Holomorphic Curves in Abelian Varieties

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dedicated to Professor Tadashi Nagano on his 60 th birthday

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Abstract

We introduce a new technique in the value distribution theory of holomorphic curves and apply it to establish a Second Main Theorem for holomorphic curves into Abelian varieties. As a corollary, we settle a conjecture of Lang: if a holomorphic curve into an Abelian variety omits an ample divisor, then it is necessarily a constant map.

1 Introduction

The value distribution theory of holomorphic curves into general complex projective algebraic varieties is not fully developed except for Cartan-Weyl-Ahlfors' theory on holomorphic curves into complex projective space (cf. [Car], [A], [Wy], [Ch], [Co-G] and [F]). We refer to [Gri1], [KS2], [L2], [L3], [R-W], [R-S], [S2], [Wo2] and [Wu] for surveys on the value distribution theory of holomorphic curves and related topics. For holomorphic curves in general projective varieties, however, there is a conjectural "Second Main Theorem" due to Lang ([L1], [L3], see also [No3]). Let $f : \boldsymbol{C} \longrightarrow X$ be a holomorphic map from the one-dimensional complex line $\boldsymbol{C}$ into a smooth complex projective algebraic variety X, which we call a *holomorphic curve* in X. Let D be a divisor in X. Write $T_{f,D}(r)$ (resp. $N_{f^*(D)}(r)$) for Nevanlinna's characteristic (resp. counting) function for the holomorphic map f with respect to the divisor D. With these definitions (see Section 2), we can state Lang's conjecture as follows:

Conjecture 1 (Lang) *Let X be a smooth complex projective algebraic variety, D a divisor with at worst simple normal crossings, E the hyperplane section and K the canonical divisor of X. Then there exixts a proper analytic subset D' of X which depends only on D such that for any holomorphic curve*

$$f : \boldsymbol{C} \longrightarrow X$$

with the non-degeneracy condition

$$f(\boldsymbol{C}) \not\subset D'$$

the following inequality holds:

$$T_{f,D}(r) + T_{f,K}(r) \leq N_{f^*(D)}(r) + O_{exc}(\log r + \log T_{f,E}(r)).$$

The O_{exc} term is called the *error term*[1].

In the case $\dim X = 1$ this statement is the classical Nevanlinna theory (cf. [A], [Ne1]). For higher dimensional equi-dimensional value distribution theory, see, for example, [Ca-G], [G-K], [L-C], [N-O], [S1,2] and [Wo1]. The case $X = P_n(\boldsymbol{C})$, $(n \geq 2)$, and D is the collection of hyperplanes in general position is contained in Cartan-Weyl-Ahlfors' theory ([Car], [A], [Wy], [Ch] and [Co-G]). Derived curves into Grassmannians are treated in [A], [Wy] and [F]. For holomorphic curves in algebraic surfaces of general type, Lu-Yau [L-Y] and Lu [Lu] obtained some degeneracy and finiteness theorems. On the other hand, Siu [Si] obtained a general defect relation under the assumption of the existence of some special meromorphic connections.

In this paper, we consider the case where X is an Abelian variety and D is an effective reduced divisor. So, let A be an Abelian variety of dimension n defined over a complex number field $\boldsymbol{C}$ and D a reduced divisor in A with at worst simple normal crossings. We further assume that D *is ample*. Later

[1]The symbol O_{exc} means that the inequality is valid for r outside a Borel set of finite linear measure. See [L-C] for details. Hereafter we do not mention this explicitly.

we will relax these conditions and consider general effective reduced divisor (see Theorem 3 and Theorem 6). Let

$$f : \boldsymbol{C} \longrightarrow A$$

be a holomorphic curve. Assume that

$$f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D),$$

i.e., the image of A is not contained in the support of D. The purpose of this paper is to prove the following inequality of the Second Main Theorem type between $T_f(r, D)$ and $N(f^*D)$:

Theorem 1 *Let $f : \boldsymbol{C} \to A$ and D be as above. Then for any positive number ϵ the following inequality holds:*

$$T_{f,D}(r) \le (1+\varepsilon)N_{f^*(D)}(r) - N_{f,Ram}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1).$$

The symbol $O_\varepsilon(1)$ represents an $O(1)$ term which depends on ε[2]. This inequality is weaker than Lang's conjectural "Second Main Theorem". However, we are able to use this inequality to settle another conjecture of Lang. Namely, we have:

Theorem 2 *Let A and D be as above. Then every holomorphic curve*

$$f : \boldsymbol{C} \longrightarrow A - \mathrm{Supp}(D)$$

degenerates to a constant map.

Clearly Conjecture 1 implies both Theorem 1 and Theorem 2. For Lang's conjecture and related problems, we refer to Griffiths (Problem F. in [Gri]) and Kobayashi (Problem D.9 in [Kob2]). See also [L3]. Special cases of Theorem 2 have been considered by Ax [Ax], Green [Gre], Ochiai [O] and Noguchi [No2]. Namely Ax proved Theorem 2 when f is a one-parameter subgroup, while Green proved Theorem 2 when D contains no nontrivial

[2]We will not write $O_\varepsilon(1)$ terms explicitly and include this in the error term, if no confusion is possible.

translated Abelian subvariety by showing that $A - D$ is complete hyperbolic and is hyperbolically embedded in A in the sense of Kobayashi [Kob1,2]. Ochiai [O] proved Theorem 2 when $D = D_1 + D_2$ where D_1 and D_2 are linearly equivalent ample divisors with no common component. Noguchi [No2] generalized Ochiai's result to the case $D = D_1 + D_2$ where D_1 and D_2 are homologously equivalent ample divisors with no common component. In [No2] Noguchi proved an inequality

$$T_{f,D}(r) \leq C\overline{N}_{f^*D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) \tag{1}$$

of the Second Main Theorem type for holomorphic curves in smooth projective algebraic variety X with "abundant" logarithmic 1-forms along a hypersurface D. Here, $\overline{N}_{f^*D}(r)$ denotes the counting function counted without counting multiplicites. In the proof of Theorem 9, the importance of the appearance of $\overline{N}_{f^*D}(r)$ will be clarified. Noguchi's proof of this inequality is based on Bloch-Ochiai's Theorem ([B],[O]; see Theorem A in [O]) on holomorphic curves into a smooth irreducible quasi projective algebraic variety with "abundant" holomorphic 1-forms. For instance, suppose A is an Abelian variety and D_1, D_2 are homologously equivalent ample divisors of A with no common component. Then the collection of logarithmic 1-forms on A consisting of linearly independent n holomorphic 1-forms and the exterior differential of the theta function with zeros along D_1 and poles along D_2 satisfies the "abundance" condition in Bloch-Ochiai's Theorem. We can then apply (1) to conclude the degeneracy of f, provided f omits Supp(D). It seems to be difficult to generalize Noguchi's method to the case where D is a general ample divisor.

In both Noguchi's arguments in [No2] and our proof of Theorem 1, the problem is reduced, in some sense, to the one-equidimensional value distribution theory. The difference lies in the fact that Noguchi makes most of "abundant" logarithmic 1-forms through pull-back argument whereas we make most of "abundant" algebraic curves in an Abelian variety through push forward argument. Moreover, our push-forward argument heavily depends on the special feature of Abelian varieties, namely, the abelian group structure (the parallel transportation or the flatness in terms of differntial geometry). So it again seems to be difficult to generalize our argument directly to holomorphic curves in general projective varieties. But if we consider the group structure of $(C^*)^n$ on the complement of $(n+1)$ hyperplanes in general position in $P_{n-1}(C)$, our argument will probably be generalized to holomorphic curves in projective spaces (see Theorem 9 and Concluding Remarks). In our proof, the Second Main Theorem for holomorphic mappings from a finite analytic covering space over C to a compact Riemann surface of genus ≥ 2 is very important. This kind of equidimensional Second Main Mheorem was first considered by Griffiths-King [G-K] (see also [S2]) and was fully developed in [No1], [St] and [L-C] (see also [No4] for non-equidimensional case)[3]. In our proof, we shall use Cauchy-Crofton type averaging argument (see Lemma 3) over and over again. This argument is based on the freedom which we find in the choice of a collection of algebraic curves in an Abelian variety. In the final step of the proof, we shall consider jet differentials up to (dim$A$)-th order of the holomorphic curve under question, just as in [O], [G-G], [No3,4] (see also [N-O]), [Gra], and apply the averaging argument to the (dim$A$)-th jet map (see Lemma 6 and Proposition 1).

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2 Review of Value Distribution Theory

In this section we introduce the usual definitions and notations in the Nevanlinna theory and state the Second Main Theorem for holomorphic mappings from a finite analytic covering space of C to a compact Riemann surface ([G-K], [S2], [No1], [No4], [St] and [L-C]).

A holomorphic mapping $\pi : X \to C$ is by definition a *finite analytic covering* over C if X is a Riemann surface and π is a surjective proper holomorphic mapping. If the fiber of π over a generic point consists of k points, we call $\pi : X \to C$ an *analytic k-covering* and write $[X : C] = k$.

Let z be a natural linear coordinate in the complex vector space C and

[3]From here on, we will just refer to it as the Second Main Theorem.

set

$$
\begin{aligned}
C(r) &= \{z \in C \,;\, | z |< r\}, \\
X(r) &= \pi^{-1}(C(r)), \\
\eta &= \tfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial) \log | z |^2 = d^c \log | z |^2 = \tfrac{d\theta}{2\pi},
\end{aligned}
$$

where $z = re^{i\theta}$ in polar coordinates. Let P be an effective divisor on a finite analytic coveriong X over C. We assume $\pi^{-1}(0) \cap P = \emptyset$ for simplicity. The *counting function* for P on X is defined by

$$
N_P(r) = \sum_{a \in X(r)} \nu_{a,P} \log \left| \frac{r}{a} \right| = \int_0^r \frac{dt}{t} n_P(t) \tag{2}
$$

where $\nu_{a,P} = \mathrm{ord}_a(P)$, i.e., $\nu_{a,P}$ is zero if $a \notin \mathrm{Supp}(P)$ and is equal to the coefficient of a if $a \in \mathrm{Supp}(P)$, and $n_P(t)$ is the degree of P in $| z |< t$ (counted with multiplicities). If the divisor P is defined by the ramification locus of $\pi : X \to C$, then we write

$$
N_P(r) = N_{\pi,Ram}(r)
$$

and if P is defined as the zero set (counting multiplicity) of the differential of a holomorphic mapping f of X into a Riemann surface, then we write

$$
N_P(r) = N_{f,Ram}(r).
$$

Let V be a smooth complex projective variety, $L \to V$ a holomorphic line bundle with a Hermitian metric $\| \cdot \|$ whose curvature form is Ω, and

$$
f : X \longrightarrow V
$$

a holomorphic curve. The *characteristic function* of f with respect to the Hermitian line bundle $(L, \| \cdot \|)$ is defined by

$$
T_{f,L,\Omega}(r) = \int_0^r \frac{dt}{t} \int_{X(t)} f^*\Omega \tag{3}
$$

Note that the characteristic function is defined purely in a differential geometric way. In particular, suppose we use the curvature form Ω' of another

Hermitian metric $\|\cdot\|'$ to define the characteristic function $T_{f,L,\Omega'}$. Then we have

$$\Omega' = \Omega + \sqrt{-1}\partial\bar{\partial}u$$

for some smooth function u on V. We then see from Jensen's formula that

$$T_{f,L,\Omega'}(r) = T_{f,L,\Omega}(r) + O(1).$$

For this reason, we write the characteristic function simply as $T_{f,L}(r)$ without indicating the curvature form used in the definition.

For $D \in \mid L \mid$ which does not contain the whole image of f, we define the *proximity function* of f with respect to D by

$$m_{f,D}(r) = \int_{\partial X(r)} \log(\frac{1}{\|\sigma \circ f\|})\pi^*\eta \tag{4}$$

where σ is a holomorphic section of L such that $(\sigma)_0 = D$ and $\|\sigma\| \leq 1$. Since $[D] = L$, we often write $T_{f,L,\Omega} = T_{f,D,\Omega}$.

Now let us assume that $f(0) \notin D$. Let $\|\cdot\|_t$ be a family of Hermitian metrics of L such that the curvature forms Ω_t converge to D as $t \to \infty$ in the sense of current. We now let $t \to \infty$ in

$$T_{f,D,\Omega}(r) = (T_{f,D,\Omega}(r) - T_{f,D,\Omega_t}(r)) + T_{f,D,\Omega_t}(r) \tag{5}$$

Note that

$$\Omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\frac{1}{\|\sigma\|_t^2})$$

and that $\|\sigma\|_t$ goes to a positive constant outside of D in the limit. Applying then Jensen's formula to the first term of the right hand side of (5), we obtain the First Main Theorem (the Nevanlinna inequality)

$$T_{f,L}(r) = m_{f,D}(r) + N_{f^*D}(r) + C \geq N_{f^*D}(r) + O(1) \tag{6}$$

where C is a constant determined by $\log\|\sigma \circ f(0)\|$. As is pointed out by Chern and Griffiths (see, for example, [Ch] and [Gri2]), the value distribution theory is considered to be a non-compact analogue of the intersection theory on projective manifolds. If C and D are an algebraic curve and a divisor in a projective manifold respectively, then the intersection number is given by the "area" of C with respect to the $(1,1)$-form representing $c_1([D])$:

$$C \cdot D = \int_C c_1([D]).$$

Similarly, the First Main Theorem (6) describes how often a holomorphic curve $f : X(r) \to V$ intersects a divisor D, in terms of the integration of $c_1([D])$ over the image of $X(r)$. The proximity function is then the correction term comming from the non-compactness and the distorsion between a smooth curvature form and the "Dirac like" curvature form. On the other hand, in the compact intersection theory, $C \cdot D > 0$ if $c_1([D]) > 0$ (i.e., $[D]$ has positive curvature). An analogue of this in the value distribution theory may have a right to be called the Second Main Theorem. So the Second Main Theorem is a statement estimating $N_{f^*D}(r)$ in terms of $T_{f,D}(r)$ *from below.* The theory is in satisfactory form only in two cases. These are the Cartan-Weyl-Ahlfors theory for holomorphic curves in $P_n(\boldsymbol{C})$ and the equidimensional theory by Griffiths and coauthors. In equidimensional theory, the curvature computations (see, for example, [Ca-G], [G-K], [S1,2], [N-O] for Nevanlinna-Carlson-Griffiths method and see [A], [Wo1], [L4], [L-C] for Ahlfors-Wong method) were very powerful. Also in the proof of our Second Main Theorem for holomorphic curves in Abelian varieties, curvature computations are very important. But it is so in a restricted way. In fact, all curvature computations are contained in the proof of the following Fact 1.

Question 1 *What is the relation between the negative curvature methods of Cowen-Griffiths* [Co-G] *and Green-Griffiths* [G-G], *the Frenet frame argument of Chern* [Ch] *and Wong* [Wo2], *and our method of the proof of Theorem 1 ?*

We now state the Second Main Theorem ([G-K], [S2], [No1], [No4], [St] and [L-C]) in one-dimensional case.

Fact 1 *Let $\pi : X \to \boldsymbol{C}$ be a finite analytic covering and $f : X \to V$ a holomorphic mapping into a compact Riemann surface V. Suppose that $\pi : X \to \boldsymbol{C}$ is unramified over* 0, *$f : X \to V$ is a non-degenerate holomorphic map such that f is unramified above* 0 *and that $f(y) \notin \mathrm{Supp}(D)$ for all $y \in X(0)$. Let $D = \sum_{j=1}^{q} p_j$ be a reduced divisor on V and L_j the line bundle* *on V defined by p_j. Let K be the canonical bundle of V and E an ample line bundle on V. Then we have*

$$T_{f,K}(r) + \sum_{j=1}^{q} T_{f,L_j}(r) - N_{f^*D}(r) + N_{f,Ram}(r) - N_{\pi,Ram}(r)$$

$$\leq\ O_{exc}(\log r + \log T_{f,E}(r)).$$

Remark 1 *The ramification terms $N_{f,Ram}(r)$ and $N_{\pi,Ram}(r)$ in the above Second Main Theorem will become inportant in applications to prove our results. For instance, we shall make essential use of $N_{f,Ram}(r)$ to prove Theorem 9.*

It should be noted that Lang and Cherry [L-C] study systematically the error term in the Second Main Theorem for holomorphic mappings $X \to V$ of a finite analytic covering space over $\boldsymbol{C}^n$ into an n-dimensional smooth projective variety V and clarifies how the covering index $[X : \boldsymbol{C}^n]$ appears in the error term. See [L-C, Theroem 5.4, pp. 164-165] and see also [Wol] and [L4] for new techniques (Ahlfors-Wong method) by which we are able to get the best understanding of the error term in the equidimensional Nevanlinna theory. To get better estimates for the error term in the Second Main Theorem for holomorphic curves will be an interesting problem.

3 Proof of Theorems

Let A be an Abelian variety of dimension n, D an ample effective reduced divisor with at worst simple normal crossings in A and $f : \boldsymbol{C} \to A$ a holomorphic curve such that $f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D)$. We choose a Hermitian metric $\|\cdot\|$ on $[D]$ (the line bundle determined by D) and write σ for a holomorphic section of $[D]$ such that $(\sigma)_0 = D$ and $\|\sigma\| \leq 1$. We may assume that the curvature form

$$\Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{1}{\|\sigma\|^2}$$

defines a *flat* metric on A.

Theorem 1 *Let $f : \boldsymbol{C} \to A$ and D be as above. Then for any positive number ϵ, the following inequality holds:*

$$T_{f,D}(r) \leq (1+\varepsilon)\, N_{f^*D}(r) - N_{f,Ram}(r) + O_{exc}(\log r + \log T_{f,D}(r)).$$

Proof of Theorem 1. We approximate the Abelian variety A by a collection of smooth algebraic curves in A as follows. Let $p \in A$ be the identity element of the group variety A. We choose $M(\geq n)$ smooth closed algebraic curves $S_1, S_2, \cdots, S_M$ in A through p such that any n tangent vectors to the S_i's at p are linearly independent and the set $\{T_p S_i\}_{i=1}^M$ forms a δ-net in $P_{n-1}(C)$ for a small $\delta > 0$. Clearly $\delta \to 0$ as $M \to \infty$. Let K_i be the canonical divisor of S_i, $g(S_i) > 1$ the genus of S_i and $e_i > 0$ the negative of the Euler number. In the following discussions, M must be chosen sufficiently large if the extrinsic geometry of D is not simple, e.g., if the second fundamental form varies very rapidly or D has normal crossing singularities. In any case, we let M to be very large in the last stage of our proof. We may assume that none of S_i's are contained in the parallel transforms of $\mathrm{Supp}(\iota D)$, where ι is the canonical involution of A, $\iota : A \to A$, $\iota(a) = -a$. We may further assume that $dz_\alpha \not\equiv 0$ on each S_i for all $\alpha = 1, \cdots, n$, in particular, $g(S_i) \geq n$. Since A is an Abelian variety rather than simply a complex torus, we can let $M \to \infty$ assuming that *all S_i's are in the same algebraic equivalence class in A.* We set $m = \mid e(S_i) \mid$ and $k = S_i \cdot D$, where $e(S_i)$ stands for the Euler characteristic of S_i. For each i, $(i = 1, \cdots, M)$, we define a proper analytic subset D_i of A in the following way. *A point $a \in A$ is contained in the analytic set D_i if and only if $a + \iota D$ and S_i have at least one multiple intersection point.* D_i is a divisor of A. To see this, we look at a holomorphic map

$$\phi_i : A \to S^k(A), \ \phi_i(a) = \{D \cap (a + \iota S_i)\},$$

where $S^k(A)$ is the k-th symmetric product of A. Let Δ be a divisor in $S^k(A)$ defined by the coincidence condition. Then we define

$$D_i = \phi_i^{-1}(\Delta).$$

Suppose $a \in D_i$. Then either S_i pass through at least one singular point of $a + \iota D$ or $(a + \iota D) \cap S_i$ consists of regular points of D and S_i is tangent to $a + \iota D$ at a regular point of D. In the latter case, for a point $a \in D_i$, there exists a point of $(a + \iota D) \cap S_i$ at which the tangent line to S_i is contained in the tangent hyperplane of $a + \iota D$ there. The proper analytic set D_i' of A defined by $D_i' = \cup_{s \in \mathrm{Sing}(D)}(s + S_i)$ is contained in D_i and the divisorial part of it comes from the component of $\mathrm{Sing}(D)$ with codimension one in

D. Higher codimensional components of $\mathrm{Sing}(D)$ contribute to D_i only by analytic set of codimension at least two. Let $X_i = X_i(f, D, S_i)$ be a finite analytic covering over $\boldsymbol{C}$ defined by

$$X_i(f, D, S_i) = \{(z, q)\,;\; z \in \boldsymbol{C},\; q \in D \text{ such that } f(z) - q \in S_i\} \subset \boldsymbol{C} \times D$$

for $i = 1, \cdots, N$. Let k_i be the covering index of the covering

$$\pi_i \,:\, X_i(f, D, S_i) \longrightarrow \boldsymbol{C}, \quad \pi_i(z, q) = z.$$

Then we have $k_i = S_i \cdot D$. Since D is effective and A is an Abelian variety, we have $k_i > 0$ and $X_i \neq \emptyset$. We define M holomorphic mappings

$$f_i \,:\, X_i(f, D, S_i) \longrightarrow S_i, \;\; f_i(z, q) = f(z) - q$$

for $i = 1, \cdots, M$. Suppose that $f(z)$ is very close to D for some $z \in \boldsymbol{C}$, i.e., $\|\sigma \circ f(z)\|$ is very small. Let $(z, q_{i\nu(i)}) \in X_i$, $(i = 1, \cdots, k_i)$, be the points over z. Then we see that for some $(z, q_{i\nu(i)})$, the point $f_i(z, q_{i\nu(i)}) = f(z) - q_{i\nu(i)} \in S_i$ is very close to the identity element $p \in S_i$ of A with respect to, for instance, the Euclidean metric on A. For general $(z, q_{i\nu(i)})$, the distance of $f(z) - q_{i\nu(i)}$ to the identity element p depends on the relation of the tangent hyperplanes of D near $f(z)$ and the tangent line of S_i at p, in other words, on how $f(z) + \iota D$ and S_i are curved near p. Let σ_i be a section of $[p]$ on S_i and $\|\cdot\|_i$ a Hermitian metric for $[p]$ such that $\|\sigma_i\|_i \leq 1$. It is clear that such $\|\cdot\|_i$'s exist uniformly. Now we look at $f(z)$ which is very close to $\mathrm{Supp}(D)$. We choose a positive integer c with $c << M$ (intuitively, c is the "number" of such S_i's which are approximately "parallel" to $f(z) + \iota D$ near the origin p). We choose c so that

$$\frac{M}{M - c} \longrightarrow 1$$

as $M \to \infty$. Then the following string of inequalities holds, up to additive $O(1)$ terms which depend on M, c and the extrinsic geometry of D in A (if $\frac{c}{M}$ becomes small, the $O(1)$ terms will become bigger), for any holomorphic mapping $f : \boldsymbol{C} \to A$ with $f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D)$:

$$
\begin{aligned}
m_{f,D}(r) &= \int_{\partial C(r)} \log(\tfrac{1}{\|\sigma\circ f\|})\eta \\
&\quad \text{[from (4)]} \\
&\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M} \int_{\partial X_i(r)} \log(\tfrac{1}{\|\sigma_i\circ f_i\|_i})\pi_i^*\eta \\
&\quad \text{[from the construction]} \\
&\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(T_{f_i,p}(r) - N_{f_i^*p}(r)) \\
&\quad \text{[from (6)]} \\
&\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(\tfrac{1}{e_i}T_{f_i,K_i}(r) - N_{f_i^*p}(r)) \\
&\quad \text{[because } c_1(K_i) = e_i\text{]} \\
&\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{\tfrac{1}{e_i}(N_{\pi_i,Ram}(r) - T_{f_i,p}(r) + N_{f_i^*p}(r) - N_{f_i,Ram}(r)) \\
&\quad -N_{f_i^*p}(r)\} + O_{exc}(\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(\log r + \log T_{f_i,p}(r))) \\
&\quad \text{[from Fact 1].}
\end{aligned}
$$

There are two reasons why we have used Fact 1 this way. First, $N_{\pi_i,Ram}(r)$ contains much information on f and second, we want $T_{f_i,p}(r)$ with a small coefficient. If D has at worst normal crossings as we assumed, then the set of points $a \in D_i$ such that $(a + \iota D) \cap S_i$ contains a singular point of $\mathrm{Supp}(D)$ forms a divisor $D_i'(\subset D_i)$ of A. If $f(z)$ passes through a normal crossing singularity of D, then an ordinary multiple point (e.g., a node) occurs in $X_i(r) \subset \boldsymbol{C} \times D$. Conversely, if D has at worst normal crossings, any ordinary multiple point of $X_i(r)$ arises in this way. We then take the normalization of $X_i(r)$ at these ordinary multiple points and write the resulting Riemann surface also as $X_i(r)$ for notational simplicity. After performing this modification, we may assume that the normal crossing singularities of D make no contribution to the ramification divisor of $\pi_i : X_i \to \boldsymbol{C}$. Suppose $f(z) \in D_i$. Then S_i and $D - f(z)$ may have several contact points. In such case, $f(z)$ is necessarily contained in the singular locus of D_i. Taking multiplicities into account, we have

$$N_{\pi_i,Ram}(r) \leq N_{f^*D_i}(r).$$

If D has at worst normal crossings, if we deform S_i slightly (if necessary), we have from the construction that

$$N_{\pi_i,Ram}(r) = N_{f^*D_i}(r).$$

We note that although D is a general ample hypersurface, the above inequality between two counting functions remains true, but we cannot obtain the equality simply by deforming S_i. It follows from the construction that

$$T_{f,D}(r) \leq \frac{1}{M-c}\sum_{i=1}^{M} T_{f_i,p}(r) + C$$

where C is a constant independent of f. This remains true for general ample reduced D. We then have from the above inequalities the following:

$$\begin{aligned} m_{f,D}(r) &\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{\tfrac{1}{e_i}N_{f^*D_i}(r) - (1-\tfrac{1}{e_i})N_{f_i^*p}(r) \\ &\quad -\tfrac{1}{e_i}N_{f_i,Ram}(r)\} - \tfrac{1}{m}T_{f,D}(r) \\ &\quad +O_{exc}(\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(\log r + \log T_{f_i,p}(r))) \end{aligned}$$

from the construction. Now we note that the counting function of a sum of divisors is the sum of the of the counting functions for each divisor. Since $D_i = mD + (D_i - mD)$, we have

$$\begin{aligned} m_{f,D}(r) &\leq \tfrac{1}{m}(\tfrac{M}{M-c}N_{f^*mD}(r) - T_{f,D}(r)) \\ &\quad -\tfrac{M}{M-c}(1-\tfrac{1}{m})N_{f^*D}(r) \\ &\quad +\tfrac{1}{m}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*D_i}(r) - (N_{f^*(mD)}(r) + N_{f_i,Ram}(r))\} \\ &\quad +O_{exc}(\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(\log r + \log T_{f_i,p}(r))). \end{aligned}$$

Therefore, from (6) and the concavity of log, we have

$$\begin{aligned} m_{f,D}(r) \ \le\ & \tfrac{1}{m}\left(\tfrac{M}{M-c}T_{f,mD}(r) - T_{f,D}(r)\right) \\ & -\tfrac{M}{M-c}\left(1-\tfrac{1}{m}\right)N_{f^*D}(r) \\ & +\tfrac{1}{m}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*D_i}(r) - (N_{f^*(mD)}(r) + N_{f_i,Ram}(r))\} \\ & +O_{exc}(\log r + \log T_{f,D}(r)) \end{aligned}$$

up to additive constants depending on M. Note that the error term is independent of ε and sufficiently large M. We infer from this and (6) that, if M is sufficiently large so that

$$\frac{M}{M-c} - \frac{1}{m} < 1,$$

there exists a positive number

$$C_{M,c,m} = \frac{1 - \frac{M}{M-c} + \frac{M}{M-c}\frac{1}{m}}{1 - \frac{M}{M-c} + \frac{1}{m}} > 1$$

which depends on n, M and the extrinsic geometry of D such that

$$\begin{aligned} T_{f,D}(r) \ \le\ & C_{M,c,m}N_{f^*D}(r) \\ & +\tfrac{1}{m}\tfrac{1}{1-\frac{M}{M-c}+\frac{1}{m}}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*D_i}(r) \\ & -(N_{f^*mD}(r) + N_{f_i,Ram}(r))\} \\ & +O_{exc}(\log r + \log T_{f,D}(r)). \end{aligned}$$

Note that for any small positive number ε, there exists an M so that $C(n,D,M) < 1+\varepsilon$. Therefore our task is to bound the quantity

$$N_{f^*D_i}(r) - (N_{f^*(mD)}(r) + N_{f_i,Ram}(r))$$

from above by

$$o_{exc}(T_{f,D}(r)) + O_{exc}(\log r + \log T_{f,D}(r)).$$

This can be done in the following way.

We define a divisor $\widetilde{D}_i$ in $P(TA) = A \times P_{n-1}(\boldsymbol{C})$ by

$$\widetilde{D}_i = \cup_{a \in A} D_a,$$

where D_a stands for the divisor of degree $k_i = S_i \cdot D$ in $P_{n-1}(\boldsymbol{C})$ defined by the images in $P_{n-1}(\boldsymbol{C})$ of the tangent hyperplanes (or tangent cones, if singular points are contained in the intersection) of ιD at $(a + \iota D) \cap S_i$. Let

$$\pi : A \times P_{n-1}(\boldsymbol{C}) \longrightarrow A$$

be the projection onto the first factor. Write $\tilde{f}$ for the canonical lifting of $f : \boldsymbol{C} \to A$

$$\tilde{f} : \boldsymbol{C} \longrightarrow P(TA) = A \times P_{n-1}(\boldsymbol{C}),$$

namely, if $\frac{\partial f}{\partial z}(z) \neq 0$, then $\tilde{f}(z)$ is defined to be $(f(z), [\frac{\partial f}{\partial z}(z)]) \in A \times P_{n-1}(\boldsymbol{C})$, where $[v]$ for $v \in \boldsymbol{C}^n - \{0\}$ denotes the image of v in $P_{n-1}(\boldsymbol{C})$. Now it is extended holomorphically across $z \in \boldsymbol{C}$ where $\frac{\partial f}{\partial z}(z) = 0$. We note that

$$f \text{ hits } D_i \iff \tilde{f} \text{ hits } \pi^* D_i$$

and

$$f_i \text{ ramifies at } (z, q) \impliedby \tilde{f} \text{ hits } \widetilde{D}_i + \pi^* D_i''$$

where

$$D_i'' = \cup_{s \in D''}(s + S_i)$$

and D'' is the codimension one non normal crossing singular locus of D. (If D has non normal crossing singular locus of codimension one, we must take the multiplicity of this singular locus into account. See the proof of Theorem 4.) This implies

$$N_{f_i, Ram}(r) \geq N_{\tilde{f}^*(\widetilde{D}_i + \pi^* D_i'')}(r).$$

Therefore, if D has at worst normal crossing singularities, we may replace the above quantity by

$$N_{\tilde{f}^* \pi^* D_i}(r) - (N_{\tilde{f}^* \pi^* mD}(r) + N_{\tilde{f}^* \widetilde{D}_i}(r)) \tag{7}$$

and dominate this by $o_{exc}(T_{f,D}(r))$. We note that we have omitted the negative term $-N_{f,Ram}(r)$ which comes from the solutions of $f'(z) = 0$.

Lemma 1 *Let $L \longrightarrow A$ be a holomorphic line bundle over an Abelian variety A. If $\Gamma \cdot L = 0$ for all curves Γ in A, then we have $c_1(L) = 0$.*

Proof. Any Abelian variety has an ètale covering which is a direct product of simple Abelian varieties. Suppose X is a product $X_1 \times \cdots \times X_r$ of projective varieties. Then $\Gamma \cdot L = 0$ for any curve $\Gamma \subset X$ if $\Gamma \cdot L = 0$ for any curve Γ which projects to a point in X_i for all indices except one i. So, it suffices to verify the assertion only for simple Abelian varieties. Now suppose A is simple. Define a holomorphic map

$$\phi_L : A \longrightarrow \mathrm{Pic}^0(A), \ \phi_L(x) = T_x^* L \otimes L^{-1},$$

where T_x denotes the translation by x and $\mathrm{Pic}^0(A)$ stands for the group of line bundles with $c_1(L) = 0$. Then ϕ_L is a homomorphism between n-dimensional Abelian varieties. Set $K = \mathrm{Ker}(\phi_L)$. Since A is simple, K is either a finite group or A. It follows from the Riemann-Roch Theroem in [M, Section 16] that

$$\deg(\phi_L) = \chi(L)^2 = \left(\frac{L^n}{n!}\right)^2 = 0.$$

So $\dim \deg(\phi_L) < n$, which means $\dim K > 0$. Therefore K must be A and $\phi_L \in \mathrm{Hom}(A, \mathrm{Pic}^0(A))$ is trivial. Since $L \in \mathrm{Pic}^0(A)$ is characterized by the property $\phi_L = 0$ (see [M, Sections 8 and 9]), we conclude $L \in \mathrm{Pic}^0(A)$, which proves Lemma 1. □

Lemma 2 *Let the situation be as above. Let H be a hyperplane in $P_{n-1}(\boldsymbol{C})$. There exists an ample hypersurface $E_i \subset A$ such that*

$$O_{A \times P_{n-1}(\boldsymbol{C})}(\widetilde{D}_i) = O_{A \times P_{n-1}(\boldsymbol{C})}(\pi^* E_i + k_i(A \times H))$$

and such that

$$c_1(O_A(D_i - (mD + E_i))) = 0.$$

Proof. Define a holomorphic line bundle $\widetilde{L} \longrightarrow A \times P_{n-1}(\boldsymbol{C})$ by

$$\widetilde{L} = O_{A \times P_{n-1}(\boldsymbol{C})}(\widetilde{D}_i - k_i(A \times H)).$$

Then it follows from the definition of $\widetilde{D}_i$ that $\widetilde{L}$ is numerically trivial along $P_{n-1}(\boldsymbol{C})$ fibers of π. So it is analytically trivial along fibers. We thus get a line bundle $L \to A$ such that $\widetilde{L} = \pi^* L$. We prove that L is ample by applying Nakai's criterion. Let $\Gamma \subset A$ be a curve. We compute $L \cdot \Gamma$. To do so, we define a holomorphic map

$$t : \Gamma \times S_i \to A, \ t(a, b) = a + b$$

and its lifting

$$\tilde{t} : \Gamma \times S_i \to A \times P_{n-1}(C), \ \tilde{t}(a, b) = (a + b, [v])$$

where v is a tangent vector of S_i at b. Let G_D be the Gauss map of D defined on the regular part $\mathrm{Reg}(D)$ of D. Let $G_D^* H$ be the Zariski closure of $G_D^{-1} H$. If the Gauss map is constant on each component of D, then D consists of codimension one Abelian subvarieties. In this case, the problem is reduced to the classical Nevanlinna theory and we have nothing to prove. So we may assume that the D has non-constant Gauss map. Then we have

$$L \cdot \Gamma = \widetilde{D}_i \cdot (\Gamma \times \{\text{point}\}) = t(\Gamma \times S_i) \cdot (G_D^* H) > 0$$

because D is ample. If we deform S_i, then the associated $\widetilde{D}_i$ will be deformed. So if we deform S_i generically, we see that $\widetilde{D}_i^{2n-2}$ is reperesented by a curve in $A \times P_{n-1}(C)$ which projects to a point in $P_{n-1}(C)$. It follows that

$$L^n > 0.$$

Therefore L is ample. The Riemann-Roch theorem and the Kodaira vanishing theorem imply that there exists an ample hypersurface $E_i \subset A$ such that

$$L = O_A(E_i).$$

Next we prove that D_i is numerically equivalent to $mD + E_i$ as divisors of A. Let $\Gamma \subset A$ be a curve. We define a codimension two analytic subvariety $\widetilde{D}$ of $A \times P_{n-1}(C)$ by

$$\widetilde{D} = \cup_{x \in D} \mathrm{P}((TD)_x)$$

where $\mathrm{P}((TD)_x)$ stands for the projectivization of the tangent cone of D at x. We can now compute the intersection number

$$
\begin{aligned}
\Gamma \cdot D_i &= \tilde{\iota}(\Gamma \times S_i) \cdot \widetilde{D} \\
&= \mid e(S_i) \mid (D \cdot \Gamma) + \iota(\Gamma \times S_i) \cdot (G_D^* H) \\
&= (mD + E_i) \cdot \Gamma .
\end{aligned}
$$

The second equality follows if we decompose $\widetilde{D}$ homologously into subvarieties which are the product of subvarieties of A and $P_{n-1}(C)$ of correct dimensions. Indeed, we have

$$\widetilde{D} \sim D \times H + (G_D^* H) \times P_{n-1}(C)$$

where $\sim$ means the homological equivalence. Since the degree of the Gauss map of a smooth curve in a complex torus is equal to the Chern number of its canonical divisor, we have

$$\tilde{\iota}(\Gamma \times S_i) \cdot (D \times H) = \mid e(S_i) \mid (D \cdot \Gamma).$$

Therefore D_i and $mD + E_i$ is numerically equivalent. Applying Lemma 1, we see

$$c_1(O_A(D_i - (mD + E_i))) = 0.$$

This completes the proof of Lemma 2. □

To estimate the quantity (7), we need several lemmas. We now formulate a variant the Cauchy-Crofton formula (see, for example, [N-O, Lemma (5.4.5) and Lemma (5.4.11), pp. 197-199]). Let X be a smooth projective variety and L a semi-ample line bundle, i.e., for all points $x \in X$ there exists $D \in H^0(X, O_X(L))$ such that $x \notin \mathrm{Supp}(D)$. Write $\Phi_{|L|} : X \to P^*$ and $\| \cdot \|$ for the associated holomorphic map and the Hermitian metric for $L \to X$ induced from the canonical Hermitian metric of the hyperplane bundle over P^*. Suppose V is a subvariety of the projective space $P = P(H^0(X, O_X(L)))$ such that for any $x \in X$ there exists a $\rho(\sigma) \in V$ with $\sigma(x) \neq 0$, where $\rho : H^0(X, O_X(L)) - \{0\} \to P$ is the projection. Let $H \to P$ be the hyperplane bundle and $\| \cdot \|$ the canonical Hermitian metric. Then the Chern form ω of $(L, \| \cdot \|)$ defines the Fubini-Study metric and the associated measure ω^n on P has volume 1. For each $x \in X$, we define a smooth function

$$P \ni \rho(\sigma) \longmapsto \frac{\|\sigma(x)\|}{\|\sigma\|} \in [0,1].$$

If D is the zero locus of a $\sigma \in H^0(X, O_X(L))$, we we identify D and σ and write the above function as $S_x(D)$. We then have

Lemma 3 *Let $L \to X$, P and $S_x(D)$ be as above. Let $f : \mathbf{C} \to X$ be a holomorphic map. Then*

$$T_{f,L}(r) = \int_{D \in V} N_{f^*D}(r) + O(1)$$

where the measure on V is the one induced from the Fubini-Study metric on P and is normalized so that $\mathrm{Vol}(V) = 1$.

Proof. It is easy to modify the proofs of [N-O, Lemma (5.4.5) and Lemma (5.4.11)] in our situation. The point is to show that the integral

$$\int_{D \in V} \log \frac{1}{S_x(D)}$$

is bounded uniformly with respect to $x \in X$. If the subvariety $V \subset P$ has the above property, then the same argument as in [N-O] works. So we omit the details. □

Lemma 4 *Let A be an n dimensional Abelian variety and D an ample divisor and let H be the hyperplane bundle on $P_{n-1}(\mathbf{C})$. Let $f : \mathbf{C} \to A$ be a holomorphic curve and $[f']$ the holomorphic curve into $P_{n-1}(\mathbf{C})$ defined by $[f'](z) = [f'(z)] \in P_{n-1}(\mathbf{C})$. Then we have*

$$T_{[f'],H}(r) = O(\log r + \log T_{f,D}(r)).$$

Proof. Jensen's formula and the concavity of the logarithm imply

$$\begin{aligned} T_{[f'],H}(r) &= \int_0^r \tfrac{dt}{t} \int_{\mathbf{C}(t)} \sqrt{-1}\partial\bar{\partial} \log \|f'\|^2 \\ &= \int_{\partial \mathbf{C}(r)} \log \|f'\| \eta - N_{f,Ram}(r) + O(1) \\ &\leq \int_{\partial \mathbf{C}(r)} \log \|f'\| \eta + O(1) \\ &\leq \tfrac{1}{2} \log \int_{\partial \mathbf{C}(r)} \|f'\|^2 \eta + O(1) \end{aligned}$$

We think of f and f' as maps into C^n. From Cauchy's integral formula we have for $z \in \partial C(r)$

$$f_i'(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial C(r+1)} \frac{f_i(\zeta)}{(\zeta - z)^2} d\zeta.$$

This implies

$$| f_i'(z) | \leq 2\pi(r+1) \int_{\partial C(r+1)} | f | \eta.$$

From the definition and Jensen's formula we have

$$\begin{aligned} T_{f,D}(r) &= \text{const.} \textstyle\int_0^r \frac{dt}{t} \int_{C(t)} \sqrt{-1}\partial\bar{\partial}(\sum_{i=1}^n | f_i |^2) \\ &= \tfrac{1}{2} \cdot \text{const.} \textstyle\int_{\partial C(r)} \sum_{i=1}^n | f_i |^2 \eta + O(1). \end{aligned}$$

Therfore there is a constant $C > 0$ independent of f such that

$$T_{[f'],H}(r) \leq C(\log(r+1) + \log T_{f,D}(r+1)).$$

The proof of [N-O, Lemma (5.2.33)] (see Equation (5.2.35)) implies that there exists a constant C' independent of f such that

$$T_{f,D}(r+1) \leq C' T_{f,.D}(r).$$

Hence there exists a constant C'' independent of f such that

$$T_{[f'],H}(r) \leq C''(\log r + \log T_{f,D}(r))$$

for all sufficiently large r. This completes the proof of Lemma 4. □

Let $\widetilde{D}$ be a subvariety of codimension two in $P(TA)$ defined by

$$\widetilde{D} = P(TD) = \bigcup_{a \in D} [\text{the projectivization of } T_a D].$$

Then every $\widetilde{D}_i \subset P(TA)$ contains $\widetilde{D}$ as a codimension one subvariety. We want to make a *base point free* system of divisors by blowing up $\widetilde{D}$. We may assume that all S_i's are algebraically equivalent to a curve S_0 and consider all $\widetilde{D}_S$ constructed from S (algebraically equivalent to S_0) and D as in the same manner as $\widetilde{D}_i$. The union $\cup_{S \equiv S_0} | \widetilde{D}_S |$ ($\equiv$ stands for the algebraic equivalence) of complete linear systems will form a subvariety W of the product of the Picard variety $\mathrm{Pic}^0(A)$ of A and some complex projective space. Let V be a subvariety of W consisting of $\widetilde{D}_S$ where S varies the algebraic equivalence class of S_0. If all S_i's are obtained by cutting A by generic $(N-n+1)$-dimensional linear subspaces, the $\mathrm{Pic}^0(A)$-component of W is necessarily a constant, because there are no nonconstant holomorphic maps of a Grassmannian into Abelian varieties. We define

$$\tilde{P}(TA) = Bl_{\widetilde{D}}(P(TA)),$$

which is the blow up of $P(TA)$ with center $\widetilde{D}$ and write $\overline{f}$ for the holomorphic curve

$$\overline{f} : \boldsymbol{C} \longrightarrow \tilde{P}(TA)$$

which coincides with $\tilde{f}$ outside the inverse image of $\widetilde{D}$. We denote the proper transform of $\widetilde{D}_i$ by the same symbol. We take the proper transforms in $\tilde{P}(TA)$ of the divisors $\widetilde{D}_S$ in V. In this way, V parametrizes a set of divisors of $\tilde{P}(TA)$ and for any $x \in \tilde{P}(TA)$ there exists a $\widetilde{D}_S \in V$ such that $x \notin \widetilde{D}_S$, i.e., V is "base point free". Now the same argument as in the proof of Lemma 3 (Cauchy-Crofton's averaging formula) implies

$$\text{Average over } V \text{ of } N_{\overline{f}^*\widetilde{D}_S}(r) \;=\; T_{\overline{f},\widetilde{D}_S}(r) + O(1).$$

Let E denote the inverse image of $\widetilde{D} \subset P(TA)$ in the blow up $\tilde{P}(TA)$. Then we have

$$T_{\overline{f},\widetilde{D}_S}(r) = T_{\tilde{f},\widetilde{D}_i}(r) - T_{\overline{f},E}(r) \le T_{\tilde{f},\widetilde{D}_i}(r) + O(1).$$

The above averaging formula implies that there exist a collection of curves $\{S_i\}$ of general position such that

$$\begin{aligned} N_{\tilde{f}^*\widetilde{D}_i}(r) &= T_{\overline{f},\widetilde{D}_i}(r) + N_{\overline{f},E}(r) - \varepsilon' T_{\overline{f},\widetilde{D}_i}(r) \\ &= T_{\tilde{f},\widetilde{D}_i}(r) - m_{\overline{f},E}(r) - \varepsilon' T_{\overline{f},\widetilde{D}_i}(r) \end{aligned}$$

for any small positive number ε' (recall that we need the general position condition for S_i's in the starting point of our proof). Thus, we have

$$N_{\tilde{f}^*\pi^*D_i}(r) - (N_{\tilde{f}^*\pi^*mD}(r) + N_{\tilde{f}^*\widetilde{D}_i}(r))$$

$$\leq -T_{\tilde{f},k(A\times H)}(r) + m_{\overline{f},E}(r) + \varepsilon' T_{\overline{f},\widetilde{D}_i}(r)$$

where we have chosen good S_i's (such a good choice may vary if r varies) and applied Lemma 3 to counting functions for $\overline{f}$ with respect to the proper transform $\widetilde{D}_i$ and also the usual Cauchy-Crofton formula to the linear system $\mid mD \mid$.

Lemma 5 *Let the notation be as above. Then there exists a positive constant C independent of f such that*

$$T_{\overline{f},\widetilde{D}_i}(r) \leq C T_{f,D}(r).$$

Proof. Applying Lemma 4, we have

$$\begin{aligned} T_{\overline{f},\widetilde{D}_i}(r) &= T_{\tilde{f},\widetilde{D}_i}(r) - T_{\overline{f},E}(r) \\ &\leq T_{\tilde{f},\widetilde{D}_i}(r) + O(1) \\ &\leq O(T_{f,D}(r) + T_{[f'],H}(r)) \\ &\leq O(T_{f,D}(r)) + O(\log r + \log T_{f,D}(r)). \end{aligned}$$

□

Lemma 6 *Let the notation be as above. Then we have*

$$m_{\overline{f},E}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1).$$

Proof. To prove Lemma 6, we use the above arguments, which is summarized as follows, successively. First, we approximate the whole group variety by a collection of curves and reduce the problem to a collection of equidimensional problems for holomorphic maps from a covering space of $\boldsymbol{C}$ to a compact Riemann surface. Then, we use the equidimensional Second Main Theorem ([G-K], [Nol], [St], [L-C]) to reduce the orginal problem to an estimate of an averaged difference of three counting functions (cf. (7)) for the canonical lifting of the original holomorphic curve to the projectivized tangent bundle with respect to several divisors in it. The first step is just what we did so far. Namely, we have just shown that Theorem 1 is equivalent to:

$$m_{f,D}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1)$$

$$\Leftrightarrow \quad m_{\tilde{f},\tilde{D}}(r) = m_{\tilde{f},E}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1),$$

where ε is an arbitraty positive number and $O_\varepsilon(1)$ is an $O(1)$ term which depends on ε. Set $f^{(1)} = \tilde{f}$ and $D^{(1)} = \widetilde{D}$. The second step is to apply our arguments to the canonical lifting

$$f^{(1)} : \boldsymbol{C} \longrightarrow P(TA).$$

At this step, we apply aur arguments with respect to the analytic set $D^{(1)} = \widetilde{D} = P(TD)$ in $P(TA)$ of codimension 2 and not with respect to a divisor. Let $\mathcal{F}$ be a family $\{F\}$ of linearly equivalent divisors on $P(TA)$ such that
(i) $D^{(1)} \subset \mathrm{Supp}(F)$ for all $F \in \mathcal{F}$,
(ii) $\cap_{F\in\mathcal{F}}\mathrm{Supp}(F) = D^{(1)}$ and
(iii) $\pi^*D \not\subset F$ for all $F \in \mathcal{F}$.
Now we think of $P(TA)$ as a compactfication of a group variety $A \times (\boldsymbol{C}^*)^{n-1}$ in a natural way. Namely,

$$P(TA) = A \times (\boldsymbol{C}^*)^{n-1} \cup A \times \Delta$$

where Δ is a collection of n hyperplanes of general position in $P_{n-1}(\boldsymbol{C})$. Let $D^{(2)} = P(TD^{(1)})$ be the projectivized tangent bundle of $D^{(1)}$. Then it is a codimension 4 analytic subvariety of $P(T(P(TA)))$ (the projectivized tangent bundle of $P(TA)$). Let $f^{(2)}$ be the holomorphic curve

$$f^{(2)} : \boldsymbol{C} \longrightarrow P(T(P(TA)))$$

which is the canonical lifting of $f^{(1)}$ naturally defined by the second jet $(f(z), f'(z), f''(z))$ of the original f. We then use the Abelian group structure of $A \times (\boldsymbol{C}^*)^{n-1}$, which is compatible with the flat metric, and apply the same arguments as in the first step. In the second step, we work on the complement of an anticanonical divisor at infinity. So we must take the contribution of the anticanonical divisor at infinity into consideration. To do so, just arguing as in the early stage of the proof of Theorem 1, we see that it suffices to estimate the contribution of the anticanonical divisor (at infinity) to the characteristic function T of $f^{(2)}$. But Lemma 4 and its natural generalization to higher derivatives impliy that this is bounded from above by $O(\log r + \log T_{f,D}(r))$. This is simply a consequence of Jensen's formula. Consequently, we have the equivalence:

$$m_{f^{(1)},D^{(1)}}(r) \le \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1)$$

$$\Leftrightarrow \quad m_{f^{(2)},D^{(2)}}(r) \le \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1).$$

Making use of the Abelian group structure of the complement of an anticanonical divisor at each step, we apply the same process successively. Thus we get the the sequence of holomorphic curves $f^{(i)}$, that of analytic sets $D^{(i)}$ of codimension 2^i in $(PT)^i(A)$ and the equivalence:

$$\text{Theorem 1} \Leftrightarrow m_{f^{(i)},D^{(i)}}(r) \le O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1).$$

Now we consider the n-th step, i.e., we look at the n-th jet differential of f. Fix a point $x \in D$. Consider all local curves in A which is tangent to D with multiplicity $\ge$ n. The union $J_x(n)$ of the n-th jet differentials at x of such curves forms an analytic subset of codimension n in the n-th jet space J_n. Since $\dim D = n-1$, the codimension of the union $J_D(n) = \cup_{x \in D} J_x(n)$ in J_n is still one. As in [G-G] and [L2], $P(J_n)$ is canonically considered to be a certain weighted projective space. Since the divisor $J_D(n)$ projects down to a divisor in $P(J_n)$, we shall use the same symbol for the projectivization of $J_D(n)$ in $P(J_n)$. The projectivization of $J_D(n)$ is still of codimension one in $P(J_n)$. On the other hand, we have the implication:

$$m_{f^{(n)},D^{(n)}}(r) \le \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1)$$

$$\Leftarrow \quad m_{J_n(f),A\times J_D(n)}(r) \le \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_{\varepsilon}(1),$$

if the image of $J_n(f)$ is not contained $A \times J_D(n)$, where $J_n(f)$ is the n-th jet map of f into the projectivized jet bundle $A \times P(J_n)$. Using the same argument as in the proof of Lemma 4, we have

$$m_{J_n(f),A\times J_D(n)}(r) \le T_{J_n(f),A\times J_D(n)}(r) \le O(\log r + \log T_{f,D}(r)),$$

if $J_n(f)(\boldsymbol{C}) \not\subset J_D(n)$. We have thus showed that if f is nondegenerate in the sense that $J_n(f)(\boldsymbol{C}) \not\subset J_D(n)$, then

$$m_{f,D}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1).$$

The additional nondegeneracy condition for f is insatisfactory. In fact, we can say more, using again the Cauchy-Crofton type averaging argument over the collection of curves $\{S_i\}$. Set

$$W_i = P(\cup_{a\in A}(\cup_{x\in D\cap(S_i+a)} J_x(n))),$$

where the symbol P means the canonical weighted projectivization of the fiber over each $a \in A$. Then, we have the implication:

$$m_{f^{(n)},D^{(n)}}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1)$$

$$\Leftarrow \quad m_{J_n(f),W_i}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1).$$

We now apply the averaging argument over $\{S_i\}$. We can thus choose generic $\{S_i\}$ so that if $f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D)$, we have

$$\begin{aligned} m_{J_n(f),W_i}(r) &\leq \varepsilon T_{f,D}(r) + T_{J_n(f),A\times J_D(n))}(r) \\ &\leq \varepsilon T_{f,D}(r) + O(\log r + \log T_{f,D}(r)) \end{aligned}$$

for any small $\varepsilon > 0$. We see this by considering Q-divisors on $A \times P(J_n)$ of the form $\delta D \times J_n + A \times P(J_D(n))$ for small positive rational number δ. This completes the proof of Lemma 6. □

Applying Lemma 5 and Lemma 6 to an inequality just before Lemma 5, we have

$$N_{\tilde{f},\pi^* D_i}(r) - (N_{\tilde{f},\pi^*[mD]}(r) + N_{\tilde{f},\tilde{D}_i}(r)) \leq C\varepsilon' T_{f,D}(r).$$

Thus we have

$$T_{f,D}(r) \leq (1+\varepsilon)N_{f,D}(r) - N_{f,Ram}(r) + O_{exc}(\log r + \log T_{f,D}(r))$$

for any small $\varepsilon > 0$, which completes the proof of Theorem 1. □

In the proof of the following proposition, we look at jet differntials as in the proof of Lemma 6. The author was inspired by Grauert's idea in [Gra]. Thanks are due to P. -M. Wong for explaining Grauert's idea to the author.

Proposition 1 *Let A, D and $f : \boldsymbol{C} \to A$ be as in Theorem 1. Assume further that $f(\boldsymbol{C})$ is not contained in a (not necessarily closed) complex subvariety $V(\neq A)$ which is not Zariski dense and is foliated by translations of a proper Abelian subvariety (e.g., $f(\boldsymbol{C})$ is not contained in a closed subvariety whose Kidaira dimension is less than its complex dimension.). Let $\overline{N}_{f^*D}(r)$ be the counting function without counting multiplicities. Then we have*

$$0 \leq N_{f^*D}(r) - \overline{N}_{f^*D}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Proof. From the proof of Lemma 6, we have

$$\overline{N}_{\tilde{f}^*E}(r) \geq \frac{1}{n-1} N_{\tilde{f}*E}(r) + O(\log r + \log T_{f,D}(r)),$$

since at the last stage of the inductive argument, not only the proximity function but also the characteristic function is bounded by $O(\log r + \log T_{f,D}(r))$. On the other hand, the middle term $N_{f^*D}(r) - \overline{N}_{f^*D}(r)$ of the consequence of Proposition 1 is dominated by $N_{\tilde{f}^*E}(r) = N_{\tilde{f}^*\tilde{D}}(r)$. Therefore, to prove Proposition 1, it suffices to show

$$\overline{N}_{\tilde{f}^*E}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Let $\mathcal{D}$ be an irreducible algebraic variety parametrizing curves (one dimensional algebraic subvarieties) in A. We shall choose $\mathcal{D}$ so that the curves parametrized by $\mathcal{D}$ have sufficiently large degree with respect to $c_1(O_A(D))$. At each $z \in \boldsymbol{C}$, we consider a jet $j(f)(z)$ of f up to order order, say, d. We choose $\mathcal{D}$ so that for each $z \in \boldsymbol{C}$, the number of curves in $\mathcal{D}$ which pass through $f(z)$ and which have $j(f)(z)$ as their d-th jet at $f(z)$ is finite, say, b. We note that a similar idea was used by Grauert in [Gra]. Let $C_1(z), \cdots, C_b(z)$ be such curves for $z \in \boldsymbol{C}$. If $\zeta \in \boldsymbol{C}$ varies around a neighborhood of $0 \in \boldsymbol{C}$, we have a collection of holomorphic curve segments $B_1(z+\zeta), \cdots, B_b(z+\zeta)$ in D by cutting $C_1(z+\zeta), \cdots, C_b(z+\zeta)$ by D. We then consider the $(n-1)$-st osculating space $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{(n-1)}(z)$, $(1 \leq i \leq b)$, in A of these curve segments (in D). This is not necessarily well-defined, because the above wedge product may vanish identically. But for the time being, we assume that it is well-defined. Now suppose $z \in \boldsymbol{C}$ is such that $\tilde{f}(z)$ hits $\widetilde{D}$. If d is chosen sufficiently large, say, $d \geq d_0$ (such d_0 depends on the extrinsic geometry of D in A only), then (the projectivization of) the $(n-1)$-st osculating space of one of $B_i(z+\zeta)$'s at $f(z)$ contains (the projectivization of) the tangent vector of f at $f(z)$. We thus have, generically, for each $z \in \boldsymbol{C}$, a finite number, say, d', of points in D such that the tangent hyperplanes coincides with those $(n-1)$-st osculating spaces of $B_1(z+\zeta), \cdots, B_b(z+\zeta)$ at $\zeta = 0$. This gives rise a holomorphic curve g into $P(H^0(P_{n-1}(\boldsymbol{C}), O(d')))$. From Lemma 4 and its natural generalization to higher derivatives, we see

$$T_{g,O(1)}(r) \leq O(\log r + \log T_{f,D}(r)).$$

We consider g as a moving target in the sense of [R-S]. The First Main Theorem for moving target [R-S] states

$$T_{[f'],d'H}(r) + T_{g,O(1)}(r) = N_{[f'],g}(r) + m_{[f'],g}(r) + O(1),$$

where $N_{[f'],g}(r)$ and $m_{[f'],g}(r)$ are the counting function and the proximity function of $[f']$ with respect to the moving target g (see [R-S]). Lemma 4 implies

$$T_{[f'],d'H}(r) \leq O(\log r + \log T_{f,D}(r)$$

if f is not constant. We thus have

$$N_{[f'],g}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Clearly we have

$$\overline{N}_{\tilde{f}^* E}(r) \leq N_{[f'],g}(r).$$

Thus, we have the desired estimate:

$$\overline{N}_{\tilde{f}^* E}(r) \leq O(\log r + \log T_{f,D}(r)),$$

if every $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{n-1}(z) \not\equiv 0$ $(1 \leq i \leq b)$ (i.e., the moving target g is well-defined) and if $f(\boldsymbol{C})$ is not contained in the moving target g. So, the above argument breakes down if

(i) $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{n-1}(z) \not\equiv 0$ for all irreducible components and the

image of $[f']$ is contained in the moving target g,or
(ii) $B_i'(z) \wedge B_i''(z) \wedge \cdots B_i^{n-1}(z) \equiv 0$ for some iorreducible component for some i, say $i = 1$.
Note that we have a freedom in choosing $\mathcal{D}$. So, if the above argument breakes down for all admissible choice of $\mathcal{D}$, we see that either
(i) $f(C)$ is contained in a proper Abelian subvariety of dimension $n-1$ in A, or
(ii) some curve segment $B_i(z+\zeta)$ in D is contained in a proper Abelian subvariety of dimension $\leq n-2$.
But this will be the case only if the image $f(C)$ is contained in a (not necessarily closed) subvariety $V \subset A$ such that
(a) for some étale covering of A, the pull back of V splits a proper Abelian subvariety A',
(b) V is tangent to D along a divisor (codimension 1 analytic set) W in V and V is foliated by the translations of a (not necessarily closed) linear subspace of A which contains the above proper Abelian subvariety A', and
(c) (after taking a suitable étale covering) W is foliated by translations of a divisor in A'.
If V is closed, it is shown in [U] that any proper subvariety of an Abelian variety whose Kodaira dimension is smaller than its complex dimension is foliated by the translations of an Abelian subvariety. This completes the proof of Proposition 1. □

We here recall the "Lemma on the logarithmic derivarive" (se, for example, [L-C], [No2] and [No3]). Let X be a compact complex manifold and $D = \sum_{k=1}^{l} m_k D_k$ (D_k are mutually distinct irreducible hypersurfaces of X) a divisor which is homologously equivalent. Then Kodaira [Kod] proved that there exists a multiplicative meromorphic function θ (a theta function) on X with divisor D. Set $\Sigma = \sum_{k=1}^{l} D_k$ and

$$\omega = d \log \theta.$$

Then ω is a well-defined meromorphic 1-form on X with logarithmic poles along Σ. Let

$$f : C \to X$$

be a holomorphic curve and set

$$\zeta(z)dz = f^*\omega$$

where z is the standard coordinate of $\boldsymbol{C}$. Then $\zeta(z)$ is a meromorphic function on $\boldsymbol{C}$ with only simple poles. We think of a meromorphic function on $\boldsymbol{C}$ a holomorphic function to $P_1(\boldsymbol{C})$ with the Fubini-Study metric, which is a curvature form of the line bundle of degree 1. Noguchi's version [No2] of the "Lemma of the logarithmic derivative" states

Fact 2 *Let X, D, f and $\zeta(z)$ be as above. Then*

$$m_{\zeta,\infty}(r) \leq O_{exc}(\log r + \log^+ T_{f,\Sigma}(r))$$

where $\log^+ v = \max\{0, \log v\}$.

Suppose that $| m_k |= 1$ for all k in the above situation. Following [No2], we introduce a "*counting function without counting multiplicities*" $\overline{N}_{f^*\Sigma}(r)$, which is defined in a similar way as $N_{f^*\Sigma}(r)$ except that we do not count the multiplicity. This means that if $f(\boldsymbol{C})$ intersects Σ at $f(z)$ with multiplicity ν, then we count this by ν in the definition of N while we count this by 1 in that of $\overline{N}$. Then we have

$$N_{\zeta^*\infty}(r) = \overline{N}_{f^*\Sigma}(r)$$

since $\zeta(z)$ has a pole if and only if $f(z) \in \Sigma$ and every pole of $\zeta(z)$ is simple. We then have a weak version of the Second Main Theorem.

$$\begin{aligned} T_{\zeta,\infty}(r) &= m_{\zeta,\infty}(r) + N_{\zeta^*\infty}(r) \\ &= m_{\zeta,\infty}(r) + \overline{N}_{f^*\Sigma}(r) \\ &\leq \overline{N}_{f^*\Sigma}(r) + O_{exc}(\log r + \log^+ T_{f,\Sigma}(r)) \end{aligned}$$

Next, we examine the effect to $T_{\zeta,\infty}(r)$ if we deform the divisor Σ (fixing f) in a linear system. So let $F_1 = F_{11} - F_{12}$ and $F_2 = F_{21} - F_{22}$ be divisors of X such that (i) every F_{ij} is a reduced divisor, (ii) F_{k1} and F_{k2} have no common components for $k = 1, 2$, (iii) $F_{11} \sim F_{12}$, $F_{21} \sim F_{22}$, and (iv) $O_X(F_{11}) = O_X(F_{21})$, $O_X(F_{12}) = O_X(F_{22})$. For each $k = 1, 2$, let θ_k be a multiplicative meromorphic function with divisor F_k. Set $\omega_k = d\log\theta_k$, which is a global meromorphic 1-form on X with logarithmic poles along $\Sigma_k = \mathrm{Supp}(F_k)$. Define for $k = 1, 2$

$$\zeta_k(z)dz = f^*\omega_k.$$

From the assumption, we have a one parameter family of divisors F_{sk}, $(k = 1, 2)$ which moves in each linear system from F_{1k} to F_{2k}. Applying Kodaira's construction for each s, we thus have a "differentiable" one parameter family of meromorphic functions $\zeta_s(z)$ on $\boldsymbol{C}$ which connects $\zeta_1(z)$ and $\zeta_2(z)$.

Proposition 2 *Let $X = A$ be an Abelian variety and Σ_1 an ample divisor in the above argument. Suppose $f(\boldsymbol{C})$ is not contained in a (not necessarily closed) subvariety ($\neq A$) of A which is not Zariski dense and is foliated by translations of a proper Abelian subvariety. Let $\zeta_k(z)$ be as above. Then*

$$| T_{\zeta_1,\infty}(r) - T_{\zeta_2,\infty}(r) | \leq O_{exc}(\log r + \log T_{f,\Sigma_1}(r))$$

and

$$| \overline{N}_{f^*\Sigma}(r) - \overline{N}_{f^*\Sigma}(r) | \leq O_{exc}(\log r + \log T_{f,\Sigma_1}(r)).$$

Proof. This follows directly from Theorem 1 and Proposition 1. □

Proposition 3 *Let H_1 and H_2 be linearly equivalent reduced divisors in an Abelian variety A. Let $f : \boldsymbol{C} \to X$ be a holomorphic curve which is not contained in a (not necessarily closed) subvariety ($\neq A$) which is not Zariski dense and is foliated by translations of a proper Abelian subvariety. Suppose further that $f(\boldsymbol{C})$ is not contained in H_1 and H_2. Then*

$$| \overline{N}_{f^*H_1}(r) - \overline{N}_{f^*H_2}(r) | \leq O_{exc}(\log^+ T_{f,H_1}(r)).$$

Proof. Applying Theorem 1 (or rather Theorem 4 which we will prove later) and Proposition 1, we get Proposition 3. □

Theorem 2 *Let A and D be as in Theorem 1. Then every holomorphic map*

$$\boldsymbol{C} \longrightarrow A - \mathrm{Supp}(D)$$

is necessarily a constant map.

Proof. Suppose that there exists a non-constant holomorphic curve $f : \boldsymbol{C} \to A - \mathrm{Supp}(D)$. From the assumption, we have $N_{f^*D}(r) = 0$. We thus have from Theorem 1 that

$$T_{f,D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r)).$$

From [No2, Lemma 4.3] (see also [N-O, Lemma (5.2.33)]), we know that if $f : \boldsymbol{C} \to A$ is a non-constant holomorphic curve, then there exists a positive constant C such that

$$T_{f,D}(r) \geq Cr^2$$

for all $r \geq 0$. (We note that our definition of the characteristic function is slightly different from that in [No2] and [N-O]. Namely, our definition is $T_{f,D}(r) = \int_0^r \frac{dt}{t} \cdots$, while in [No2] and [N-O], $T_{f,D}(r) = \int_1^r \frac{dt}{t} \cdots$.) This clearly contradicts to the above inequality. Hence $f : \boldsymbol{C} \to A - \mathrm{Supp}(D)$ is a constant map. □

Theorem 3 *Let A be an Abelian variety and D an irreducible smooth hypersurface in A. Then every holomorphic curve $f : \boldsymbol{C} \to A - \mathrm{Supp}(D)$ is algebraically degenerate.*

Proof. From Theorem 2 we may assume that D is not an ample divisor. From [W, Chapt. VI], there exists an Abelian variety A' of positive dimension n', an ample irreducible smooth hypersurface D' in A' and a surjective homomorphism $\rho : A \to A'$ such that $D = \rho^* D'$. Then the holomorphic curve

$$\rho \circ f : \boldsymbol{C} \longrightarrow A'$$

omits $\mathrm{Supp}(D')$. Therefore $\rho \circ f$ is necessarily a constant mapping. This implies that the image $f(\boldsymbol{C})$ is contained in a translation of a proper Abelian subvariety. In particular $f : \boldsymbol{C} \to A - \mathrm{Supp}(D)$ is algebraically degenerate. □

Since we aimed to prove a weak version of Conjecture 1 when X is an Abelian variety A, we have so far assumed that D has at worst simple normal crossings. Although this assumption makes the proof of Theorem 1 quite simple, this is in fact redundant in our argument in the proof of Theorem 1. For instance, we have

Theorem 4 *Let A be an Abelian variety and D an ample effective reduced divisor, where we make no assumption on the singularities of D. Let $f : C \to A$ be a holomorphic mapping such that $f(C) \not\subset \mathrm{Supp}(D)$. Then for any positive constant ϵ, the following inequality holds:*

$$T_{f,D}(r) \leq (1+\epsilon)N_{f^*D}(r) - N_{f,Ram}(r) + O_{exc}(\log r + \log T_{f,D}(r)).$$

Proof. We have only to choose a collection of M algebraic curves in A so that this takes care of all singularities of D. This is possible by considering the tangent cones of D at singularities. Suppose that $f(z)$ is very clase to D. If we choose M sufficiently large, then the integrand of $m_{f,D}(r)$ is bounded from above, up to an additive constant depending on M, by the average of those of $m_{f_i,p}(r)$'s which has tangent directions at p outside of a cone-like neighborhoods in A of the tangent cones of D near $f(z)$. Here, we compare tangent directions and tangent cones at different points by parallel translating objects to p. Note that these cone-like neighborhoods can be made closer and closer to the tangent cones of D if $f(z)$ gets nearer to D. So, since D is compact, the same string of inequalities as in the proof of Theorem 1 holds, except for the last two steps, if we choose M algebraic curves in A so that the collection of tangent directions at p forms a sufficiently dense set of points in $P_{n-1}(C)$ and put $\frac{1}{M-c}$ before the summation over $i = 1, \cdots, M$, where c indicates the number of i's such that the tangent directions at p of which are contained in thin cone-like neighborhoods of tangent cones of D. If $f(z)$ is very close to D, we can make the cone-like neighborhood very thin and in particular the quantity $\frac{M}{M-c}$ is very close to 1. Now we examine the last two steps in the string of inequalities in the proof of Theorem 1. We take the normalization of $X_i(r)$ over all ordinary multiple points coming from the normal crossing singularities of D. After performing this modification, we see that only the divisorial part of D'_i which comes from the codimension one non normal crossing singularities contributes to the counting function of the ramification divisor of $\pi_i : X_i \to C$ by the quantity

$$\sum_{\alpha} m_\alpha \frac{1}{m} \frac{1}{M-c} \sum_{i=1}^{m} N_{f^*\overline{D}''_{i,\alpha}}(r) \tag{8}$$

where $D''_{i,\alpha}$ are irreducible components of the divisor D''_i of A constructed by parallel translating the codimension one non normal crossing singular locus of D by elements of S_i, and m_α stands for $\sum$(multiplicity $-$ 1) where the sum is taken over all local components of $X_i(r)$ lying over z such that $f(z) \in D''_{i,\alpha}$ and the "multiplicity" means the multimplicity of the singularity corresponding to $D''_{i,\alpha}$ on each local irreducible component of $X_i(r)$. As in the proof of Theorem 1, we see that the contribution from D_i are bounded above by

$$\frac{M}{M-c} T_{f,D}(r) + \frac{1}{m}\frac{1}{M-c}\sum_{i=1}^{m}(N_{f^*D_i}(r) - N_{f^*mD}(r))$$

But the additional contribution (8) is bounded above (in fact, canceled) by a part of the counting function of the ramification divisor of $f_i : X_i(r) \longrightarrow S_i$ which appears in the Second Main Theorem (Fact 1) ([G-K], [S2], [No1], [No2], [St] and [L-C]) with minus sign. Indeed, suppose that $f(z_0) + \iota D$ intersects S_i at $f(z_0) - q$ where $q \in \mathrm{Sing}(D)$. Take a generic point of the codimension one component of $\mathrm{Sing}(D)$ and cut D'_i at this point by a two dimensional ball B transversally in A. Then we get a curve in B with a non-ordinarry singularity at the origin. Take the normalization of $X_i(r)$ here and pick a component corresponding to a component with a non-ordinary singularity. It is now clear that the restriction of f_i on this component ramifies at the point corresponding to the singularity. Therefore the last stage of the string of inequalities which bounds $m_{f,D}(r)$ becomes

$$\begin{aligned} m_{f,D}(r) \ \leq \ & (\tfrac{M}{M-c} - \tfrac{1}{m})T_{f,D}(r) - \tfrac{M}{M-c}(1 - \tfrac{1}{m})N_{f^*D}(r) \\ & -\tfrac{1}{m}\tfrac{M}{M-c}N_{f,Ram}(r) + \tfrac{1}{m}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*(D_i+\sum_\alpha m_\alpha D''_{i,\alpha})}(r) \\ & -(N_{f^*mD}(r) + N_{\tilde{f}^*(\tilde{D}_i+\pi^*\sum_\alpha m_\alpha D''_{i,\alpha})}(r))\} \\ & +O_{exc}(\log r + \log T_{f,D}(r)). \end{aligned}$$

Compare this with the last stage of the string of inequalities in the proof of Theorem 1. Now the same argument as in the proof of Theorem 1 completes the proof of Theorem 4. □

Theorem 5 *Let A and D be as in Theorem 4. Then every holomorphic curve*

$$f : \boldsymbol{C} \longrightarrow A$$

which omits $\mathrm{Supp}(D)$ *is necessarily a constant mapping.*

Theorem 6 *Let A be an Abelian variety and D an irreducible reduced divisor. Then every holomorphic curve $f : \boldsymbol{C} \rightarrow A - \mathrm{Supp}(D)$ is necessarily contained in a translated proper Abelian subvariety and hence algebraically degenerate.*

Proof. We use the same notation as in the proof of Theorem 3. Since D is not necessarily smooth, the ample divisor D' in A' ([W, Chapt. VI]) is not necessarily smooth. But Theorem 5 implies that the holomorphic curve $\rho \circ f : \boldsymbol{C} \rightarrow A' - \mathrm{Supp}(D)$ is a constant maping. This implies Theorem 6. □

Remark 2 *Bloch-Ochiai's Theorem* ([O],[N-O]) *(completed by Green, Kawamata* [Ka] *and Green-Griffiths* [G-G]) *implies that the Zariski closure of any holomorphic curve $f : \boldsymbol{C} \rightarrow A$ is necessarily a translated Abelian subvariety.*

We refer to [KS1,2], [L2,3] and [N-O] for definitions and related topics of hyperbolicity. We say that a (possibly non compact) complex manifold X is *Brody-hyperbolic* if there exists no non-constant holomorphic curve $f : \boldsymbol{C} \rightarrow X$. Theorem 5 implies that if A and D is in Theorem 4, then $A - D$ is Brody hyperbolic. Clearly (Kobayashi-)hyperbolicity implies Brody-hyperbolicity. Brody [Br] proved that for compact complex manifolds, the notions of (Kobayashi-)hyperbolicity and Brody-hyperbolicity are equivalent. As a corollary of the proof of Theorem 4, we have

Theorem 7 *Let A and D be as in Theorem 4. Then $A - D$ is Kobayashi-hyperbolic.*

Proof. From the proof of Theorem 4 and Wong-Lang-Cherry's estimates for error terms ([Wo1], [L-C]), there exists an explicit estimate for error terms. By an argument in [L-C, Remarks after Theorem 8.2, pp.106-107] and [N-O, Lemma(5.2.33)], our Second Main Theorem then implies the existence of the estimate of the radius r on which there exists a non-trivial holomorphic curve $f : \boldsymbol{C}(r) \rightarrow A - D$ with a given initial tangent vector $f'(0)$ (cf. proof of Theorem 2). □

It is then natural to ask

Question 2 *Is $A - D$ complete hyperbolic ?*

See also [Kob2, Problem D9]. Green [Gre] proved that if D contains no non-trivial translates of Abelian subvarieties, then $A - D$ is complete hyperbolic and hyperbolically embedded in A, i.e., the Kobayashi metric of $A - D$ dominates a Hermitian metric of A. Green [Gre] also points out that when D contains non-trivial translates of Abelian subvarieties, then $A - D$ is not hyperbolically embedded in A.

Let $f : \boldsymbol{C} \to V$ be a non-constant holomorphic curve into a smooth projective variety V and D a reduced divisor of Y. We assume that $f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D)$. We introduce the *Nevanlinna defect* of f with respect to D:

$$\delta_f(D) = 1 - \limsup_{r\to\infty} \frac{N_{f^*D}(r)}{T_{f,D}(r)}.$$

It follows from $T_{f,D}(r) \geq (\mathrm{const.}) \log r$ and (6) that

$$0 \leq \delta_f(D) \leq 1$$

and

$$\delta_f(D) = 1$$

if f omits $\mathrm{Supp}(D)$.

Theorem 8 *Let f be a holomorphic curve into an Abelian variety A and D an ample reduced divisor in A. Suppose that $f(\boldsymbol{C}) \not\subset \mathrm{Supp}(D)$. Then $\delta_f(D) = 0$.*

Proof. From Theorem 4 we have

$$1 - \frac{N_{f^*D}(r)}{T_{f,D}(r)} \leq \varepsilon \frac{N_{f^*D}(r)}{T_{f,D}(r)} + \frac{O_{exc}(\log r + \log T_{f,D}(r))}{T_{f,D}(r)}$$

Since $T_{f,D}(r) \geq C \log r$, (6) implies that $\lim_{r\to\infty} \frac{N_{f^*D}(r)}{T_{f,D}(r)} \leq 1$. By [No2, Lemma 4.3], we have $T_{f,D}(r) \geq Cr^2$ for all large r. So the lower limit of the second term in the right hand side of the above inequality is 0 as $r \longrightarrow \infty$. Taking the lower limit of the both sides of the above inequality, we have

$$1 - \limsup_{r\to\infty} \frac{N_{f^*D}(r)}{T_{f,D}(r)} \leq \varepsilon$$

for all small $\varepsilon > 0$. This implies Theorem 8. □

Theorem 8 means that any algebraically non-degenerate holomorphic curve into an Abelian variety meets homologously equivalent ample divisors equaly often and as often as possible.

The following result generalizes Bloch-Ochiai's Theorem ([B],[O],[Ka],[G-G]) in the case where the irregularity of a projective variety of general type under question is equal to its dimension.

Theorem 9 *Let V be a complex projective algebraic variety and $\pi : V \longrightarrow A$ be a finite branched covering over an Abelian variety A. Suppose that the branch locus of π with its reduced structure is an ample divisor in A. Then there exist no non-constant holomorphic curves in V except for those $f : \mathbf{C} \longrightarrow A$ such that $f(\mathbf{C})$ is contained in the ramification locus of π. In particular, let W be a smooth projective variety of general type with $q(W) = \dim W (= n)$ and $\omega_1, \cdots, \omega_n$ a set of linearly independent holomorphic one forms on W. Suppose $\omega_1 \wedge \cdots \wedge \omega_n \not\equiv 0$. Then any holomorphic curve $f : \mathbf{C} \longrightarrow W$ is algebraically degenerate and the image $f(\mathbf{C})$ is contained either in the proper analytic subset of W defined by $\omega_1 \wedge \cdots \wedge \omega_n = 0$ or in the inverse image $\pi^{-1}(Z) \subset V$ of a proper subvariety Z of A whose Kodaira dimension is strictly smaller than its dimension.*

Proof. Suppose $g : \mathbf{C} \longrightarrow V$ is a non-constant holomorphic curve whose image is not contained in the ramification locus of π. Suppose further that $\pi \circ g(\mathbf{C})$ is not contained in a (not necessarily closed) subvariety of A which is not Zariski dense and is foliated by translations of a proper Abelian subvariety, i.e., the Zariski closure of $\pi \circ g(\mathbf{C})$ is not a proper subvariety whose Kodaira dimension is less than its complex dimension. Under this assumption, we can use Proposition 1. Set $f = \pi \circ g : \mathbf{C} \longrightarrow A$. We apply the argument of the proof of Theorem 1 to f. Let D be the branch locus of π with reduced structure. The point here is that the ramification term $N_{f_i,Ram}(r)$ includes contributions from the ramification divisor of π. We use the same notations as in the proof of Theorem 1. Then, from Lemma 5, Proposition 1 and the assumption, we have, for suitable choice of S_i, the following estimate:

$$\begin{aligned} N_{\tilde{f}^*\tilde{D}_i}(r) &= T_{\tilde{f},\tilde{D}_i}(r) + N_{\tilde{f}^*E}(r) - \varepsilon' T_{\tilde{f},\tilde{D}_i}(r) \\ &= T_{\tilde{f},\tilde{D}_i}(r) - T_{\tilde{f},E}(r) + N_{\tilde{f}^*E}(r) - \varepsilon' T_{\tilde{f},\tilde{D}_i}(r) \\ &\geq T_{\tilde{f},\tilde{D}_i}(r) + N_{g^*R}(r) - \varepsilon' C T_{f,D}(r) + O(\log T_{f,D}(r)) \end{aligned}$$

where R is the ramification locus in V of the covering $\pi : V \to A$. Therefore the quantity

$$N_{f^*(D_i + \sum_\alpha m_\alpha D''_{i,\alpha})}(r) - (N_{f^* mD}(r) + N_{\tilde{f}^*(\tilde{D}_i + \pi^* \sum_\alpha m_\alpha D''_{i,\alpha})}(r))$$

is bounded above by

$$-N_{g^*R}(r) + O_{exc}(\epsilon' C T_{f,D}(r) + \log T_{f,D}(r)).$$

It follows from our Second Main Theorem type inequality that

$$\begin{aligned} N_{g^*R}(r) &= T_{g,R}(r) - m_{g,R}(r) \\ &\geq C' T_{f,D}(r) - m_{f,D}(r) \\ &\geq C' T_{f,D}(r) - \varepsilon N_{f^*D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) \\ &\geq C'' T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) \end{aligned}$$

where C' and C'' are positive constants independent of f. Combining these facts, we have

$$\begin{aligned} m_{f,D}(r) \leq\ & (\tfrac{M}{M-c} - \tfrac{1}{m}) T_{f,D}(r) - \tfrac{M}{M-c}(1 - \tfrac{1}{m}) N_{f^*D}(r) \\ & - \tfrac{MC''}{M-c} T_{f,D}(r) + O_{exc}(\varepsilon' C \log T_{f,D}(r) + \log T_{f,D}(r)). \end{aligned}$$

We thus have

$$\frac{1-\frac{M}{M-c}+\frac{1}{m}+\frac{MC''}{M-c}-\varepsilon' C}{1-\frac{M}{M-c}+\frac{M}{M-c}\frac{1}{m}} T_{f,D}(r) - N_{f^*D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r)).$$

From this (taking M sufficiently large and ε' sufficiently small) and (6), we infer

$$T_{f,D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r)).$$

This implies

$$T_{f,D}(r) \leq O_{exc}(\log r).$$

This forces f to be a rational map (see, for example, [L3] and [N-O]). But there exists no non-constant rational map of $\boldsymbol{C}$ to an Abelian variety. Therefore f must be a constant mapping. This contradicts the assumption that g is non-constant. We note that this proof establishes a sharp Second Main Theorem type inequality for algebraically non degenerate holomorphic curve into V (cf. [O]). □

We can generalize this argument for holomorphic curves into algebraic subvarieties of an Abelian variety and give a new geometric proof of Bloch-Ochiai's Theorem [KR].
Thus, we should investigate holomorphic curves into a more general W without any assumption on $q(W)$. For this, we must develope the value distribution theory into varieties of general type. This will be a much harder problem and something is yet to be done (see [Lu] and [L-Y]).

We can apply our method of the proof of Theorem 1 to holomorphic curves in Abelian varieties defined in the punctured disk $\Delta^* = \{z \in \boldsymbol{C}; \mid z \mid \geq r_0\}$ in $P_1(\boldsymbol{C})$ with puncture ∞. In particular, Theorems 1 and 4, i.e., the Second Main Theorems, hold for holomorphic curves from a punctured disc into Abelian varieties. Arguing as in [No3, Theorem(4.5)], we have the following extension theorem of big Picard type, which generalizes a result of Green [Gre].

Theorem 10 *Let A be an Abelian variety and D an ample reduced divisor. Then any holomorphic mapping $f : \Delta^* \to A - D$ has a holomorphic extension $\overline{f} : \Delta^* \cup \{\infty\} \to A$ if $f(\Delta^*) \not\subset D$.*

Proof. We define $T_{f,D}(r)$, $N_{f^*D}(r)$ and $m_{f,D}(r)$ for holomorphic curves $f : \Delta^* \to A$ as in [No3, Section 1, pp.215-217]. Then we have

$$T_{f,D}(r) \leq (1+\varepsilon)N_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r))$$

by the same argument as in Theorems 1 and 4. This is so, because:
(i) the Second Main Theorem (Fact 1) ([G-K], [S2], [No2,4], [St] and [L-C]) still holds in our situation if we take $X_i^* := \pi^{-1}(\Delta^*)$ as a domain of f_i,
(ii) the First Main Theorem [No3, Equation(1.4)] for holomorphic curves $f : \Delta^* \to X$ into a smooth projective variety X is of the form

$$T_{f,D}(r) = N_{f^*D}(r) + m_{f,D}(r) + O(\log r)$$

(iii) the Cauchy-Crofton averaging formula still holds for holomorphic curves $f : \Delta^* \to X$ in the form

$$T_{f,D}(r) = \int_{D \in V} N_{f^*D}(r) + O(\log r)$$

where we assume the condition of Lemma 3 on V.

Now suppose f omits D. Then we have

$$T_{f,D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r)).$$

We may assume that f is non constant. Then $T_{f,D}(r)$ is a convex increasing function in $\log r$. Therefore we have

$$T_{f,D}(r) = O(\log r).$$

By [No3,(2.10),(2.11)] and [Ne2, I, p.369], there exists a holomorphic extension $\overline{f} : \Delta \to A$. This completes the proof of Theorem 10. □

4 Concluding Remarks

The proofs of Theorem 1 and Theorem 9 suggest the following approach to the general Second Main Theorem (Conjecture 1):
(i) First prove Conjecture 1 for holomorphic curves into $P_n(C)$ with divisor D not necessarily linear.
(ii) Next, we realize a given projective variety X as a finite branched covering

$\pi : X \to P_n(C)$. Let $f : C \to X$ be a holomorphic curve. We then apply the "Second Main Theorem" to the holomorphic curve $\pi \circ f : C \to P_n(C)$ and look at the ramification term as in the proof of Theorem 9 and use the ramification formula of canonical divisors.

It shoud be noted that recently Eremenko and Sadin (preprint, "Second Main Teorem of holomorphic curves and non-linear divisors") obtained an unintegrated version of the Second Main Theorem for holomorphic curves in projective spaces but its integrated form is weaker than conjecture 1. Shiffman [S3] conjectured a version of the Second Main Theorem for holomorphic curves into projective spaces, which was a special case of Conjecture 1. He also gave a new geometric proof of Cartan-Ahlfors' Second Main Theorem [Car] by extending a given holomorphic curve to a map $F : C \times P_{n-1}(C) \to P_n(C)$ and placing the problem into the equidimensional theory of Carlson-Griffiths [Ca-G]. The author conjectures that Conjecture 1 for holomorphic curves into projective spaces can be proved by our geometric method using the Abelian group structure of $(C^*)^n$ posessed by the complement of $(n+1)$ hyperplanes in general position (which is an anticanonical divisor) in $P_n(C)$. We have two metrics, i.e., the Fubini-Study metric g_{F-S} of $P_n(C)$ and the natural complete flat metric g_{flat} of $(C^*)^n$. The parallel translation with respect to g_{flat} will cause the distortion between these two metrics. The term $T_{f,K}(r)$ in the right hand side of Conjecture 1 will probably come from the anticanonical divisor at infinity (the metric distorsion caused by the parallel translation by an element at infinity !).

Finally, we remark that Vojta [V] observed Nevanlinna-Diophantine connection. If we are able to find sufficiently "simple" proofs for the Second Main Theorem (especially for holomorphic curves) in Nevanlinna theory, it may become helpfull to guess the corresponding steps to access the corresponding conjecture in Diophantine approximations through Vojta's dictionary.

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