The cohomology ring of the symmetric space $F_4 I$

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Abstract

We determine the intersection numbers and the ring structure of the rational cohomology of the symmetric space $F_4/(Sp(3)Sp(1))$ by using index theory and its quaternion-Kähler structure.

1 Introduction

Recall that an oriented connected irreducible Riemannian $4n$-manifold M is called a *quaternion-Kähler manifold*, $n \geq 2$, if its linear holonomy is contained in the group $Sp(n)Sp(1)$. Examples of such manifolds were given in [7], where Wolf showed that each compact centerless Lie group G is the isometry group of a quaternion-Kähler symmetric space given as the conjugacy class of a copy of $Sp(1)$ in G determined by a highest root of G. Thus, the symmetric space

$$F_4 I = \frac{F_4}{Sp(3)Sp(1)}$$

is a 28-dimensional quaternion-Kähler manifold.

Although the cohomology of homogeneous spaces has been extensively studied, and the integral cohomology of $F_4 I$ was determined in [3], here we give a description of the rational cohomology ring $H^*(F_4 I, \mathbb{Q})$ in terms of classes determined by the quaternion-Kähler structure of this manifold. The motivation for this work is the need to understand the topological structure of general quaternion-Kähler manifolds, whose rational cohomology we conjecture to be generated by a small number of cohomology classes. This is

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indeed the case for the space F_4I as its Poincaré polynomial shows

$$\begin{aligned} P_{F_4I}(t) &= (1+t^4+t^8+t^{12}+t^{16}+t^{20})(1+t^8) \\ &= 1+t^4+2t^8+2t^{12}+2t^{16}+2t^{20}+t^{24}+t^{28} \end{aligned}$$

The note is organized as follows. In Section 2 we compute the intersection pairings of the relevant characteristic classes arising from the quaternion-Kähler structure of F_4I (see Theorem 2.1). In Section 3 we determine the ring structure of $H^*(F_4I,\mathbb{Q})$ by using the intersection numbers (see Theorem 3.1). In Section 4, as a corollary of our calculations, we compute explicitly the Pontrjagin classes and numbers of F_4I, which may be of use in other geometrical contexts. In Section 5, we revisit Ishitoya and Toda's result [3] on the torsion free part of the integral cohomology of F_4I in terms of our characteristic classes.

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2 Intersection numbers

The holonomy group $Sp(7)Sp(1) \subset SO(28)$ of a 28-dimensional quaternion-Kähler manifold M determines the following factorization of the complexified tangent bundle [6]

$$TM_c = E \otimes H, \tag{1}$$

where the fibres of the (locally defined) bundles E and H are isomorphic to the standard representations $\mathbb{C}^{14}$ and $\mathbb{C}^2$ of $Sp(7)$ and $Sp(1)$ respectively.

Furthermore, for F_4I, the representation E decomposes further under $Sp(3) \subset Sp(7)$

$$E = \textstyle\bigwedge_0^3 \tilde{E} \tag{2}$$

where $\tilde{E} \cong \mathbb{C}^6$ is the standard representation of $Sp(3)$, and $\bigwedge_0^p \tilde{E}$ denotes the irreducible representation of $Sp(3)$ obtained as the primitive subspace of $\bigwedge^p \tilde{E}$ with respect to wedging by a symplectic form. Furthermore, the faithful 26-dimensional representation of F_4 also decomposes under $Sp(3)Sp(1)$

$$26 = \textstyle\bigwedge_0^2 \tilde{E} + \tilde{E} \otimes H, \tag{3}$$

where the left hand side now denotes a rank 26 trivial bundle on F_4I (cf. [1]). Note that (2) implies that the characteristic classes of E are given in terms of those of the rank 6 bundle $\tilde{E}$, and (3) implies relations between the characteristic classes of $\tilde{E}$ and H.

More precisely, by computing the first three components of the Chern character of $\bigwedge_0^2 \tilde{E} + \tilde{E} \otimes H$ and equating them to zero we find that

$$\begin{aligned} c_2(\tilde{E}) &= u, \\ c_6(\tilde{E}) &= c_4(\tilde{E}) u, \end{aligned}$$

where $u = -c_2(H)$. This provides us with two candidates for the generators of $H^*(F_4 I)$: u in dimension 4 and $c_4(\tilde{E})$ in dimension 8. From now on, we shall denote

$$c_4 = c_4(\tilde{E}).$$

Thus, our first task is to compute the pairings

$$u^7, \quad c_4 u^5, \quad c_4^2 u^3, \quad c_4^3 u, \tag{4}$$

where the notation really means the evaluation of representatives of such 28-dimensional cohomology classes on the fundamental cycle of $F_4 I$.

In order to compute such pairings, we will make use of a Hilbert polynomial given by the index of certain twisted Dirac operators [6, 5]. More precisely, we will use the polynomial in q given by

$$f(q) = \mathrm{ind}(\partial\!\!\!/ \otimes S^q H) = \langle \widehat{A} \cdot \mathrm{ch}(S^q H), [F_4 I] \rangle,$$

where $\widehat{A}$ denotes the $\widehat{A}$-genus of the manifold, ch denotes the Chern character and $S^q H$ denotes the q^{th} symmetric power of H.

On the one hand, due to (1), (2) and (3), the coefficients of $f(q)$ are linear combinations of the intersection pairings in (4). Namely,

$$\begin{aligned} f(q) = {} & \frac{u^7 q^{15}}{1307674368000} + \frac{u^7 q^{14}}{87178291200} + \frac{u^7 q^{13}}{37362124800} - \frac{u^7 q^{12}}{2874009600} \\ & + \left(\frac{u^5 c_4}{4105728000} - \frac{u^7}{522547200} \right) q^{11} + \left(\frac{u^7}{2612736000} + \frac{u^5 c_4}{373248000} \right) q^{10} \\ & + \left(\frac{229 u^7}{10973491200} + \frac{59 u^5 c_4}{10973491200} \right) q^9 + \left(\frac{13 u^7}{406425600} - \frac{13 u^5 c_4}{406425600} \right) q^8 \\ & + \left(-\frac{151 u^7}{3657830400} - \frac{149 u^5 c_4}{457228800} + \frac{221 u^3 {c_4}^2}{18289152000} \right) q^7 \end{aligned}$$

$$+\left(-\frac{113\,u^5\,c_4}{81648000}+\frac{221\,u^3\,{c_4}^2}{2612736000}-\frac{31\,u^7}{522547200}\right)q^6$$

$$+\left(-\frac{17\,u^5\,c_4}{18711000}+\frac{1037\,u^3\,{c_4}^2}{9580032000}+\frac{107\,u^7}{1368576000}\right)q^5$$

$$+\left(-\frac{1751\,u^3\,{c_4}^2}{5748019200}+\frac{2603\,u^5\,c_4}{359251200}-\frac{1751\,u^7}{5748019200}\right)q^4$$

$$+\left(\frac{739163\,u^5\,c_4}{52306974720}+\frac{402959\,u\,{c_4}^3}{7846046208000}-\frac{3201281\,u^3\,{c_4}^2}{784604620800}-\frac{385673\,u^7}{523069747200}\right)q^3$$

$$+\left(-\frac{13528111\,u^3\,{c_4}^2}{1307674368000}+\frac{1237813\,u^5\,c_4}{261534873600}+\frac{3721\,u^7}{20922789888}+\frac{402959\,u\,{c_4}^3}{2615348736000}\right)q^2$$

$$+\left(\frac{2713\,u^7}{4828336128}-\frac{3383123\,u^3\,{c_4}^2}{980755776000}+\frac{535039\,u\,{c_4}^3}{7846046208000}-\frac{769633\,u^5\,c_4}{140107968000}\right)q$$

$$+\left(\frac{12899\,u^7}{373621248000}+\frac{294779\,u^3\,{c_4}^2}{93405312000}-\frac{12899\,u\,{c_4}^3}{373621248000}-\frac{294779\,u^5\,c_4}{93405312000}\right).$$

On the other hand, these indices can be seen as holomorphic Euler characteristics of the twistor space

$$Z = Z(F_4 I) = \frac{F_4}{Sp(3)U(1)}$$

of $F_4 I$ by twistor transform [6, 5]. Namely,

$$\begin{aligned}\mathrm{ind}\,(\not\partial \otimes S^q H) &= \chi(Z, \mathcal{O}(L^{(q-7)/2})) \\ &= \sum_{i=0}^{15} (-1)^i \dim H^i(Z, \mathcal{O}(L^{(q-7)/2})),\end{aligned}$$

where L is a positive line bundle over Z which restricted to the $\mathbb{CP}^1$-fibers is $\mathcal{O}(2)$. These holomorphic Euler characteristics can be computed by means of the Bott-Borel-Weil theorem and the Weyl dimension formula as follows [4].

Let $R(\mathfrak{sp}(3) \oplus \mathfrak{u}(1))$ be the set of roots of $Sp(3)U(1) \subset F_4$, R^+ be the set of positive roots of F_4 with $R(\mathfrak{sp}(3) \oplus \mathfrak{u}(1))$ generated by simple roots,

$\delta = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. Let $V(\lambda)$ be an irreducible representation for $Sp(3)U(1)$ with highest weight $\lambda \in R(\mathfrak{sp}(3)\oplus\mathfrak{u}(1))$ and $\mathbf{V}(\lambda)$ the corresponding homogeneous vector bundle on F_4I. By the Bott-Borel-Weil theorem and the Weyl dimension formula [4]

$$\chi(Z, \mathcal{O}(\mathbf{V}(\lambda))) = (-1)^s \prod_{\alpha\in R^+} \frac{\langle\alpha, \delta+\lambda\rangle}{\langle\alpha,\delta\rangle},$$

where

$$s = \sharp\{\alpha \in \mathbb{R}^+ \mid \langle\lambda+\delta, \alpha\rangle < 0\}.$$

More precisely, let $\mathfrak{H}$ be the Cartan subalgebra of $(\mathfrak{f}_4)_c$ spanned by the following basic roots

$$\{\alpha_1 = (1,-1,0,0),\ \alpha_2 = (0,1,-1,0)\ \alpha_3 = (0,0,2,0),\ \alpha_4 = (-1,-1,-1,1)\}.$$

The coordinates have been chosen so that $\mathfrak{sp}(3)$ has the Cartan subalgebra spanned by $\{\alpha_1,\alpha_2,\alpha_3\}$ which is orthogonal to the maximal root $\rho = (0,0,0,2)$. In this case $\delta = (3,2,1,8)$. The roots coming from $Sp(3)$ are thus embedded canonically in the first three coordinates and the one coming from $U(1)$ corresponds to the last coordinate.

The bundle $L^{(q-7)/2}$ corresponds to $\frac{q-7}{2}(0,0,0,2)$. When adding δ we get $(3,2,1,q+1)$. Therefore

$$\begin{aligned} f(q) = \chi(Z(F_4I), \mathcal{O}(L^{(q-7)/2})) &= \frac{1}{8583708672000}q^{15} + \frac{1}{572247244800}q^{14} \\ &+\frac{1}{245248819200}q^{13} - \frac{13}{245248819200}q^{12} - \frac{59}{204374016000}q^{11} \\ &+\frac{1}{11147673600}q^{10} + \frac{253}{78033715200}q^{9} + \frac{13}{2890137600}q^{8} - \frac{1111}{111476736000}q^{7} \\ &-\frac{541}{22295347200}q^{6} + \frac{23}{9083289600}q^{5} + \frac{8567}{245248819200}q^{4} + \frac{4751}{357654528000}q^{3} \\ &-\frac{29}{1907490816}q^{2} - \frac{1}{113541120}q \end{aligned}$$

Equating the coefficients of the two expressions of the polynomial $f(q)$ we get the intersection pairings, which show a remarkable symmetry.

Theorem 2.1 *Let $u = -c_2(H)$ and $c_4 = c_4(\tilde{E})$ where H and $\tilde{E}$ are the locally defined bundles by the isotropy factors of F_4I. The intersection numbers are the following*

$$u^7 = \frac{39}{256}, \quad c_4u^5 = \frac{3}{256}, \quad c_4^2u^3 = \frac{3}{256}, \quad c_4^3u = \frac{39}{256}.$$

$\square$

3 Cohomology ring

Armed with the intersection numbers of Theorem 2.1 and the Poincaré polynomial of F_4I, we can now compute the generators of $H^*(F_4I)$ and their relations.

- In dimension 4: u is non-degenerate, so it is non-zero in $H^4(F_4I)$.
- In dimension 8: We have two classes u^2 and c_4. Suppose

  $$au^2 + bc_4 = 0.$$

  Then

  $$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc_4^2u^3 &= 0, \\ ac_4^2u^3 + bc_4^3u &= 0, \end{aligned}$$

  which has no non-trivial solutions for a and b when we substitute the intersection numbers. Therefore, u^2 and c_4 generate $H^8(F_4I)$.
- In dimension 12: We have two classes u^3 and c_4u. Suppose

  $$au^3 + bc_4u = 0.$$

  Then we get the same system of equations as above

  $$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc_4^2u^3 &= 0, \\ ac_4^2u^3 + bc_4^3u &= 0, \end{aligned}$$

  which has no non-trivial solutions for a and b. Therefore, u^3 and c_4u generate $H^{12}(F_4I)$.
- In dimension 16: We have three classes u^4, c_4u^2 and c_4^2. Since $H^{16}(F_4I)$ is 2-dimensional, we must find the relation between these classes. Suppose

  $$au^4 + bc_4u^2 + c_4^2 = 0.$$

  Then we get

  $$\begin{aligned} au^7 + bc_4u^5 + c_4^2u^3 &= 0, \\ ac_4u^5 + bc_4^2u^3 + c_4^3u &= 0, \end{aligned}$$

which have a unique solution

$$a = 1, \quad b = -14,$$

so that

$$c_4^2 = -u^4 + 14c_4u^2.$$

Moreover, u^4 and c_4u^2 are linearly independent since

$$au^4 + bc_4u^2 = 0$$

implies

$$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc_4^2u^3 &= 0, \end{aligned}$$

whose only solution is the trivial one. Therefore, u^4 and c_4u^2 generate $H^{16}(F_4I)$.

- In dimension 20: We have three classes u^5, c_4u^3 and c_4^2u. Suppose

$$au^5 + bc_4u^3 + c_4^2u = 0.$$

Then

$$\begin{aligned} au^7 + bc_4u^5 + c_4^2u^3 &= 0, \\ ac_4u^5 + bc_4^2u^3 + c_4^3u &= 0, \end{aligned}$$

which have a unique solution

$$a = 1, \quad b = -14.$$

Thus,

$$c_4^2u = -u^5 + 14c_4u^3,$$

which comes from the relation found in dimension 16. Moreover, u^5 and c_4u^3 are linearly independent since

$$au^5 + bc_4u^3 = 0$$

implies

$$\begin{aligned} au^7 + bc_4u^5 &= 0 \\ ac_4u^5 + bc_4^2u^3 &= 0 \end{aligned}$$

whose only solution is the trivial one. Therefore, u^5 and c_4u^3 generate $H^{20}(F_4I)$.

- In dimension 24: We have four classes u^6, c_4u^4, $c_4^2u^2$ and c_4^3. In this case, $H^{24}(F_4I)$ is 1-dimensional and we see that if

$$au^6 + c_4u^4 = 0,$$

then

$$a = -\frac{1}{13},$$

and the other classes can all be put in terms of u^6

$$\begin{aligned} 13c_4u^4 &= u^6 \\ 13c_4^2u^2 &= u^6 \\ c_4^3 &= u^6. \end{aligned}$$

Hence, we have proved the following.

Theorem 3.1 *Let $u = c_2(H)$ and $c_4 = c_4(\tilde{E})$ where H and $\tilde{E}$ are the locally defined bundles by the isotropy factors of F_4I. The rational comohomology ring of F_4I is*

$$H^*(F_4I, \mathbb{Q}) = \mathbb{R}[u, c_4] / \left(c_4^2 + u^4 - 14c_4u^2, u^6 - 13c_4u^4\right)$$

□

4 Pontrjagin classes and numbers

As a corollary of the intersection numbers and relations we obtain the Pontrjagin numbers of F_4I.

Theorem 4.1 *The Pontrjagin numbers of F_4I are given as follows:*

$$\begin{aligned} p_7 &= 348, \\ p_1^7 &= 2496, \\ p_2^3p_1 &= 8424, \\ p_2p_3p_1^2 &= 4932, \\ p_2^2p_3 &= 5904, \\ p_3^2p_1 &= 3972, \\ p_2^2p_1^3 &= 6192, \\ p_4p_2p_1 &= 4842, \end{aligned}$$

$$
\begin{aligned}
p_3p_1^4 &= 3048, \\
p_2p_1^5 &= 3888, \\
p_6p_1 &= 2091, \\
p_4p_3 &= 2832, \\
p_5p_2 &= 2718, \\
p_4p_1^3 &= 4188, \\
p_5p_1^2 &= 3246,
\end{aligned}
$$

where p_i denotes the i^{th} Pontrjagin class of F_4I.

Proof. This follows from the relations described in the previous section and

$$
\begin{aligned}
p_1 &= 4u \\
p_2 &= 26u^2 - 14c_4 \\
p_3 &= 84u^3 - 76c_4u \\
p_4 &= 281u^4 + 1866c_4u^2 + 65c_4^2 \\
&= 216u^4 + 2776c_4u^2 \\
p_5 &= 720u^5 + 7376c_4u^3 + 576c_4^2u \\
&= 144u^5 + 15440c_4u^3 \\
p_6 &= 1620u^6 + 11864c_4u^4 + 12724c_4^2u^2 - 80c_4^3 \\
&= 44608c_4u^4 \\
p_7 &= 3200u^7 + 10624c_4^2u^3 + 5760c_4u^5 - 2176c_4^3u. \\
&= 348
\end{aligned}
$$

□

5 Torsion-free part of the integral cohomology of F_4I

We can go a little further by revisiting the following result of Ishitoya and Toda [3] about the torsion-free part of the integral cohomology of F_4I.

Theorem 5.1 [3] *The torsion-free part of the integral cohomology of F_4I can be described as follows*

$$
H^*(F_4I,\mathbb{Z})_{tf} = \frac{\mathbb{Z}[f_4,f_8,f_{12}]}{(f_4^3 - 12f_4f_8 + 8f_{12}, f_4f_{12} - 3f_8^2, f_8^3 - f_{12}^2)},
$$

where $\deg(f_i) = i$, $i = 4, 8, 12$.

First, let us observe that $4u = p_1(F_4I)$ is integral and indivisible. If $4u = m\xi$ with $\xi \in H^4(F_4I, \mathbb{Z})$ an indivisible class and m an non-zero integer, then

$$\left(\frac{4u}{m}\right)^7 = \frac{4^3 39}{m^7}$$

should be an integer, which can only happen if $m = \pm 1$. Thus, let us set

$$f_4 = 4u.$$

Taking the relations in Theorem 5.1 we are able to deduce

$$f_{12} = -\frac{1}{8}f_4^3 + \frac{3}{2}f_4 f_8,$$

$$f_8^2 = -\frac{1}{24}f_4^4 + \frac{1}{2}f_4^2 f_8,$$

$$f_4^6 = \frac{104}{11}f_4^4 f_8,$$

so that

$$u^5 f_8 = \frac{33}{128},$$

$$u^3 f_8^2 = \frac{7}{16},$$

$$u f_8^3 = \frac{3}{4}.$$

By setting $f_8 = au^2 + bc_4$ we get three equations

$$\begin{aligned} a^2u^7 + 2abc_4u^5 + b^2c_4^2u^3 &= \frac{7}{16}, \\ au^7 + bc_4u^5 &= \frac{33}{128}, \\ a^3u^7 + 3a^2bc_4u^5 + 3ab^2c_4^2u^3 + b^3c_4^3u &= \frac{3}{4}, \end{aligned}$$

i.e.

$$\begin{aligned} \frac{39}{256}a + \frac{3}{256}b &= \frac{33}{128}, \\ \frac{39}{256}a^2 + \frac{3}{128}ab + \frac{3}{256}b^2 &= \frac{7}{16}, \\ \frac{39}{256}a^3 + \frac{9}{256}a^2b + \frac{9}{256}ab^2 + \frac{39}{256}b^3 &= \frac{3}{4}. \end{aligned}$$

with unique solution

$$a = \frac{5}{3}, \quad b = \frac{1}{3},$$

i.e.

$$f_8 = \frac{5}{3}u^2 + \frac{1}{3}c_4.$$

It is interesting to notice that

$$6f_8 = 10u^2 + 2c_4 = c_4(\tilde{E} \otimes H),$$

so that this class has a geometrical interpretation.

Furthermore, the relations in Theorem 5.1 become

$$\begin{aligned} 0 &= 0, \\ -\frac{1}{3}u^4 + \frac{14}{3}c_4u^2 - \frac{1}{3}c_4^2 &= 0, \\ 96u^6 - 1248u^4c_4 &= 0, \end{aligned}$$

which are in fact just a multiple of the two relations we already had in rational cohomology.

Thus we can rewrite the Theorem 5.1 as follows.

Theorem 5.2 *The torsion-free part of the integral cohomology of F_4I can be described as follows*

$$H^*(F_4I, \mathbb{Z})_{tf} = \frac{\mathbb{Z}[4u, f_8]}{(3f_8^2 + 32u^4 - 24u^2f_8, -26624u^4f_8 + 45056u^6)},$$

where $f_8 = 5/3u^2 + 1/3c_4$ is an integral class, and $f_{12} = -8u^3 + 6uf_8 = 2u^3 + 2c_4u$.

This result can be used to reinterpret the integral cohomology ring of the twistor space $Z(F_4I)$, which is torsion free. In [2], they calculated such a cohomology ring using a Schubert calculus approach. It may be interesting to investigate the geometry arising from that description in combination with the geometry encoded in the Chern classes u and c_4.

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