Max-Planck-Institut für Mathematik
Bonn

Period(d)ness of L-values

by

Wadim Zudilin

Max-Planck-Institut für Mathematik
Preprint Series 2012 (42a)

Period(d)ness of L-values

Wadim Zudilin

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

School of Mathematical and Physical Sciences
The University of Newcastle
Callaghan, NSW 2308
Australia

MPIM 12-42a

Period(d)ness of L-values

Wadim Zudilin*

School of Mathematical and Physical Sciences,
The University of Newcastle, Callaghan, NSW 2308, Australia

April 2013

Abstract

In our recent work with Rogers on resolving some of Boyd's conjectures on two-variate Mahler measures, a new analytical machinery was introduced to write the values $L(E,2)$ of L-series of elliptic curves as periods in the sense of Kontsevich and Zagier. Here we outline, in slightly more general settings, the novelty of our method with Rogers, and provide two illustrative period evaluations of $L(E,2)$ and $L(E,3)$ for a conductor 32 elliptic curve E.

1 Introduction

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients [5]. Without much harm, the three appearances of the adjective "rational" can be replaced by "algebraic". The set of periods $\mathcal{P}$ is countable and admits a ring structure. But what is probably most exciting about the set — it contains a lot of "important" numbers, mathematical constants like π [2] and $\zeta(3)$ [11].

The extended period ring $\widehat{\mathcal{P}} := \mathcal{P}[1/\pi] = \mathcal{P}[(2\pi i)^{-1}]$ (rather than the period ring $\mathcal{P}$ itself) contains many natural examples, like values of generalised hypergeometric functions [1] at algebraic points and special L-values. For example, a general theorem [5] due to Beilinson and Deninger–Scholl states that the (non-critical) value of the L-series attached to a cusp form $f(\tau)$ of weight k at a positive integer $m \ge k$ (cf. formula (2) below) belongs to $\widehat{\mathcal{P}}$. In spite of the effective nature of the proof of the theorem, computing these L-values as periods remains a difficult problem even for particular examples; it is this odd difficulty which lets us refer to the property of being a period as "period(d)ness". Most such computations are motivated by (conjectural) evaluations of the logarithmic Mahler measures of multi-variate polynomials.

*This work is supported by the Max Planck Institute for Mathematics (Bonn, Germany) and the Australian Research Council (grant DP110104419).

With the purpose of establishing such evaluations in the two-variate case, Rogers and the present author [8] have developed a machinery for writing the L-values $L(f,2)$ attached to cusp forms $f(\tau)$ of weight 2 as periods, the machinery which is different from that of Beilinson. In this note, we give an overview of the method of [8, 9] on a particular example of $L(E,2)$ in Section 2, and then attempt in Section 3 to describe a general algorithm behind the method. In Section 4 we present an example of evaluating $L(E,3)$ as a period, a computation we failed to find in the existing literature. Finally, in Section 5 we demonstrate that the two particular evaluations discussed can be further reduced to a hypergeometric form; such reduction is not expected to be available for general special L-values and so far is known for very few instances. In the examples of Sections 2, 4 (and 5), E stands for an elliptic curve of conductor 32. There are at least two reasons for choosing this conductor. First of all, it is not discussed in our joint work [8, 9], and secondly, the modular parameterisations involved are sufficiently classical and remarkably simple.

Throughout the note we keep the notation $q=e^{2\pi i\tau}$ for τ from the upper half-plane $\operatorname{Im}\tau>0$, so that $|q|<1$. Our basic constructor of modular forms and functions is Dedekind's eta-function

$$\eta(\tau):=q^{1/24}\prod_{m=1}^{\infty}(1-q^m)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$$

with its modular involution

$$\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau). \tag{1}$$

We also set $\eta_k:=\eta(k\tau)$ for short.

For functions of τ or $q=e^{2\pi i\tau}$ we use the differential operator

$$\delta:=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}=q\,\frac{\mathrm{d}}{\mathrm{d}q}$$

and denote by δ^{-1} the corresponding anti-derivative normalised by 0 at $\tau=i\infty$ (or $q=0$):

$$\delta^{-1}f=\int_0^q f\,\frac{\mathrm{d}q}{q}.$$

In particular, for a modular form $f(\tau)=\sum_{n=1}^{\infty}a_nq^n$, whose expansion vanishes at infinity, we have

$$L(f,m)=\frac{1}{(m-1)!}\int_0^1 f\log^{m-1}q\,\frac{\mathrm{d}q}{q}=\sum_{n=1}^{\infty}\frac{a_n}{n^m}=(\delta^{-m}f)|_{q=1} \tag{2}$$

whenever the latter sum makes sense.

The generalised hypergeometric function is defined by the series

$${}_{k+1}F_k\left(\begin{matrix}a_0,\,a_1,\,\dots,\,a_k\\ b_1,\,\dots,\,b_k\end{matrix}\;\middle|\;z\right)=\sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n\cdots(a_k)_n}{(b_1)_n\cdots(b_k)_n}\,\frac{z^n}{n!}$$

in the disc $|z| < 1$; here $(a)_n := \Gamma(a+n)/\Gamma(a) = \prod_{m=0}^{n-1}(a+m)$ denotes Pochhammer's symbol. Details about the analytic continuation and integral representations of the function can be found in Bailey's classical treatise [1]; relevant references are made explicit in Section 5.

2 $L(E, 2)$

For a conductor 32 elliptic curve E, the L-series is known to coincide with that for the cusp form $f(\tau) := \eta_4^2\eta_8^2$.

Note the (Lambert series) expansion

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m\ge1}\left(\frac{-4}{m}\right)\frac{q^m}{1-q^{2m}} = \sum_{\substack{m,n\ge1\\ n \text{ odd}}}\left(\frac{-4}{m}\right)q^{mn} = \sum_{m,n\ge1} a(m)b(n)q^{mn},$$

$$\text{where} \quad a(m) := \left(\frac{-4}{m}\right), \quad b(n) := n \bmod 2,$$

and $\left(\frac{-4}{m}\right)$ denotes the quadratic residue character modulo 4.

Then

$$\begin{aligned} f(it) &= \frac{\eta_8^4}{\eta_4^2}\frac{\eta_4^4}{\eta_8^2}\bigg|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=it} \cdot \frac{1}{2t}\frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=i/(32t)} \\ &= \frac{1}{2t}\sum_{m_1,n_1\ge1} a(m_1)b(n_1)e^{-2\pi m_1 n_1 t}\sum_{m_2,n_2\ge1} b(m_2)a(n_2)e^{-2\pi m_2 n_2/(32t)}, \end{aligned} \tag{3}$$

where $t > 0$ and the modular involution (1) was used.

Now,

$$\begin{aligned} L(E,2) = L(f,2) &= \int_0^1 f\,\log q\,\frac{\mathrm{d}q}{q} = -4\pi^2\int_0^\infty f(it)t\,\mathrm{d}t \\ &= -2\pi^2\int_0^\infty \sum_{m_1,n_1,m_2,n_2\ge1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\qquad \times \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)\mathrm{d}t \\ &= -2\pi^2 \sum_{m_1,n_1,m_2,n_2\ge1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\qquad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)\mathrm{d}t. \end{aligned}$$

Here comes the crucial transformation of purely analytical origin: we make the change of variable $t = n_2u/n_1$. This does not change the form of the integrand but affects the differential, and we obtain

$$\begin{aligned}
L(E,2) &= -2\pi^2 \sum_{m_1,n_1,m_2,n_2\ge 1} \frac{a(m_1)b(n_1)b(m_2)a(n_2)n_2}{n_1} \\
&\qquad \times \int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_2u+\frac{m_2n_1}{32u}\biggr)\biggr)\mathrm{d}u \\
&= -2\pi^2 \int_0^\infty \sum_{m_1,n_2\ge 1} a(m_1)a(n_2)n_2 e^{-2\pi m_1 n_2 u} \\
&\qquad\qquad \times \sum_{m_2,n_1\ge 1} \frac{b(m_2)b(n_1)}{n_1} e^{-2\pi m_2 n_1/(32u)}\mathrm{d}u.
\end{aligned}$$

The first double series in the integrand corresponds to

$$\sum_{m,n\ge 1} a(m)a(n)n\,q^{mn} = \sum_{m,n\ge 1}\biggl(\frac{-4}{mn}\biggr)n\,q^{mn} = \sum_{n\ge 1} n\biggl(\frac{-4}{n}\biggr)\frac{nq^n}{1+q^{2n}} = \frac{\eta_2^4\eta_8^4}{\eta_4^4},$$

while the second one is

$$\begin{aligned}
\sum_{m,n\ge 1}\frac{b(m)b(n)}{n}\,q^{mn} &= \sum_{m,n\ge 1}\frac{q^{mn}}{n}-\frac{q^{(2m)n}}{n}-\frac{q^{m(2n)}}{2n}+\frac{q^{(2m)(2n)}}{2n} \\
&= \frac{1}{2}\sum_{m,n\ge 1}\frac{2q^{mn}-3q^{2mn}+q^{4mn}}{n} \\
&= -\frac{1}{2}\,\log\prod_{m\ge 1}\frac{(1-q^m)^2(1-q^{4m})}{(1-q^{2m})^3} = -\frac{1}{2}\,\log\frac{\eta_1^2\eta_4}{\eta_2^3},
\end{aligned}$$

hence

$$L(E,2) = \pi^2\int_0^\infty \frac{\eta_2^4\eta_8^4}{\eta_4^4}\bigg|_{\tau=iu}\cdot\log\frac{\eta_1^2\eta_4}{\eta_2^3}\bigg|_{\tau=i/(32u)}\mathrm{d}u.$$

Applying the involution (1) to the eta quotient under the logarithm sign we obtain

$$L(E,2) = \pi^2\int_0^\infty \frac{\eta_2^4\eta_8^4}{\eta_4^4}\,\log\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\bigg|_{\tau=iu}\mathrm{d}u.$$

Now comes the modular magic: choosing a particular modular function $x(\tau) := \eta_2^4\eta_8^2/\eta_4^6$, which ranges from 0 to 1 when τ ranges from 0 to $i\infty$, one can easily verify that

$$\frac{1}{2\pi i}\,\frac{x\,\mathrm{d}x}{2\sqrt{1-x^4}} = -\frac{\eta_2^4\eta_8^4}{\eta_4^4}\,\mathrm{d}\tau \quad\text{and}\quad \biggl(\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\biggr)^2 = \frac{1-x}{1+x}.$$

Thus, we arrive at the following result.

Theorem 1. *For an elliptic curve E of conductor* 32,

$$L(E,2) = \frac{\pi}{8}\int_0^1 \frac{x}{\sqrt{1-x^4}}\,\log\frac{1+x}{1-x}\,\mathrm{d}x = 0.9170506353\dots.$$

3 General L-values

To summarise our evaluation of $L(E,2)=L(f,2)$ in Section 2, we first split $f(\tau)$ into a product of two Eisenstein series of weight 1 and at the end we arrive at a product of two Eisenstein(-like) series $g_2(\tau)$ and $g_0(\tau)$ of weights 2 and 0, respectively, so that $L(f,2)=c\pi L(g_2g_0,1)$ for some rational c. The latter object is doomed to be a period as $g_0(\tau)$ is a logarithm of a modular function, while $2\pi i\,g_2(\tau)\,\mathrm{d}\tau$ is, up to a modular function multiple, the differential of a modular function, and finally any two modular functions are connected by an algebraic relation over $\overline{\mathbb{Q}}$.

The method can be further formalised to more general settings, and it is this extension which we attempt to outline in this section.

For two *bounded* sequences $a(m)$, $b(n)$, we refer to an expression of the form

$$g_k(\tau)=a+\sum_{m,n\ge1}a(m)b(n)n^{k-1}q^{mn} \tag{4}$$

as an Eisenstein-like series of weight k, especially in the case when $g_k(\tau)$ is a modular form of certain level, that is, when it transforms sufficiently 'nicely' under $\tau\mapsto -1/(N\tau)$ for some positive integer N. This automatically happens when $g_k(\tau)$ is indeed an Eisenstein series (for example, when $a(m)=1$ and $b(n)$ is a Dirichlet character modulo N of designated parity, $b(-1)=(-1)^k$), in which case $\widehat{g}_k(\tau):=g_k(-1/(N\tau))(\sqrt{-N}\tau)^{-k}$ is again an Eisenstein series. It is worth mentioning that the above notion makes perfect sense in the case $k\le0$ as well. Indeed, modular units, or weak modular forms of weight 0, that are the logarithms of modular functions are examples of Eisenstein-like series $g_0(\tau)$. Also, for $k\le0$ examples are given by Eichler integrals, the $(1-k)$ th τ-antiderivatives of holomorphic Eisenstein series of weight $2-k$, a consequence of the famous lemma of Hecke [12, Section 5].

Suppose we are interested in the L-value $L(f,k_0)$ of a cusp form $f(\tau)$ of weight $k=k_1+k_2$ which can be represented as a product (in general, as a linear combination of several products) of two Eisenstein(-like) series $g_{k_1}(\tau)$ and $\widehat{g}_{k_2}(\tau)$, where the first one vanishes at infinity ($a=g_{k_1}(i\infty)=0$ in (4)) and the second one vanishes at zero ($\widehat{g}_{k_2}(i0)=0$). (The vanishing happens because the product is a cusp form!) In reality, we need the series $g_{k_2}(\tau):=\widehat{g}_{k_2}(-1/(N\tau))(\sqrt{-N}\tau)^{-k_2}$ to be Eisenstein-like:

$$g_{k_1}(\tau)=\sum_{m,n\ge1}a_1(m)b_1(n)n^{k_1-1}q^{mn}\quad\text{and}\quad g_{k_2}(\tau)=\sum_{m,n\ge1}a_2(m)b_2(n)n^{k_2-1}q^{mn}.$$

We have

$$\begin{aligned}
L(f,k_0)=L(g_{k_1}\widehat{g}_{k_2},k_0)&=\frac{1}{(k_0-1)!}\int_0^1 g_{k_1}\widehat{g}_{k_2}\log^{k_0-1}q\,\frac{\mathrm{d}q}{q}\\
&=\frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!}\int_0^\infty g_{k_1}(it)\widehat{g}_{k_2}(it)t^{k_0-1}\,\mathrm{d}t\\
&=\frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty g_{k_1}(it)g_{k_2}(i/(Nt))t^{k_0-k_2-1}\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \int_0^\infty \sum_{m_1,n_1\ge1} a_1(m_1)b_1(n_1)n_1^{k_1-1}e^{-2\pi m_1n_1t} \\
&\qquad \times \sum_{m_2,n_2\ge1} a_2(m_2)b_2(n_2)n_2^{k_2-1}e^{-2\pi m_2n_2/(Nt)}t^{k_0-k_2-1}\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \sum_{m_1,n_1,m_2,n_2\ge1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1-1}n_2^{k_2-1} \\
&\qquad \times \int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_1t+\frac{m_2n_2}{Nt}\biggr)\biggr)t^{k_0-k_2-1}\mathrm{d}t;
\end{aligned}
$$

the interchange of integration and summation is legitimate because of the exponential decay of the integrand at the endpoints. After performing the change of variable $t=n_2u/n_1$ and interchanging summation and integration back again we obtain

$$
\begin{aligned}
L(f,k_0) &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \sum_{m_1,n_1,m_2,n_2\ge1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1+k_2-k_0-1}n_2^{k_0-1} \\
&\qquad \times \int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_2u+\frac{m_2n_1}{Nu}\biggr)\biggr)u^{k_0-k_2-1}\mathrm{d}u \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \int_0^\infty \sum_{m_1,n_2\ge1} a_1(m_1)b_2(n_2)n_2^{k_0-1}e^{-2\pi m_1n_2u} \\
&\qquad \times \sum_{m_2,n_1\ge1} a_2(m_2)b_1(n_1)n_1^{k_1+k_2-k_0-1}e^{-2\pi m_2n_1/(Nu)}u^{k_0-k_2-1}\mathrm{d}u \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \int_0^\infty g_{k_0}(iu)g_{k_1+k_2-k_0}(i/(Nu))u^{k_0-k_2-1}\,\mathrm{d}u.
\end{aligned}
$$

Assuming a modular transformation of the Eisenstein-like series $g_{k_1+k_2-k_0}(\tau)$ under $\tau\mapsto-1/(N\tau)$, we can realise the resulting integral as $c\pi^{k_0-k_1}L(g_{k_0}\widehat{g}_{k_1+k_2-k_0},k_1)$, where c is algebraic (plus some extra terms when $g_{k_1+k_2-k_0}(\tau)$ is an Eichler integral). Alternatively, if $g_{k_0}(\tau)$ transforms under the involution, we perform the transformation and switch to the variable $v=1/(Nu)$ to arrive at $c\pi^{k_0-k_2}L(\widehat{g}_{k_0}g_{k_1+k_2-k_0},k_2)$. In both cases we obtain an identity which relates the starting L-value $L(f,k_0)$ to a different 'L-value' of a modular-like object of the same weight.

The case $k_1=k_2=1$ and $k_0=2$, discussed in [8, 9] and in Section 2 above, allows one to reduce the L-values to periods. As we will see in Section 4, the period(d)ness can be achieved in a more general situation, based on the fact that Eichler integrals are related to solutions of inhomogeneous linear differential equations.

4 $L(E,3)$

To manipulate with $L(E,3)$ for a conductor 32 elliptic curve, we use again $L(E,3) = L(f,3)$ with $f(\tau) = \eta_4^2\eta_8^2$ and write the decomposition in (3) as

$$f(it) = \frac{1}{2t}\sum_{m_1,n_1\ge1} b(m_1)a(n_1)e^{-2\pi m_1n_1t}\sum_{m_2,n_2\ge1} b(m_2)a(n_2)e^{-2\pi m_2n_2/(32t)}.$$

Then

$$\begin{aligned}
L(E,3) = L(f,3) &= \frac{1}{2}\int_0^1 f\,\log^2 q\,\frac{\mathrm{d}q}{q} = 4\pi^3\int_0^\infty f(it)t^2\,\mathrm{d}t \\
&= 2\pi^3\int_0^\infty \sum_{m_1,n_1,m_2,n_2\ge1} b(m_1)a(n_1)b(m_2)a(n_2) \\
&\qquad\times\exp\biggl(-2\pi\biggl(m_1n_1t+\frac{m_2n_2}{32t}\biggr)\biggr)t\,\mathrm{d}t \\
&= 2\pi^3\sum_{m_1,n_1,m_2,n_2\ge1} b(m_1)a(n_1)b(m_2)a(n_2) \\
&\qquad\times\int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_1t+\frac{m_2n_2}{32t}\biggr)\biggr)t\,\mathrm{d}t
\end{aligned}$$

(here we perform the change of variable $t = n_2u/n_1$)

$$\begin{aligned}
&= 2\pi^3\sum_{m_1,n_1,m_2,n_2\ge1} \frac{b(m_1)a(n_1)b(m_2)a(n_2)n_2^2}{n_1^2} \\
&\qquad\times\int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_2u+\frac{m_2n_1}{32u}\biggr)\biggr)u\,\mathrm{d}u \\
&= 2\pi^3\int_0^\infty \sum_{m_1,n_2\ge1} b(m_1)a(n_2)n_2^2e^{-2\pi m_1n_2u} \\
&\qquad\times\sum_{m_2,n_1\ge1}\frac{b(m_2)a(n_1)}{n_1^2}e^{-2\pi m_2n_1/(32u)}\,u\,\mathrm{d}u.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sum_{m,n\ge1} b(m)a(n)n^2q^{mn} &= \sum_{\substack{m,n\ge1\\ m\text{ odd}}}\biggl(\frac{-4}{n}\biggr)n^2q^{mn} = \frac{\eta_2^8\eta_8^4}{\eta_4^6}, \\
\sum_{m,n\ge1} b(m)a(n)m^2q^{mn} &= \sum_{\substack{m,n\ge1\\ m\text{ odd}}}\biggl(\frac{-4}{n}\biggr)m^2q^{mn} = \frac{\eta_4^{18}}{\eta_2^8\eta_8^4},
\end{aligned}$$

so that

$$r(\tau) := \sum_{m,n\ge1}\frac{b(m)a(n)}{n^2}\,q^{mn} = \delta^{-2}\biggl(\frac{\eta_4^{18}}{\eta_2^8\eta_8^4}\biggr). \tag{5}$$

Continuing the previous computation,

$$L(E,3) = 2\pi^3 \int_0^\infty \frac{\eta_2^8\eta_8^4}{\eta_4^6}\bigg|_{\tau=iu} \cdot r(i/(32u))\, u \,\mathrm{d}u$$

(we apply the involution to the eta quotient)

$$= \frac{\pi^3}{8} \int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\, r(\tau)\bigg|_{\tau=i/(32u)} \frac{\mathrm{d}u}{u^2}$$

(we change the variable $u = 1/(32v)$)

$$= 4\pi^3 \int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\, r(\tau)\bigg|_{\tau=iv} \mathrm{d}v.$$

This is so far the end of the algorithm we have discussed in Section 3. In order to show that the resulting integral is a period we require to do one step more. As in Section 2 we make a modular parameterisation; this time we take the modular function $x(\tau) := 4\eta_2^4\eta_8^8/\eta_4^{12}$ which ranges from 0 to 1 when τ goes from $i\infty$ to 0. Then

$$\delta x = \frac{4\eta_2^{12}\eta_8^8}{\eta_4^{16}}, \quad (1-x^2)^{1/4} = \frac{\eta_2^4\eta_8^2}{\eta_4^6}, \quad s(x) := \frac{(1-\sqrt{1-x^2})^2}{x(1-x^2)^{3/4}} = \frac{16\eta_4^{10}\eta_{16}^8}{\eta_2^8\eta_8^{10}}.$$

Furthermore, the substitution $z = x^2(\tau)$ into the hypergeometric function

$$F(z) := {}_2F_1\left(\begin{matrix}\frac12, \frac12\\ 1\end{matrix}\,\bigg|\, z\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-zy^2)}}$$

results in the modular form

$$\varphi(\tau) := F(x^2) = \sum_{n=0}^\infty \binom{2n}{n}^2 \left(\frac{x}{4}\right)^{2n} = \frac{\eta_4^{10}}{\eta_2^4\eta_8^4}$$

of weight 1. Because $F(z)$ (along with $F(1-z)$) satisfy the hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (1-2z)\frac{\mathrm{d}F}{\mathrm{d}z} - \frac14 F = 0,$$

it is not hard to write down the corresponding linear second order differential operator

$$\mathcal{L} := x(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1-3x^2)\frac{\mathrm{d}}{\mathrm{d}x} - x$$

(in terms of x) such that $\mathcal{L}\varphi = 0$.

With this notation in mind, we obtain

$$\begin{aligned} L(E,3) &= \pi^3 \int_0^\infty \frac{\eta_4^{10}\eta_{16}^8}{\eta_2^8\eta_8^{10}}\, \varphi(\tau)\, r(\tau)\delta x\bigg|_{\tau=iv} \mathrm{d}v \\ &= \frac{\pi^3}{16}\int_0^\infty s(x(\tau))\,\varphi(\tau)\, r(\tau)\delta x\bigg|_{\tau=iv} \mathrm{d}v, \end{aligned} \tag{6}$$

and at this point we make an observation that the function $h(\tau) := 4\varphi(\tau)r(\tau)$ solves the inhomogeneous differential equation

$$\mathcal{L}h = \frac{1}{1-x^2} \qquad \left(\text{which is nothing but [10, 13]} \quad \frac{\delta^2 r}{\delta x\cdot\varphi} = \frac{\eta_4^{24}}{4\eta_2^{16}\eta_8^8}\right),$$

so that it can be written as an integral using the method of variation of parameters:

$$\begin{aligned} h &= \frac{\pi}{2}\left(F(x^2)\int\frac{F(1-x^2)}{1-x^2}\,\mathrm{d}x - F(1-x^2)\int\frac{F(x^2)}{1-x^2}\,\mathrm{d}x\right) \\ &= \frac{\pi x}{2}\int_0^1 \frac{F(x^2)F(1-x^2w^2)-F(1-x^2)F(x^2w^2)}{1-x^2w^2}\,\mathrm{d}w. \end{aligned}$$

This implies that

$$L(E,3) = \frac{\pi^2}{128}\int_0^1 s(x)\,h(x)\,\mathrm{d}x,$$

an expression which can be clearly transformed into a (complicated) real integral.

The recipe of expressing Eisenstein series of negative weight via solutions of nonhomogeneous linear differential equations is standard [13] and applicable in any situation similar to the one considered above. The Eisenstein series (5) of weight -1 however possesses a different treatment because of a special formula due to Ramanujan [4, eq. (2·2)]:

$$r(\tau) = \sum_{\substack{m,n\ge1\\ m\text{ odd}}}\left(\frac{-4}{n}\right)\frac{q^{mn}}{n^2} = \frac{\tilde{x}\,G(-\tilde{x}^2)}{4F(-\tilde{x}^2)},$$

where $\tilde{x}(\tau) := 4\eta_8^4/\eta_2^4$, $F(-\tilde{x}^2) = \eta_2^4/\eta_4^2$ and

$$G(z) := {}_3F_2\left(\begin{matrix}1,\,1,\,1\\ \frac{3}{2},\,\frac{3}{2}\end{matrix}\,\middle|\, z\right) = \frac{1}{4}\int_0^1\!\!\int_0^1 \frac{(1-x_1)^{-1/2}(1-x_2)^{-1/2}}{1-zx_1x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

The latter integral also gives the analytic continuation of the hypergeometric ${}_3F_2$-series to the domain $\operatorname{Re} z < 1$ (see [6, Lemma 2]); the change $y = (1-x_1)^{1/2}$, $w = (1-x_2)^{1/2}$ translates the integral into the form

$$G(z) = \int_0^1\!\!\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-z(1-y^2)(1-w^2)}.$$

Rolling back to the modular function $x(\tau) = 4\eta_2^4\eta_8^8/\eta_4^{12}$ and noting that $\tilde{x} = x/\sqrt{1-x^2}$ we may now write (8) as

$$L(E,3) = \frac{\pi^3}{64}\int_0^\infty \frac{s(x(\tau))\,x(\tau)}{1-x(\tau)^2}\,G\left(-\frac{x(\tau)^2}{1-x(\tau)^2}\right)\delta x\bigg|_{\tau=iv}\mathrm{d}v.$$

After performing the modular substitution $x = x(\tau)$ we finally arrive at

Theorem 2. *For an elliptic curve E of conductor* 32*,*

$$\begin{aligned} L(E,3) &= \frac{\pi^2}{128}\int_0^1 \frac{(1-\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x\int_0^1\!\!\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-x^2(1-(1-y^2)(1-w^2))} \\ &= 0.9826801478\ldots. \end{aligned}$$

5 Hypergeometric evaluations

A remarkable feature of the integrals in Theorems 1 and 2 is the possibility to reduce them further to hypergeometric functions [3].

Theorem 3. *For an elliptic curve E of conductor* 32,

$$L(E,2)=\frac{\pi^{1/2}\Gamma(\frac14)^2}{96\sqrt2}\,{}_3F_2\left(\begin{matrix}1,\,1,\,\frac12\\ \frac74,\,\frac32\end{matrix}\,\middle|\,1\right)+\frac{\pi^{1/2}\Gamma(\frac34)^2}{8\sqrt2}\,{}_3F_2\left(\begin{matrix}1,\,1,\,\frac12\\ \frac54,\,\frac32\end{matrix}\,\middle|\,1\right), \tag{7}$$

$$\begin{aligned}L(E,3)&=\frac{\pi^{3/2}\Gamma(\frac14)^2}{768\sqrt2}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac74,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right)+\frac{\pi^{3/2}\Gamma(\frac34)^2}{32\sqrt2}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac54,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right)\\&\quad+\frac{\pi^{3/2}\Gamma(\frac14)^2}{256\sqrt2}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac34,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right).\end{aligned} \tag{8}$$

Proof. In the integral representation for $L(E,2)$ in Theorem 1, write

$$\log\frac{1+x}{1-x}=\frac23x^3\,{}_2F_1\left(\begin{matrix}\frac34,\,1\\ \frac74\end{matrix}\,\middle|\,x^4\right)+2x\,{}_2F_1\left(\begin{matrix}\frac14,\,1\\ \frac54\end{matrix}\,\middle|\,x^4\right)$$

and change the variable $x^4=x_0$ to get

$$\begin{aligned}L(E,2)&=\frac{\pi}{48}\int_0^1\left(x_0^{1/4}\,{}_2F_1\left(\begin{matrix}\frac34,\,1\\ \frac74\end{matrix}\,\middle|\,x_0\right)+3x_0^{-1/4}\,{}_2F_1\left(\begin{matrix}\frac14,\,1\\ \frac54\end{matrix}\,\middle|\,x_0\right)\right)(1-x_0)^{-1/2}\mathrm{d}x_0\\&=\frac{\pi^{3/2}}{48}\frac{\Gamma(\frac54)}{\Gamma(\frac74)}\,{}_3F_2\left(\begin{matrix}\frac34,\,\frac54,\,1\\ \frac74,\,\frac74\end{matrix}\,\middle|\,1\right)+\frac{\pi^{3/2}}{16}\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\,{}_3F_2\left(\begin{matrix}\frac14,\,\frac34,\,1\\ \frac54,\,\frac54\end{matrix}\,\middle|\,1\right);\end{aligned}$$

the representation (7) now follows from application of Thomae's transformation [1, Eq. 3.2.(1)] to the both ${}_3F_2$ series.

In the integral of Theorem 2, let $x_0=x^2$, $x_1=1-y^2$ and $x_2=1-w^2$:

$$\begin{aligned}L(E,3)=\frac{\pi^2}{1024}\int_0^1&x_0^{-1/2}\big((1-x_0)^{-3/4}-2(1-x_0)^{-1/4}+(1-x_0)^{1/4}\big)\,\mathrm{d}x_0\\&\times\int_0^1\!\!\int_0^1\frac{(1-x_1)^{-1/2}(1-x_2)^{-1/2}}{1-x_0(1-x_1x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

First consider the integral (see [1, Eqs. 1.5.(1) and 1.4.(1)])

$$\begin{aligned}\int_0^1\frac{x_0^{-1/2}(1-x_0)^{a-1}}{1-x_0z}\,\mathrm{d}x_0&=\frac{\Gamma(\frac12)\,\Gamma(a)}{\Gamma(a+\frac12)}\,{}_2F_1\left(\begin{matrix}1,\,\frac12\\ a+\frac12\end{matrix}\,\middle|\,z\right)\\&=\frac{\Gamma(\frac12)\,\Gamma(a-1)}{\Gamma(a-\frac12)}\,{}_2F_1\left(\begin{matrix}1,\,\frac12\\ 2-a\end{matrix}\,\middle|\,1-z\right)\\&\quad+\Gamma(a)\,\Gamma(1-a)\,(1-z)^{a-1}{}_2F_1\left(\begin{matrix}a-\frac12,\,a\\ a\end{matrix}\,\middle|\,1-z\right)\\&=\frac{\sqrt\pi\,\Gamma(a-1)}{\Gamma(a-\frac12)}\,{}_2F_1\left(\begin{matrix}1,\,\frac12\\ 2-a\end{matrix}\,\middle|\,1-z\right)+\frac{\pi}{\sin\pi a}\,\frac{(1-z)^{a-1}}{z^{a-1/2}}\end{aligned}$$

for $z = 1 - x_1 x_2$. Secondly,

$$\int_0^1\!\!\int_0^1 (1-x_1)^{-1/2}(1-x_2)^{-1/2}\,{}_2F_1\left(\begin{matrix}1,\ \frac12\\ 2-a\end{matrix}\,\middle|\, x_1x_2\right)\mathrm{d}x_1\,\mathrm{d}x_2$$
$$= 4\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac12\\ 2-a,\ \frac32,\ \frac32\end{matrix}\,\middle|\, 1\right)$$

and

$$\int_0^1\!\!\int_0^1 \frac{x_1^{a-1}(1-x_1)^{-1/2}x_2^{a-1}(1-x_2)^{-1/2}}{(1-x_1x_2)^{a-1/2}}\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$= \left(\frac{\Gamma(\frac12)\,\Gamma(a)}{\Gamma(a+\frac12)}\right)^2 {}_3F_2\left(\begin{matrix}a,\ a,\ a-\frac12\\ a+\frac12,\ a+\frac12\end{matrix}\,\middle|\, 1\right)$$
$$= 2\pi^{1/2}\Gamma(a)\Gamma(\tfrac32-a)\cdot {}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac32-a\\ 1,\ \frac32\end{matrix}\,\middle|\, 1\right),$$

where Thomae's transformation [1, Eq. 3.2.(1)] is used on the last step.

The computation above means that if we take

$$I_1(a) := \frac{\pi^{5/2}\Gamma(a-1)}{256\Gamma(a-\frac12)}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac12\\ 2-a,\ \frac32,\ \frac32\end{matrix}\,\middle|\, 1\right),$$
$$I_2(a) := \frac{\pi^{7/2}\Gamma(a)\,\Gamma(\frac32-a)}{512\sin\pi a}\,{}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac32-a\\ 1,\ \frac32\end{matrix}\,\middle|\, 1\right),$$

then

$$L(E,3) = \big(I_1(\tfrac14) - 2I_1(\tfrac34) + I_1(\tfrac54)\big) + \big(I_2(\tfrac14) - 2I_2(\tfrac34) + I_2(\tfrac54)\big).$$

For $I_2(\frac34)$, the Watson–Whipple summation [1, Eq. 3.3.(1)] results in

$${}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac34\\ 1,\ \frac32\end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(\frac12)\,\Gamma(\frac54)}{\Gamma(\frac34)},$$

so that $I_2(\frac34) = \pi^5/1024$. Furthermore,

$$I_2(\tfrac14) + I_2(\tfrac54) = \frac{\pi^{7/2}\Gamma(\frac14)\,\Gamma(\frac54)}{256\sqrt2}\left({}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac54\\ 1,\ \frac32\end{matrix}\,\middle|\, 1\right) - {}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac14\\ 1,\ \frac32\end{matrix}\,\middle|\, 1\right)\right)$$
$$= \frac{\pi^{7/2}\Gamma(\frac14)^2}{1024\sqrt2}\sum_{n=0}^{\infty}\frac{(\frac12)_n^2\cdot\big((\frac54)_n - (\frac14)_n\big)}{(1)_n(\frac32)_n n!}$$
$$= \frac{\pi^{7/2}\Gamma(\frac14)^2}{1024\sqrt2}\sum_{n=1}^{\infty}\frac{(\frac12)_n^2(\frac54)_{n-1}}{(1)_n(\frac32)_n (n-1)!}$$
$$= \frac{\pi^{7/2}\Gamma(\frac14)^2}{1024\sqrt2\cdot 6}\,{}_3F_2\left(\begin{matrix}\frac32,\ \frac32,\ \frac54\\ 2,\ \frac52\end{matrix}\,\middle|\, 1\right)$$
$$= \frac{\pi^{7/2}\Gamma(\frac14)^2}{1024\sqrt2\cdot 6}\,\frac{\Gamma(\frac12)\,\Gamma(\frac14)\,\Gamma(\frac74)}{\Gamma(\frac54)^2} = \frac{\pi^5}{512},$$

where again the Watson–Whipple summation was applied.

To summarise, $L(E,3) = I_1(\frac{1}{4}) - 2I_1(\frac{3}{4}) + I_1(\frac{5}{4})$, which is exactly equation (8). $\square$

Theorem 3 produces amazingly similar hypergeometric forms of $L(E,2)$ and $L(E,3)$. In the notation

$$F_k(a) := \frac{\pi^{k-1/2}\Gamma(a)}{2^{3k-1}\Gamma(a+\frac{1}{2})}\ {}_{k+1}F_k\left(\begin{matrix} \overbrace{1,\ \dots,\ 1}^{k \text{ times}},\ \frac{1}{2} \\ a+\frac{1}{2},\ \underbrace{\frac{3}{2},\ \dots,\ \frac{3}{2}}_{k-1 \text{ times}} \end{matrix}\ \middle|\ 1\right),$$

relations (7) and (8) can be alternatively written as

$$L(E,2) = F_2(\tfrac{5}{4}) + F_2(\tfrac{3}{4}) \quad \text{and} \quad L(E,3) = F_3(\tfrac{5}{4}) + 2F_3(\tfrac{3}{4}) + F_3(\tfrac{1}{4}). \tag{9}$$

In view of the known formula

$$L(E,1) = \frac{\pi^{-1/2}\Gamma(\frac{1}{4})^2}{8\sqrt{2}} = \frac{\pi^{-1/2}\Gamma(\frac{1}{4})^2}{24\sqrt{2}}\ {}_3F_2\left(\begin{matrix} 1,\ \frac{1}{2} \\ \frac{7}{4} \end{matrix}\ \middle|\ 1\right) = 2F_1(\tfrac{5}{4}),$$

we can conclude that, for $k = 1$, 2 or 3, the L-value $L(E,k)$ can be written as a (simple) $\mathbb{Q}$-linear combination of $F_k(\frac{7}{4} - \frac{m}{2})$ for $m = 1, \dots, k$. However this pattern does not seem to work for $k > 3$.

The formulae (9) in turn can be transformed into the period representations of $L(E,2)$ and $L(E,3)$ which differ from the ones in Theorems 2 and 3:

$$\begin{aligned} L(E,2) &= \frac{\pi}{16}\int_0^1 \frac{1+\sqrt{1-x^2}}{(1-x^2)^{1/4}}\,\mathrm{d}x \int_0^1 \frac{\mathrm{d}y}{1-x^2(1-y^2)}, \\ L(E,3) &= \frac{\pi^2}{128}\int_0^1 \frac{(1+\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x \int_0^1\!\!\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-x^2(1-y^2)(1-w^2)}. \end{aligned}$$

An interesting problem is identifying the (linear combination of the) hypergeometric series involved in the right-hand side of (8) with a linear combinations of 3-variable Mahler measures. There are related results in [7], although not written hypergeometrically enough.

A crucial ingredient in deducing the integral representation in Theorem 2 is the hypergeometric form of an Eisenstein series of negative weight given in [4]. Are there other results of this type?

Acknowledgements. I am thankful to Anton Mellit, Mat Rogers, Evgeny Shinder, Masha Vlasenko and James Wan for fruitful conversations on the subject, and to Don Zagier for his encouragement to isolate the L-series transformation part from [8, 9]. I am grateful to the anonymous referee for his careful reading of an earlier version and for his suggesting some useful corrections.

The principal part of this work was done during my visit in the Max Planck Institute for Mathematics (Bonn). I would like to thank the staff of the institute for hospitality and enjoyable working conditions.

References

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Math. **32** (Cambridge Univ. Press, Cambridge, 1935); 2nd reprinted edn. (Stechert-Hafner, New York–London, 1964).

[2] J. L. Berggren, J. M. Borwein and P. Borwein, *Pi: A source book*, 3rd edn. (Springer, New York 2004).

[3] J. M. Borwein and R. E. Crandall, Closed forms: what they are and why we care, *Notices Amer. Math. Soc.* **60** (2013), 50–65.

[4] W. Duke, Some entries in Ramanujan's notebooks, *Math. Proc. Camb. Phil. Soc.* **144** (2008), 255–266.

[5] M. Kontsevich and D. Zagier, Periods, in: *Mathematics unlimited — 2001 and beyond* (Springer, Berlin, 2001), 771–808.

[6] Yu. V. Nesterenko, Integral identities and constructions of approximations to zeta-values, *J. Théor. Nombres Bordeaux* **15** (2003), 535–550.

[7] M. D. Rogers, A study of inverse trigonometric integrals associated with three-variable Mahler measures, and some related identities, *J. Number Theory* **121** (2006), 265–304.

[8] M. Rogers and W. Zudilin, From L-series of elliptic curves to Mahler measures, *Compositio Math.* **148** (2012), 385–414.

[9] M. Rogers and W. Zudilin, On the Mahler measure of $1+X+1/X+Y+1/Y$, *Intern. Math. Research Notices* **2013** (2013), in press (`doi: 10.1093/imrn/rns285`), 22 pp.

[10] E. Shinder and M. Vlasenko, Linear Mahler measures and double L-values of modular forms, *Preprint* `http://arxiv.org/abs/1206.1454` (2012), 22 pp.

[11] A. van der Poorten, A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$, *Math. Intelligencer* **1** (1978/79), 195–203.

[12] A. Weil, Remarks on Hecke's lemma and its use, in: *Algebraic number theory*, Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto 1976 (Japan Soc. Promotion Sci., Tokyo 1977), 267–274.

[13] Y. Yang, Apéry limits and special values of L-functions, *J. Math. Anal. Appl.* **343** (2008), 492–513.