ.

We can now estimate the number of hyperplanes in which the critical locus of appartment Σ' is contained, by the number of boundary walls in all regularity components of Σ'. But this is not bigger than $N_1 \cdot N_2$, and the Proposition 1.8 follows.

3.7. Notation. Following [Brown] VI.6, view the Coxeter complex Σ as a vector space with fixed zero vector $o \in \Sigma$. Denote by $\mathcal{H}$ the set of all hyperplanes in Σ through o, parallel to reflection hyperplanes of Σ. Then the components of the space $\Sigma \setminus \bigcup \mathcal{H}$ are called *open cone sectors* of Σ based at o. Using vector space notation, if P is an open cone sector based at o, then for any $x \in \Sigma$ the set of form $x + P$ will be called an open cone sector based at x. By *cone sectors* of Σ we will mean closures of open cone sectors. Observe that for each cone sector P based at o, the set $-P$ is also a cone sector and it is called to be *opposite* to P.

Given two chambers C_1, C_2 of a Coxeter complex Σ, denote by $\mathcal{B}(C_1, C_2)$ the combinatorial convex hull of $C_1 \cup C_2$, i.e. the subcomplex consisting of all chambers appearing in minimal galleries from C_1 to C_2.

3.8. Lemma. Let C_1, C_2 be two chambers of a euclidean Coxeter complex Σ, $x_1 \in C_1$

and $x_2 \in C_2$ two points of Σ (not necessarily vertices), and let $x_1 + P$ be a cone sector containing vector $\overrightarrow{x_1 x_2}$. Then
(a) $(x_1 + P) \cap (x_2 - P) \subset \mathcal{B}(C_1, C_2)$;
(b) for any $x \in (x_2 + P)$ it holds $x_2 \in (x_1 + P) \cap (x - P)$;
(c) $(x_2 + P) \subset Ray(C_1, C_2)$.

Proof: Part (a) follows from the Lemma in [Brown] VI.6, page 164; part (b) is obvious. To prove part (c), consider a point $x \in (x_2 + P)$, and a chamber C containing x. Then by (a) and (b) we have

$$x_2 \in (x_1 + P) \cap (x - P) \subset \mathcal{B}(C_1, C),$$

and thus there is a minimal gallery from C_1 to C passing through C_2. But this means that $C \subset Ray(C_1, C_2)$, and part (c) of Lemma 3.8 follows.

3.9. Proof of Proposition 3.4.
It follows directly from Lemma 3.8 (c).

§4. Proof of Proposition 1.9.

The proof of Proposition 1.9 is contained in 4.4. We precede it by stating Lemmas 4.2 and 4.3, necessary in the proof. We skip proofs of these Lemmas until 4.5 and 4.6.

4.1. Property. Let Σ be a Coxeter complex of type $\widetilde{A}_n$. Let $v, w \in \Sigma^{(0)}$ be any two vertices of Σ, and denote by T a translation (in Σ viewed as a euclidean space), which transforms v to w. Then T induces a combinatorial authomorphism of the chamber complex Σ.

The property stated above distinguishes Coxeter complexes of type $\widetilde{A}_n$ in the class of all euclidean Coxeter complexes. This property is well known, and can be seen easily i.e. in the explicit model of the Coxeter complex, as described in [Brown] VI.1F, page 147. The proofs of the following Lemmas 4.2 and 4.3 are based on this Property.

4.2. Lemma. Let Σ be a Coxeter complex of type $\widetilde{A}_n$, and d_Σ the metric on $\Sigma^{(0)}$, as defined in 0.2. Let L be a straight line contained in the 1–skeleton $\Sigma^{(1)}$ (in Σ vieved as a euclidean, or at least affine space), and view L as a subcomplex of Σ. Then for any two vertices $v, w \in L^{(0)}$ we have

$$d_\Sigma(v, w) = d_L(v, w).$$

4.3. Lemma. Let v, w be any adjacent vertices in a Coxeter complex Σ of type $\widetilde{A}_n$, and let T be the translation of Σ transforming v to w. Moreover, let C be any chamber of Σ containing edge $(T^{-1}v, v)$. Then

$$\varepsilon_\Sigma(v, T^n v) = n \cdot \varepsilon_\Sigma(v, w) = dist_\Sigma(C, T^n C).$$

Recall that ε_Σ used above, is the useful distance–like function defined in 1.3.

4.4. Proof of Propsition 1.9.

Let Σ be an appartment in Δ through vertex v. Consider a vertex $w \in \Sigma^{(0)}$, adjacent to v, for which the value of $\varepsilon_\Delta(v, w)$ is maximal among vertices adjacent to v, and put $m := \varepsilon_\Delta(v, w)$. Since for any two vertices $v_1, v_2 \in \Delta^{(0)}$ it holds $\varepsilon_\Delta(v_1, v_2) = \varepsilon_\Sigma(v_1, v_2)$, for any appartment Σ containing v_1 and v_2, it is clear that we can choose $w \in \Sigma^{(0)}$ as above. Denote by T the translation of Σ transforming v to w, and by C any chamber of Σ containing the edge $(T^{-1}v, v)$. Finally, consider the folding map $\varphi : \Delta \to \Sigma$, defined as in 1.4, with respect to the chamber C.

We will procede with a serie of claims under above notation.

Claim 1. $\varphi^{-1}(T^n v) \subset S_n(v)$.

Proof: For each $u \in \varphi^{-1}(T^n v)$ there is a polygonal path of length n in $\Delta^{(1)}$, joining u to v, namely the path lifted from the straight path in $\Sigma^{(1)}$ joining v and $T^n v$. If there were a shorter path, it would project by φ onto a shorter path in $\Sigma^{(1)}$ joining v and $T^n v$, which is impossible in view of Lemma 4.2. Thus the Claim follows.

The above argument could be shortened a little by reffering to Lemma 1.11 (a).

Claim 2. $\#\varphi^{-1}(T^n v) = q^{mn}$, where q is the thickness of Δ.

Proof: According to Lemma 1.11 (b), we have $\#\varphi^{-1}(x) = q^k$, where $k = \min\{\lambda(D) : D \in cham\,\Sigma \cap st_\Sigma(x)\}$. But in our case $\lambda(D) = dist_\Sigma(C, D)$, and thus the Claim follows from Lemma 4.3.

Claim 3. For any vertex $u \in \Sigma^{(0)}$ we have $\#\varphi^{-1}(u) \leq q^{m \cdot d_\Sigma(v,u)}$.

Proof: Since $\varepsilon_\Sigma(u_1, u_2) \leq m$ for any two vertices at distance 1, by the triangle inequality for ε_Σ we get $\varepsilon_\Sigma(u_1, u_2) \leq m \cdot d_\Sigma(u_1, u_2)$, for arbitrary vertices $u_1, u_2 \in \Sigma^{(0)}$. Since moreover $\min\{\lambda(D) : D \in cham\,\Sigma \cap st(u)\} \leq \varepsilon_\Sigma(v, u)$, the Claim follows easily from Lemma 1.11 (b).

Recall the following Fact, which we have already used in the proof of Lemma 2.1.

Fact. There exist constants $C_1 > 0$ and $d \in \mathbf{N}$, such that $\#S_n^\Sigma(v) \leq C_1(1+n)^d$.

Now, we estimate using above Claims, the Fact, and Lemma 1.11 (a):

$$\#B_N^\Delta(v) = \sum_{n=0}^{N} \#S_n^\Delta(v) \leq \sum_{n=0}^{N} (\#S_n^\Sigma(v)) \cdot q^{mn} \leq \sum_{n=0}^{N} C_1(1+n)^d q^{mn} \leq C_1(1+N)^d \sum_{n=0}^{N} q^{mn} =$$

$$= C_1(1+N)^d \frac{q^{m(N+1)} - 1}{q^m - 1} < C_1 \frac{q^m}{q^m - 1}(1+N)^d q^{mN} < C_1 \frac{q^m}{q^m - 1}(1+N)^d (\#S_N^\Delta(v)) =$$

$$= C'(1+N)^d (\#S_N^\Delta(v)).$$

Thus the Proposition 1.9 follows.

4.5. Proof of Lemma 4.2.

The Lemma follows easily from the following.

Claim. View Σ as a euclidean space. Then the orthogonal projection of any edge e of $\Sigma^{(1)}$ onto the straight line containing any other edge f of $\Sigma^{(1)}$, is not longer than f.

We will first derive the Lemma, and then prove the Claim.

Choose an arbitrary polygonal path in $\Sigma^{(1)}$ joining v with w, and project it orthogonally onto L. Since the images of path edges under this projection are not longer than the edges of L itself, it follows that the combinatorial length of the path is not smaller than the combinatorial length of the straight path joining v with w in L. By this the Lemma follows.

Proof of Claim: By Property 4.1, we may assume that edges e and f have common vertex p; let then $f = (p, q)$ and $e = (p, r)$. Denote by s the vertex symmetric to q with respect to p, and by H_q, H_s the hyperplanes in Σ, orthogonal to the line through p, q and s, passing through q and s respectively (note that these hyperplanes needn't be contained in the 2–skeleton of Σ). Then it is enough to show that all vertices of Σ adjacent to p lie in the closed streep F between parallel hyperplanes H_q and H_s. We will show that the subcomplex $st_\Sigma(p)$ is contained in F.

It is well known that the star $st_\Sigma(p)$ is convex (see e.g. [Brown], Exercise 2, page 15). Thus it is enough to show that H_q and H_s are the supporting hyperplanes for $st_\Sigma(p)$. But since it is a polytope, it is enough to show that its any boundary vertex t adjacent to q (or s) lies in F. If t is such a vertex, then (p, q, t) is a 2–simplex of Σ, and thus its angles are not bigger than $\pi/2$ (cf. [Bourbaki] V.3.5, Lemma 6 (ii)). But then $t \in F$, and the Claim follows, thus finishing also the proof of Lemma 4.2.

4.6. Proof of Lemma 4.3.

We start with the following.

Claim 1. $\varepsilon_\Sigma(v, w) = d_\Sigma(T^{-1}C, C)$.

Proof: First observe that among the chambers of $st_\Sigma(w)$, the closest (with respect to the gallery metric $dist_\Sigma$) to $T^{-1}(C)$ is C, since $T^{-1}C$ lies in a simplicial sector centered at w and determined by C, and thus the number of reflection hyperplanes in Σ separating it from $T^{-1}C$ is the smallest.

Second, note that $dist_\Sigma(T^{-1}C, C)$ does not depend on the choice of chamber C containing (v, w), since the Coxeter subgroup fixing the line through v and w acts transitively on the set of such chambers, and moreover this subgroup commutes with the translation T.

Finally, consider the sector S containing C, being the intersection of halfspaces bounded by hyperplanes of those faces of C which contain (v, w). Then both chambers C and $T^{-1}C$ are contained in this sector, and moreover, any reflection hyperplane through v that meets interior of this sector, separates C from $T^{-1}C$. But this means that any chamber of $S \cap st_\Sigma(v)$ other than $T^{-1}C$, is closer to C, since it is separated by less number of reflection hyperplanes.

Combining the above three observations, the Claim follows.

Claim 2. $dist_\Sigma(C, T^{n+1}C) = dist_\Sigma(C, T^nC) + dist_\Sigma(T^nC, T^{n+1}C)$.

Proof: This follows from the observation, that chamber T^nC lies in the convex hull of the sum of chambers C and $T^{n+1}C$, and thus a minimal gallery from C to $T^{n+1}C$ passes through T^nC (compare [Brown], Exercise 2 on page 15).

From Claim 2 it follows by induction that $dist_\Sigma(C, T^nC) = n \cdot dist_\Sigma(C, TC)$. Now, since according to the proof of Claim 1, any chamber of $S \cap st_\Sigma(v)$ other than $T^{-1}C$ is closer to C, it follows from triangle inequality that any such chamber is closer to $T^{n-1}C$ also. But this means that

$$\varepsilon_\Sigma(v, T^nv) = dist_\Sigma(T^{-1}C, T^{n-1}C) = dist_\Sigma(C, T^nC),$$

and the Lemma follows.

References.

[Bourbaki] N. Bourbaki, *Groupes et algebres de Lie, Chapitres 4–6*, Masson, Paris, 1981.

[Brown] K. Brown, *Buildings*, Springer–Verlag, New York, 1989.

[Cartwright] D. Cartwright, A. Mantero, T. Steger, A. Zappa, *Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I, II*, Geometriae Dedicata 47 (1993), 143–166, 167–223.

[Tits] J. Tits, *Buildings of spherical type and finite BN–pairs*, Lecture Notes in Math. 386, Springer–Verlag, 1974.