Cocycles on the Gauge Group and the Algebra of Chern-Simons Class

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Abstract

We consider here generalizations of Chern-Simons classes and related algebraic problems. We describe a new class of algebras whose elements contain Chern and generalized Chern-Simons classes. There is a Poisson bracket in these algebras similar to [Kon]. Using this bracket we construct a graded Lie algebra containing differential forms representing Chern and Chern-Simons classes. We develop an algebraic model for the action of the gauge group and describe how elements of algebra corresponding to the secondary characteristic classes change under this action. At the end we give new explicit formulas for cocycles on a gauge group and for corresponding cocycles on the Lie algebra constructed using our explicit formulas for generalized Chern-Simons classes given in the Appendix.

There are several approaches to combinatorial formulas for characteristic classes. These approaches are due to Gelfand-Gabrielov-Losik, MacPherson, Patodi, Ranicki-Sullivan, Cheeger, Brylinski, and others. One of the main problems in the field is the problem of explicit description of secondary characteristic classes and difference cocycles (cf. Chern-Simons [ChS], Cheeger, Bott-Shulman-Stasheff [BSS], Youssin [You], and others).

One of the other main problems related to combinatorial formulas for characteristic classes is the problem of writing topological invariants using local data given as fields of geometric objects. For Chern classes, curvature enables us to write these formulas using ordinary Chern-Weil theory.

One of the approaches to the local formulas for topological invariants uses formal power series of geometric objects and construction of the universal field and variational bicomplex (see Gelfand-Kazhdan-Fuks [GKF]).

Let us illustrate the problem of writing topological invariants using local data. Suppose we have an oriented graph consisting of points and arrows. To each vertex we associate a complex vector space E_i, and to each arrow we associate a linear mapping s_{ij}, as in the

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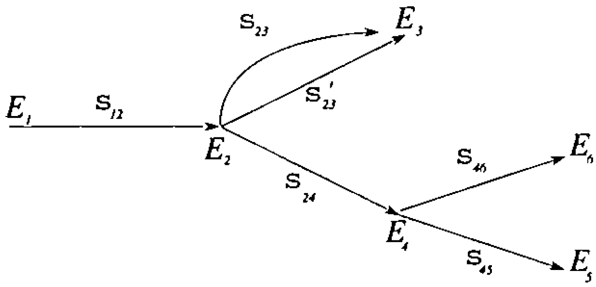
picture below. We suppose that the graph is parametrized by a manifold X; this means that $E_i(x)$ is a vector bundle and $s_{ij}(x)$ is a bundle map.



It is important to describe "Chern classes" and "secondary Chern classes" for such objects using local differential geometry. For a graph consisting of one vertex, the Chern-Weil theory gives ordinary Chern classes.

Consider first a graph with one vertex and a loop. The vertex corresponds to an n-dimensional vector bundle over a manifold X and the loop corresponds to an automorphism σ of the bundle. In topological K-theory, this data (the bundle and its automorphism) defines an element of $K_1(X)$. We can write in this case the usual Chern class for $K_1(X)$. For the trivial bundle, this class is given by

$$\mathrm{ch}_m(\sigma) = \frac{(-1)^{m-1}(m-1)!}{(2m-1)!}\mathrm{tr}(\sigma^{-1}\cdot d\sigma)^{2m-1}.$$

This class can also be defined as a secondary Chern class associated to the two connections $\nabla = d$ and $\nabla^\sigma = d + \sigma^{-1}d\sigma$. Locally, σ is a nondegenerate $n \times n$ matrix of functions of $x \in X$ defining the automorphism σ.

Let us now return to the problem of explicit calculation of Chern-Simons classes and illustrate it by a simple example.

It is well known that the Chern-Simons class constructed from two connections ω_0 and ω_1 has the form

$$\mathrm{ch}_2^1(\omega_0,\omega_1) = \frac{1}{2}\mathrm{tr}\big((\omega_0 d\omega_0 + \frac{2}{3}\omega_0^3) - (\omega_1 d\omega_1 + \frac{2}{3}\omega_1^3) + d(\omega_0\omega_1)\big),$$

where the ω_i are matrices of 1-forms which define the connections. For higher secondary classes, the calculations give longer polynomials. For example,

$$\mathrm{ch}_3^1(\omega_0,\omega_1) = \tfrac{1}{2}\mathrm{tr}[(\tfrac{1}{5}a_0^5 + \tfrac{1}{3}a_0b_0b_0 + \tfrac{1}{2}a_0^3b_0) - (\tfrac{1}{5}a_1^5 + \tfrac{1}{3}a_1b_1b_1 + \tfrac{1}{2}a_1^3b_1) + \tfrac{1}{6}d((a_1a_0 - a_0a_1)(b_0+b_1) - (a_0^2 + a_1^2 + \tfrac{1}{2}a_0a_1)a_0a_1)],$$

where $a_i = \omega_i$ and $b_i = d\omega_i$.

Thus, even in the simplest case for "normal" secondary characteristic classes (i.e. for a graph consisting of one point and zero arrows), formulas are lengthy and we have to handle the combinatorics of long noncommutative expressions. We introduce here the algebraic language that allows us to do this easily.

The construction of the algebra of Chern classes. We construct first a graded free associative algebra A generated by elements a_i of degree 1 and elements b_i of degree 2 ($i = 1, ..., N$), together with a differential d, such that $da_i = b_i$, $db_i = 0$. Then we consider the space V of cyclic words of A (i.e. a complex vector space with a basis consisting of equivalence classes of monomials, where two monomials obtained by cyclic permutation are considered to be the same up to a sign). Strictly speaking, V is the factor space of A by the subspace $\{PQ - (-1)^{|P||Q|}QP\}$ spanned by the graded commutators of all monomials in A.

That is,

$$V = A/\{PQ - (-1)^{|P||Q|}QP\}.$$

We also refer to monomials in A as *words* and to their equivalence classes as *cyclic words* (they form a basis of V).

The sign $\stackrel{\mathrm{tr}}{=}$ distinguishes equality up to cyclic permutation (i.e. equality in V) from equality in the algebra A, which is denoted by $=$.

Let us now define a Poisson bracket on V. The Poisson bracket gives V the structure of a graded Lie algebra. A partial derivative $\frac{\partial}{\partial z}P$, or $\partial_z P$, where $z = a_i$ or b_i, is defined on monomials by the following rule: we take a monomial P and look at all appearances of z in it. For each letter z which appears in P we cyclically permute the word P so that this letter z becomes the first letter of the permuted word. We then delete this letter z. The sum of the resulting monomials will be a partial derivative.

For example, suppose A is generated by two letters a and b. Then

$$\begin{aligned}\partial_a(aaabab) &= \cancel{a}aabab + (-1)^{1\cdot 7}\cancel{a}ababa + (-1)^{2\cdot 6}\cancel{a}babaa + (-1)^{5\cdot 3}\cancel{a}baaab \\ &= aabab - ababa + babaa - baaab. \\ \partial_b(aaabab) &= (-1)^{3\cdot 5}\cancel{b}abaaa + (-1)^{2\cdot 6}\cancel{b}aaaba = aaaba - abaaa.\end{aligned}$$

The partial derivative is well-defined in the space of cyclic words V.

Now we may define the Poisson bracket of P, $Q \in V$ to be

$$\{P, Q\} \stackrel{\mathrm{tr}}{=} \sum_{i=1}^{N}(\partial_{a_i}P \cdot \partial_{b_i}Q + (-1)^{|P||Q|}\partial_{a_i}Q \cdot \partial_{b_i}P).$$

In order to make sense of this formula. we take any of preimages of P and Q in A, also denoted by P and Q, and apply the operators ∂_{a_i} and ∂_{b_i} to these preimages. We multiply

$\partial_{a_i} P$ and $\partial_{b_i} P$ in A and only then do we take the corresponding cyclic element in V. This is independent of our choices.

Theorem 0. *The Poisson bracket is well defined on V and is linear in its arguments. It has the property*

$$\{P,\{Q,R\}\}+(-1)^{|P|(|Q|+|R|)}\{Q,\{R,P\}\}+(-1)^{|R||\{Q,P\}|}\{R,\{P,Q\}\}\overset{\mathrm{tr}}{=}0.$$

($\mathbf{Z}_2$*-graded Jacobi identity*).

Let us now define algebraic analogs of Chern and Chern-Simons classes. We fix an index i, so a_i and b_i are denoted by a and b. Choose a positive integer k. Consider the image in V of a polynomial $\frac{1}{k!}(a^2+b)^k \in A$. We call this image the k-th Chern character associated to a (recall $b = da$), and denote it by $\mathrm{ch}_k(a)$:

$$\mathrm{ch}_k(a) \overset{\mathrm{tr}}{=} \tfrac{1}{k!}(a^2+b)^k.$$

This corresponds to the differential form

$$\mathrm{tr}\,\tfrac{1}{k!}(\omega^2+d\omega)^k.$$

Now we define algebraic analogs of the Chern-Simons classes ch_k^1. These are the cyclic images in V of the following polynomials in A:

$$\tfrac{1}{(k-1)!}a\big(\tfrac{1}{k}b^{k-1}+\tfrac{1}{k+1}\Sigma_{1,k-2}+\tfrac{1}{k+2}\Sigma_{2,k-3}+\ldots+\tfrac{1}{2k-1}a^{2(k-1)}\big),$$

where $\Sigma_{p,q}$ is the sum of all possible noncommutative monomials in a^2 and b with p appearances of a^2 and q appearances of b. For example, $\Sigma_{1,2} = a^2bb + ba^2b + bba^2$. We shall call the corresponding element of V the $k-$th secondary Chern character (Chern–Simons class) and denote it by $\mathrm{ch}_k^1(a)$:

$$\mathrm{ch}_k^1(a) \overset{\mathrm{tr}}{=} \tfrac{1}{(k-1)!}a\big(\tfrac{1}{k}b^{k-1}+\tfrac{1}{k+1}\Sigma_{1,k-2}+\tfrac{1}{k+2}\Sigma_{2,k-3}+\ldots+\tfrac{1}{2k-1}a^{2(k-1)}\big).$$

Example. The second Chern character in V is

$$\mathrm{ch}_2(a) \overset{\mathrm{tr}}{=} \tfrac{1}{2}(a^2+b)^2 \overset{\mathrm{tr}}{=} \tfrac{1}{2}(b^2+2a^2b).$$

($a^4 \overset{\mathrm{tr}}{=} 0$). This corresponds to the differential form

$$\tfrac{1}{2}\mathrm{tr}(\omega^2+d\omega)^2.$$

The Chern-Simons class in V is

$$\mathrm{ch}_2^1(a) \stackrel{\mathrm{tr}}{=} \tfrac{1}{2}(ab + \tfrac{2}{3}a^3).$$

This corresponds to the Chern-Simons differential 3-form

$$\tfrac{1}{2}\mathrm{tr}(\omega d\omega + \tfrac{2}{3}\omega^3).$$

Theorem 1.

1. Chern characters are closed elements of the cyclic space V:

$$d\,\mathrm{ch}_k(a) \stackrel{\mathrm{tr}}{=} 0.$$

2. The differential in V of the Chern–Simons class defined by the above formula is the k-th Chern character:

$$d\,\mathrm{ch}_k^1(a) \stackrel{\mathrm{tr}}{=} \mathrm{ch}_k(a).$$

We develop a similar theory for the algebra with generators depending on a real parameter t. Let A be a free associative algebra generated by elements $a(t)$, $\dot{a}(t)$ of degree 1 and elements $b(t)$, $\dot{b}(t)$ of degree 2, where $t \in [0,1]$. This algebra has a differential d such that

$$d\,a(t) = b(t), \quad d\,\dot{a}(t) = \dot{b}(t), \quad db(t) = 0, \quad d\,\dot{b}(t) = 0.$$

We shall multiply the elements of our algebra A by polynomials in t and by a differential 1-form dt. We may treat dt formally as an element of degree 1 that commutes with $b(t)$ and $\dot{b}(t)$ and anticommutes with $a(t)$ and $\dot{a}(t)$, also $(dt)(dt) = 0$. Polynomials in t commutes with elements of A. We define another differential $\delta = d_t$ that is formally not included in the algebraic structure of A. It maps elements $a(t)$ and $b(t)$ of A into a product of a formal element dt and $\dot{a}(t)$ and $\dot{b}(t)$ correspondingly.

$$d_t a(t) = dt\,\dot{a}(t) \quad \text{and} \quad d_t b(t) = dt\,\dot{b}(t).$$

Furthermore, there is a derivative in t for the expressions in a and b given by the formal rule $\frac{d}{dt}a = \dot{a}$ and $\frac{d}{dt}b = \dot{b}$. We are not going to apply d_t to expressions involving $\dot{a}$ and $\dot{b}$ (otherwise we would have to introduce letters with more dots, but we do not want to deal with this). We do not want to make our algebra very complicated at this point. It is just an abstraction of the algebra of matrices of 1 and 2 forms on a manifold X, when these forms depend on a real parameter t.

From this algebra A we construct as before the factor space

$$V = A/\{CB - (-1)^{|C||B|}BC\},$$

where $\{CB - (-1)^{|C||B|}BC\}$ is the subspace of A generated by commutators. We call V the *space of cyclic words* and we denote the equality in V by $\overset{\mathrm{tr}}{=}$.

Consider a path $a(t)$, $0 \leq t \leq 1$, connecting a_0 and a_1 and consider

$$\mathrm{ch}^1_k(a(t)) \overset{\mathrm{def}}{=} \int_0^1 [(d_t + d)a(t) + a(t)^2]^k.$$

We can now write an explicit formula for $\mathrm{ch}^1_k(a(t))$.

In addition, to $\Sigma_{p,q}$ as defined above, we also need to consider the expression $\Sigma_{\dot{n},p,q}$ which denotes the sum of all possible noncommutative monomials in $\dot{a}$, a^2 and b, where $\dot{a}$ appears n times, a^2 appears p times and b appears q times.

Theorem 2. *The secondary characteristic class defined by a path $a(t)$ is given by*

$$\mathrm{ch}^1_k(a(t)) \overset{\mathrm{tr}}{=} \tfrac{1}{(k-1)!}\Big(a\big(\tfrac{1}{k}b^{k-1} + \tfrac{1}{k+1}\Sigma_{1,k-2} + \tfrac{1}{k+2}\Sigma_{2,k-3} + \ldots + \tfrac{1}{2k-1}a^{2(k-1)}\big)\Big|_1^0$$
$$+ \tfrac{1}{(k-1)!}\int_0^1 dt d\big(a\big(\tfrac{1}{k}\Sigma_{\dot{1},0,k-2} + \tfrac{1}{k+1}\Sigma_{\dot{1},1,k-3} + \ldots + \tfrac{1}{2k-1}\Sigma_{\dot{1},k-2,0}\big)\big).$$

The gauge group action on the algebra of Chern classes. Consider now the action of the gauge group on connections (or on elements a_i of the algebra A corresponding to connections).

The gauge transformation acts in a natural way:

$$g : a \to g^{-1}dg + g^{-1}ag.$$

Let us denote by c the expression $dg \cdot g^{-1}$.

Theorem 3. *Under the gauge transformation on the space of cyclic words in the letters g, g^{-1}, dg, dg^{-1}, a_i, and b_i, the Chern-Simons class $\mathrm{ch}^1_k(a)$ transforms in the following way:*

$$\mathrm{ch}^1_k(a) \mapsto \tilde{\mathrm{ch}}^1_k \overset{\mathrm{tr}}{=} \tfrac{1}{(k-1)!}(a+c)\sum_{\alpha+\beta+\gamma=k-1}(-1)^\gamma \frac{(k+\beta-1)!\gamma!}{(k+\beta+\gamma)!}\Sigma[(b)^\alpha, (a^2)^\beta, (u)^\gamma],$$

where $u = ca + ac + c^2$ and $\Sigma[(b)^\alpha, (a^2)^\beta, (u)^\gamma]$ is the sum of all possible words in which b, a^2, and u appear α times, β times, and γ times respectively.

Cocycles on the gauge group. Let M be a smooth real m-dimensional manifold. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra. Let $E \to M$ be a principal G bundle with a base manifold M and let $G(E)$ be its gauge group. If we trivialize the bundle E over an open set $U \subset M$ then the trivialization gives us an isomorphism between the gauge group and the group of G valued functions on U.

We use ideas of Faddeev, Reiman and Semyonov-Tian-Shanskii [FRS] in the construction of cocycles on the gauge group of the bundle E. We construct new cocycles on the group and its Lie algebra.

Let us fix a connection $B(x)$, $x \in M$ on a bundle E. Locally we can look at B as a $\mathfrak{g}$ valued 1-form on M. Let us take $n+1$ connections $\omega_0, \ldots, \omega_n$ which are gauge equivalent to B under the action of elements $g_0, \ldots, g_n$ of the gauge group.

$$\omega_0(x) = g_0^{-1} dg_0 + g_0^{-1} B g_0,$$

$$\ldots$$

$$\omega_n(x) = g_n^{-1} dg_n + g_n^{-1} B g_n.$$

Here d is a differential with respect to $x \in M$. Consider a product space $M \times \Delta_n$ where Δ_n is a standard n-dimensional simplex:

$$\Delta_n = \{(t_0, t_1, \ldots, t_n) | t_0 + t_1 + \ldots + t_n = 1, t_0, t_1, \ldots, t_n > 0\}.$$

On $M \times \Delta_n$ define a a connection

$$\omega(x,t) = t_0 \omega_0(x) + \cdots + t_n \omega_n(x). \tag{0}$$

Let d be a differential with respect to $x \in M$ and δ be a differential with respect to $t \in \Delta_n$. Then $(d+\delta)$ is the total differential on $M \times \Delta_n$. The curvature of $\omega(x,t)$ is

$$R_\omega = (d+\delta)\omega + \omega^2$$

We shall use here a standard definition of coboundary operator in group cohomology. Let G be a group and V be a left G module. Let $A^p(G,V)$ be the space of functions on $\underbrace{G \times \ldots \times G}_{p+1 \quad times}$ taking values in V. For $\alpha(h_0, \ldots, h_n) \in A^p(G,V)$ define the action of $g \in G$ as

$$(g\alpha)(h_0, \ldots h_n) = g(\alpha(g^{-1}h_0, \ldots, g^{-1}h_n)).$$

Define a differential as

$$(\partial \alpha)(h_0, \ldots h_n) \stackrel{\text{def}}{=} \sum (-1)^i \alpha(h_0, \ldots, \hat{h_i}, \ldots, h_n).$$

Definition. Let $P(R_1, \ldots, R_N)$ be an invariant symmetric N form. We define cochains $a_n(g_0, \ldots, g_n)$ on the group G as

$$a_n(g_0, \ldots, g_n) = \int_{\Delta_n} P(R_\omega, \ldots, R_\omega)$$

Cochains $a_n(g_0, \ldots, g_n)$ are differential $(2N-n)$-forms on M depending on $g_0, \ldots, g_n$.

The important property of these cochains with values in differential forms is that

$$(\partial a_{n-1})(g_0, \ldots, g_n) = -d\, a_n(g_0, \ldots, g_n).$$

Proposition 1. *If V is a closed submanifold of M of dimension $(2N-n)$ then*

$$a_n^V(g_0, \ldots, g_n) \overset{\text{def}}{=} \int_V a_n(g_0, \ldots, g_n)$$

is a cocycle on the group $G(E)$ (in the sense of cohomology of groups). If V is homologous to V' then cocycles a_n^V and $a_n^{V'}$ are homologous.

Cocycles on the Lie algebra of the gauge group.

Proposition 1. *The following cochains on the Lie algebra correspond to cochains*

$$a_n(1, g_1, \ldots, g_n)$$

on the gauge group (here N =degree of P).

$\bar{a}_n(X_1, \ldots, X_n) =$

$$= \frac{N!}{(N-n)!}(-1)^{\frac{(n-1)}{2}} P\big(\underbrace{(X_1 \cdot B), \ldots, (X_n \cdot B), \overbrace{R_B, \ldots, R_B}^{N-n \quad times}}_{N \ times}\big).$$

Here vector fields $X_1, \ldots, X_n$ are the elements of the Lie algebra of the gauge group, $(X \cdot B)$ is the Lie derivative of the connection B along the vector field X and $R_B = dB + B^2$ is the curvature of connection B.

Remark. We are working here with matrix groups. The Lie derivative of B is $(X \cdot B) = dX + XB - BX$. X is an element of the Lie algebra of the Gauge group, so it is a matrix of 0-forms, B is a connection, so it is a matrix of 1-forms. And BX and XB are just products of the matrices of 0 and 1-forms. So $(X \cdot B) = dX + XB - BX$*) is a matrix of 1-forms.

*) We assume that the connection B is transformed by the infinitesimal gauge transformations e^{-sX} as $e^{-sX} de^{sX} + e^{-sX} B e^{sX}$.

Example 1. Elements of the group are 1, g_1, g_2. Take $P = \frac{1}{2!}\mathrm{tr}R^2$. The cocycle on the Lie algebra is

$$Const \cdot \mathrm{tr}\big((X_1 \cdot B)(X_2 \cdot B)\big).$$

Take $P = \frac{1}{3!}\mathrm{tr}R^3$. Take $P = \frac{1}{3!}\mathrm{tr}R^3$. The cocycle on the Lie algebra is

$$Const \cdot \mathrm{tr}\big((X_1 \cdot B)(X_2 \cdot B)(dB + B^2)\big).$$

Definition. Let us consider k connections $A_1, \ldots, A_k$ on the bundle over M and let us fix a connection B. We construct $n+1$ connections that are gauge equivalent to B using transformations $g_0, \ldots, g_n$ in the gauge group G. Altogether we have $k+n+1$ connections $A_1, \ldots, A_k, g_0^{-1}dg_0 + g_0^{-1}Bg_0, \ldots, g_n^{-1}dg + g_n^{-1}Bg_n$. We construct on the group G a cochain $c_n(A_1, \ldots, A_k | g_0, \ldots, g_n)$. As before, we consider a connection ω on $M \times \Delta_{k+n}$, where

$$\Delta_{k+n} = \{\tau_1 + \ldots + \tau_k + t_0 + \ldots + t_n = 1 | \tau_i, t_p \geq 0\}$$

is a standard $k+n$ dimensional simplex. This connection ω linearly approximates our $k+n+1$ connections that are placed in the vertices of the simplex. We consider an invariant N form P and and define

$$c_n(A_1, \ldots, A_k | g_0, \ldots, g_n) \stackrel{\mathrm{def}}{=} \int_{\Delta_{n+k}} P(R_\omega, \ldots, R_\omega)$$

Cochains $c_n(A_1, \ldots, A_k | g_0, \ldots, g_n)$ are $(2N - n - k)$-forms on M depending on $g_0, \ldots, g_n$ and on k connections $A_1, \ldots, A_k$ that are "external fields".

Theorem 4. *The cochain $\bar{c}_n(A_1, \ldots, A_k | X_1, \ldots, X_n)$ on the Lie algebra that corresponds to a cochain*

$$c_n(A_1, \ldots, A_k | 1, g_1, \ldots, g_n)$$

on the Lie group G can be written as

$$\int_{\{\tau_0 + \ldots \tau_k = 1\}} \int_{\{t_0 + \ldots + t_n \leq 1\}} \frac{1}{n!}\tau_0^n P\Big(d\tau_1(A_1 - B), \ldots, d\tau_k(A_k - B), dt_1(X_1 \cdot B), \ldots$$

$$\ldots, dt_n(X_n \cdot B), R_\tau, \ldots, R_\tau\Big),$$

where

$$R_\tau = \tau_0 dB + \tau_1 dA_1 + ...\tau_k dA_k + (\tau_0 B + \tau_1 A_1 + ... + \tau_k A_k)^2,$$

$(X_i \cdot B)$ is the Lie derivative of the connection B along the vector field X_i and $\tau_0 = 1 - \tau_1 - ... - \tau_k$.

Example 2. Let

$$P(R_1, R_2, R_3, R_4) = \sum_{\sigma\in S_4} \mathrm{tr}(R_{\sigma(1)} R_{\sigma(2)}, R_{\sigma(3)}, R_{\sigma(4)})$$

Then

$$\bar{c}_1(A_1, A_2, B, X_1) = \frac{1}{2} \int_{t_1=0}^{t_1=1} \int_{\Delta_2} \tau_0^2 d\tau_1 (A_1 - B) d\tau_2 (A_2 - B) dt_1 (X_1 \cdot B)$$

$$\left(\tau_0 dB + \tau_1 dA_1 + \tau_2 dA_2 (\tau_0 B + \tau_1 A_1 + \tau_2 A_2)^2\right).$$

In fact let

$y_1 = A_1 - B,$

$y_2 = A_2 - B,$

$y_3 = X_1 \cdot B = dX_1 + BX_1 + X_1 B,$

$y_4 = R_\tau = \frac{1}{20} dB + \frac{1}{60} dA_1 + \frac{1}{60} dA_2 + \frac{1}{30} B^2 + \frac{1}{180} A_1^2 +$
$\frac{1}{180} A_2^2 + \frac{1}{120}(A_1 B + BA_1) + \frac{1}{120}(A_2 B + BA_2) + \frac{1}{360}(A_2 A_1 + A_1 A_2).$

Then cocycle on the Lie algebra is

$$\bar{c}_1(A_1, A_2, B, X_1) =$$

$$= 4\ \mathrm{tr}(y_1 y_3 y_4 y_2 + y_1 y_3 y_2 y_4 + y_1 y_4 y_3 y_2 - y_1 y_2 y_3 y_4 - y_1 y_4 y_2 y_3 - y_1 y_2 y_4 y_3)$$

We used here

$$\int_{\Delta^2} \tau_0^3 = \frac{1}{20}, \quad \int_{\Delta^2} \tau_0^2 \tau_1 = \frac{1}{60}, \quad \int_{\Delta^2} \tau_0^2 \tau_1^2 = \frac{1}{180},$$

$$\int_{\Delta^2} \tau_0^4 = \frac{1}{30}, \quad \int_{\Delta^2} \tau_0^3 \tau_1 = \frac{1}{120}, \quad \int_{\Delta^2} \tau_0^2 \tau_1 \tau_2 = \frac{1}{360},$$

Example 3.

$$\bar{c}_1(A, B, X_1) = 3\ \mathrm{tr}(y_1 y_2 y_3 + y_1 y_3 y_2),$$

where

$$\begin{aligned}
y_1 &= A - B,\\
y_2 &= X \cdot B = dX + XB - BX,\\
y_3 &= \int\limits_{\{\tau_0+\tau_1=1\}} \tau_0(\tau_0 dB + \tau_1 dA + \tau_0^2 B^2 + \tau_1 A^2 + \tau_0\tau_1(AB + BA)) =\\
&= \tfrac{1}{3} dB + \tfrac{1}{6} dA + \tfrac{1}{4} B^2 + \tfrac{1}{12} A^2 + \tfrac{1}{12}(AB + BA).
\end{aligned}$$

Appendix 1. Explicit formulas for secondary classes ch_k^2

Let G be a Lie group with finitely many components and let $\mathfrak{g}$ be its Lie algebra (we shall take $G = GL(n, \mathbf{C})$, or $GL(n, \mathbf{R})$ so $\mathfrak{g}$ consists of all $n \times n$ matrices). Let X be a C^∞ oriented manifold and let E_G be a principal G bundle over X provided with N connections $\nabla_1, \ldots, \nabla_N$, $(N \neq n)$. Let us associate to E_G an n-dimensional vector bundle $\pi : E \to X$ and connections $\omega_0(x), \omega_1(x), \ldots, \omega_N(x)$ on this bundle given by their matrices of 1-forms $\omega_0(x), \omega_1(x), \ldots, \omega_N(x) \in \Omega^1(X) \otimes \mathfrak{g}$. Let Δ_I be a k-dimensional simplex with vertices $i_0, \ldots, i_k \subset \{0, 1, \ldots, N\}$, $\Delta_I \cong \{t_0 + t_1 + \ldots + t_k = 1,\, t_0, \ldots, t_k > 0\}$. Consider a connection $\omega(t)$, $t \in \Delta_I$ such that $\omega(t) = t_0\omega_{i_0} + \ldots + t_k\omega_{i_k}$. Then we define a secondary characteristic class

$$\mathrm{ch}_m^k(\omega_{i_0} \ldots \omega_{i_k}) = \mathrm{tr} \int_\Delta [d(t_0\omega_{i_0} + \ldots + t_k\omega_{i_k}) + (t_0\omega_{i_0} + \ldots + t_k\omega_{i_k})^2]^m,$$

where d is the total differential (with respect to x and t on $\Delta_I \times X$ [GGL]) and in the expression under the integral we take only summands with k dt's and forget about the other summands.

The following relation holds:

$$d\,\mathrm{ch}_m^k(\omega_{i_0} \ldots \omega_{i_k}) = \sum_{p=1}^{k} (-1)^p \mathrm{ch}_m^{k-1}(\omega_{i_0} \ldots \hat{\omega}_{i_p} \ldots \omega_{i_k}).$$

We are interested in the explicit expressions for ch_m^k and the relations between them.

Let Δ be a 2-simplex $t_0 + t_1 + t_2 = 1$, $t_0, t_1, t_2 > 0$. Consider 3 connections a_0, a_1, a_2 on the bundle $E \to M$. Consider the connection $a(t)$, $t \in \Delta$ such that $a(t) = t_0a_0 + t_1a_1 + t_2a_2$. Then

$$\mathrm{ch}_k^2 = \frac{1}{k!}\mathrm{tr} \int_\Delta [d(t_0a_0 + t_1a_1 + t_2a_2) + (t_0a_0 + t_1a_1 + t_2a_2)^2]^k =$$

$$= \frac{1}{k!}\mathrm{tr} \int_\Delta [(dt_0a_0 + dt_1a_1 + dt_2a_2) + (t_0b_0 + t_1b_1 + t_2b_2) + (t_0a_0 + t_1a_1 + t_2a_2)^2]^k$$

$$= \frac{1}{k!}\mathrm{tr} \int_\Delta (P + Q)^k,$$

where

$$P = dt_0a_0 + dt_1a_1 + dt_2a_2,$$

$$Q = (t_0 b_0 + t_1 b_1 + t_2 b_2) + (t_0 a_0 + t_1 a_1 + t_2 a_2)^2$$

and both P and Q are of degree 2. We are interested only in summands with 2 letters P and $(k-2)$ letters Q because only they will give a 2-form in dt which can be can be integrated over the 2-simplex Δ. Let us take $t_0 = 1 - t_1 - t_2$, then $dt_0 = -(dt_1 + dt_2)$. When integrating over Δ we in fact integrate over the triangle $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 1 - t_1$.

$$\int_\Delta (\ldots) \stackrel{\text{def}}{=} \int_\Delta dt_1 dt_2 (\ldots) = \int_0^1 dt_1 \int_0^{1-t_1} dt_2 (\ldots).$$

We have

$$P^2 \stackrel{\text{tr}}{=} (dt_0 a_0 + dt_1 a_1 + dt_2 a_2)^2 = dt_1 dt_2 [a_1, a_0] + dt_1 dt_2 [, a_2] + dt_1 dt_2 [a_2, a_1] =$$

$$= dt_1 dt_2 ([a_1, a_0] + [a_0, a_2] + [a_2, a_1]).$$

For future calculations we shall use the following integral:

$$\int_\Delta t_0^\alpha t_1^\beta t_2^\gamma \, dt_1 dt_2 = \frac{\alpha! \, \beta! \, \gamma!}{(\alpha + \beta + \gamma + 2)!}$$

Calculation of $\mathrm{ch}_k^2(a_0, a_1, a_2)$.

Case $k = 2$,

$$\mathrm{ch}_2^2(a_0, a_1, a_2) \stackrel{\text{tr}}{=} \frac{1}{2!} \int_\Delta PP \stackrel{\text{tr}}{=} (a_1 a_0 + a_0 a_2 + a_2 a_1) \stackrel{\text{tr}}{=} \frac{1}{2}(-A_1 A_2 + A_2 A_1),$$

where $A_1 = a_1 - a_0, A_2 = a_2 - a_0$.

Case $k = 3$,

$$\mathrm{ch}_3^2(a_0, a_1, a_2) \stackrel{\text{tr}}{=} \frac{1}{3!} \int_\Delta (PPQ + PQP + QPP) \stackrel{\text{tr}}{=} \frac{1}{3!} \, 3 \int_\Delta PPQ$$

$$\stackrel{\text{tr}}{=} \frac{1}{3!} 3((a_1 a_0 - a_0 a_1) + (a_0 a_2 - a_2 a_0) + (a_2 a_1 - a_1 a_2)) \cdot$$

$$\cdot \left(\tfrac{1}{6}(b_0 + b_1 + b_2) + \tfrac{1}{12}((a_0^2 + a_1^2 + a_2^2) + \right.$$

$$\left. + \frac{1}{24}((a_0 a_1 + a_1 a_0) + (a_1 a_2 + a_2 a_1) + (a_0 a_2 + a_2 a_0)) \right)$$

Really

$$\int_\Delta (t_0 b_0 + t_1 b_1 + t_2 b_2)(t_0 a_0 + t_1 a_1 + t_2 a_2)^2 \stackrel{\text{tr}}{=} \frac{1}{6}(b_0 + b_1 + b_2) +$$

$$\int_\Delta (t_0^2 a_0^2 + t_1^2 a_1^2 + t_2^2 a_2^2) + \int_\Delta (t_0 t_1 \Sigma[a_0, a_1] + t_1 t_2 \Sigma[a_1, a_2] + t_0 t_2 \Sigma[a_0, a_2]) \stackrel{\text{tr}}{=}$$

$$\stackrel{\text{tr}}{=} \frac{1}{6}(b_0 + b_1 + b_2) + \frac{1}{12}(a_0^2 + a_1^2 + a_2^2) + \frac{1}{24}(\Sigma[a_0, a_1] \; + \Sigma[a_1, a_2] + \Sigma[a_0, a_2]),$$

because

$$\int_\Delta t_i = \frac{1!}{3!} = \frac{1}{6}, \quad \int_\Delta t_i^2 = \frac{2!}{4!} = \frac{1}{12}, \quad \int_\Delta t_i t_j = \frac{1!}{4!} = \frac{1}{24},$$

where $i, j = 1, 2, 3$ and $i \neq j$.

Case $k = 3$ and $a_2 = 0$

$$\mathrm{ch}_3^2(a_0, a_1, 0) \stackrel{\text{tr}}{=} \frac{1}{3!} \int (\ldots)^3 \stackrel{\text{tr}}{=}$$

$$\stackrel{\text{tr}}{=} \tfrac{1}{3!} 3(a_1 a_0 - a_0 a_1)(\tfrac{1}{12}(a_0^2 + a_1^2) + \tfrac{1}{24}(a_0 a_1 + a_1 a_0) + \tfrac{1}{6}(b_0 + b_1)) \stackrel{\text{tr}}{=}$$

$$\stackrel{\text{tr}}{=} \tfrac{1}{3!} \tfrac{3}{6}(a_1 a_0 - a_0 a_1)((b_0 + b_1) + \tfrac{1}{2}(a_0^2 + a_1^2 + \tfrac{1}{2}(a_0 a_1 + a_1 a_0)))$$

This coincides with the exact term obtained in the example of section 5:

$\mathrm{ch}_3^1(a(t)) \stackrel{\text{tr}}{=} \frac{1}{3!}(ab^2 + \frac{3}{2}a^3 b + \frac{3}{5}a^5)\Big|_0^1 + d\frac{1}{2}\int_0^1 dt\,\left(\frac{1}{3}(a\dot{a}b - \dot{a}ab) + \frac{1}{2}a^3\dot{a}\right)$
if we take $a(t) = ta_0 + (1 - t)a_1$.

Case $k = 4$,

$$\mathrm{ch}_4^2 = \frac{1}{4!} \int (PPQQ + PQPQ + PQQP + QPPQ + QPQP + QQPP) =$$

$$= \frac{1}{4!} \int (4PPQQ + 2PQPQ),$$

Let us take $P = dt_0a_0 + dt_1a_1 + dt_2a_2$, $Q = B + A^2$, where $A = (t_0a_0 + t_1a_1 + t_2a_2)$ and $B = (t_0b_0 + t_1b_1 + t_2b_2)$.

$$4\int_\Delta P^2Q^2 = 4\int_\Delta P^2(B+A^2)^2 = 4\int_\Delta P^2(B^2+BA^2+A^2B+A^4) =$$

$$= 4([a_1a_0]+[a_0a_2]+[a_2a_1])\bigg(\frac{1}{12}(b_0^2+b_1^2+b_2^2)+\frac{1}{24}(\Sigma[b_0,b_1]+\Sigma[b_1,b_2]+\Sigma[b_0,b_2])+$$

$$\sum\int_\Delta t_it_jt_k\Sigma[b_i,(a_ja_k)]+\sum\int_\Delta t_it_jt_kt_l\Sigma[a_i,a_j,a_k,a_l]\bigg)$$

$$2\int_\Delta PQPQ = 2\int_\Delta P(B+A^2)P(B+A^2) =$$

$$= 2\int_\Delta PA^2PA^2 + 2\int_\Delta PBPB + 2\int_\Delta 2PBPA^2 =$$

$$= 4\sum\int_\Delta t_it_jt_kt_la_1(a_ia_j)a_2(a_ka_l)+$$

$$+4\sum\int_\Delta t_it_jt_k[a_1b_ia_2(a_ja_k)-a_2b_ia_1(a_ja_k)]+4\sum\int_\Delta t_it_ja_1b_ia_2b_j$$

Case $k=4$, $a_0=0$:

$$\mathrm{ch}_4^2(0,a_1,a_2) = \frac{1}{6}\bigg(\frac{1}{12}(a_2a_1-a_1a_2)(b_1^2+b_2^2)+\frac{1}{24}(a_2a_1-a_1a_2)(b_1b_2+b_2b_1)+$$

$$+\frac{1}{12}a_1b_1a_2b_1+\frac{1}{12}a_1b_2a_2b_2+\frac{1}{24}a_1b_1a_2b_2+\frac{1}{24}a_1b_2a_2b_1+$$

$$+\frac{1}{20}(a_1^2a_2a_1-a_1a_2a_1^2)b_1+$$

$$+\frac{1}{60}(a_2a_1a_2a_1-a_1a_2a_1a_2+a_1^2a_2^2-a_2^2a_1^2-a_2^2a_1a_2+a_2a_1a_2^2)b_1+$$

$$+\frac{1}{20}(a_2a_1a_2^2 - a_2^2a_1a_2)b_2+$$

$$+\frac{1}{60}(a_2a_1a_2a_1 - a_1a_2a_1a_2 + a_1^2a_2^2 - a_2^2a_1^2 - a_1a_2a_1^2 + a_1^2a_2a_1)b_2+$$

$$+\frac{1}{30}(a_2^5a_1 + a_2a_1^5) + \frac{1}{60}a_2a_1a_2a_1^3 - \frac{1}{60}a_1a_2a_1a_2^3+$$

$$+\frac{1}{180}(a_2^3a_1^3 - a_1a_2a_1^2a_2^2 + a_2a_1a_2^2a_1^2 - a_1a_2a_1a_2a_1a_2)\Big). \quad []$$

Proof.

$$\int_\Delta P^2Q^2 = \int_\Delta [a_2,a_1]((t_1^2b_1^2 + t_2^2b_2^2) + t_1t_2\Sigma[b_1,b_2]+$$

$$+t_1t_1t_1\Sigma[b_1,a_1^2] + t_2t_1t_1\Sigma[b_2,a_1^2] + t_1t_2t_2\Sigma[b_1,a_2^2] + t_2t_2t_2\Sigma[b_2,a_2^2]+$$

$$+t_1t_1t_2\Sigma[b_1,(a_1a_2)] + t_1t_1t_2\Sigma[b_1,(a_2a_1)] + t_2t_1t_2\Sigma[b_2,(a_1a_2)]+$$

$$+t_2t_1t_2\Sigma[b_2,(a_2a_1)] + t_2t_2t_1t_1\Sigma[a_1,a_1,a_2,a_2] + t_2t_2t_2t_1\Sigma[a_1,a_2,a_2,a_2]+$$

$$+t_2t_1t_1t_1\Sigma[a_2,a_1,a_1,a_1] + t_1^4a_1^4 + t_2^4a_2^4)$$

$$= (a_2a_1 - a_1a_2)\Big(\frac{1}{12}(b_1^2 + b_2^2) + \frac{1}{24}(b_1b_2 + b_2b_1)+$$

$$+\frac{1}{20}\Sigma[b_1,a_1^2] + \frac{1}{60}\Sigma[b_2,a_1^2] + \frac{1}{20}\Sigma[b_2,a_2^2] + \frac{1}{60}\Sigma[b_1,a_2^2]+$$

$$+\frac{1}{60}\Sigma[b_1,(a_1a_2)] + \frac{1}{60}\Sigma[b_1,(a_2a_1)]+$$

$$+\frac{1}{60}\Sigma[b_2,(a_1a_2)] + \frac{1}{60}\Sigma[b_2,(a_2a_1)]+$$

$$\frac{1}{180}\Sigma[a_1,a_1,a_2,a_2]+\frac{1}{120}\Sigma[a_1,a_2,a_2,a_2]+\frac{1}{120}\Sigma[a_2,a_1,a_1,a_1]+\frac{1}{30}a_1^4+\frac{1}{30}a_2^4\Big)=$$

$$=(a_2a_1-a_1a_2)\bigg(\frac{1}{12}(b_1^2+b_2^2)+\frac{1}{24}(b_1b_2+b_2b_1)\bigg)+$$

$$+(a_2a_1^3-a_1a_2a_1^2+a_1^2a_2a_1-a_1^3a_2)(\frac{1}{20}b_1+\frac{1}{60}b_2)+$$

$$+(-a_1a_2^3+a_2a_1a_2^2-a_2^2a_1a_2+a_2^3a_1)(\frac{1}{20}b_2+\frac{1}{60}b_1)+$$

$$+\frac{1}{60}(2a_2a_1a_2a_1b_1-2a_1a_2a_1a_2b_1)+\frac{1}{60}(2a_2a_1a_2a_1b_2-2a_1a_2a_1a_2b_2)+$$

$$+\frac{1}{180}(2a_2^3a_1^3+2a_2a_1^2a_2^2a_1+2a_2a_1a_2a_1a_2a_1+2a_2a_1a_2^2a_1^2)+$$

$$\frac{1}{120}(4a_2a_1a_2a_1^3+\underbrace{2a_2a_1^2a_2a_1^2}_{\overset{\mathrm{tr}}{=}0})+\frac{1}{120}(-4a_1a_2a_1a_2^3-\underbrace{2a_1a_2^2a_1a_2^2}_{\overset{\mathrm{tr}}{=}0})+$$

$$+\frac{1}{30}(2a_2^5a_1+2a_2a_1^5)$$

where

$$\int_\Delta t_i^4=\frac{4!}{6!}=\frac{1}{30},\quad \int_\Delta t_i^2t_j^2=\frac{2!2!}{6!}=\frac{1}{180},\quad \int_\Delta t_i^3t_j=\frac{3!}{6!}=\frac{1}{120},$$

$i,j=1,2,3$ and $i\neq j$.

$$2\int_\Delta PQPQ\overset{\mathrm{tr}}{=}4\bigg(\sum\int_\Delta t_it_jt_kt_la_1(a_ia_j)a_2(a_ka_l)+$$

$$+\sum\int_\Delta t_it_jt_k[a_1(a_ja_k)a_2b_i-a_2(a_ja_k)a_1b_i]+\sum\int_\Delta t_it_ja_1b_ia_2b_j\bigg).$$

But

$$\sum \int_{\Delta} t_i t_j t_k t_l a_1 (a_i a_j) a_2 (a_k a_l) =$$

$$= \frac{1}{30}(a_2^5 a_1 + a_2 a_1^5) + \frac{2}{120} a_1^3 a_2 a_1 a_2 + \frac{2}{120} a_1 a_2 a_1 a_2^3 +$$

$$+ \frac{1}{180}(a_1^3 a_2^3 + a_1 a_2 a_1^2 a_2^2 - a_2 a_1 a_2^2 a_1^2 + a_1 a_2 a_1 a_2 a_1 a_2),$$

$$\sum \int_{\Delta} t_i t_j t_k [a_1 (a_j a_k) a_2 b_i - a_2 (a_j a_k) a_1 b_i] =$$

$$= \frac{1}{20}(a_1^3 a_2 - a_2 a_1^3) b_1 + \frac{1}{60}(-a_2 a_1 a_2 a_1 + a_1 a_2 a_1 a_2 + a_1^2 a_2^2 - a_2^2 a_1^2 + a_1 a_2^3 - a_2^3 a_1) b_1 +$$

$$+ \frac{1}{20}(a_1 a_2^3 - a_2^3 a_1) b_2 + \frac{1}{60}(-a_2 a_1 a_2 a_1 + a_1 a_2 a_1 a_2 + a_1^2 a_2^2 - a_2^2 a_1^2 + a_1^3 a_2 - a_2 a_1^3) b_2,$$

$$\sum \int_{\Delta} t_i t_j a_1 b_i a_2 b_j = \frac{1}{12} a_1 b_1 a_2 b_1 + \frac{1}{12} a_1 b_2 a_2 b_2 + \frac{1}{24} a_1 b_1 a_2 b_2 + \frac{1}{24} a_1 b_2 a_2 b_1 . []$$

Appendix 2. Formulas for the secondary class $\mathrm{ch}_k^1(a_0, a_1)$

Now let us write several convenient formulas for a secondary class

$$\mathrm{ch}_k^1(a_0, a_1) \stackrel{\mathrm{def}}{=} \frac{1}{k!} \int_0^1 ((dt_0 a_0 + dt_1 a_1) + (t_0 b_0 + t_1 b_1) + (t_0 a_0 + t_1 a_1)^2).$$

Theorem.

$$\mathrm{ch}_k^1(a_0, a_1) \stackrel{\mathrm{tr}}{=} \frac{1}{(k-1)!} A \sum_{m+s+r=k-1} \frac{1}{2m+s+1} \Sigma_{(m,s,r)}$$

where $A = a_1 - a_0$, $R = b_0 + a_0^2$, $S = (b_1 - b_0) + (a_0 A + A a_0)$, $M = A^2$ *and* $\Sigma_{(m,s,r)}$ *is the sum of all possible words with* m *letters* M, s *letters* S *and* r *letters* R.

Sketch of the proof.

$$\mathrm{ch}^1_k(a_0,a_1) \stackrel{\mathrm{tr}}{=} \frac{1}{k!}\int_0^1 ((dt_0a_0+dt_1a_1)+(t_0b_0+t_1b_1)+(t_0a_0+t_1a_1)^2) \stackrel{\mathrm{tr}}{=}$$

$$= \frac{1}{k!}\int_0^1 (dt_1 \underbrace{(a_1-1_0)}_{A} + \underbrace{(b_0+a_0^2)}_{R} +$$

$$+t_1 \underbrace{[(b_1-b_0)+(a_0(a_1-a_0)+(a_1-a_0)a_0)]}_{S} + t_1^2\underbrace{(a_1-a_0)^2}_{M})^k$$

$$\stackrel{\mathrm{tr}}{=} \frac{1}{k!}\int_0^1 (dt\,A+R+tS+t^2M)^k \stackrel{\mathrm{tr}}{=} \frac{1}{k!}\int_0^1 kdtA(R+tS+t^2M)^{k-1} \stackrel{\mathrm{tr}}{=}$$

$$\stackrel{\mathrm{tr}}{=} \frac{1}{k!}kA \sum_{m+s+r=k-1} \frac{1}{2m+s+1}\Sigma_{(m,s,r)}. \qquad \square$$

We can write it also as

$$\mathrm{ch}^1_k(a_0,a_1) = \tfrac{1}{(k-1)!}A\cdot \sum_{k-1=\alpha+\beta+\gamma+\delta+\varepsilon} \frac{(2\alpha+\beta+\delta)!(2\gamma+\beta+\varepsilon)!}{(2\alpha+2\beta+2\gamma+\varepsilon+\delta)!}\Sigma(\alpha,\beta,\gamma,\delta,\varepsilon),$$

where $\Sigma(\alpha,\beta,\gamma,\delta,\varepsilon)$ is the sum of all possible words with α letters a_0^2, β letters $a_0a_1+a_1a_0$, γ letters a_1^2, δ letters b_0, and ε letters b_1.

Appendix 3. Formulas for secondary classes $\mathrm{ch}^n_n(a_0,\dots,a_n)$ and $\mathrm{ch}^n_{n+1}(a_0,\dots,a_n)$.

Let us write now formulas for some other secondary characteristic classes.

Theorem.

$$\mathrm{ch}^n_n(a_0,\dots,a_n) = \frac{(-1)^{n(n-1)/2}}{n!}\sum_{\sigma\in S_n}(-1)^{sign(\sigma)}A_{\sigma(1)}\dots A_{\sigma(n)} =$$

$$= \frac{(-1)^{n(n-1)/2}}{n!}\sum_{\mu\in S_{n+1}}(-1)^{sign(\mu)}\hat{a}_{\mu(0)}a_{\mu(1)}\dots a_{\mu(n)}$$

$$\mathrm{ch}^n_{n+1}(a_0,\ldots,a_n) = \left(\frac{(-1)^{n(n-1)/2}}{n!}\sum_{\mu\in S_{n+1}}(-1)^{sign(\mu)}\hat{a}_{\mu(0)}a_{\mu(1)}\ldots a_{\mu(n)}\right)\cdot$$

$$\cdot\left(\frac{1}{(n+1)!}(b_0+\ldots+b_n)+\frac{2}{(n+2)!}(a_0^2+\ldots+a_n^2)+\frac{1}{(n+2)!}\sum_{i\neq j}(a_ia_j+a_ja_i)\right).$$

where $A_i = a_i - a_0$. Summations are over all permutations σ in the symmetric group S_n on $\{1,\ldots,n\}$, or over all permutations μ in the symmetric group S_{n+1} on $\{0,1,\ldots,n\}$.

Example.

$\mathrm{ch}^3_3(a_0,a_1,a_2,a_3) =$

$$= \frac{1}{3!}(-A_1A_2A_3 + A_1A_3A_2 - A_2A_3A_1 + A_2A_1A_3 - A_3A_1A_2 + A_3A_2A_1) =$$

$$= \frac{1}{3!}(-a_1a_2a_3 + a_1a_3a_2 - a_2a_3a_1 + a_2a_1a_3 - a_3a_1a_2 + a_3a_2a_1 -$$

$$+a_0a_2a_3 - a_0a_3a_2 + a_2a_3a_0 - a_2a_0a_3 + a_3a_0a_2 - a_3a_2a_0 +$$

$$+a_1a_0a_3 - a_1a_3a_0 + a_0a_3a_1 - a_0a_1a_3 + a_3a_1a_0 - a_3a_0a_1 +$$

$$+a_1a_2a_0 - a_1a_0a_2 + a_2a_0a_1 - a_2a_1a_0 + a_0a_1a_2 - a_0a_2a_1).$$

$\mathrm{ch}^3_4(a_0,a_1,a_2,a_3) =$

$$= \frac{1}{3!}(-A_1A_2A_3 + A_1A_3A_2 - A_2A_3A_1 + A_2A_1A_3 - A_3A_1A_2 + A_3A_2A_1)\times$$

$$\times\left(\frac{1}{4!}(b_0+b_1+b_2+b_3)+\frac{2}{6!}(a_0^2+a_1^2+a_2^2+a_3^2)+\frac{1}{6!}\sum_{i\neq j}(a_ia_j+a_ja_i)\right).$$

Proof.

$$\mathrm{ch}^n_n(a_0,\ldots,a_n) \overset{\mathrm{tr}}{=} \frac{1}{n!}\int_\Delta (dt_0a_0+\ldots+dt_na_n)^n =$$

taking $t_0 = 1 - t_1 - t_2 - \ldots - t_n$

$$= \int_\Delta \frac{1}{n!}(dt_1(a_1 - a_0) + \ldots + dt_n(a_n - a_0))^n = \frac{1}{n!}\int_\Delta (dt_1 A_1 + \ldots + dt_n A_n)^n =$$

$$= \frac{1}{n!}\int_\Delta dt_1 \ldots dt_n (-1)^{1+2+\ldots+(n-1)} A_1 \ldots A_n + \ldots +$$

$$+ \frac{1}{n!}\int_\Delta dt_{\sigma(1)} \ldots dt_{\sigma(n)} (-1)^{1+2+\ldots+(n-1)} A_{\sigma(1)} \ldots A_{\sigma(n)} + \ldots =$$

$$= \frac{(-1)^{n(n-1)/2}}{n!} \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} A_{\sigma(1)} \ldots A_{\sigma(n)} =$$

$$= \frac{(-1)^{n(n-1)/2}}{n!} \sum_{\mu \in S_{n+1}} (-1)^{sign(\mu)} \hat{a}_{\mu(0)} a_{\mu(1)} \ldots a_{\mu(n)}.$$

$\mathrm{ch}^n_{n+1}(a_0, \ldots, a_n) \overset{\mathrm{tr}}{=}$

$$\overset{\mathrm{tr}}{=} \tfrac{1}{(n+1)!} \int_\Delta \big((dt_0 a_0 + \ldots + dt_n a_n) + (t_0 b_0 + \ldots + t_n b_n) + (t_0 a_0 + \ldots + t_n a_n)^2\big)^{n+1} \overset{\mathrm{tr}}{=}$$

$$\overset{\mathrm{tr}}{=} \tfrac{(n+1)}{(n+1)!} \int_\Delta (dt_0 a_0 + \ldots + dt_n a_n)^n \big((t_0 b_0 + \ldots + t_n b_n) + (t_0 a_0 + \ldots + t_n a_n)^2\big) =$$

$$= \frac{1}{n!}\int_\Delta (dt_1 A_1 + \ldots + dt_n A_n)^n (t_0 b_0 + \ldots + t_n b_n) + (t_0 a_0 + \ldots + t_n a_n)^2) =$$

$$= \left(\frac{(-1)^{n(n-1)/2}}{n!} \sum_{\mu \in S_{n+1}} (-1)^{sign(\mu)} \hat{a}_{\mu(0)} a_{\mu(1)} \ldots a_{\mu(n)} \right) \cdot$$

$$\cdot \left(\frac{1}{(n+1)!}(b_0 + \ldots + b_n) + \frac{2}{(n+2)!}(a_0^2 + \ldots + a_n^2) + \frac{1}{(n+2)!} \sum_{i \neq j} (a_i a_j + a_j a_i) \right)$$

We use

$$\int_{\Delta^n} t_0^{\alpha_0} \dots t_n^{\alpha_n} = \frac{\alpha_0! \dots \alpha_n!}{(\alpha_0 + \dots + \alpha_n + n)!},$$

$$\text{so} \quad \int_{\Delta^n} t_i = \frac{1}{(n+1)!}, \quad \int_{\Delta^n} t_i^2 = \frac{2}{(n+2)!}, \quad \int_{\Delta^n} t_i t_j = \frac{1}{(n+2)!}.$$

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