CELL-LIKE MAPS AND TOPOLOGICAL STRUCTURES

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Abstract. We show that there are homotopy equivalences $h : N \to M$ between closed manifolds which are induced by cell-like maps $p : N \to X$ and $q : M \to X$ but which are not homotopic to homeomorphisms. The phenomenon is based on construction of cell-like maps that kill certain $\mathbb{L}$-classes. In dimension > 5 we identify all such homotopy equivalences to M with a torsion subgroup $\mathcal{S}^{CE}(M)$ of the topological structure group $\mathcal{S}(M)$. In the case of simply connected M with finite $\pi_2(M)$, the subgroup $\mathcal{S}^{CE}(M)$ coincides with the odd torsion in $\mathcal{S}(M)$. For general M, the group $\mathcal{S}^{CE}(M)$ admits a description in terms of the second stage of the Postnikov tower of M. As an application, we show that there exist a contractibility function ρ and a precompact subset $\mathcal{C}$ of Gromov-Hausdorff space such that for every $\epsilon > 0$ there are nonhomeomorphic Riemannian manifolds with contractibility function ρ which lie within ϵ of each other in $\mathcal{C}$.

1. Introduction

In [17] Grove and Petersen proved that for every n the class of n-dimensional closed Riemannian manifolds with sectional curvature bounded below by κ, volume bounded below by v, and the diameter bounded above by D contains only finitely many homotopy types. The main technical lemma in their paper shows that there is a uniform "contractibility function" which applies to all manifolds in such a class.

Definition 1.1. A continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$ and $\rho(t) \geq t$ for all t is a *contractibility function* for a metric space X if there is $R > 0$ such that for each $x \in X$ and $t \leq R$, the t-ball $B_t(x)$ centered at x can be contracted to a point in the $\rho(t)$-ball $B_{\rho(t)}(x)$.

In a second paper, [18], Grove, Petersen, and Wu showed that there are only finitely many homeomorphism types in such class for $n \neq 3$. G. Perelman has removed the $n \neq 3$ restriction [23]. Perelman's work shows (as does the earlier work of Grove-Petersen-Wu for $n \neq 3$) that there is $\epsilon > 0$ so that if M and M' satisfy the stated restrictions on curvature, diameter, and volume with the Gromov-Hausdorff distance $d_{GH}(M, M') < \epsilon$, then M and M' must be homeomorphic. In this paper we show that this is not the case when we relax the

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hypotheses to simply require that the manifolds in question have a common contractibility function.

Our example goes as follows (Theorem 6.2). We construct nonhomeomorphic manifolds M and M' and cell-like maps $q : M \to X$ and $q' : M' \to X$ (see Corollary 2.13). Using a mapping cylinder construction, we construct two sequences of Riemannian manifolds converging to X (supplied with some metrics and a fixed contractibility function) in Gromov-Hausdorff space. One sequence consists of manifolds homeomorphic to M, the other to M'. Such manifolds M and M' must be simple homotopy equivalent by [11] and must have the same rational Pontrjagin classes by Corollary 2.9. One such example can be constructed in which M is the boundary of a regular neighborhood of a Moore space in $\mathbb{R}^8$. We note that by Quinn's uniqueness of resolution theorem it follows that X must be infinite dimensional. Thus, this example ultimately contains a construction of the first author of a cell-like map that raises dimension to infinity [4].

Our main theorem classifies all such examples. We give a complete description saying when a given n-manifold M, $n > 5$, admits a cell-like map $q : M \to X$ together with a twin cell-like map $q' : M' \to X$ such that the induced homotopy equivalence $M' \to M$ is not homotopic to a homeomorphism. If M is simply connected with finite π_2, our classification depends only on odd torsion characteristic class information. The general classification involves surgery theory and the second stage of the Postnikov tower of M. (Theorem 2.4 and Theorem 2.7). The second stage of the Postinikov tower turns out to be relevant because of acyclicity results in the K-theory of Eilenberg-MacLane spaces [1, 2].

2. Surgery and cell-like maps

Definition 2.1.

(i) A compact subset X of the Hilbert cube Q is said to be *cell-like* if for every open neighborhood U of X in Q, the inclusion $X \to U$ is nullhomotopic. This is a topological property of X and is the Čech analog of "contractible". The space $\sin(1/x)$-with-a-bar is an example of a cell-like set which is not contractible.

(ii) A map $f : Y \to Z$ between compact metric spaces is *cell-like or CE* if for each $z \in Z$, $f^{-1}(z)$ is cell-like. The empty set is not considered to be cell-like, so cell-like maps must be surjective.

Cell-like maps with domain a compact manifold or finite polyhedron are weak homotopy equivalences over every open subset of the range. That is, if $c : M \to X$ is cell-like, then for every open $U \subset X$, $c| : c^{-1}(U) \to U$ is a weak homotopy equivalence. The range space of such a cell-like map always has finite cohomological dimension. If the range has finite covering dimension, then c is a homotopy equivalence over every open set.

LIFTING PROPERTY: Let $f : M \to X$ be a cell-like map with M an ANR space. Given a space W with $\dim W < \infty$, a closed subset $A \subset W$, a map $g : W \to X$, and a map $h : A \to M$ with $fh = g|_A$, there is a map $\bar{h} : W \to M$ extending h such that g is homotopic to $f\bar{h}$ rel A:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] & M \arrow[d, "f"] \\ W \arrow[r, "g"] \arrow[ur, "\bar{h}"] & X. \end{tikzcd}$$

See [21] for details.

Definition 2.2. A homotopy equivalence $f : N \to M$ is *realized by cell-like maps* if there exist a space X and cell-like maps $c_1 : N \to X$, $c_2 : M \to X$ so that the diagram

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "CE"'] & & M \arrow[dl, "CE"] \\ & X & \end{tikzcd}$$

homotopy commutes. We will also say that such a homotopy equivalence *factors through cell-like maps* and we will call such manifolds N and M *CE-related.*

In view of the lifting property, every pair of cell-like maps $c_1 : N \to X$, $c_2 : M \to X$ of ANR-spaces induces a unique homotopy equivalence $f : N \to M$. Theorem D of [12] implies that f is in fact a simple homotopy equivalence. We note that the simple homotopy theory is well-defined for ANRs (see [3]) and hence for topological manifolds.

If $\dim X < \infty$, Quinn's uniqueness of resolutions theorem implies that this homotopy equivalence is homotopic to a homeomorphism ([25]).

Two simple homotopy equivalences of closed manifolds $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are called equivalent if there is a homeomorphism $h : N_1 \to N_2$ that produces a homotopy commutative diagram. We recall that the set $\mathcal{S}(M)$ of classes of simple homotopy equivalences $f : N \to M$ is called the set of *topological structures* on M. The structure set $\mathcal{S}(M)$ is functorial and has an abelian group structure defined either by Siebenmann periodicity [20] or by algebraic surgery theory [27]. Ranicki's theory gives the following useful formula.

Proposition 2.3. *Let M^n be a closed topological n-manifold, $n \geq 5$ and let $h : M \to N$ be a simple homotopy equivalence, $[h] \in \mathcal{S}(N)$. Then the isomorphism $h_* : \mathcal{S}(M) \to \mathcal{S}(N)$ is defined by the formula*

$$h_*([f]) = [h \circ f] - [h].$$

By $\mathcal{S}^{CE}(M) \subset \mathcal{S}(M)$ we denote the subset of structures realized by cell-like maps.

Theorem 2.4. *Let M^n be a closed simply connected topological n-manifold with finite $\pi_2(M)$, $n > 5$. Then $\mathcal{S}^{CE}(M)$ is the odd torsion subgroup of $\mathcal{S}(M)$.*

We recall the Sullivan-Wall surgery exact sequence [29] for topological manifolds:

$$L_{n+1}(\mathbb{Z}\pi_1(M)) \to \mathcal{S}(M) \xrightarrow{\eta} [M, \mathrm{G/Top}] \xrightarrow{\theta} L_n(\mathbb{Z}\pi_1(M)).$$

The homomorphism η is called the *normal invariant* and the homomorphism θ is called the *surgery obstruction.* The Sullivan-Wall surgery exact sequence was extended by Quinn and Ranicki to the following functorial exact sequence:

$$\dots L_{n+1}(\mathbb{Z}\pi_1(M)) \to \mathcal{S}_n(M) \xrightarrow{\eta'} H_n(M;\mathbb{L}) \xrightarrow{\theta'} L_n(\mathbb{Z}\pi_1(M))\dots$$

where $H_n(M;\mathbb{L}) = H^0(M;\mathbb{L}) = [M, \mathrm{G/Top}] \times \mathbb{Z}$ and $\eta'|_{\mathcal{S}(M)} = \eta$. The homomorphism θ' is called the *assembly map* for M. This sequence is defined and functorial when M is a finite polyhedron and this was extended to more general spaces in [31]. We write $L_n = L_n(\mathbb{Z})$ and recall that $L_n = \mathbb{Z}$ if $n = 4k$, $L_n = \mathbb{Z}_2$ if $n = 4k+2$, and $L_n = 0$ for odd n.

In general, Ranicki's algebraic surgery functor gives us a long exact sequence

$$\dots \to \mathcal{S}_n(P,\, Q) \to H_n(P,\, Q;\mathbb{L}) \to L_n(\mathbb{Z}\pi_1 P,\, \mathbb{Z}\pi_1 Q) \to \dots$$

for any CW pair $(P,\, Q)$. If P happens to be a compact n-dimensional manifold, then $\mathcal{S}_n(P)$ is the usual rel boundary structure set when P has nonempty boundary and differs from the usual geometrically defined structure set by at most a $\mathbb{Z}$ in the closed case. We also have a long exact sequence

$$\dots \to \mathcal{S}_{n+1}(P,\, Q) \to \mathcal{S}_n(Q) \to \mathcal{S}_n(P) \to \mathcal{S}_n(P,\, Q) \to \dots$$

where for an n-dimensional manifold with boundary $(P,\, \partial P)$, $\mathcal{S}_n(P,\, \partial P)$ is the *not* rel boundary structure set. There is also a long exact sequence of L-groups.

All of these sequences are 4-periodic. If $Q \to P$ induces an isomorphism on π_1, then $\mathcal{S}_k(P,Q) \cong H_k(P,\, Q;\mathbb{L})$ because the Wall groups $L_*(\mathbb{Z}\pi_1 P,\, \mathbb{Z}\pi_1 Q)$ are zero. Composing this isomorphism with the boundary map in Ranicki's exact sequence, we have a homomorphism $\partial' : H_{k+1}(P,\, Q;\mathbb{L}) \to \mathcal{S}_k(Q)$. For a closed n-manifold there is a split epimorphism $p : \mathcal{S}_n(M) \to \mathcal{S}(M) \to 0$ with the kernel $\mathbb{Z}$ or 0 depending on M. The following statement is classical. Apply the $\pi - \pi$ theorem to the pair (M, pt).

Proposition 2.5. *For a simply connected closed n-manifold M the (reduced) normal invariant $\bar{\eta} : \mathcal{S}_n(M) \to \bar{H}_n(M, \mathbb{L})$ is an isomorphism.*

To state the main theorem for non-simply connected manifolds we need the following.

Definition 2.6. If K is a CW complex, let $E_2(K)$ be the CW complex obtained from K by attaching cells in dimensions 4 and higher to kill the homotopy groups of K in dimensions 3 and above. Thus, $K \subset E_2(K)$, $\pi_i(E_2(K)) = 0$ for $i \geq 3$, and $E_2(K) - K$ consists of cells of dimension ≥ 4. Note that $E_2(K)$ will not, in general, be a finite complex. The space $E_2(K)$ is called *the second stage of the Postnikov tower of* K.

Let M be a closed n-manifold. We denote by

$$\delta : H_{n+1}(E_2(M), M; \mathbb{L}) \to \mathcal{S}(M)$$

the composition:

$$H_{n+1}(E_2(M), M; \mathbb{L}) \cong \mathcal{S}_{n+1}(E_2(M), M; \mathbb{L}) \xrightarrow{\partial} \mathcal{S}_n(M) \xrightarrow{p} \mathcal{S}(M).$$

Let $\phi : A \to B$ be a homomorphism of abelian groups. By $\phi^T : \operatorname{Tor} A \to \operatorname{Tor} B$ we denote the restriction $\phi|_{\operatorname{Tor} A}$ of ϕ to the torsion subgroups and by $\phi_{[p]} : A_{[p]} \to B_{[p]}$ denote the localization of ϕ away from p.

Theorem 2.7. *Let M^n be a closed topological n-manifold, $n > 5$. Then $\mathcal{S}^{CE}(M) = \operatorname{im}(\delta^T_{[2]})$. In particular, $\mathcal{S}^{CE}(M)$ is a subgroup of the odd torsion of $\mathcal{S}(M)$.*

We note that Theorem 2.4 is a consequence of Theorem 2.7.

Corollary 2.8. *Let $f_* : \mathcal{S}(M) \to \mathcal{S}(N)$ be the induced homomorphism for a continuous map $f : M \to N$ between two closed n-manifolds, $n > 5$. Then $f_*(\mathcal{S}^{CE}(M)) \subset \mathcal{S}^{CE}(N)$.*

Corollary 2.9. *Let $f : N \to M$ be a homotopy equivalence between closed n-manifolds with $n \geq 6$ that is realized by cell-like maps. Then f preserves rational Pontryagin classes.*

Corollary 2.10. *'To be CE-related' is an equivalence relation on closed n-manifolds, $n > 5$.*

Proof. We prove transitivity. Let M_1 be CE-related to M_2 and M_2 CE-related to M_3. Let $h_1 : M_1 \to M_2$ and $h_2 : M_2 \to M_3$ be corresponding homotopy equivalences. It suffices to show that the composition $h_2 h_1$ is realized by cell-like maps. In view of Corollary 2.8 we have $(h_2)_*([h_1]) \in \mathcal{S}^{CE}(M_3)$ and hence by the formula for the induced homomorphism (Proposition 2.3) we obtain that $[h_2 h_1] = [h_2] + (h_2)_*([h_1]) \in \mathcal{S}^{CE}(M_3)$. □

In special cases, it is not hard to understand the map $H_{n+1}(E_2(M), M; \mathbb{L}) \to \mathcal{S}(M)$ well enough to get concrete "rigidity" and "flexibility" results. We begin with two typical rigidity statements.

Corollary 2.11. *If M^n is a closed manifold with $n \geq 6$ and either*

(i) *M is aspherical or*
(ii) *M is homotopy equivalent to a complex projective space*

then any homotopy equivalence $f : N \to M$ that factors through cell-like maps is homotopic to a homeomorphism.

Proof. If M is aspherical, then $M = E_2(M)$ and $H_{n+1}(E_2(M), M; \mathbb{L}) = 0$, so structures in the image of $H_{n+1}(E_2(M), M; \mathbb{L}) = 0$ are trivial.

If M is homotopy equivalent to CP^k, then $E_2(M) = CP^\infty$. But $H_{n+1}(CP^\infty, CP^k; \mathbb{L}) = \lim_{\ell \to \infty} H_{n+1}(CP^\ell, CP^k; \mathbb{L})$, which has no odd torsion, so no nontrivial element of $\mathcal{S}(M)$ can be the image of an odd torsion element. See Lemma 2.12 below. □

Lemma 2.12. *If (K, L) is a finite CW pair and $H_*(K, L; \mathbb{Z})$ has no torsion, then $H_*(K, L; \mathbb{L})$ has no odd torsion.*

Proof. For finite CW pairs, $H_n(K, L; \mathbb{L}) \otimes \mathbb{Q} \cong \oplus_k H_{n-4k}(K, L; \mathbb{Q})$. Comparing this to the Atiyah-Hirzebruch spectral sequence gives the result, since there can be no nonzero differentials between terms of the form $H_p(K, L; L_{4k})$ at the E_2-level. □

Corollary 2.13. *There are closed nonhomeomorphic 7-dimensional manifolds M and N which are CE-related.*

Proof. Let $p \geq 5$ be a prime number. The Moore complex $P = S^3 \cup_p B^4$ can be PL-embedded in $\mathbb{R}^8$ (see for example [9]). Let $M = \partial W$ be the boundary of a regular neighborhood W of P. Then M is 2-connected since every homotopy in W of a sphere of dimension ≤ 2 to a constant map can be pushed off the core P by general position and retracted to M. By the Lefchsetz duality, $H_3(W, M) = H^5(W) = H^5(P) = 0$ and $H_4(W, M) = H^4(W) = H^4(P) = \mathbb{Z}_p$. The exact sequence of the pair (W, M) turns into the following:

$$0 \to \mathbb{Z}_p \to H_3(M) \to \mathbb{Z}_p \to 0.$$

By the Atiyah-Hirzebruch spectral sequence $\bar{H}_3(M; \mathbb{L})$ consists of nontrivial p-torsion. Take a nontrivial p-torsion element $\alpha \in \bar{H}_7(M; \mathbb{L}) \cong \bar{H}_3(M; \mathbb{L})$. Let $\beta = p(\eta')^{-1}(\alpha)$ where $\eta' : \mathcal{S}_7(M) \to \bar{H}_7(M; \mathbb{L})$ is an isomorphism by Proposition 2.5 and $p : \mathcal{S}_n(M) \to \mathcal{S}(M)$ is the projection. Since the kernel of p is torsion free, $\beta \neq 0$. Thus, by Theorem 2.4, β defines a homotopy equivalence $f : N \to M$ that belongs to $\mathcal{S}^{CE}(M)$.

It remains to show that N is not homeomorphic to M. Rationally the normal invariant measures the difference in L-polynomials. Since, $L_1 = \frac{1}{3}p_1$ where p_1 is an integral Pontryagin class, for 7-dimensional manifolds L-classes live in cohomology with coefficients in $\mathbb{Z}[\frac{1}{3}]$. Then in our case the normal invariant with coefficients in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ is the difference of L_1-classes. By the construction M is stably parallizable and hence it has zero Pontryagin classes. Since $p \geq 5$, the normal invariant $\eta(f)$ with coefficients in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ is nonzero. Hence the first class $L_1(N) \neq 0$. Novikov's theorem on the topological invariance of rational Pontryagin classes in fact proves the topological invariance of L-classes. Then by Novikov's theorem these manifolds cannot be homeomorphic. □

Corollary 2.14. *If L and L' are homotopy equivalent odd order lens spaces, then $L \times S^{4k+3}$ and $L' \times S^{4k+3}$ are CE-related. It follows that $L \times S^{4k+3}$ and $L' \times S^{4k+3}$ can be deformed to be arbitrarily close to each other in Gromov-Hausdorff space and that this deformation can be performed through Riemannian metrics which preserve a contractibility function. There exist homotopy equivalent odd order lens spaces L and L' so that $L \times S^5$ and $L' \times S^5$ are not CE related.*

The proof of this Corollary will appear elsewhere.

3. Cell-like maps that kill $\mathbb{L}$-classes

We need the following facts [1], [2], [32].

Theorem 3.1. $\widetilde{KO}_*(K(\pi, n); \mathbb{Z}_p) = 0$, *$n \geq 3$, for any group π and* $\widetilde{KO}_*(K(\pi, n)) = 0$, *$n \geq 2$, for finite π.*

Let $M(p)$ denote the $\mathbb{Z}_p$ Moore spectrum. The chain of homotopy equivalences of spectra for odd p,

$$\widetilde{KO}_* \wedge M(p) \sim \widetilde{KO}_*[\frac{1}{2}] \wedge M(p) \sim \mathbb{L}[\frac{1}{2}] \wedge M(p) \sim \mathbb{L} \wedge M(p)$$

implies the following:

Corollary 3.2. *Let p be odd, then $\bar{H}_*(K(\mathbb{Z}_p, 2); \mathbb{L} \wedge M(p)) = 0$ where $\mathbb{L} \wedge M(p)$ is $\mathbb{L}$-theory with coefficients in $\mathbb{Z}_p$.*

We recall that for an extraordinary homology theory given by a spectrum $\mathbb{E}$ of CW complexes there are the following Universal Coefficient Formulas for coefficients $\mathbb{Z}_p$ and $\mathbb{Q}$:

$$0 \to H_n(K; \mathbb{E}) \otimes \mathbb{Z}_p \to H_n(K; \mathbb{E} \wedge M(p)) \to H_{n-1}(K; \mathbb{E}) * \mathbb{Z}_p \to 0$$

and

$$H_n(K; \mathbb{E}_{(0)}) = H_n(K; \mathbb{E}) \otimes \mathbb{Q}.$$

Here $H * \mathbb{Z}_p = \{c \in H \mid pc = 0\}$ and $\mathbb{E}_{(0)}$ denotes the localization at 0. Let $X = \varprojlim\{K_i\}$ be a compact metric space presented as the inverse limit of finite polyhedra. By $\check{H}_*(X; \mathbb{E}) = \varprojlim\{H_*(K_i, \mathbb{E})\}$ we denote the Čech $\mathbb{E}$-homology. The Steenrod homology $H_n(X; \mathbb{E})$ of X fits into the following exact sequence

$$0 \to \lim{}^1\{H_{n+1}(K_i; \mathbb{E})\} \to H_n(X; \mathbb{E}) \to \check{H}_n(X; \mathbb{E}) \to 0.$$

If $H_k(pt; \mathbb{E})$ is finitely generated for each k, the Mittag-Lefler condition holds with rational or finite coefficients, so we have

$$H_n(X; \mathbb{E} \wedge M(p)) = \check{H}_n(X; \mathbb{E} \wedge M(p)) \quad \text{and} \quad H_n(X; \mathbb{E}_{(0)}) = \check{H}_n(X; \mathbb{E}_{(0)}).$$

In the case of $\mathbb{Z}_p$-coefficients we obtain an exact sequence which is natural in X:

$$(**) \quad 0 \to \varprojlim(H_n(K_i; \mathbb{E}) \otimes \mathbb{Z}_p) \to H_n(X; \mathbb{E} \wedge M(p)) \xrightarrow{\phi'} \check{H}_{n-1}(X; \mathbb{E}) * \mathbb{Z}_p.$$

Lemma 3.3. *Let M be a simply connected finite complex with finite $\pi_2(M)$. Then for every element $\gamma \in H_k(M; \mathbb{L})$ of odd order p there exists an odd torsion element $\alpha \in H_{k+1}(E_2(M), M; \mathbb{L})$ such that $\partial(\alpha) = \gamma$ where ∂ is the connecting homomorphism in the exact sequence of the pair $(E_2(M), M)$.*

Proof. Note that $E_2(M) = K(\pi_2(M), 2)$.

If $\pi_2(M) = 0$, the space $E_2(M)$ is contractible and the lemma is trivial.

If $\pi_2(M)$ is torsion, then in view of Theorem 3.1, $\bar{H}_*(E_2(M); \mathbb{L} \wedge M(p)) = 0$. Then by the Universal Coefficient diagram

$$\begin{tikzcd}
H_{k+2}(E_2(M), M; \mathbb{L} \wedge M(p)) \arrow[r, "epi"] \arrow[d, "iso"', "\partial"] & H_{k+1}(E_2(M), M; \mathbb{L}) * \mathbb{Z}_p \arrow[r, "mono"] & H_{k+1}(E_2(M), M; \mathbb{L}) \arrow[d, "\partial"] \\
H_{k+1}(M; \mathbb{L} \wedge M(p)) \arrow[r, "epi"] & H_k(M; \mathbb{L}) * \mathbb{Z}_p \arrow[r, "mono"] & H_k(M; \mathbb{L})
\end{tikzcd}$$

we obtain the required result. □

Let $q : M \to X$ be a cell-like map. According to Proposition 3.5 for every map $h : M \to E_2(M)$ there is a map $g : X \to E_2(M)$ such that $g \circ q$ is homotopic to h. In particular, there is an induced map $\tilde{g} : M_q \to M_h$ between their mapping cylinders, $\tilde{g}|_M = id_M$, $\tilde{g}|_X = g$. We apply this when h is the inclusion $j : M \subset E_2(M)$ and denote the induced map by $i : M_q \to M_j$. Denote by

$$i_* : H_*(M_q, M; \mathbb{L}) \to H_*(E_2(M), M; \mathbb{L})$$

the induced homomorphism for the Steenrod $\mathbb{L}$-homology groups [10], [19].

Theorem 3.4. *Let M^n be a closed connected topological n-manifold, $n \geq 6$, let p be odd, and let $\beta \in H_*(E_2(M), M; \mathbb{L} \wedge M(p))$, then there exist a cell-like map $q : M \to X$ and an element $\widehat{\beta} \in H_*(M_q, M; \mathbb{L} \wedge M(p))$ such that $i_*(\widehat{\beta}) = \beta$.*

The proof of Theorem 3.4 will follow Lemma 3.11. The following proposition is proven in [30] Appendix B.

Proposition 3.5. *Let E be a CW complex with trivial homotopy groups $\pi_i(E) = 0$, $i \geq k$ for some k, and let $q : X \to Y$ be a cell-like map between compacta. Then q induces a bijection of the homotopy classes $q^* : [Y, E] \to [X, E]$.*

Lemma 3.6. *Let M^n be a closed connected topological n-manifold, $n \geq 6$. If $\alpha \in H_*(E_2(M), M; \mathbb{L})$ is an odd torsion element, then there exist a cell-like map $q : M \to X$ and an odd order element $\widehat{\alpha} \in H_*(M_q, M; \mathbb{L})$ such that $i_*(\widehat{\alpha}) = \alpha$.*

Proof. Let $\alpha \in H_k(E_2(M), M; \mathbb{L})$ be an element of order p where p is an odd number. Then by the universal coefficient formula, there is an epimorphism

$$\phi : H_{k+1}(E_2(M), M; \mathbb{L} \wedge M(p)) \to H_k(E_2(M), M; \mathbb{L}) * \mathbb{Z}_p.$$

Note that $H * \mathbb{Z}_p = \{c \in H \mid pc = 0\}$ so that there is an inclusion $H * \mathbb{Z}_p \subset H$ which is natural in H. Thus, $\alpha \in H_k(E_2(M), M; \mathbb{L}) * \mathbb{Z}_p$. Hence, there is an element $\beta \in H_{k+1}(E_2(M), M; \mathbb{L} \wedge M(p))$ such that $\phi(\beta) = \alpha$. By Theorem 3.4 there exist a cell-like map $q : M \to X$ and an element $\widehat{\beta}$ such that $i_*(\widehat{\beta}) = \beta$. The commuting diagram of universal coefficient formulas gives (see **)

$$\begin{tikzcd}
H_{k+1}(M_q, M; \mathbb{L} \wedge M(p)) \arrow[r, "\phi'"] \arrow[d, "i_*"] & H_k(M_q, M; \mathbb{L}) * \mathbb{Z}_p \arrow[r, "\subset"] & H_k(M_q, M; \mathbb{L}) \arrow[d, "i_*"] \\
H_{k+1}(E_2(M), M; \mathbb{L} \wedge M(p)) \arrow[r, "\phi"] & H_k(E_2(M), M; \mathbb{L}) * \mathbb{Z}_p \arrow[r, "\subset"] & H_k(E_2(M), M; \mathbb{L})
\end{tikzcd}$$

which implies that $i_*(\hat{\alpha}) = \alpha$ where $\widehat{\alpha} = \phi'(\widehat{\beta})$ is an element of order p. □

REMARK. By Proposition 3.5 a cell-like map induces a rational isomorphism on $\mathbb{L}$-homology. Therefore, the group $H_*(M_q, M; \mathbb{L})$ consists of torsions.

Theorem 3.7. *Let M^n be a closed simply connected topological n-manifold, $n \geq 6$, with $\pi_2(M)$ finite. Then for every odd torsion element $\gamma \in H_*(M; \mathbb{L})$ there is a cell-like map $q : M \to X$ such that $q_*(\gamma) = 0$.*

Proof. By Lemma 3.3 there is an odd torsion element $\alpha \in H_*(E_2(M), M; \mathbb{L})$ such that $\partial(\alpha) = \gamma$. By Theorem 3.6 there exists a cell-like map $q : M \to X$ and an element $\widehat{\alpha} \in H_*(M_q, M; \mathbb{L})$ such that $i_*(\widehat{\alpha}) = \alpha$. Then the commutative diagram

$$\begin{tikzcd} H_{*+1}(M_q, M; \mathbb{L}) \arrow[r] \arrow[d, "i_*"] & H_*(M; \mathbb{L}) \arrow[r, "q_*"] \arrow[d, "="] & H_*(X; \mathbb{L}) \arrow[d] \\ H_{*+1}(E_2(M), M; \mathbb{L}) \arrow[r] & H_*(M; \mathbb{L}) \arrow[r] & H_*(E_2(M); \mathbb{L}) \end{tikzcd}$$

implies that $q_*(\gamma) = 0$. □

REMARK. Without the finiteness assumption on $\pi_2(M)$ one can show that q kills an element $\gamma \otimes 1_{\mathbb{Z}_p}$ with $\mathbb{Z}_p$ coefficients.

We recall that the cohomological dimension of a topological space X with respect to the coefficient group G is the following number

$$c - \dim_G X = \max\{n \mid \check{H}^n(X, A; G) \neq 0 \ \text{ for some closed } \ A \subset X\}.$$

A map of pairs $f : (X, L) \to (Y, L)$ is called *strict* if $f(X - L) = Y - L$ and $f|_L = id_L$.

The following theorem is taken from [6] (Theorem 7.2). For $G = \mathbb{Z}$ it can be found in [7].

Theorem 3.8. *Let $\tilde{h}_*$ be a reduced generalized homology theory. Suppose that $\tilde{h}_*(K(G, n)) = 0$ for some countable abelian group G. Then for any finite polyhedral pair (K, L) and any element $\alpha \in \tilde{h}_*(K, L)$ there is a compactum $Y \supset L$ and a strict map $f : (Y, L) \to (K, L)$ such that*

(i) *$c - \dim_G(Y - L) \leq n$;*
(ii) *$\alpha \in \operatorname{im}(f_*)$.*

The following is a relative version of Theorem 3 from [5].

Lemma 3.9. *Let (Y, L) be a pair of compacta such that*

$$c - \dim_{\mathbb{Z}_p}(Y - L) \leq 2 \quad \text{and} \quad c - \dim_{\mathbb{Z}[\frac{1}{p}]}(Y - L) \leq 2.$$

Then there is a strict cell-like map $g : (Z, L) \to (Y, L)$ such that $\dim(Z - L) \leq 3$ and $\dim(Z - L)^2 \leq 5$.

Lemma 3.10. *Let (Z, M) be a compact pair such that $\dim(Z - M)^2 \leq 2n - 1$ and let M be a manifold of dimension $2n$. Suppose there is a retraction $\rho : Z \to M$. Then ρ is homotopic $\operatorname{rel} M$ to a retraction $r : Z \to M$ with $r|_{(Z-M)}$ one-to-one.*

Proof. The condition $\dim X^2 \leq 2n - 1$ for a compact metric space X implies that every continuous map $\phi : X \to M$ to a $2n$-dimensional manifold can be approximated by embedding [8],[28]. Moreover, the space of embeddings $\operatorname{Emb}(X, M)$ is a dense G_δ in the space of mappings $\operatorname{Map}(X, M)$. The same argument shows that under the condition $\dim(Z - M)^2 \leq 2n - 1$ the space of retraction-embeddings $\operatorname{rEmb}(Z, M)$ is dense in the space of retractions $\operatorname{Ret}(Z, M)$. □

The following lemma is proven in [7] Lemma 3.7.

Lemma 3.11. *Let Z be a compact and $r : Z \to M$ be a retraction with $r|_{(Z-M)}$ one-to-one. Let $g : (Z, M) \to (Y, M)$ be a continuous map which is identity over M. Then the decomposition of M whose nondegenerate elements are $r(g^{-1}(y))$ is upper semicontinuous.*

Proof of Theorem 3.4.

We consider the generalized homology theory $h_* = \mathbb{L} \wedge M(p)$, i.e., $\mathbb{L}$-theory with coefficients in $\mathbb{Z}_p$.

Let $\beta \in h_{k+1}(E_2(M), M)$. There is a finite complex K, $M \subset K \subset E_2(M)$, and an element $\gamma \in h_*(K, M)$ such that γ is taken to β by the inclusion homomorphism.

Note that $\tilde{h}_*(K(\mathbb{Z}[\frac{1}{p}], 2)) = 0$ since the $\mathbb{L}$-theory of this space is p-divisible. Taking into account Corollary 3.2 and Theorem 3.1 we can state that $\tilde{h}_*(K(G, 2)) = 0$ for $G = \mathbb{Z}_p \oplus \mathbb{Z}[\frac{1}{p}]$. Then we apply Theorem 3.8 to (K, M) and γ with this G to obtain $f : (Y, M) \to (K, M)$ satisfying the conditions (i)-(ii) of Theorem 3.8. Condition (i) allows us to apply Lemma 3.9 to obtain a cell-like map $g : (Z, M) \to (Y, M)$ with $\dim(Z - M) \le 3$ and $\dim(Z - M)^2 \le 5$.

Because $E_2(M) - M$ has no cells of dimension ≤ 3, there is a homotopy of $f \circ g$ rel M that sweeps $Z - M$ to M. Thus, $f \circ g$ is homotopic to a retraction $\rho : Z \to M$. By Lemma 3.10, $f \circ g$ is homotopic rel M to a retraction $r : Z \to M$ which is one-to one on $Z - M$. By Lemma 3.11 the decomposition of M into $r(g^{-1}(y))$ and singletons defines a cell-like map $q : M \to X$ such that there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "r"] \arrow[d, "g"] & M \arrow[d, "q"] \\ Y \arrow[r, "r'"] & X \end{tikzcd}$$

By Proposition 3.5 there is a map $g' : X \to E_2(M)$ such that $g' \circ q$ is homotopic to the inclusion $M \subset E_2(M)$. Hence $f \circ g \sim r \sim g' \circ q \circ r = g' \circ r' \circ g$. Since g is cell-like, the map f is homotopic to $g' \circ r'$ by Proposition 3.5. Then there is a homotopy commutative diagram of the mapping cylinders

$$\begin{tikzcd} M_{j'} \arrow[r, "r'"] \arrow[d, "f"] & M_q \arrow[dl, "i"] \\ M_j & \end{tikzcd}$$

where $j : M \to E_2(M)$ and $j' : M \to Y$ are the embeddings. For Steenrod h_*-homology this gives us the following diagram:

$$\begin{tikzcd} \tilde{h}_*(Y, M) \arrow[r, "="] \arrow[d, "f_*"] & \tilde{h}_*(M_{j'}, M) \arrow[r, "r'_*"] \arrow[d] & \tilde{h}_*(M_q, M) \arrow[dl, "i_*"] \\ \tilde{h}_*(E_2(M), M) \arrow[r] & \tilde{h}_*(M_j, M) & \end{tikzcd}$$

By condition (ii) of Theorem 3.8 there is $\gamma' \in \tilde{h}_*(Y, M)$ such that $f_*(\gamma') = \gamma$. Then $i_*(\widehat{\beta}) = \beta$ where $\widehat{\beta} = r'_*(\gamma')$. □

4. Continuously controlled topology and cell-like maps of simply connected manifolds

Let $g : Z \to X$ be a proper map and let $Y = \bar{X} - X$ be the corona of a compactification $\bar{X}$ of X. Then there is a natural compactification $\bar{Z}$ of Z with corona Y such that the map g extends to a strict map $\bar{g} : (\bar{Z}, Y) \to (\bar{X}, Y)$. We recall that a map of pairs $f : (Z, Y) \to (Z', Y)$ is strict if $(Z - Y) \subset Z' - Y$ and $f|_Y = id_Y$. A proper homotopy $f_t : Z \to X$ which is strict at each level is called *strict* if the homotopy $\bar{f}_t : (\bar{Z}, Y) \to (\bar{X}, Y)$ is continuous.

Let X be a locally compact space compactified by a compact corona $Y = \bar{X} - X$. A proper map $f : Z \to X$ is a *strict homotopy equivalence* if there is a proper map $g : X \to Z$ such that $g \circ f$ and $f \circ g$ are strict homotopic to $id_{\bar{Z}}$ and $id_{\bar{X}}$ respectively where Z is given a compactification as above.

Definition 4.1.

(i) Let X be an open manifold and let Y be a compact corona of a compactification $\bar{X}$ of X. Two strict homotopy equivalences $f : W \to X$ and $f' : W' \to X$ are *equivalent* if there is a homeomorphism $h : W \to W'$ such that $f = f' \circ h$.

(ii) The set of the equivalence classes of strict homotopy equivalences of manifolds is called the set of *continuously controlled structures* on X at Y and it is denoted as $\mathcal{S}^{cc}(\bar{X}, Y)$.

We note that if $\tilde{X}$ is another compactification of X with a compact corona Y' which is dominated by $\bar{X}$, i.e., there is a continuous map $\phi : \bar{X} \to \tilde{X}$ which is the identity on X, then there is a map $\phi_* : \mathcal{S}^{cc}(\bar{X}, Y) \to \mathcal{S}^{cc}(\tilde{X}, Y')$.

Definition 4.2. A pair (X, Y) is said to be *locally 1-connected at* Y if for each $y \in Y$ and neighborhood U of y in X there is a smaller neighborhood V of y in X so that the inclusion-induced map $\pi_1(V - Y) \to \pi_1(U - Y)$ is zero.

Proposition 4.3. *Let X be an open manifold of dimension $n \geq 5$ compactified by a compact corona Y in such a way that the pair $(\bar{X}, Y)$ is locally 1-connected. Then there is a surgery exact sequence*

$$\cdots \to \bar{H}_n(Y; \mathbb{L}) \to \mathcal{S}^{cc}(\bar{X}, Y) \to [X, \mathrm{G/Top}] \to \tilde{H}_{n-1}(Y; \mathbb{L})$$

which is natural with respect to compactification dominations. Here $\bar{H}_*(-; \mathbb{L})$ is reduced Steenrod $\mathbb{L}$-homology.

Proof. This sequence can be obtained by adjusting the bounded surgery theory of [14] to a continuously controlled case. It is presented in [22] in a form where the side terms are

Ranincki-Wall L-groups of the continuously controlled additive category $\mathcal{B}(\bar{X}, Y; \mathbb{Z})$. Theorem 2.4 of [22] states that these terms are in fact the reduced Steenrod $\mathbb{L}$-homology groups of the corona.

The naturality follows from the definition of the continuously controlled category . $\square$

A UV^1-map is a proper surjection with Čech simply connected point-inverses. See [21] for details. Let M be a closed simply connected n-manifold and let $q : M \to Y$ be a UV^1-map. Then the mapping cone C_q is a compactification of $M \times \mathbb{R}$ by $Y_+ = Y \sqcup pt$ which is locally 1-connected at Y_+. Since $(C_q - Y_+)$ is homotopy equivalent to M and $\bar{H}_*(Y_+; \mathbb{L}) = H_*(Y; \mathbb{L})$, the controlled surgery exact sequence turns into the following

$$\cdots \to H_{n+1}(Y; \mathbb{L}) \to \mathcal{S}^{cc}(C_q, Y_+) \to [M, \mathrm{G/Top}] \to H_n(Y; \mathbb{L}).$$

Let $\tilde{\mathbb{L}}$ be the connected cover of the spectrum $\mathbb{L}$. Note that it is a loop spectrum and $\tilde{L}_0 = \mathrm{G/Top}$. By Poincare duality, $[M, \mathrm{G/Top}] = H^0(M, \tilde{\mathbb{L}}) = H_n(M, \tilde{\mathbb{L}})$. The n-th homotopy group $\mathcal{S}^{cc}_n(C_q, Y_+)$ of the fiber of the controlled assembly map of spectra $\mathbb{H}_*(M; \mathbb{L}) \to \mathbb{H}_*(Y; \mathbb{L})$ differs from $\mathcal{S}^{cc}(C_q, Y_+)$ by at most a copy of $\mathbb{Z}$.

The next proposition follows from Browder's $M \times \mathbb{R}$ theorem and the h-cobordism theorem.

Proposition 4.4. *Let M be simply connected and let ΣM denote the unreduced suspension over M with the suspension points S^0. Then $\mathcal{S}^{cc}(\Sigma M, S^0) \cong \mathcal{S}(M)$.*

We note that the suspension ΣM can be treated as the mapping cone of the constant map. Therefore for every surjective map $q : M \to Y$ there is an induced map

$$\mathcal{S}^{cc}(C_q, Y_+) \to \mathcal{S}^{cc}(\Sigma M, S^0)$$

which we call the *forget control* map.

Proposition 4.5. *Let $q : M \to X$ be a UV^1 map of a simply connected n-manifold and let M_q be the mapping cylinder, then there is a commutative diagram:*

$$\begin{tikzcd}
\mathcal{S}^{cc}(C_q, X_+) \arrow[r, "split-mono"] \arrow[d, "forget"] & \mathcal{S}^{cc}_{n+1}(C_q, X_+) \arrow[r, "\tilde{\eta}"] \arrow[d, "forget"] & H_{n+1}(M_q, M; \mathbb{L}) \arrow[d, "\bar{\partial}"] \\
\mathcal{S}(M) \arrow[r, "split-mono"] & \mathcal{S}_n(M) \arrow[r, "\bar{\eta}"] & \bar{H}_n(M; \mathbb{L})
\end{tikzcd}$$

where $\tilde{\eta}$ and $\bar{\eta}$ are isomorphisms.

Proof. Consider the diagram of spectra

$$\begin{tikzcd}
\mathbb{H}_{*+1}(M_q, M; \mathbb{L}) \arrow[r] \arrow[d, "\bar{\partial}"] & \mathbb{H}_*(M; \mathbb{L}) \arrow[r] \arrow[d, "id"] & \mathbb{H}_*(X; \mathbb{L}) \arrow[d, "const"] \\
\bar{\mathbb{H}}_*(M; \mathbb{L}) \arrow[r] & \mathbb{H}_*(M; \mathbb{L}) \arrow[r] & \mathbb{H}_*(pt; \mathbb{L})
\end{tikzcd}$$

and compare it in dimension n with the diagram defined by the quotient map $p : C_q \to \Sigma M$ that collapses X to a point

$$\begin{tikzcd}
\mathcal{S}^{cc}(C_q, X_+) \arrow[r] \arrow[d, "p_*"] & {[M, \mathrm{G/Top}]} \arrow[r] \arrow[d, "="] & H_n(X;\mathbb{L}) \arrow[d] \\
\mathcal{S}^{cc}(C_{const}, pt_+) \arrow[r] & {[M, \mathrm{G/Top}]} \arrow[r] & H_n(pt;\mathbb{L})
\end{tikzcd}$$

to obtain the required commutative diagram. By Proposition 2.5 $\bar\eta$ is an isomorphism. The exact sequence of pair (M_q, M) implies that $\bar\eta$ is an isomorphism.

□

Proposition 4.6. *Let $q : M \to X$ be a cell-like map of a simply connected closed manifold M. Then*

(1) *$\mathcal{S}^{cc}(C_q, X_+)$ is generated by strict maps $f : (C_p, X) \to (C_q, X)$ where $p : N \to X$ is a cell-like map.*
(2) *The forget control map takes f to the homotopy equivalence $h : N \to M$ which factors through the cell-like maps q and p.*
(3) *$\eta' : \mathcal{S}^{cc}(C_q, X_+) \to H_{n+1}(M_q, M;\mathbb{L})$ is an isomorphism.*

Proof. (1) Follows from Quinn's end theorem [25] and the h-cobordism theorem.
(2) Obvious.
(3) We omit the proof of this fact since we do not use it in the paper.

□

Proof of Theorem 2.4. ($\mathrm{Tor}^{odd}(\mathcal{S}(M)) \subset \mathcal{S}^{CE}(M)$.)
We are given an odd torsion element $\alpha \in \mathcal{S}(M)$. We denote by the same letter α the corresponding element of $\mathcal{S}_n(M)$. Let $\gamma = \bar\eta(\alpha) \in H_n(M;\mathbb{L})$. By Theorem 3.7 there is a cell-like map $q : M \to X$ such that $q_*(\gamma) = 0$. Consider the diagram of Proposition 4.5. There is an element $\widehat\gamma \in H_{n+1}(M_q, M;\mathbb{L})$ such that $\bar\partial(\widehat\gamma) = \gamma$. Let $\alpha' = \bar\eta^{-1}(\widehat\gamma)$ and let $\widehat\alpha \in \mathcal{S}^{cc}(C_q, X_+)$ be the projection of α'. Since α is the image of $\widehat\alpha$ under the forgetful map, by Proposition 4.6 (2) we obtain that $\alpha \in \mathcal{S}^{CE}(M)$.
($\mathrm{Tor}^{odd}(\mathcal{S}(M)) \supset \mathcal{S}^{CE}(M)$.)
Suppose that $c : N \to X$ and $q : M \to X$ are cell-like maps and that $f : N \to M$ is a homotopy equivalence such that $q \circ f \simeq c$.

$$\begin{tikzcd}
N \arrow[rr, "f"] \arrow[dr, "c"', "CE"] & & M \arrow[dl, "q", "CE"'] \\
& X &
\end{tikzcd}$$

We consider the diagram of Proposition 4.5

$$\begin{tikzcd}
\mathcal{S}^{cc}(C_q, X_+) \arrow[r] \arrow[d, "forget"] & \mathcal{S}^{cc}_{n+1}(C_q, X_+) \arrow[r, "\tilde{\eta}"] \arrow[d, "forget"] & H_{n+1}(M_q, M; \mathbb{L}) \arrow[d, "\bar{\partial}"] \\
\mathcal{S}(M) \arrow[r] & \mathcal{S}_n(M) \arrow[r, "\bar{\eta}"] & \bar{H}_n(M; \mathbb{L})
\end{tikzcd}$$

By Vietoris-Begle theorem a cell-like map induces an isomorphism of ordinary cohomology or Steenrod homology with any coefficients (see Proposition 3.5). Therefore $H_n(M;\mathbb{L}) \to H_n(X;\mathbb{L})$ is an isomorphism rationally, and hence, the image of $H_{n+1}(M_q, M;\mathbb{L})$ in $H_n(M;\mathbb{L})$ is a torsion group. Since $\mathbb{L}$ is an Eilenberg-MacLane spectrum at 2, $H_n(M;\mathbb{L}) \to H_n(X;\mathbb{L})$ is an isomorphism at 2 and hence the image of $H_{n+1}(M_q, M;\mathbb{L})$ in $H_n(M;\mathbb{L})$ is odd torsion. By Proposition 4.6 $[f]$ is the image of $[c] \in \mathcal{S}^{cc}(C_q, X_+)$ under the forgetful map. Then $[f] = (\bar{\eta})^{-1}\bar{\partial}(\gamma)$ is an odd torsion element where $\gamma = \tilde{\eta}([c]) \in H_{n+1}(M_q, M;\mathbb{L})$. □

5. Continuous control near the corona

The proof of following proposition is based on diagram chasing. Since the proposition can be considered as a definition of the homomorphism ∂', we leave the proof to the reader.

Proposition 5.1. *Let (P, Q) be a CW pair with the inclusion isomorphism $\pi_1(Q) \to \pi_1(P) = \pi$. Then the homomorphism $\partial' : H_{n+1}(P, Q;\mathbb{L}) \to \mathcal{S}_n(Q)$ defined in §2 coincides with the n-homotopy group homomorphism generated by the map of the homotopy fibers of the following fibrations of spectra*

$$\begin{tikzcd}
\mathbb{H}_*(Q;\mathbb{L}) \arrow[r] \arrow[d, "="] & \mathbb{H}_*(P;\mathbb{L}) \arrow[d, "A_P"] \\
\mathbb{H}_*(Q;\mathbb{L}) \arrow[r] & \mathbb{L}_*(\mathbb{Z}\pi)
\end{tikzcd}$$

where A_P is the assembly map for P.

We consider $(P, Q) = (E_2(M), M)$ where M is a closed manifold M and $E_2(M)$ is its second Postnikov stage. We recall the notation $\delta = p \circ \partial'$ where $p : \mathcal{S}_n(M) \to \mathcal{S}(M)$ is the projection.

To prove Theorem 2.7 we need a germ version of continuously controlled surgery theory.

Definition 5.2.

(i) Let N be an open manifold and let Y be a compact corona of a compactification $\bar{N}$ of (one end of) N. A *strict homotopy equivalence near* Y is a proper map $f : W \to N$ onto a neighborhood of Y such that there are neighborhoods $U \supset V$ of Y in $\bar{N}$ and $U' \supset V'$ of Y in $\bar{W}$ with $f(V') \subset V$ and a strict map $g : V \to V'$ such that $g \circ f|_{V'}$ is strict proper homotopic to the inclusion $V' \subset U'$ and $f \circ g$ is strict proper homotopic to the inclusion $V \subset U$.

(ii) Two strict homotopy equivalences near Y, $f : W \to N$ and $f' : W' \to N$ are *equivalent* if there is a neighborhood V of Y in $\bar{W} = W \cup Y$ and a strict map $h : (V, Y) \to (\bar{W}', Y)$ which is an open imbedding and $f' \circ h : V - Y \to N$ is strict homotopic to $f|_V$.

(iii) The set of the equivalence classes of strict homotopy equivalences of manifolds near Y is called the set of *germs of continuously controlled structures* on N at Y and it is denoted as $\mathcal{S}^{cc}(\bar{N}, Y)_\infty$.

One can define germs of homotopy classes $[N, \mathrm{G/Top}]_\infty$ of maps at Y and the corresponding L-groups and form a surgery exact sequence. This was done in §15 of [14] in the case of the bounded control. These results can be translated to the continuous control setting as in [22]. We state the result here in the case when $N = M \times (0,1)$ and $\bar{N}$ is an open mapping cylinder $\overset{\circ}{M}_q$ of a cell-like map $q : M \to Y$ of a closed orientable manifold.

Proposition 5.3. *Let $q : M \to Y$ be a cell-like map of a closed orientable n-manifold, then there is an exact sequence*

$$\cdots \to \bar{H}_{n+1}(Y; \mathbb{L}) \to \mathcal{S}^{cc}(\overset{\circ}{M}_q, Y)_\infty \to [M, \mathrm{G/Top}] \to \bar{H}_n(Y; \mathbb{L}).$$

In view of Proposition 4.6 forget control defines a map $\phi : \mathcal{S}^{cc}(\overset{\circ}{M}_q, Y)_\infty \to \mathcal{S}(M)$. Moreover, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{S}^{cc}(\overset{\circ}{M}_q, Y)_\infty \arrow[r] \arrow[d, "\phi"] & {[M, \mathrm{G/Top}]} \arrow[r] \arrow[d] & \bar{H}_n(Y; \mathbb{L}) \arrow[d, "A"] \\
\mathcal{S}(M) \arrow[r] & {[M, \mathrm{G/Top}]} \arrow[r] & L_n(\mathbb{Z}\pi_1(M)).
\end{tikzcd}$$

Here A is the assembly map for Y.

Proposition 5.4. *Let $q : M \to Y$ be a cell-like map of a closed n-manifold, then the forget control map $\phi : \mathcal{S}^{cc}(\overset{\circ}{M}_q, Y)_\infty \to \mathcal{S}(M)$ factors as*

$$\mathcal{S}^{cc}(\overset{\circ}{M}_q, Y)_\infty \xrightarrow{j} H_{n+1}(M_q, M; \mathbb{L}) \xrightarrow{i_*} H_{n+1}(E_2(M), M; \mathbb{L}) \xrightarrow{\delta} \mathcal{S}(M)$$

where j is a monomorphism with the cokernel $\mathbb{Z}$ or 0.

Proof. Proposition 3.5 defines a map $g : X \to E_2(M)$ such that $g \circ q$ is homotopic to the inclusion $M \to E_2(M)$. We consider the diagram of fibrations of spectra

$$\begin{tikzcd}
\mathbb{H}_*(M; \mathbb{L}) \arrow[r] \arrow[d, "="] & \mathbb{H}_*(X; \mathbb{L}) \arrow[d, "g_*"] \\
\mathbb{H}_*(M; \mathbb{L}) \arrow[r] \arrow[d, "="] & \mathbb{H}_*(E_2(M); \mathbb{L}) \arrow[d, "A_E"] \\
\mathbb{H}_*(M; \mathbb{L}) \arrow[r] & \mathbb{L}_*(\mathbb{Z}\pi).
\end{tikzcd}$$

In the dimension n we have the homomorphism of homotopy groups of the fibers

$$H_{n+1}(M_q, X; \mathbb{L})_\infty \xrightarrow{i_*} H_{n+1}(E_2(M), M; \mathbb{L}) \xrightarrow{\partial'} \mathcal{S}_n(M)$$

where $H_{n+1}(M_q, M; \mathbb{L})$ differs from $\mathcal{S}^{cc}(\overset{\circ}{M}_q, X)_\infty$ by a potential summand $\mathbb{Z}$. In fact, one can argue that in this case they agree. Then the result follows in view of Proposition 5.1. $\square$

Proof of Theorem 2.7. ($\mathcal{S}^{CE}(M) \supset im(\delta^T_{[2]})$.)

We are given an odd torsion element $\alpha \in H_{n+1}(E_2(M), M; \mathbb{L})$ with $\delta(\alpha) = [f] \in \mathcal{S}_n(M)$ where δ is the composition

$$H_{n+1}(E_2(M), M; \mathbb{L}) \cong \mathcal{S}_{n+1}(E_2(M), M) \to \mathcal{S}_n(M) \to \mathcal{S}(M).$$

By Corollary 3.6, there exist a cell-like map $q : M \to X$ and an odd torsion element $\widehat{\alpha} \in H_{n+1}(M_q, M; \mathbb{L}) = \mathcal{S}^{cc}_{n+1}(\overset{\circ}{M}_q, X)_\infty$ so that α is the image of $\widehat{\alpha}$ under the inclusion-induced map $i_* : H_{n+1}(M_q, M; \mathbb{L}) \to H_{n+1}(E_2(M), M; \mathbb{L})$. Since $\widehat{\alpha}$ has finite order and $j : \mathcal{S}^{cc}(\overset{\circ}{M}_q, X)_\infty \to \mathcal{S}^{cc}_{n+1}(\overset{\circ}{M}_q, X)_\infty$ is an isomorphism on torsion subgroups, $\widehat{\alpha} = j(\alpha')$, where $\alpha' \in \mathcal{S}^{cc}(\overset{\circ}{M}_q, X)_\infty$. By Propostion 5.4 $\phi(\alpha') = [f]$. Let $g : W \to M \times (0,1)$ be a representative for α'. By Quinn's end of maps theorem [25] we may assume that $W = N \times (0,1)$ and $\bar{W} = M_p$, where $p : N \to X$ is cell-like. Thus, $[f] = \phi(\alpha')$ is realized by cell-like maps p and q.

($\mathcal{S}^{CE}(M) \subset im(\delta^T_{[2]})$.

Suppose that $c : N \to X$ and $q : M \to X$ are cell-like maps and that $f : N \to M$ is a homotopy equivalence such that $q \circ f \simeq c$.

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "c"', "CE"] & & M \arrow[dl, "q", "CE"'] \\ & X & \end{tikzcd}$$

As above, there is an inclusion-induced map $p : X \to E_2(M)$ and the forgetful map $H_{n+1}(M_q, M; \mathbb{L}) \cong \mathcal{S}^{cc}_\infty(M_q, X) \to \mathcal{S}(M)$ factors through $H_{n+1}(E_2(M), M; \mathbb{L})$. It therefore suffices to show that the image of $H_{n+1}(M_q, M; \mathbb{L})$ in $H_{n+1}(E_2(M), M; \mathbb{L})$ is an odd torsion group. By Vietoris-Begle theorem a cell-like map induces an isomorphism of ordinary cohomology or Steenrod homology with any coefficients (see Proposition 3.5). Therefore $H_*(M; \mathbb{L}) \to H_*(X; \mathbb{L})$ is an isomorphism rationally, and hence, the image of $H_*(M_q, M; \mathbb{L})$ in $H_*(E_2(M), M; \mathbb{L})$ is a torsion. Since at 2 $\mathbb{L}$ is an Eilenberg-MacLane spectrum, $H_*(M; \mathbb{L}) \to H_*(X; \mathbb{L})$ is an isomorphism at 2 and hence $H_*(E_2(M), M; \mathbb{L})$ is an odd torsion.

This half of the theorem is true for all n.

6. Pushing manifolds together in Gromov-Hausdorff space

Definition 6.1.

(i) If X and Y are compact subsets of a metric space Z, the *Hausdorff distance* between X and Y is

$$d_H(X,Y) = \inf\{\epsilon > 0 \mid X \subset N_\epsilon(Y), Y \subset N_\epsilon(X)\}.$$

(ii) If X and Y are compact metric spaces the *Gromov-Hausdorff distance* from X to Y is

$$d_{GH}(X,Y) = \inf_Z\{d_H(X,Y) \mid X, Y \subset Z\}.$$

(iii) Let $\mathcal{CM}$ be the set of isometry classes of compact metric spaces with the Gromov-Hausdorff metric.

(iv) Let $\mathcal{M}^{man}(n,\rho)$ be the set of all $(X,d) \in \mathcal{CM}$ such that X is a topological n-manifold with (topological) metric d with contractibility function ρ.

It is well-known that $\mathcal{CM}$ is a complete metric space (see [16] or [24] for an exposition).

Theorem 6.2. (i) *If $n \neq 3$ and $X \in \mathcal{CM}$ is in the closure of $\mathcal{M}^{man}(n,\rho)$, then there is an $\epsilon > 0$ so that there are only finitely many homeomorphism types of manifolds $M \in \mathcal{M}^{man}(n,\rho)$ with $d_{GH}(M,X) < \epsilon$. If $d_{GH}(M,X)$, $d_{GH}(M',X) < \epsilon$, then there exists a simple homotopy equivalence $h : M' \to M$ which preserves rational Pontryagin classes.*

(ii) *There exist a contractibility function ρ, nonhomeomorphic manifolds M and N, and a compact metric space X such that every ϵ-neighborhood of X in $\mathcal{CM}$ contains manifolds lying in $\mathcal{M}^{man}(n,\rho)$ and homeomorphic to both M and N.*

Proof. Part (i) is Theorem 2.10 of [10].

Let M and N be from Corollary 2.13 and let $q : M \to X$ and $p : N \to X$ be cell-like maps. By the main result of [13] there are a contractibility function ρ, sequences of Riemannian metrics $\{d_i^M\}$ and $\{d_i^N\}$ on M and N respectively lying in $\mathcal{M}^{man}(n,\rho)$ and converging in $\mathcal{CM}$ to (X,d) for some metric d. □

Let $\overline{\mathcal{M}^{man}(n,\rho)}$ be the closure of $\mathcal{M}^{man}(n,\rho)$ in the Gromov-Hausdorff space $\mathcal{CM}$ and $\partial\mathcal{M}^{man}(n,\rho)$ be the boundary. For a compact metric space X we will denote its isometry class by the same letter $X \in \mathcal{CM}$.

Theorem 6.3. *Suppose that the isometry type of a metric space X belongs to $\partial\mathcal{M}^{man}(n,\rho)$. Then there is $\epsilon > 0$ such every two manifolds $M, N \in B_\epsilon(X) \cap \mathcal{M}^{man}(n,\rho)$ from ϵ-neighborhood of X in $\mathcal{CM}$ are CE-related.*

Definition 6.4. A map $f : M \to X$ has a δ-lifting property in dimensions $\leq k$. If for every PL pair (P, Q), $\dim P \leq k$ for every commutative diagram

$$\begin{tikzcd} Q \arrow[r, "g'"] \arrow[d] & M \arrow[d, "f"] \\ P \arrow[r, "g"] \arrow[ur, "\bar{g}"] & X. \end{tikzcd}$$

there is a map $\bar{g} : P \to M$ extending g' such that $dist(f\bar{g}, g) < \delta$.

Proposition 6.5. *Let X be a locally k-connected space for $k > n$, then there exists $\delta > 0$ such that every map $f : M \to X$ from a compact n-dimensional ANR with the δ-lifting property in dimensions $\leq n + 1$ is a weak homotopy equivalence in dimension n (i.e., is $n + 1$-connected). Furthermore, it induces isomorphisms of the Steenrod homology groups $f_* : H_i(M) \to H_i(X)$ for $i \leq n$.*

Proof. The weak homotopy equivalence in dimension n easy follows from the liting property. Then the fact follows for the singular homology. We note that the Steenrod homologies coincide with the singular homologies in this case. □

Proposition 6.6. *Let $X \in \partial\mathcal{M}^{man}(n, \rho)$, then for every $\delta > 0$ there exists $\epsilon > 0$ such that every $M \in \mathcal{M}^{man}(n, \rho)$ with $d_{GH}(M, X) < \epsilon$ there is a map $f : M \to X$ with the δ-lifting property in dimensions $\leq n + 1$.*

Proof. The space X is locally k-connected for all finite k (see [12]). Then for small ϵ a map $f : M \to X$ can be constructed by induction by means of a small triangulation on M (If M does not admit a triangulation, one can use a CW complex structure). Given $\delta_0 > 0$, we may assume that $d(x, f(x)) < \delta_0$. Clearly, for a proper choice of δ_0 the map f will have the δ-lifting property. □

Proposition 6.7. *Let $X \in \partial\mathcal{M}^{man}(n, \rho)$ then there exists $\epsilon > 0$ such that every $M \in \mathcal{M}^{man}(n, \rho)$ with $d_{GH}(M, X) < \epsilon$ there is a map $f : M \to X$ such that $f_* : H_*(M; \mathbb{L}_{(2)}) \to H_*(X; \mathbb{L}_{(2)})$ is an isomorphism.*

Proof. We note that $\mathbb{L}_{(2)}$ is an Eilenberg-MacLane spectrum. We take ϵ from Proposition 6.6. Then Proposition 6.5 and the fact that $H_i(M) = H_i(X) = 0$ for $i > n$ imply the required result. □

Proof of Theorem 6.3.
We take ϵ from Proposition 6.7. Let $c : N \to X$ and $q : M \to X$ be corresponding maps. We may assume that there is a homotopy lift $f : N \to M$ of c which is a homotopy equivalence.

As it was shown in [12] that X can be presented as the limit space of an inverse sequence of polyhedra $\{K_i, p_i\}$ such that each map $p_i : X \to K_i$ is $n + 3$-connected. Moreover, we may assume that every bonding map $p_i^{i+1} : K_{i+1} \to K_i$ is $(\dim K_i + 3)$-connected (see P4, page 98 of [12]). Since $p_i \circ q$ is 2-connected, the space $E_2(M)$ can be constructed out of

K_i by killing higher dimensional homotopy groups. Thus the inclusion $M \subset E_2(M)$ can be factored through X and P_i (for large i). Hence there is a commutative diagram

$$(*)\quad \begin{array}{ccccc} H_{n+1}(P_{i+1}, M; \mathbb{L}) & \xrightarrow{\cong} & \mathcal{S}_{n+1}(P_i, M) & \longrightarrow & \mathcal{S}_n(M) \\ \downarrow & & \downarrow & & \downarrow{=} \\ H_{n+1}(E_2(M), M; \mathbb{L}) & \xrightarrow{\cong} & \mathcal{S}_{n+1}(E_2(M), M) & \longrightarrow & \mathcal{S}_n(M). \end{array}$$

By Proposition 6.7 we obtain that $H_*(X, M; \mathbb{L}_{(2)}) = 0$ for the Steenrod homology. Hence

$$\lim_{\leftarrow} H_*(P_i, M; \mathbb{L}_{(2)}) = 0 \ \text{ and } \ \lim_{\leftarrow}{}^1 H_*(P_i, M; \mathbb{L}_{(2)}) = 0.$$

By Theorem 2.6 of [12] the structure $[f]$ defined by $f : N \to M$ belongs to the kernel G_i of the induced map $(p_i q)_* : \mathcal{S}_n(M) \to \mathcal{S}_n(P_i)$ for all sufficiently large i. There is a morphism of inverse sequences

$$\begin{array}{ccccc} \dots & & \dots & & \\ \downarrow & & \downarrow & & \\ H_{n+1}(P_{i+1}, M; \mathbb{L}) & \xrightarrow{\psi_{i+1}} & G_{i+1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle (p_i^{i+1})_*} & & \downarrow{\scriptstyle \xi_i^{i+1}} & & \\ H_{n+1}(P_i, M; \mathbb{L}) & \xrightarrow{\psi_i} & G_i & \longrightarrow & 0. \end{array}$$

such that ξ_i^{i+1} are inclusions and $[f] \in G_i$ for all i. We tensor it with $\mathbb{Z}_{(2)}$ and take the inverse limit. Since $\lim^1_{\leftarrow} H_*(P_i, M; \mathbb{L}_{(2)}) = 0$ and $H_*(P_i, M; \mathbb{L}_{(2)}) = H_*(P_i, M; \mathbb{L}) \otimes \mathbb{Z}_{(2)}$, we obtain an epimorphism

$$0 = \lim_{\leftarrow} H_{n+1}(P_i, M; \mathbb{L}_{(2)}) \to \lim_{\leftarrow} G_i \otimes \mathbb{Z}_{(2)} \to 0.$$

Therefore $\lim_{\leftarrow} G_i \otimes \mathbb{Z}_{(2)} = 0$ and hence $[f]$ is an odd torsion. In view of the above diagram (*) it suffices to show that show that $\phi_i^{-1}([f]) \cap Tor(H_{n+1}(P_i, M; \mathbb{L}) \neq \emptyset$ for some large i.

Since q induces isomorphism of rational homology we obtain

$$\lim_{\leftarrow} H_*(P_i, M; \mathbb{Q}) = 0 \ \text{ and } \ \lim_{\leftarrow}{}^1 H_*(P_i, M; \mathbb{Q}) = 0.$$

The later implies that rationally the system is Mittag-Lefler. Thus, we may assume that the bonding maps $(p_i^{i+1})_*$ take all elements of $H_{n+1}(P_{i+1}, M; \mathbb{L})$ to the torsions of $H_{n+1}(P_i, M; \mathbb{L})$. Thus,

$$\emptyset \neq (p_i^{i+1})_*(\phi_{i+1}([f]) \subset \phi_i^{-1}([f]) \cap Tor(H_{n+1}(P_i, M; \mathbb{L})).$$

We obtain that our odd torsion element $[f]$ is the image of torsion element from $H_{n+1}(E_2(M), M; \mathbb{L})$. Then it is an image of an odd torsion element. $\square$

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