Some homological properties of the category $\mathcal{O}$

Volodymyr Mazorchuk

Abstract

In the first part of this paper the projective dimension of the structural modules in the BGG category $\mathcal{O}$ is studied. This dimension is computed for simple, standard and costandard modules. For tilting and injective modules an explicit conjecture relating the result to Lusztig's $\mathbf{a}$-function is formulated (and proved for type A). The second part deals with the extension algebra of Verma modules. It is shown that this algebra is in a natural way $\mathbb{Z}^2$-graded and that it has two $\mathbb{Z}$-graded Koszul subalgebras. The dimension of the space Ext^1 into the projective Verma module is determined. In the last part several new classes of Koszul modules and modules, represented by linear complexes of tilting modules, are constructed.

2000 Mathematics Subject Classification: 16E10; 16E30; 16G99; 17B10

1 Introduction

The Bernstein-Gelfand-Gelfand category $\mathcal{O}$, [BGG1], associated with a triangular decomposition of a semi-simple complex finite-dimensional Lie algebra is an important and intensively studied object in modern representation theory. It has many very beautiful properties and symmetries. For example it is equivalent to the module category of a standard Koszul quasi-hereditary algebra and is Ringel self-dual. Its principal block is even Koszul self-dual. Powerful tools for the study of the category $\mathcal{O}$ are Kazhdan-Lusztig's combinatorics, developed in [KL], and Soergel's combinatorics, worked out in [So1]. These two machineries immediately give a lot of information about the numerical algebraic and homological invariants of simple, projective, Verma and tilting modules in $\mathcal{O}$ respectively. However, many natural questions about such invariants are still open. The present paper answers some of them.

The paper starts with a description of notation and preliminary results in Section 2. The rest is divided into three parts. The first part of this is Section 3, which is dedicated to the study of homological dimension for structural modules in the principal block $\mathcal{O}_0$ of $\mathcal{O}$. By *structural* I mean projective, injective, simple, standard (Verma), costandard (dual Verma), and tilting modules respectively. In some cases the result is rather expected. Some estimates go back to the original paper [BGG1]. For simple and standard modules the result can be deduced from Soergel's Koszul self-duality of $\mathcal{O}$. However, to my big surprise I failed to find more elementary arguments in the available literature. Here I present an explicit answer for simple, standard and costandard modules, and a proof, which does not even uses the Kazhdan-Lusztig conjecture. However, the shortest "elementary" argument I could come up with uses some properties of Arkhipov's twisting functors, established in [AS]. Things become really interesting when one tries to compute the projective dimension of an indecomposable tilting module. Although the projective dimension of the characteristic tilting module in $\mathcal{O}_0$ is well-known (see for example [MO1]), it seems that nobody has tried to determine the projective dimension of an indecomposable tilting module. A very surprizing conjecture based on several examples and Theorem 11, which says that the projective dimension of an indecomposable tilting module is a function, constant on two-sided cells, suggests that this dimension is given by Lusztig's $\mathbf{a}$-function from [Lu1]. This conjecture is proved here for type A (Theorem 15), which might be considered as a good evidence that the result should be true in general. However, I have no idea how to approach this question in the general case and my arguments from type A certainly can't be transfered. The determination of the projective dimension for injective modules reduces to that of tilting modules. As a "bonus" we also give a formula for the projective dimension of Irving's shuffled Verma modules in Proposition 18.

In Section 4 we study the extension algebra of standard modules in $\mathcal{O}_0$. This is an old open problem, where really not that much is known. The only available conjecture about the numerical description of such extensions, formulated in [GJ, Section 5], is known to be false ([Bo]), and the only explicit partial results I was able to find is the ones obtained in [GJ, Ca]. Here I follow the philosophy of [DM], where it was pointed out that the extension algebra of standard modules is naturally $\mathbb{Z}^2$-graded. This $\mathbb{Z}^2$-grading is obtained from two different $\mathbb{Z}$-gradings: the first one which comes from the category of graded modules, and the second one which comes from the derived category. Koszul self-duality of $\mathcal{O}_0$ induces a non-trivial automorphism of this $\mathbb{Z}^2$-graded algebra, which swaps the $\mathbb{Z}$-graded subalgebras of homomorphisms and linear extensions, see Theorem 21. This allows one to calculate linear extensions between standard modules, in particular, to reprove the main result from [Ca]. A surprizing corollary here is that by far not all projectives from the linear projective resolution of a standard module give rise to a non-trivial linear extension with the standard module, determined by this projective. In [DM] it was shown that in the multiplicity free case the extension algebra of standard modules is Koszul (with respect to the $\mathbb{Z}$-grading, which is naturally induced by the $\mathbb{Z}^2$-grading mentioned above). I do not think that this is true in the general case since I do not believe that the extension algebra of standard modules is generated in degree 1. However, I think it is reasonable to expect that the subalgebra of this extension algebra, generated by all elements of degree 1, is Koszul. To support this it is shown that the $\mathbb{Z}$-graded subalgebra of all *homomorphisms* between standard modules is Koszul, see Proposition 27. As the last result of Section 4 I explicitly determine the dimension of the Ext^1 space from a standard module to a projective standard module, see Theorem 32. From my point of view, the answer is again surprizing.

In [MO2, MOS] one finds an approach to Koszul duality using the categories of linear complexes of projective or tilting modules. For the category $\mathcal{O}_0$ this approach can be used to get quite a lot of information, see [Ma, MO2, MOS]. In particular one can prove the Koszul duality of various functors and various algebras, associated to $\mathcal{O}_0$. A very important class of modules for Koszul algebras is the class of the so-called *Koszul modules*. These are modules with linear projective resolutions. Such modules have a two-folded origin, namely, they are both modules over the original algebra and over its Koszul dual (via the corresponding linear resolution). In Section 5 I show for several natural classes of modules from $\mathcal{O}_0$ that they are either Koszul or can be represented in the derived category by a linear complex of tilting modules (which roughly means that they correspond to Koszul modules for the Ringel dual of $\mathcal{O}_0$). The latter property seems to be more "natural" for the category $\mathcal{O}_0$. For example, while only the simple and the standard modules are Koszul, it turns out that all simple, standard, costandard and shuffled Verma modules are represented by linear complexes of tilting modules (for the latter statement see Theorem 35). As an extension of this list we also show that some structural modules from the parabolic subcategories also have at least one of these properties, when considered as objects in the original category $\mathcal{O}_0$.

2 Notation and preliminaries

Let $\mathfrak{g}$ denote a semi-simple finite-dimensional Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\mathcal{O}$ denote the corresponding BGG-category $\mathcal{O}$, defined in [BGG1]. Let $\mathcal{O}_0$ denote the *principal block* of $\mathcal{O}$, that is the indecomposable direct summand of $\mathcal{O}$, containing the trivial module. Let W be the Weyl group of $\mathfrak{g}$ which acts on $\mathfrak{h}^*$ in the usual way $w(\lambda)$ and via the dot-action $w \cdot \lambda$. The category $\mathcal{O}_0$ contains the Verma modules $M(w \cdot 0)$, $w \in W$. For $w \in W$ we set $\Delta(w) = M(w \cdot 0)$ and let $L(w)$ denote the unique simple quotient of $\Delta(w)$. Further, $P(w)$ is the indecomposable projective cover of $L(w)$ and $I(w)$ is the indecomposable injective envelope of $L(w)$. We set $L = \oplus_{w \in W} L(w)$ and analogously for all other structural modules.

The category $\mathcal{O}_0$ is a highest weight category in the sense of [CPS], in particular, associated to $L(w)$ we also have the costandard module $\nabla(w)$, and the indecomposable tilting module $T(w)$ (see [Ri]). If $\star$ is the standard duality on $\mathcal{O}$, we have $\nabla(w) \cong \Delta(w)^\star$ and $T(w) \cong T(w)^\star$. For $w \in W$ by $l(w)$ we denote the length of w. Let w_0 denote the longest element of W. By $\leq$ we denote the Bruhat order on W. For $w \in W$ let $\theta_w : \mathcal{O}_0 \to \mathcal{O}_0$ denote the indecomposable projective functor, corresponding to w, see [BG, Theorem 3.3]. In particular, if $s \in W$ is a simple reflection, then θ_s is the translation functor through the s-wall (see [GJ, Section 3]). We have $\theta_w P(e) \cong P(w)$ ([BG, Theorem 3.3]) and $\theta_w T(w_0) \cong T(w_0 w)$ ([CI, Theorem 3.1]).

If $\mathcal{X}^\bullet$ is a complex and $n \in \mathbb{Z}$, by $\mathcal{X}^\bullet[n]$ we will denote the n-th shifted complex, that is the complex, satisfying $(\mathcal{X}^\bullet[n])^i \cong \mathcal{X}^{i+n}$ for all $i \in \mathbb{Z}$. We also use the standard notation $\mathcal{D}^b(A)$, $\mathcal{L}\,\mathrm{F}$ and $\mathcal{R}\,\mathrm{F}$ to denote the bounded derived category, and the left and right derived functors respectively.

Let $A = \mathrm{End}_{\mathcal{O}}(P)^{\mathrm{op}}$ be the associative algebra of $\mathcal{O}_0$. This means that $\mathcal{O}_0$ is equivalent to the category $A-$mod of finitely generated left A-modules. This algebra is Koszul ([So1, Theorem 18]) and we denote by $\mathtt{A}$ the associated positively graded algebra. Denote by $\mathtt{A}-$gmod the category of all finitely generated graded left $\mathtt{A}$-modules. For $w \in W$ we denote by $\mathtt{L}(w)$ the standard graded lift of $L(w)$, concentrated in degree 0; and by $\mathtt{P}(w)$ and $\mathtt{I}(w)$ the corresponding lifts of $P(w)$ and $I(w)$ respectively such that the maps $P(w) \twoheadrightarrow L(w)$ and $L(w) \hookrightarrow I(w)$ become homogeneous of degree 0. Further we fix graded lifts $\mathtt{\Delta}(w)$ and $\nabla(w)$ such that the obvious maps $P(w) \twoheadrightarrow \Delta(w)$ and $\nabla(w) \hookrightarrow I(w)$ become homogeneous of degree 0. Finally, we fix the graded lift $\mathtt{T}(w)$ such that the map $\Delta(w) \hookrightarrow T(w)$ becomes homogeneous of degree 0. In general, we will try to follow the conventions of [MOS, Introduction] and refer the reader to that paper for details. In particular, a graded lift of a module, M, will be usually denoted by $\mathtt{M}$. For $k \in \mathbb{Z}$ we denote by $\langle k \rangle$ the functor of shifting the grading as follows: if $\mathtt{M} = \oplus_{i \in \mathbb{Z}} \mathtt{M}_i$ then $\mathtt{M}\langle k \rangle_i = \mathtt{M}_{i+k}$. A complex $\mathcal{X}^\bullet$ of graded projective (respectively injective or tilting) modules is called *linear* provided that $\mathcal{X}^i \in \mathrm{add}(\mathtt{P}\langle i \rangle)$ (respectively $\mathtt{I}\langle i \rangle$ and $\mathtt{T}\langle i \rangle$) for all $i \in \mathbb{Z}$. By $\mathfrak{LC}(\mathtt{P})$ (respectively $\mathfrak{LC}(\mathtt{I})$ or $\mathfrak{LC}(\mathtt{T})$) we denote the category, whose objects are all linear (bounded) complexes of projective (respectively injective and tilting) modules, and morphisms are all possible morphisms of complexes of graded modules. For general information about the categories of linear complexes and their applications, see [MO2, MOS].

3 Projective dimensions of structural modules in $\mathcal{O}_0$

As we already mentioned, the category $\mathcal{O}_0$ is a highest weight category. All simple, standard, costandard, projective, injective and tilting modules play various important roles in this structure. Our first natural question is to determine the projective dimension of all these (indecomposable) structural modules. We will write p.d.(M) for the *projective dimension* of a module, M, and denote by gl.dim. the *global* (or *homological*) *dimension* of an algebra or its module category. As an obvious result here one can mention p.d.$(P(w)) = 0$ for all $w \in W$.

3.1 Standard and simple modules

It turns out that determining the projective dimension of standard and simple modules in $\mathcal{O}_0$ is the easiest part of the task. Actually, first estimates for these dimensions were already obtained in the original paper [BGG1].

Proposition 1. *([BGG1, Section 7])*

(i) p.d.$(\Delta(w)) \leq l(w)$.

(ii) p.d.$(L(w)) \leq 2l(w_0) - l(w)$.

(iii) gl.dim.$\mathcal{O}_0 \leq 2l(w_0)$.

Proof. Obviously, p.d.$(\Delta(e)) = 0$ since $\Delta(e) = P(e)$. As we have already mentioned, $\mathcal{O}_0$ is a highest weight category with respect to the Bruhat order on W. In particular, this means that the kernel of the natural projection $P(w) \twoheadrightarrow \Delta(w)$ has a filtration with subquotients $\Delta(w')$, $l(w') < l(w)$. Hence

$$\text{p.d.}(\Delta(w)) \leq \max_{w' : l(w') < l(w)} \{\text{p.d.}(\Delta(w'))\} + 1,$$

which implies (i) by induction.

Since $\Delta(w_0) = L(w_0)$, the formula of (ii) for $w = w_0$ is just a special case of (i). Consider now the short exact sequence $X \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$. Then

X has a filtration with subquotients of the form $L(w')$, $l(w') > l(w)$. Hence one obtains

$$\mathrm{p.d.}(L(w)) \leq \max_{w': l(w') > l(w)} \{\mathrm{p.d.}(L(w'))\} + 1,$$

which implies (ii) by induction.

(iii) is an immediate corollary from (ii). $\square$

Further, in the last remark in [BGG1] it is mentioned that one can show that $\mathrm{gl.dim.}\mathcal{O}_0 = 2l(w_0)$. The shortest argument I know, which does this, is the following:

Proposition 2. $\mathrm{p.d.}(L(e)) \geq 2l(w_0)$*, in particular,* $\mathrm{gl.dim.}\mathcal{O}_0 = 2l(w_0)$.

Proof. Consider the BGG-resolution

$$0 \to M_{l(w_0)} \to M_{l(w_0)-1} \to \cdots \to M_1 \to M_0 \to L(e) \to 0$$

of $L(e)$, see [BGG2, Theorem 10.1], and let $\mathcal{M}^\bullet$ be the corresponding complex of (direct sums of) Verma modules, whose only non-zero homology is $\mathrm{H}^0(\mathcal{M}^\bullet) \cong L(e)$. Every non-zero map $f : \Delta(w_0) \to \nabla(w_0)$ induces a non-zero map $\overline{f} : \mathcal{M}^\bullet \to (\mathcal{M}^\bullet)^\star[2l(w_0)]$. Since $\dim \mathrm{Hom}_{\mathcal{O}}(\Delta(w), \nabla(w')) = \delta_{w,w'}$ by [Ri, Section 3], it follows that $\overline{f}$ is not homotopic to 0. Since $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(w), \nabla(w')) = 0$ for all $i > 0$ by [Ri, Theorem 4], from [Ha, Chapter III(2), Lemma 2.1] it follows that $\mathrm{Ext}^{2l(w_0)}_{\mathcal{O}}(L(e), L(e)) \neq 0$. Thus we get $\mathrm{p.d.}(L(e)) \geq 2l(w_0)$. The latter and Proposition 1(iii) imply $\mathrm{gl.dim.}\mathcal{O}_0 = 2l(w_0)$. $\square$

Now let us show that the estimates in Proposition 1(i) and Proposition 1(ii) are in fact the exact values. Already this becomes slightly tricky, especially for simple modules. Here we present a uniform approach, which works for both standard and simple modules, and is based on certain properties of the so-called *twisting functors* on $\mathcal{O}_0$. Some other approaches will be discussed in remarks at the end of this subsection. For $w \in W$ let $\mathrm{T}_w : \mathcal{O}_0 \to \mathcal{O}_0$ denote the corresponding twisting functor, see [Ar, AS]. Let further $\mathrm{G}_{w^{-1}} : \mathcal{O}_0 \to \mathcal{O}_0$ denote the right adjoint of T_w. The functor $\mathrm{G}_{w^{-1}}$ is isomorphic ([KM, Corollary 6]) to Joseph's completion functor defined in [Jo]. We start with the case of standard modules since the proof is more direct in this case.

Proposition 3. $\mathrm{Ext}^{l(w)}_{\mathcal{O}}(\Delta(w), L(e)) \neq 0$*, in particular,* $\mathrm{p.d.}(\Delta(w)) = l(w)$.

Proof. We do induction on $l(w)$. If $w = e$ the statement is obvious. If s is a simple reflection such that $l(sw) > l(s)$, we have

$$
\begin{array}{lll}
\mathrm{Ext}_{\mathcal{O}}^{l(sw)}(\Delta(sw), L(e)) & = & \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(sw), L(e)[l(sw)]) & = & \text{(by [AS, (2.3)])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathrm{T}_s\Delta(w), L(e)[l(sw)]) & = & \text{(by [AS, Theorem 2.2])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathrm{T}_s\Delta(w), L(e)[l(sw)]) & = & \text{(by [AS, Corollary 4.2])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(w), \mathcal{R}\mathrm{G}_s L(e)[l(sw)]) & = & \text{(by [AS, Corollaries 4.2 and 6.2])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(w), L(e)[l(sw)-1]) & = & \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(w), L(e)[l(w)]) & = & \\
\mathrm{Ext}_{\mathcal{O}}^{l(w)}(\Delta(w), L(e)) & \neq & 0
\end{array}
$$

by induction. The statement now follows from Proposition 1(i). □

Remark 4. Another way to prove the formula for the projective dimension of standard modules from Proposition 3 is to use [So1, Theorem 18], [ADL, Proposition 2.7] and [Ir1, 3.5]. A disadvantage in this case is the fact that so far there is no purely algebraic proof of [So1, Theorem 18], whereas the results from [AS] used in the proof of Proposition 3 can be proved algebraically.

Remark 5. Yet another way to prove the formula for the projective dimension of standard modules from Proposition 3 is to observe, using translation functors, that p.d.$\Delta(w_0)$ coincides with the projective dimension of the characteristic tilting module in $\mathcal{O}_0$. Then [MO1, Corollary 2] and Proposition 2 imply p.d.$(\Delta(w_0)) = l(w_0)$. For any $w \in W$ and a simple reflection $s \in W$ such that $l(ws) > l(w)$ there is a short exact sequence $\Delta(ws) \hookrightarrow \theta_s\Delta(w) \twoheadrightarrow \Delta(w)$. Since θ_s is exact and maps projectives to projectives, we have p.d.$(\theta_s\Delta(w)) \leq$ p.d.$(\Delta(w))$. This implies p.d.$(\Delta(ws)) \leq$ p.d.$(\Delta(w))+1$ and the second statement of Proposition 3 follows by induction from the extreme cases $w = e$ and $w = w_0$ for which it is already established.

Now we move to the case of simple modules.

Proposition 6. $\mathrm{Ext}_{\mathcal{O}}^{2l(w_0)-l(w)}(L(w), L(e)) \neq 0$, *in particular,* p.d.$(L(w)) = 2l(w_0) - l(w)$.

Proof. Again the second statement follows from the first statement and Proposition 1(ii). Since $L(w_0) = \Delta(w_0)$, in the case $w = w_0$ the first statement follows from Proposition 3. Now we use the inverse induction on $l(w)$. Let $s \in W$ be a simple reflection such that $l(sw) < l(w)$. Let

$m = 2l(w_0) - l(w)$. Using the results of [AS] we have:

$$\begin{array}{lll}
\mathrm{Ext}_{\mathcal{O}}^{m+1}(\mathrm{T}_s L(w), L(e)) & = & \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathrm{T}_s L(w), L(e)[m+1]) & = & \text{(by [AS, Theorems 2.2 and 6.1])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathrm{T}_s L(w), L(e)[m+1]) & = & \text{(by [AS, Corollary 4.2])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(L(w), \mathcal{R}\mathrm{G}_s L(e)[m+1]) & = & \text{(by [AS, Corollary 4.2 and 6.2])} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(L(w), L(e)[m]) & = & \\
\mathrm{Ext}_{\mathcal{O}}^{m}(L(w), L(e)). & &
\end{array}$$

From the inductive assumption we thus get $\mathrm{Ext}_{\mathcal{O}}^{m+1}(\mathrm{T}_s L(w), L(e)) \neq 0$. From [AS, Lemma 2.1(3)] and the right exactness of T_s it follows that all composition subquotients of $\mathrm{T}_s L(w)$ are either of the form $L(sw)$ or of the form $L(w')$, where $l(w') > l(sw)$. From the inductive assumption we have $\mathrm{p.d.}(L(w')) \leq m < \mathrm{p.d.}(X)$, which implies $\mathrm{p.d.}(L(sw)) = \mathrm{p.d.}(X) = m + 1$. This completes the proof. □

Remark 7. Another way to prove the second statement of Proposition 6 is to use [So1, Theorem 18], reducing the question to the Loewy length of some projective module in $\mathcal{O}_0$. This Loewy length can then be estimated using the results from [Ir1].

3.2 Costandard modules

An easy corollary from Proposition 6 is the following formula for projective dimensions of costandard modules:

Proposition 8. $\mathrm{p.d.}(\nabla(w)) = 2l(w_0) - l(w)$.

Proof. For $w = w_0$ we have $\nabla(w_0) = L(w_0)$ and the statement follows from Proposition 6. Now we use the inverse induction on $l(w)$. Let s be a simple reflection such that $l(ws) < l(w)$. Then we have the short exact sequence $\nabla(w) \hookrightarrow \theta_s \nabla(w) \twoheadrightarrow \nabla(ws)$. Since θ_s is exact and preserves projectives, we have $\mathrm{p.d.}(\theta_s \nabla(w)) \leq \mathrm{p.d.}(\nabla(w))$, which implies $\mathrm{p.d.}(\nabla(ws)) \leq \mathrm{p.d.}(\nabla(w)) + 1 = 2l(w_0) - l(ws)$. On the other hand, for the short exact sequence $L(ws) \hookrightarrow \nabla(ws) \twoheadrightarrow X$ we have that all simple subquotients of X have the form $L(w')$, where $l(w') > l(ws)$. Hence, by the inductive assumption, we have $\mathrm{p.d.}(X) < 2l(w_0) - l(ws)$, which implies that $\mathrm{p.d.}(\nabla(ws)) = \mathrm{p.d.}(L(ws))$. The claim follows. □

Remark 9. Another way to prove Proposition 8 is to use twisting functors and the results of [AS], analogously to the proofs of Proposition 3 and Proposition 6.

Remark 10. It is worth mentioning that all the results so far are obtained without using the Kazhdan-Lusztig conjecture (=Theorem).

3.3 Injective and tilting modules

We are now left to consider the cases of injective and tilting modules. It turns out that these are by far more complicated than the others. Firstly, we will be forced to use the Kazhdan-Lusztig conjecture. Secondly, we will not be able to obtain a description so explicit as above in all cases, and even in the cases when an explicit description is obtained, the result is formulated in terms of Kazhdan-Lusztig's combinatorics. To shorten our notation for $w \in W$ we set

$$\mathfrak{t}(w) := \text{p.d.}(T(w)), \quad \mathfrak{i}(w) := \text{p.d.}(I(w)).$$

Our main observation about $\mathfrak{t}(w)$ and $\mathfrak{i}(w)$ is the following:

Theorem 11. *(a) Both, $\mathfrak{t}$ and $\mathfrak{i}$, are constant on the right cells of W.*

(b) Both, $\mathfrak{t}$ and $\mathfrak{i}$, are constant on the left cells of W.

(c) Both, $\mathfrak{t}$ and $\mathfrak{i}$, are constant on the two-sided cells of W.

Proof. The statement (c) follows immediately from (a) and (b).

Proof of the statement (a). As a consequence of the Kazhdan-Lusztig conjecture, for $w \in W$ and a simple reflection, $s \in W$, we have (see e.g. [Ir2, Corollary 5.2.4]):

$$\theta_s\theta_w = \begin{cases} \theta_w \oplus \theta_w, & \text{if } ws < w; \\ \theta_{ws} \oplus \bigoplus_{y<w, ys<y} \mu(y,w)\theta_y, & \text{if } ws > w, \end{cases} \tag{1}$$

where $\mu(y,w)$ is Kazhdan-Lusztig's μ-function (see [Ir2, 2.1] or [KL]).

By [BG, Theorem 3.3] we have $\theta_w P(e) \cong P(w)$ and hence $\theta_w I(e) \cong I(w)$ since θ_w obviously commutes with $\star$. Now let $w \in W$ and a simple reflection $s \in W$ be such that $ws > w$. Since θ_s is exact and sends projectives to projectives, applying θ_s to the projective resolution of $I(w) = \theta_w I(e)$ and using (1) we obtain that $\mathfrak{i}(ws) \leq \mathfrak{i}(w)$ and $\mathfrak{i}(y) \leq \mathfrak{i}(w)$ for all y such that $y < w$, $ys < y$ and $\mu(y,w) \neq 0$. In particular, it follows that $\mathfrak{i}$ is monotone with respect to the right pre-order on W (see e.g. [BB, 6.2] for details) and thus $\mathfrak{i}$ must be constant on the right cells.

Since $x \mapsto w_0x$ is a bijection on the right cells (see e.g. [BB, Corollary 6.2.10]), we have that for $\mathfrak{t}$ the arguments are just the same as for $\mathfrak{i}$, as soon as one makes the obvious observation that $\theta_w T(w_0) \cong T(w_0w)$. $\square$

Proof of the statement (b). The statement (b) is the "left hand-side version" of the statement (a). We would like to prove it using analogous arguments, however, for this we will need a "right hand-side version" of (1).

Lemma 12.

$$\theta_w\theta_s = \begin{cases} \theta_w \oplus \theta_w, & \textit{if } sw < w; \\ \theta_{sw} \oplus \bigoplus_{y<w, sy<y} \mu(y,w)\theta_y, & \textit{if } sw > w, \end{cases} \tag{2}$$

Proof. Let $\mathcal{H}$ denote the Hecke algebra of W equipped with the standard basis $(H_w)_{w\in W}$. Then there is a unique antiautomorphism σ of $\mathcal{H}$ satisfying $\sigma(H_s) = H_s$ for any simple reflection s. Now (2) is obtained from (1) by applying σ. $\square$

Let $s \in W$ be a simple reflection and $w \in W$. Applying θ_w to the short exact sequence $\Delta(sw_0) \hookrightarrow T(sw_0) \twoheadrightarrow \Delta(w_0)$ and observing that $\Delta(sw_0) = \mathrm{G_s}\Delta(w_0)$ (the dual of [AS, (2.3)]) and $\mathrm{G_s}\theta_w = \theta_w\mathrm{G_s}$ (the dual of [AS, Theorem 3.2]), we get

$$\mathrm{G_s}T(w_0w) \hookrightarrow \theta_w\theta_sT(w) \twoheadrightarrow T(w_0w). \tag{3}$$

We claim that $\mathrm{p.d.}(\mathrm{G_s}T(w_0w)) \leq \mathrm{p.d.}(T(w_0w))$. Indeed, let us denote $\mathrm{p.d.}(T(w_0w)) = m$. Then for all $i > m$ we have

$$\begin{array}{lll} \mathrm{Ext}^i_{\mathcal{O}}(\mathrm{G_s}T(w_0w), L) & = & \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathrm{G_s}T(w_0w), L[i]) & = & \text{(by [AS, Theorems 2.2])} \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{R}\mathrm{G_s}T(w_0w), L[i]) & = & \text{(by [AS, Corollary 4.2])} \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0w), \mathcal{L}\mathrm{T_s}L[i]). & & \end{array}$$

The length of a minimal projective resolution of $T(w_0w)$ is m. By [AS, Theorems 2.1], the non-zero homology of $\mathcal{L}\mathrm{T_s}L[i]$ can occur only in positions $-i$ or $-i-1$. Since $i > m$ it follows from [Ha, Chapter III(2), Lemma 2.1] that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0w), \mathcal{L}\mathrm{T_s}L[i]) = 0$ and thus $\mathrm{p.d.}(\mathrm{G_s}T(w_0w)) \leq \mathrm{p.d.}(T(w_0w))$.

From the previous paragraph and the short exact sequence (3) we derive the inequality $\mathrm{p.d.}(\theta_w\theta_sT(w_0)) \leq \mathrm{p.d.}(T(w_0w))$. Now from (2) it follows that $\mathrm{p.d.}(T(w_0y)) \leq \mathrm{p.d.}(T(w_0w))$ for each y such that $y < w$, $sy < y$ such that $\mu(y,w) \neq 0$. In particular, it follows that $\mathfrak{t}$ is monotone with respect to the left pre-order on W (see e.g. [BB, 6.2] for details) and thus $\mathfrak{t}$ must be constant on the left cells. Again, for $\mathfrak{i}$ the proof is analogous. $\square$

$\square$

Example 13. If $\mathfrak{g}$ is of type A_2, we have $W = \{e, s, t, st, ts, sts = tst\}$ with the following decomposition into two-sided cells: $\{e\} \cup \{s, t, st, ts\} \cup \{sts\}$. One easily computes the following table of values for $\mathfrak{t}$ and $\mathfrak{i}$:

w	e	s	t	st	ts	sts
$\mathfrak{t}(w)$	0	1	1	1	1	3
$\mathfrak{i}(w)$	6	2	2	2	2	0

There is a well-know integral function on W, constant on two-sided cells, namely Lusztig's function $\mathbf{a} : W \to \mathbb{Z}$, defined in [Lu1]. If $w \in W$ is an involution, then $\mathbf{a}(w) = l(w) - 2\delta(w)$, where $\delta(w)$ is the degree of the Kazhdan-Lusztig polynomial $P_{1,w}$, which, together with the property of being constant on two-sided cells, completely determines $\mathbf{a}$, since every two-sides cell contains a (distinguished) involution, see [Lu1, Lu2] for details. In particular, if W_S is a parabolic subgroup of W and w_0^S is the longest element in W_S, we have $\mathbf{a}(w_0^S) = l(w_0^S)$. Comparing the values of $\mathbf{a}$ with Example 13 and other examples leads to the following conjecture:

Conjecture 14. *For all $w \in W$ we have*

(a) $\mathfrak{t}(w) = \mathbf{a}(w)$;

(b) $\mathfrak{i}(w) = 2\mathbf{a}(w_0 w)$.

Theorem 15. *Conjecture 14 is true if $\mathfrak{g} = \mathfrak{sl}_n$.*

Proof. We start by proving Conjecture 14(a).

First we observe that in the case $\mathfrak{g} = \mathfrak{sl}_n$ every two-sided cell of W contains an element of the form w_0^S, where W_S is a parabolic subgroup of W. Indeed, from [BB, Theorem 6.5.1] we have that there is a bijection between the two-sided cells of $\mathcal{S}_n$ and partitions of n. Using [BB, Theorem 6.5.1] and [Sa, Theorem 3.6.6] one gets that the two-sided cell of $W \cong \mathcal{S}_n$, corresponding to the partition $\lambda \vdash n$, consists of all $w \in \mathcal{S}_n$, which correspond to standard tableaux of shape λ via the Robinson-Schensted correspondence. Now if w_0^S is the longest element in some parabolic subgroup of type λ, a direct calculation shows that the Robinson-Schensted correspondence associates with w_0^S the partition, which is conjugate to λ. As a corollary we get that every two-sided cell indeed contains some w_0^S.

Fix now some two-sided cell, say $\mathcal{C}$, and assume that it contains w_0^S for some S. Because of the properties of $\mathbf{a}$, listed above, Conjecture 14(a) would follow if we would prove that $\text{p.d.}(T(w_0^S)) = l(w_0^S)$. Assume further that W_S corresponds to the partition λ. From [BB, Theorem 6.2.10] and [Sa, Theorem 3.2.3] we get that $\mathcal{C}$ also contains an element of the form $w_0 w_0^{S'}$, where S' corresponds to the conjugate λ' of λ.

Decompose $\theta_{w_0^{S'}} = \theta^{out}_{w_0^{S'}} \theta^{on}_{w_0^{S'}}$, where $\theta^{on}_{w_0^{S'}}$ is the translation onto the "most singular" S'-wall, and $\theta^{out}_{w_0^{S'}}$ is the translation out of this wall. Let further the $w_0^{S'}$-singular block $\mathcal{O}_\mu$ be the image of $\theta^{on}_{w_0^{S'}}$, applied to $\mathcal{O}_0$. Finally, let X denote the simple Verma module in $\mathcal{O}_\mu$. Then $\theta^{on}_{w_0^{S'}} T(w_0 w_0^{S'}) \cong X^{\oplus |W_{S'}|}$ and $\theta^{out}_{w_0^{S'}} X \cong T(w_0 w_0^{S'})$. Since translation functors are exact and preserve projectives, we get $\text{p.d.}(T(w_0 w_0^{S'})) = \text{p.d.}(X)$.

The Koszul dual of $\mathcal{O}_\mu$ is the regular block of the S'-parabolic category $\mathcal{O}^{\mathfrak{p}}$, see [BGS, Theorem 3.10.2]. In particular, via the Koszul duality p.d.(X) becomes equal to $m-1$, where m is the Loewy length of the projective generalized Verma module in $\mathcal{O}_0^{\mathfrak{p}}$. By [IS, Corollary 3.1], since $w_0^{S'}$ corresponds to the partition λ', $m-1$ is equal to length of the longest element in some parabolic subgroup of W corresponding to the partition conjugate to λ', that is to λ. We finally get that

$$\mathfrak{t}(w_0^S) = \mathfrak{t}(w_0 w_0^{S'}) = \text{p.d.}(X) = l(w_0^S).$$

Now we prove Conjecture 14(b) using Conjecture 14(a). In fact, after Conjecture 14(a) is proved, one has only to show that $\mathfrak{i}(w_0^S) = 2\mathfrak{t}(w_0 w_0^S)$. We again decompose $\theta_{w_0^S} = \theta^{out}_{w_0^S}\theta^{on}_{w_0^S}$. We have the singular simple Verma module X such that $\theta^{out}_{w_0^S}X \cong T(w_0 w_0^S)$ (and $\theta^{on}_{w_0^S}T(w_0 w_0^S) \cong X^{\oplus |W_S|}$). We also have the singular dominant dual Verma module Y such that $\theta^{out}_{w_0^S}Y \cong I(w_0^S)$ (and $\theta^{on}_{w_0^S}I(w_0^S) \cong Y^{\oplus |W_S|}$). In particular, we have p.d.$(T(w_0 w_0^S)) = $ p.d.$(X) = m$ and p.d.$(I(w_0^S)) = $ p.d.$(Y) = n$. So we have to show that $n = 2m$. Taking the Koszul dual we get that $m+1$ equals the Loewy length of the projective standard module in some regular block of the parabolic category $\mathcal{O}^{\mathfrak{p}}$.

Let Z denote the simple socle of Y. Then the projective dimension of Z equals, via Koszul duality, to $x-1$, where x is the Loewy length of some projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$. By [MS1, Theorem 5.2(1)], all projective-injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ have the same Loewy length. By [MS1, Theorem 5.2(2)], the projective generator of $\mathcal{O}_0^{\mathfrak{p}}$ is a submodule of a projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$. It follows that projective-injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ have the maximal possible Loewy length. Thus p.d.(Z) equals the global dimension of $\mathcal{O}_0^{\mathfrak{p}}$. Since Z is in the socle of Y and has the maximal possible projective dimension, from the long exact sequence in homology it follows that $n = $ p.d.$(Y) = $ p.d.$(Z) = x-1$. Now $n = x-1 = 2m$ follows from [IS, Corollary 3.1]. This completes the proof. $\square$

Remark 16. The main difficulty to extend the above arguments to the case of arbitrary $\mathfrak{g}$ seems to be the fact that, in general, not every two-sided cell contains some element of the form w_0^S. In fact, Jian-yi Shi has informed me that in type D_4 some two-sided cell with $\mathbf{a}$-value 7 does not contain any such element. I have no idea how to estimate the values of $\mathfrak{t}$ and $\mathfrak{i}$ on elements of such cells. In the general case I can not even prove that $\mathfrak{t}(s) = 1$ for a simple reflection $s \in W$.

Remark 17. The functor $\mathrm{T} = \mathrm{T}_{w_0}$ is exactly the version of Arkhipov's functor used in [So2] to establish Ringel's self-duality of $\mathcal{O}$. In particular,

$\mathrm{T}P(w) \cong T(w_0 w)$ for all $w \in W$. Using [AS, Corollary 4.2], for every $w \in W$ and $i \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0 w), L[i]) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathrm{T}P(w), L[i]) = \\ = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(P(w), \mathcal{R}\mathrm{G}L[i]).$$

This shows that Conjecture 14 is closely connected to the understanding of $\mathcal{R}\mathrm{G}$ applied to simple modules, that is to the understanding of the homology of the complex $\mathrm{G}\mathcal{I}^\bullet$, where $\mathcal{I}^\bullet$ is an injective resolution of L. We remark that $\mathcal{I}^\bullet$ is a projective object in $\mathfrak{LC}(\mathtt{I})$; and $\mathrm{G}\mathcal{I}^\bullet$ is a projective object in the category $\mathfrak{LC}(\mathtt{T})$ (see [MOS, Proposition 11]). These categories will appear later on in the paper, where we will also try study the connection mentioned above in more details.

3.4 Shuffled Verma modules

There is a very special class of modules in $\mathcal{O}_0$, called *shuffled Verma modules*, which were introduced in [Ir3] as modules, corresponding to the principal series modules. Using [AL, Section 3] for $x, y \in W$ we define the corresponding shuffled Verma module

$$\Delta(x, y) = \mathrm{T}_x \Delta(y)$$

(as these modules are defined using the twisting functors, sometimes they are also called *twisted Verma modules*, however, we will use the name *shuffled Verma modules* as in the original paper [Ir3]). In particular, using [AS, (2.3) and Theorem 2.3] for any $w \in W$ we have

$$\begin{array}{rclrcl} \Delta(e,w) & \cong & \Delta(w), & \qquad \Delta(w,w_0) & \cong & \nabla(ww_0), \\ \Delta(w,e) & \cong & \Delta(w), & \qquad \Delta(w_0,w) & \cong & \nabla(w_0 w). \end{array}$$

For shuffled Verma modules we have the following statement, which includes Proposition 3 and Proposition 8 as special cases:

Proposition 18. *For $x, y \in W$ we have* $\mathrm{p.d.}(\Delta(x, y)) = l(x) + l(y)$.

Proof. First let us prove that $\mathrm{p.d.}(\Delta(x, y)) \leq l(x) + l(y)$ by induction on $l(x)$. If $x = e$, the statement follows from Proposition 8. Let now $x = sz$, where s is a simple reflection and $l(z) < l(x)$. Since $\Delta(x, y) = \mathrm{T}_s\Delta(z, y)$, for $i > l(x) + l(y)$ we have

$$\begin{array}{lcl} \mathrm{Ext}^i_{\mathcal{O}}(\mathrm{T}_s\Delta(z, y), L) & = & \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathrm{T}_s\Delta(z, y), L[i]) & = & \text{(by [AS, Theorems 2.2])} \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathrm{T}_s\Delta(z, y), L[i]) & = & \text{(by [AS, Corollary 4.2])} \\ \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(z, y), \mathcal{R}\mathrm{G_s}L[i]). & & \end{array} \tag{4}$$

By the assumption of induction we know that the projective resolution of $\Delta(z,y)$ has length at most $l(x)+l(y)-1$. By the dual of [AS, Theorems 2.2], non-zero homology of $\mathcal{R}\mathrm{G}_{\mathrm{s}}L[i]$ can occur only in positions $-i,-i+1 < -(l(x)+l(y)-1)$. Hence, using [Ha, Chapter III(2), Lemma 2.1], we get that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(z,y),\mathcal{R}\mathrm{G}_{\mathrm{s}}L[i])=0$.

Now it is enough to observe that $\mathrm{Ext}_{\mathcal{O}}^{l(x)+l(y)}(\Delta(x,y),L(e))\neq 0$. We use induction on $l(x)+l(y)$. If $l(x)=0$, this is proved in Proposition 3. If $l(x)>1$ this follows from the inductive assumption and (4) using [AS, Corollary 2.2]. This completes the proof. $\square$

Remark 19. Twisted tilting modules $\mathrm{T}_xT(y)$, $x,y\in W$, were studied in [St2]. One can also consider the twisted projective modules $\mathrm{T}_xP(y)$, $x,y\in W$ (for $x=w_0$ the latter coincide with the usual tilting modules). It is a natural question to determine the projective dimension of these modules. However, this question seems to be even more complicated than the corresponding question for the usual tilting modules. The main reason is that, in contrast to the usual tilting modules, for twisted tilting or twisted projective modules the function of projective dimension will be constant only on the appropriate right cells, but not on the two-sided cells in the general case.

4 On the extension algebra of standard modules

4.1 Setup for Koszul quasi-hereditary algebras

Let $\Bbbk$ be an algebraically closed field. Let $\mathtt{A}=\oplus_{i\in\mathbb{Z}}\mathtt{A}_i$ be a *positively graded* $\Bbbk$-algebra, that is $\dim\mathtt{A}_i=0$ for all $i<0$; $\dim\mathtt{A}_i<\infty$ for all i; and $\mathtt{A}_0=\oplus_{\lambda\in\Lambda}\Bbbk e_\lambda$, where $1=\sum_{\lambda\in\Lambda}e_\lambda$ is a fixed decomposition of 1 into a sum of pairwise orthogonal primitive idempotents. We denote by $\mathtt{A}^!$ the *quadratic dual* of $\mathtt{A}$, see e.g. [MO2, Section 6].

Let $\mathtt{A}-$fgmod denote the category of all graded $\mathtt{A}$-modules with finite-dimensional graded components. Morphisms in this category are homogeneous maps of degree 0 between graded modules. Under our assumptions, this category contains several natural classes of modules. To each $\lambda\in\Lambda$ there correspond the graded projective module $\mathtt{P}(\lambda)=\mathtt{A}e_\lambda$, its simple quotient $\mathtt{S}(\lambda)$, and the injective hull $\mathtt{I}(\lambda)$ of $\mathtt{S}(\lambda)$. Assume further that $\mathtt{A}$ is quasi-hereditary with respect to some order $\leq$ on Λ. Then we also have the corresponding graded standard module $\Delta(\lambda)$, the graded costandard module $\nabla(\lambda)$, and the graded tilting modules $\mathtt{T}(\lambda)$, (see for example [Zh]). As before we set $\mathtt{P}=\oplus_{\lambda\in\Lambda}\mathtt{P}(\lambda)$ and analogously for all other types of modules. We have that the canonical surjections $\mathtt{P}(\lambda) \twoheadrightarrow \Delta(\lambda) \twoheadrightarrow \mathtt{S}(\lambda)$ and $\mathtt{T}(\lambda) \twoheadrightarrow \nabla(\lambda)$, and the canonical injections $\mathtt{S}(\lambda) \hookrightarrow \nabla(\lambda) \hookrightarrow \mathtt{I}(\lambda)$ and $\Delta(\lambda) \hookrightarrow \mathtt{T}(\lambda)$ are morphisms in $\mathtt{A}-$fgmod. As before $\langle k\rangle$ denotes the shift of grading.

Denote by $\mathfrak{LC}(\mathtt{P})$ (resp. $\mathfrak{LC}(\mathtt{T})$) the category, whose objects are all complexes $\mathcal{X}^\bullet$ such that $\mathcal{X}^i \in \mathrm{add}(\mathtt{P}\langle i\rangle)$ (resp. $\mathrm{add}(\mathtt{T}\langle i\rangle)$) for all i, and morphisms are all morphisms of complexes. From the positivity of the grading it follows that the only homotopy between two objects of $\mathfrak{LC}(\mathtt{P})$ is the trivial one. The grading on $\mathtt{A}$ automatically induces a grading on the *Ringel dual* $R(\mathtt{A}) = \mathrm{End}_{\mathtt{A}}(\mathtt{P})^{\mathrm{op}}$. If this grading is positive (which is not true in general), then the only homotopy between two objects of $\mathfrak{LC}(\mathtt{T})$ is the trivial one (see [MO2, Section 6]). The category $\mathfrak{LC}(\mathtt{P})$ is equivalent to $\mathtt{A}^!-$fgmod and the category $\mathfrak{LC}(\mathtt{T})$ is equivalent to $R(\mathtt{A})^!-$fgmod, see e.g. [MO2, Section 6].

Assume now that both the minimal titling coresolution of Δ and the minimal tilting resolution of ∇ are objects in $\mathfrak{LC}(\mathtt{T})$. In particular, this implies (see [MO2, Theorem 7]) that $\mathtt{A}$ is standard Koszul in the sense of [ADL]. Hence the algebra $R(\mathtt{A})^!$ is quasi-hereditary. Certainly $R(\mathtt{A})^!$ inherits a grading. Finally, we assume that the induced grading on $R(R(\mathtt{A})^!)$ is positive (which means that $\mathtt{A}$ is *balanced* in the sense of [MO2, Section 6]).

4.2 Bigraded extension algebra of standard modules

Consider the full subcategory of $\mathcal{D}^b(\mathtt{A}-\mathrm{fgmod})$, whose objects are $\Delta(\lambda)\langle i\rangle[j]$, where $\lambda \in \Lambda$, $i, j \in \mathbb{Z}$. The group $\mathbb{Z}^2$ acts freely on this category by shifting the grading and the position in the complex. This induces a canonical $\mathbb{Z}^2$-grading on the (originally *ungraded*) Yoneda Ext-algebra $\mathrm{Ext}^*_A(\Delta)$, see e.g. [DM]. This $\mathbb{Z}^2$-graded algebra has two natural $\mathbb{Z}$-graded subalgebras. The first one the $\mathbb{Z}$-graded algebra $\mathrm{End}^*_A(\Delta)$ of all homomorphisms between graded standard modules obtained in the folowing way: Consider the full subcategory of $\mathcal{D}^b(\mathtt{A}-\mathrm{fgmod})$, whose objects are $\Delta(\lambda)\langle i\rangle$, where $\lambda \in \Lambda$, $i \in \mathbb{Z}$. The group $\mathbb{Z}$ acts freely on this category by shifting the garding. $\mathrm{End}^*_A(\Delta)$ is the $\mathbb{Z}$-graded algebra obtained as the quotient of this action. The second subalgebra is the $\mathbb{Z}$-graded algebra $\mathrm{Lext}^*_A(\Delta)$ of all *linear extensions* defined in the folowing way: Consider the full subcategory of $\mathcal{D}^b(\mathtt{A}-\mathrm{fgmod})$, whose objects are $\Delta(\lambda)\langle i\rangle[-i]$, where $\lambda \in \Lambda$, $i \in \mathbb{Z}$. The group $\mathbb{Z}$ acts freely on this category via $\langle i\rangle[-i]$, $i \in \mathbb{Z}$. $\mathrm{Lext}^*_A(\Delta)$ is the $\mathbb{Z}$-graded algebra obtained as the quotient of this action. Our main general result in this section is the following fairly obvious observation, which, however, will have some interesting applications to the category $\mathcal{O}$.

Proposition 20. *Let* $\mathtt{A}$ *be balanced. Then the Yoneda extension algebras of standard modules for* $\mathtt{A}$ *and* $R(\mathtt{A})^!$ *are canonically isomorphic as* $\mathbb{Z}^2$*-graded* *algebras. This isomorphism induces the following isomorphisms of $\mathbb{Z}$-graded subalgebras:*

$$\begin{aligned}\mathrm{End}^*_A(\Delta) &\cong \mathrm{Lext}^*_{R(\mathtt{A})^!}(\Delta),\\ \mathrm{Lext}^*_A(\Delta) &\cong \mathrm{End}^*_{R(\mathtt{A})^!}(\Delta).\end{aligned}$$

Proof. Since $\mathtt{A}$ is balanced, then both $\mathtt{A}$ and $R(\mathtt{A})$ are quasi-hereditary and Koszul. The Ringel and Koszul dualities induce equivalences between the corresponding bounded derived categories of graded modules. By [MO2, Theorem 9], standard modules for $\mathtt{A}$ and $R(\mathtt{A})^!$ can be identified via these dualities. The first part of the claim follows. The second part follows from the identification of standard modules, given in [MO2, Theorem 9]. $\square$

4.3 Applications to the category $\mathcal{O}$

Proposition 20 can immediately be applied to the graded algebra $\mathtt{A}$ of the principal block of the category $\mathcal{O}$. Namely, in the notation of Section 2 we have.

Theorem 21. *(a) There is a* non-trivial *automorphism of the $\mathbb{Z}^2$-graded algebra $\mathrm{End}^*_A(\Delta)$, which swaps $\mathrm{End}^*_A(\Delta)$ and $\mathrm{Lext}^*_A(\Delta)$. In particular, the $\mathbb{Z}$-graded algebras $\mathrm{End}^*_A(\Delta)$ and $\mathrm{Lext}^*_A(\Delta)$ are isomorphic.*

(b) $\mathrm{Ext}^i_{\mathtt{A}}(\Delta(x),\Delta(y)\langle j\rangle) \cong \mathrm{Ext}^{i+j}_{\mathtt{A}}(\Delta(w_0y^{-1}w_0),\Delta(w_0x^{-1}w_0)\langle -j\rangle)$ for all elements $x,y\in W$.

Proof. $\mathtt{A}$ is both Koszul self-dual ([So1, Theorem 18]) and Ringel self dual ([So2, Corollary 2.3]). Hence the first statement follows directly from Proposition 20. The second statement follows by tracking the correspondence induced by these self-dualities on primitive idempotents and [MOS, Theorem 21(ii)]. $\square$

The latter statement has some interesting corollaries. The first one describes the linear extensions between standard modules:

Corollary 22. *For $x,y\in W$, we have:*

$$\mathrm{Ext}^i_{\mathtt{A}}(\Delta(x),\Delta(y)\langle -i\rangle) \cong \begin{cases}\mathbb{C}, & x\geq y \text{ and } l(x)-l(y)=i;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Theorem 21 reduces the statement to the analogous statement for homomorphisms between Verma modules. We know that the positive grading on $\mathtt{A}$ induces a positive grading on Verma modules. Furthermore, we also know when homomorphisms between Verma modules do exist, and that the homomorphism space between Verma modules is at most one-dimensional (see [Di, Section 7]). Moreover, all Verma modules have the same simple socle. So, to get the explicit formula above one has to compare the lengths of their graded filtrations, which can be done using, for example, [St1, Section 5]. $\square$

Remark 23. From Corollary 22 it follows that the assertion of [MO2, Theorem 6] requires some additional assumptions, for example it is sufficient to make [DM, Assumptions (I) and (II)].

Another corollary is the following result of Carlin (see [Ca, (3.8)]):

Corollary 24. *For $x, y \in W$, $x \geq y$, we have* $\mathrm{Ext}_A^{l(x)-l(y)}(\Delta(x), \Delta(y)) \cong \mathbb{C}$.

Proof. Since A is quasi-hereditary with respect to the Bruhat order on W, the projective modules, occurring at the position $l(y) - l(x)$ in the minimal (linear) projective resolution of $\Delta(x)$, have indexes w such that $l(w) \leq l(y)$. At the same time all simple modules in the radical of $\Delta(y)$ have indexes u such that $l(u) > l(y)$. Hence any non-zero element in the space $\mathrm{Ext}_A^{l(x)-l(y)}(\Delta(x), \Delta(y))$ must belong to $\mathrm{Ext}_{\mathtt{A}}^{l(x)-l(y)}(\Delta(x), \Delta(y)\langle l(y) - l(x)\rangle)$. Now the statement follows from Corollary 22. $\square$

Remark 25. Using the parabolic-singular Koszul duality from [BGS, Ba2] and [MO2, Appendix] one obtains that the extension algebras of standard modules for parabolic and corresponding singular blocks (respectively, pairs of corresponding parabolic-singular blocks) are also isomorphic as bigraded algebras. This isomorphism again swaps the subalgebra of homomorphisms with the subalgebra of linear extensions.

4.4 Several graded subalgebras of the extension algebra of standard modules

We continue to study the $\mathbb{Z}^2$-graded extension algebra $\mathrm{Ext}_A^*(\Delta)$ of the block $\mathcal{O}_0$. From the quasi-heredity of $\mathtt{A}$ we immediately obtain the following vanishing condition: $\mathrm{Ext}_{\mathtt{A}}^i(\Delta, \Delta\langle j\rangle) \neq 0$ implies $i \geq 0$ and $j \geq -i$. It follows that the following induces a natural *positive* $\mathbb{Z}$-grading on $\mathtt{E} := \mathrm{Ext}_A^*(\Delta)$ (in the sense of [MOS, 2.1]):

$$\mathtt{E}_k = \bigoplus_{2i+j=k} \mathrm{Ext}_A^i(\Delta, \Delta\langle j\rangle), \quad k \in \mathbb{Z}.$$

In particular, both $\mathrm{End}_A^*(\Delta)$ and $\mathrm{Lext}_A^*(\Delta)$ become $\mathbb{Z}$-graded subalgebras of $\mathtt{E}$ in the natural way.

Remark 26. The natural $\mathbb{Z}$-grading on $\mathtt{E}$ given by the degree of the extension is *not positive* since the zero component of this grading (the subalgebra of all homomorphisms) is not a semi-simple subalgebra in the general case.

Our first result here is the following Koszulity statement for the subalgebra of all homomorphisms.

Proposition 27. *The algebra* $\mathrm{End}_A^*(\Delta)$ *is Koszul.*

Proof. First I claim that, as a $\mathbb{Z}$-graded algebra, the algebra $\mathrm{End}_A^*(\Delta)$ is isomorphic to the incidence algebra of the poset W with respect to $\leq$. Let us describe $\mathrm{End}_A^*(\Delta)$ via some quiver with relations. For $x, y \in W$, $x \geq y$, we have a unique up to scalar injection $\Delta(x) \hookrightarrow \Delta(y)$. In particular, we can identify each $\Delta(w)$, $w \in W$, with the corresponding submodule of $\Delta(e)$. For each $w \in W$ let v_w denote some generator of $\Delta(w)$, which we fix. If $x, y \in W$, $x \geq y$, let $\varphi_{x,y} : \Delta(x) \to \Delta(y)$ denote the homomorphism, such that $\varphi_{x,y}(v_x) = v_x$. Then, by [Di, Theorem 7.6.23], the arrows in the quiver of $\mathrm{End}_A^*(\Delta)$ are $\varphi_{x,y}$ such that $x = sy$, where s is a reflection (not necessarily simple). From the definition of $\varphi_{x,y}$ we have that these arrows obviously satisfy all relevant commutativity relations. Hence $\mathrm{End}_A^*(\Delta)$ is a quotient of the incidence algebra of the poset $(W, \geq)$. It follows that the two algebras coincide because they obviously have the same dimension.

Now we recall that the Möbius function of the poset $(W, \geq)$ was determined in [Ve]. It equals $(-1)^{l(x)-l(y)}$ for $x \geq y$. Hence, the Koszulity of the corresponding incidence algebra follows from [Yu, Theorem 1]. This completes the proof. $\square$

In [DM] it is shown that in the multiplicity-free cases the $\mathbb{Z}$-graded algebra $\mathtt{E}$ is Koszul with respect to the positive grading introduced above. This and Proposition 27 motivate the following conjecture:

Conjecture 28. *The subalgebra of* $\mathtt{E}$ *generated by* $\mathtt{E}_0$ *and* $\mathtt{E}_1$ *is Koszul.*

Remark 29. I do not know if $\mathtt{E}_0$ and $\mathtt{E}_1$ generate the whole $\mathtt{E}$ in general. I do believe that they do not. Elements from $\mathtt{E}_1$ correspond to "naive" extensions, which do not take into account the multiplicities given by the Kazhdan-Lusztig combinatorics. Additionally, the numerical structure of extensions between Verma modules seems to be really complicated, see [GJ, Ca, Bo].

4.5 Some remarks on extensions between Verma modules

As already mentioned, the description of the algebra $\mathtt{E}$, and even of the dimensions $\dim \mathrm{Ext}^i_{\mathtt{A}}(\Delta(x), \Delta(y)\langle j\rangle)$ seems to be a very complicated problem, see [GJ, Ca, Bo]. A very easy observation reduces this problem to the description of certain properties of the funtor $\mathcal{L}\mathrm{T}_x$:

Proposition 30. *Let $x, y \in W$ and $i, j \in \mathbb{Z}$. Then*

$$\begin{array}{rcl}\dim \mathrm{Ext}^i_{\mathtt{A}}(\Delta(x), \Delta(y)\langle j\rangle) & = & [\mathcal{R}^i\mathrm{G}_{x^{-1}}\Delta(y)\langle j\rangle : L(e)] \\ & = & [\mathcal{L}_i\mathrm{T}_{x^{-1}}\nabla(y)\langle -j\rangle : L(e)].\end{array}$$

Proof. Taking into account that twisting functors are gradable (see [MO2, Appendix] or [FKS, page 28]), we compute:

$$\begin{array}{lcl}\mathrm{Ext}^i_{\mathtt{A}}(\Delta(x), \Delta(y)\langle j\rangle) & = & \\ \mathrm{Hom}_{\mathcal{D}^b(\mathtt{A})}(\Delta(x), \Delta(y)\langle j\rangle[i]) & = & (\text{by [AS, (2.3)]}) \\ \mathrm{Hom}_{\mathcal{D}^b(\mathtt{A})}(\mathrm{T}_x\Delta(e), \Delta(y)\langle j\rangle[i]) & = & (\text{by [AS, Theorem 2.2]}) \\ \mathrm{Hom}_{\mathcal{D}^b(\mathtt{A})}(\mathcal{L}\mathrm{T}_x\Delta(e), \Delta(y)\langle j\rangle[i]) & = & (\text{by [AS, Corollary 4.2]}) \\ \mathrm{Hom}_{\mathcal{D}^b(\mathtt{A})}(\Delta(e), \mathcal{R}\mathrm{G}_{x^{-1}}\Delta(y)\langle j\rangle[i]) & = & (\Delta(e) - \text{projective}) \\ [\mathcal{R}^i\mathrm{G}_{x^{-1}}\Delta(y)\langle j\rangle : L(e)] & = & (\text{by duality}) \\ [\mathcal{L}_i\mathrm{T}_{x^{-1}}\nabla(y)\langle -j\rangle : L(e)]\,. & & \end{array}$$

□

Remark 31. Since both, the twisting and the shuffling functors, are autoequivalences of $\mathcal{D}^b(\mathcal{O}_0)$ (see [AS, Corollary 4.2] and [MS2, Theorem 5.7]), we have

$$\begin{array}{ll}\mathrm{Ext}^i_{\mathtt{A}}(\Delta(sx), \Delta(sy)) = \mathrm{Ext}^i_{\mathtt{A}}(\Delta(x), \Delta(y)) & \text{if} \quad sx > x, sy > y; \\ \mathrm{Ext}^i_{\mathtt{A}}(\Delta(xs), \Delta(ys)) = \mathrm{Ext}^i_{\mathtt{A}}(\Delta(x), \Delta(y)) & \text{if} \quad xs > x, ys > y.\end{array}$$

Since both, the twisting and the shuffling functors, are gradable, the above formula admits a natural graded analogue. In many cases, but not in all, this formula can be applied to reduce extensions to the case of extensions into the projective standard module. In particular, the latter extensions deserve special attention.

Here we would like to present one application of the above technique, which gives (from my point of view) a fairly unexpected description of the Ext^1-space into the projective standard module. For $x \in W$ with a fixed reduced decomposition $x = s_1 \cdots x_k$ we denote by $\overline{l}(x)$ the number of different simple reflections occurring in this reduced decomposition (for example $\overline{l}(sts) = 2$ if s and t do not commute). Since any two reduced decompositions can be obtained from each other by applying braid relations only, it follows that $\overline{l}(x)$ does not depend on the reduced decomposition of x.

Theorem 32.

$$\dim \mathrm{Ext}^1_{\mathtt{A}}(\Delta(x), \Delta(e)\langle j\rangle) = \begin{cases} \bar{l}(x), & \text{if } j = l(x) - 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We start with a special case:

Lemma 33. *The statement of Theorem 32 is true in the case $x = w_0$.*

Proof. Let $\Delta(e) \hookrightarrow X \twoheadrightarrow \Delta(w_0)$ be a non-split extension. Since $\Delta(w_0)$ is simple and $\Delta(e)$ has simple socle $L(w_0)$ it follows that X has simple socle $L(w_0)$. In particular, $X \hookrightarrow P(w_0)$. Since both $\Delta(e)$ and $\Delta(w_0)$ have central characters it follows that X is annihilated by the second power of the corresponding maximal ideal of the center. By [Ba1, Proposition 2.12], this means that X is a submodule of the submodule $Y \subset P(w_0)$, which is uniquely determined via $\Delta(e) \hookrightarrow Y \twoheadrightarrow \bigoplus_{s:l(s)=1} \Delta(s)$. Since each $\Delta(s)$ has simple socle $\Delta(w_0)$ and no other occurrences of $\Delta(w_0)$ in the composition series, we have that X is even a submodule of the submodule Z of Y such that $\Delta(e) \hookrightarrow Z \twoheadrightarrow \Delta(w_0)^{\oplus k}$, where $k = |\{s : l(s) = 1\}|$. Since Z has simple socle, it follows that $\dim \mathrm{Ext}^1_A(\Delta(w_0), \Delta(e))$ equals the number of simple roots, which obviously equals $\bar{l}(w_0)$. Now the necessary statement follows by tracking the grading using [St1] and [MO2, Appendix]. □

Now we go to the general case. Our strategy is: we first establish a lower bound and then prove that it is in fact the real value. As it was already done in Lemma 33, it is easier to prove the ungraded version and then just track the necessary grading using [St1] and [MO2, Appendix].

Set $y = x^{-1}$ and observe that $\bar{l}(x) = \bar{l}(y)$. Now consider the the short exact sequence

$$0 \to \Delta(w_0) \to P(w_0) \to Coker \to 0. \tag{5}$$

Note that $P(w_0)$ is injective. Let α denote the natural transformation from ID to G_y given by [KM, 2.3]. Observe that α is injective on all modules from (5) since they all have Verma flags (this follows, for example, from the dual of [AS, Proposition 5.4]). Further note that α is an isomorphism on both $\Delta(e)$ and $P(w_0)$ because of the projectivity of these two modules (by the dual of [KM, Corollary 9]). Now, applying G_y to (5) yields to the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
\Delta(e) \arrow[r, hook] \arrow[d, "\wr"] & P(w_0) \arrow[r, two heads] \arrow[d, "\wr"] & Coker \arrow[d, hook, "\alpha_{Coker}"] & \\
\mathrm{G}_y\Delta(e) \arrow[r, hook] & \mathrm{G}_yP(w_0) \arrow[r, "f"] & \mathrm{G}_yCoker \arrow[r, two heads] \arrow[d, two heads] & \mathcal{R}^1\mathrm{G}_y\Delta(e) \\
& & X &
\end{tikzcd}$$

From this diagram we have that the heads of the image of both f and α_{Coker} are isomorphic to $L(w_0)$ and that the multiplicity of $L(w_0)$ in both X and $\mathcal{R}^1\mathrm{G}_y\Delta(e)$ is 0. This implies that the kernel of both f and α_{Coker} is the trace of $P(w_0)$ in G_yCoker, in particular, $X = \mathcal{R}^1\mathrm{G}_y\Delta(e)$.

Let now $Y = \oplus_{s:l(s)=1}\Delta(s)$. Then we have the following short exact sequence: $Y \hookrightarrow Coker \twoheadrightarrow Coker'$, where again all modules have Verma flags. Applying G_y and using the Snake Lemma gives the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, hook, "\alpha_Y"] & Coker \arrow[r, two heads] \arrow[d, hook, "\alpha_{Coker}"] & Coker' \arrow[d, hook, "\alpha_{Coker'}"] \\
\mathrm{G}_yM \arrow[r, hook] \arrow[d, two heads] & \mathrm{G}_yCoker \arrow[r] \arrow[d, two heads] & \mathrm{G}_yCoker' \\
Z \arrow[r, hook] & \mathcal{R}^1\mathrm{G}_y\Delta(e) &
\end{tikzcd} \tag{6}$$

Let S_1 denote the set of all simple roots which appear in a reduced expression of y, and let S_2 denote the set of all other simple roots. From the dual of [AS, Theorem 2.3] we get

$$Z \cong \Delta(e)^{\oplus|S_1|} \oplus \bigoplus_{s\in S_2} \Delta(s).$$

In particular, we obtain that $[Z : L(e)] = |S_1|$ and hence $[\mathcal{R}^1\mathrm{G}_y\Delta(e) : L(e)] \geq |S_1|$ because of the third row of (6). This is our lower bound.

Now to prove that this lower bound gives the exact value, we write $\mathrm{G}_{w_0} = \mathrm{G}_z\mathrm{G}_y$, where $z = w_0x$ and note that the natural transformation from ID to G_{w_0} can be obviously written as the composition of the natural transformation from ID to G_y with the natural transformation from ID to G_z, the latter being restricted to the image of G_y. This implies that the diagram (6) can be extended to the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
Y \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, hook, bend left] & & Coker \arrow[rr, two heads] \arrow[dd, hook] \arrow[dr, hook, bend left] & & Coker' \arrow[dd, hook] \arrow[dr, hook, bend left] & \\
& \mathrm{G}_{w_0}Y \arrow[rr, hook] \arrow[dd, two heads, bend left] & & \mathrm{G}_{w_0}Coker \arrow[rr] \arrow[dd, two heads, bend left] & & \mathrm{G}_{w_0}Coker' \arrow[dd, two heads, bend left] \\
\mathrm{G}_yY \arrow[rr, hook] \arrow[dd, two heads] \arrow[ur, hook] & & \mathrm{G}_yCoker \arrow[rr] \arrow[dd, two heads] \arrow[ur, hook] & & \mathrm{G}_yCoker' \arrow[dd, two heads] \arrow[ur, hook] & \\
& Z' \arrow[rr, hook] & & \mathcal{R}^1\mathrm{G}_{w_0}\Delta(e) \arrow[rr] & & M' \\
Z \arrow[rr, hook] \arrow[ur, hook] & & \mathcal{R}^1\mathrm{G}_y\Delta(e) \arrow[rr] \arrow[ur, hook] & & M \arrow[ur, hook] &
\end{tikzcd}
$$

Assume now that there is an extra occurrence of $L(e)$ in $\mathcal{R}^1\mathrm{G}_y\Delta(e)$. This occurrence gives us a homomorphism from $\Delta(e)$ to $\mathcal{R}^1\mathrm{G}_y\Delta(e)$, which induces a non-zero homomorphism from $\Delta(e)$ to M. Since M embeds into M' and the diagram commutes, our homomorphism defines a homomorphism from $\Delta(e)$ to $\mathcal{R}^1\mathrm{G}_{w_0}\Delta(e)$, which induces a non-zero homomorphism from $\Delta(e)$ to M'. On the other hand we know that $[\mathcal{R}^1\mathrm{G}_{w_0}\Delta(e) : L(e)] = \bar{l}(w_0)$ by Lemma 33. From the previous paragraph we also know that $[Z' : L(e)] = \bar{l}(w_0)$. This gives us a contradiction and completes the proof for the ungraded case. As we have mentioned above, the graded version follows easily just tracking the grading. $\square$

Remark 34. Combined with Theorem 21(b), Theorem 32 gives information about some higher Ext-spaces.

5 Modules with linear resolutions

The category $\mathcal{LC}(\mathtt{P})$ realizes the category of graded modules over the Koszul dual of $\mathtt{A}$ (which is isomorphic to $\mathtt{A}$ by [So1, Theorem 18]). Verma modules over $\mathtt{A}$ have linear projective resolutions. These resolutions, in turn, are costandard objects in the category $\mathcal{LC}(\mathtt{P})$. In other words, this means that costandard modules are Koszul dual to standard modules (but not vice versa). Analogously, since the Ringel dual of $\mathtt{A}$ is isomorphic to $\mathtt{A}$ as well by [So2, Corollary 2.3], costandard modules are also Ringel dual to standard modules (but not vice versa).

The category $\mathcal{LC}(\mathtt{T})$ realizes the category of graded modules over the Ringel dual of the Koszul dual of $\mathtt{A}$ (which is isomorphic to $\mathtt{A}$ by above). Since the algebra $\mathtt{A}$ is standard Koszul (see [ADL, Section 3]), standard $\mathtt{A}$-modules admit linear tilting coresolutions and costandard $\mathtt{A}$-modules admit linear tilting resolutions, see [MO2, Theorem 7]. In an analogy to the previous paragraph, from this one obtains that both standard and costandard modules are Koszul-Ringel self-dual. From [MO2, Theorem 9] it also follows that simple and tilting $\mathtt{A}$-modules are Koszul-Ringel dual to each other (now in the symmetric way). A natural question then is: *Which other classes of modules can be represented by linear complexes of tilting modules?* (Such modules then in some sense “live” in the category $\mathcal{LC}(\mathtt{T})$). In this section we present several classes of such modules. In particular, quite surprizingly it turns our that all shuffled Verma modules have the above property. In what follows we will use the term *tilting linearizable modules* for those modules, which are isomorphic to some linear complexes of tilting modules in $\mathcal{D}^b(\mathtt{A}-\text{fgmod})$.

5.1 Shuffled Verma modules

To start with we have to define graded lifts of shuffled Verma modules. Let $\mathrm{T}_w : \mathtt{A}-\text{gmod} \to \mathtt{A}-\text{gmod}$ be the graded lift of T_w, see [MO2, Appendix] or [FKS, page 28]. We define the graded lifts of shuffled Verma modules as follows:

$$\Delta(x,y) = \mathrm{T}_x\Delta(y).$$

Theorem 35. *For every $x, y \in W$ the module $\Delta(x,y)$ is tilting linearizable.*

Remark 36. The motivation for this statement is a compilation of several results. [MO2, Theorem 9] and [MO2, Corollary 14] say that in the category $\mathcal{LC}(\mathtt{T}) \cong \mathtt{A}-\text{gmod}$ (which is a kind of “Koszul-Ringel dual” to $\mathtt{A}-\text{gmod}$) standard and costandard $\mathtt{A}$-modules remain standard and costandard respectively, and simple and tilting modules interchange. According to [AL], shuffled Verma modules can be equivalently described using twisting and shuffling functors, the latter being Koszul dual to each other by [MOS, 6.5]. So it becomes natural to ask whether the set of shuffled Verma modules might be “Koszul-Ringel self-dual”. The proof of Theorem 35, presented below, shows that this is indeed the case. Observe that it is very easy to see on examples that this class is neither “Ringel self-dual” nor “Koszul self-dual” in general.

Proof. The idea of the proof of Theorem 35 is to compile the results mentioned in Remark 36. The problem is to extend the “Koszul duality” of shuffling and twisting functors from [MOS, 6.5] to the “Koszul-Ringel duality” of these functors. For this we will need some notation.

Let $\mathrm{K} : \mathcal{D}^b(\mathtt{A}-\mathrm{gmod}) \to \mathcal{D}^b(\mathfrak{L}\mathfrak{C}(\mathtt{P}))$ denote the Koszul duality functor from [MOS, 5.4] (restricted to bounded complexes). Essentially this functor is given by taking the inner Hom-functor with a direct sum of all indecomposable projective objects from $\mathcal{D}^b(\mathfrak{L}\mathfrak{C}(\mathtt{P}))$.

By [AS, Theorem 2.2] and [So2, Theorem 6.6] for the functor T_{w_0} we have that $\mathrm{T}_{w_0} : \mathcal{D}^b(\mathfrak{L}\mathfrak{C}(\mathtt{P})) \to \mathcal{D}^b(\mathfrak{L}\mathfrak{C}(\mathtt{T}))$ is an equivalence, which sends indecomposable projective objects from $\mathfrak{L}\mathfrak{C}(\mathtt{P})$ to the corresponding indecomposable projective objects from $\mathfrak{L}\mathfrak{C}(\mathtt{T})$. This allows us to define the Koszul-Ringel duality functor $\overline{\mathrm{K}} : \mathcal{D}^b(\mathtt{A}-\mathrm{gmod}) \to \mathcal{D}^b(\mathfrak{L}\mathfrak{C}(\mathtt{T}))$ as follows: $\overline{\mathrm{K}} = \mathcal{L}\mathrm{T}_{w_0}\mathrm{K}$.

By [MOS, 6.4], translation and Zuckerman functors on $\mathtt{A}-\mathrm{gmod}$ and $\mathfrak{L}\mathfrak{C}(\mathtt{P})$ respectively are Koszul dual to each other with respect to the Koszul duality K. Since $\mathcal{L}\mathrm{T}_{w_0}$ commutes with translation functors by [AS, Theorem 3.2], it follows that translation and Zuckerman functors on $\mathtt{A}-\mathrm{gmod}$ and $\mathfrak{L}\mathfrak{C}(\mathtt{T})$ respectively are Koszul-Ringel dual to each other with respect to the Koszul-Ringel duality $\overline{\mathrm{K}}$. Now, repeating the arguments from the proof of [MOS, Theorem 39] one shows that twisting and shuffling functors on $\mathtt{A}-\mathrm{gmod}$ and $\mathfrak{L}\mathfrak{C}(\mathtt{T})$ respectively are Koszul-Ringel dual to each other with respect to the Koszul-Ringel duality $\overline{\mathrm{K}}$. This means that for any $w \in W$ we have

$$\mathcal{L}\mathrm{T}_w \cong \overline{\mathrm{K}}^{-1}\,\mathcal{L}\overline{\mathrm{C}}_{w^{-1}}\,\overline{\mathrm{K}}, \tag{7}$$

where $\overline{\mathrm{C}}_{w^{-1}}$ denotes the corresponding shuffling functor (see [Ir3] and [MS2, 5.1]).

The rest is now easy. Verma modules in $\mathtt{A}-\mathrm{gmod}$ and $\mathfrak{L}\mathfrak{C}(\mathtt{T})$ correspond via $\overline{\mathrm{K}}$ by [MO2, Theorem 9]. Verma modules are acyclic for twisting functors by [AS, Theorem 2.2] and for shuffling functors by [MS2, Proposition 5.3]. Hence from (7) for $x, y \in W$ we have

$$\Delta(x,y) = \mathrm{T}_x\Delta(y) = \overline{\mathrm{K}}^{-1}\,\overline{\mathrm{C}}_{w^{-1}}\,\overline{\mathrm{K}}\Delta(y).$$

Now, since the functor $\overline{\mathrm{C}}_{w^{-1}}$ is defined already on $\mathfrak{L}\mathfrak{C}(\mathtt{T})$, it follows that its value on $\overline{\mathrm{K}}^{-1}\Delta(y) \in \mathfrak{L}\mathfrak{C}(\mathtt{T})$ is again an object from $\mathfrak{L}\mathfrak{C}(\mathtt{T})$. The necessary claim follows. $\square$

5.2 Standard modules in $\mathcal{O}_0^{\mathfrak{p}}$

Let now $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be a parabolic subalgebra and $W^{\mathfrak{p}}$ the corresponding parabolic subgroup of W. Let $\mathcal{O}_0^{\mathfrak{p}}$ denote the full subcategory of $\mathcal{O}_0$, consisting of $U(\mathfrak{p})$-locally finite modules. Then simple objects of $\mathcal{O}_0$ have the form $L(w)$, where w is the shortest representative in a coset from $W^{\mathfrak{p}}\backslash W$. We will denote the set of such representatives by $W(\mathfrak{p})$. Let $A^{\mathfrak{p}}$ denote the quotient of A such that $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to the category of $A^{\mathfrak{p}}$-modules. Then $A^{\mathfrak{p}}$ is quasi-hereditary ([RC]) and inherits a positive grading $\mathtt{A}^{\mathfrak{p}}$ for $\mathtt{A}$, with respect to which it is standard Koszul ([BGS, ADL]). To indicate object of $A^{\mathfrak{p}}$ we will add the superscript $\mathfrak{p}$ to the standard notation. As $\mathtt{A}^{\mathfrak{p}}$ is standard Koszul, the standard modules $\mathtt{\Delta}^{\mathfrak{p}}(w)$, $w \in W(\mathfrak{p})$, have linear projective resolutions over $\mathtt{A}^{\mathfrak{p}}$. They also have linear tilting coresolutions over $\mathtt{A}^{\mathfrak{p}}$. Surprizingly enough, these properties are preserved if one makes the step from $\mathtt{A}^{\mathfrak{p}}$ to $\mathtt{A}$.

Proposition 37. *Let $w \in W(\mathfrak{p})$. Then, considered as an $\mathtt{A}$-module, the module $\mathtt{\Delta}^{\mathfrak{p}}(w)$ has a linear projective resolution and is tilting linearizable.*

Proof. The module $\Delta^{\mathfrak{p}}(w)$ is obtained via parabolic induction (from $\mathfrak{p}$ to $\mathfrak{g}$) from a simple finite-dimensional $\mathfrak{p}$-module. This simple finite-dimensional $\mathfrak{p}$-module has a BGG-resolution (over the Levi factor of $\mathfrak{p}$), which is obviously linear. The parabolic induction then maps this BGG-resolution to a linear resolution of $\Delta^{\mathfrak{p}}(w)$ by standard modules over $\mathtt{A}$. Each standard $\mathtt{A}$-module has a linear projective resolution and a linear tilting coresolution. These resolutions can be glued in the standard way to obtain linear projective resolution of $\Delta^{\mathfrak{p}}(w)$ and a linear complex of tilting modules isomorphic to $\Delta^{\mathfrak{p}}(w)$ respectively. □

Remark 38. I do not see any immediate connection between the linear projective resolutions of $\Delta^{\mathfrak{p}}(w)$ as $\mathtt{A}^{\mathfrak{p}}$- and $\mathtt{A}$-modules.

Remark 39. Applying T_{w_0} to the Verma resolution of $\Delta^{\mathfrak{p}}(e)$ constructed in the proof of Proposition 37 one obtains that $\mathcal{L}\mathrm{T}_{w_0}\Delta^{\mathfrak{p}}(e) \cong L(w_0^{\mathfrak{p}}w_0)[l(w_0^{\mathfrak{p}})]$. This allows one to compute the images of the simple modules $L(w_0^{\mathfrak{p}}w_0)$ under the (derived) Ringel duality functor $\mathrm{Hom}_A(T, {}_-)$. It is not clear how to compute these images for other $L(x)$. This question reduces to understanding the homology of the tilting objects in $\mathcal{LC}(\mathtt{P})$ or of the projective objects in $\mathcal{LC}(\mathtt{T})$.

Remark 40. Dually, costandard modules in a regular parabolic block admit a linear injective coresolution, when viewed as modules in the regular block of $\mathcal{O}$. Moreover, they are also tilting linearizable.

5.3 Projective modules in $\mathcal{O}_0^{\mathfrak{p}}$

Proposition 41. *Let $w \in W(\mathfrak{p})$. Then, considered as an $\mathtt{A}$-module, the module $\mathtt{P}^{\mathfrak{p}}(w)$ has a linear projective resolution.*

Proof. The module $\mathtt{P}^{\mathfrak{p}}(w)$ is obtained from $\mathtt{P}(w)$ by applying the $\mathfrak{p}$-Zuckerman functor. Analogously to [MOS, 6.4] one shows that the $\mathfrak{p}$-Zuckerman functor is Koszul dual to the translation functor through the $W^{\mathfrak{p}}$-wall. The latter functor preserves $\mathcal{LC}(\mathtt{P})$. Hence, translating the simple object $\mathtt{P}(w)$ of $\mathcal{LC}(\mathtt{P})$ through the $W^{\mathfrak{p}}$-wall we will get a linear complex of projective modules, which has only one non-zero homology, namely the one in the position 0, which is, moreover, isomorphic to $\mathtt{P}^{\mathfrak{p}}(w)$. The statement is proved. □

Remark 42. Dually, injective modules in a regular parabolic block of $\mathcal{O}$ admit linear injective coresolutions when viewed as modules in $\mathcal{O}$.

5.4 Tilting modules in $\mathcal{O}_0^{\mathfrak{p}}$

Proposition 43. *Let $w \in W(\mathfrak{p})$. Then, considered as an $\mathtt{A}$-module, the module $\mathtt{T}^{\mathfrak{p}}(w)$ is tilting linearizable.*

Proof. Apply $\mathcal{L}\mathrm{T}_{w_0}$ to the linear projective resolution of $\mathtt{P}^{\mathfrak{p}}(x)$, $x \in W(\mathfrak{p})$, constructed in Proposition 41, and follow the arguments of [MS1, Proposition 4.4]. □

Remark 44. From Propositions 41 and 43 it follows that projective tilting modules in $\mathcal{O}_0^{\mathfrak{p}}$ both admit a linear projective resolution in $\mathcal{O}$ and are tilting linearizable. However, one has to note that a module in $\mathcal{O}_0^{\mathfrak{p}}$, which is at the same time projective and tilting, has in the general case different graded lifts as a projective and as a tilting module.

5.5 Some other classes of modules

There are some other classes of modules, which are known to have linear projective resolutions (respectively, which are tilting linearizable). In [Ma, Proposition 4.1] it is shown that modules, obtained by translating standard modules in singular blocks out of the wall, admit linear projective resolutions. It is not difficult to show that they are also tilting linearizable. In [Ma, Theorem 8.1 and Corollary 8.1] it is shown that one more class of modules (the "wrong-sided" analogue of modules, obtained by translating standard modules in singular blocks out of the wall) admits both a linear projective resolution and a linear tilting coresolution.

The algebra $\mathtt{A}$ is an $\mathtt{A}$-$\mathtt{A}$ bimodule and thus can be considered as an object of the category $\mathcal{O}_0$ for the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ (this realization was used, in particular, in [Ba1]). The hereditary chain of the quasi-hereditary algebra $\mathtt{A}$ is, by definition, a bimodule Verma flag for $\mathtt{A}$. From the natural grading on $\mathtt{A}$ we get that the heads of all the Vermas occurring in this flag are concentrated in degree 0. Hence, we can glue linear projective resolutions (or linear tilting coresolutions) of these Verma modules in the standard way to obtain a linear projective resolution (resp. a linear tilting coresolutions) of the bimodule $\mathtt{A}$. As a corollary one immediately obtains a formula for computing Hochschild cohomology of $\mathtt{A}$ with coefficients in semi-simple modules.

Acknowledgments The research was partially supported by The Royal Swedish Academy of Sciences, The Swedish Research Council and The Swedish Foundation for International Cooperation in Research and Higher Education (STINT). An essential part of this paper was written during the visit of the author to MPIM at Bonn. The financial support and hospitality of MPIM is gratefully acknowledged. I would like to thank Catharina Stroppel and Olexandr Khomenko for several very stimulating discussions and comments to the preliminary draft of the paper, and to Jian-yi Shi for the information about two-sided cells he provided.

References

[ADL] *I. Ágoston, V. Dlab, E. Lukács*, Quasi-hereditary extension algebras. Algebr. Represent. Theory **6** (2003), no. 1, 97–117.

[AL] *H. Andersen, N. Lauritzen*, Twisted Verma modules. Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), 1–26, Progr. Math., **210**, Birkhäuser Boston, Boston, MA, 2003.

[AS] *H. Andersen, C. Stroppel*, Twisting functors on $\mathcal{O}$. Represent. Theory **7** (2003), 681-699.

[Ar] *S. Arkhipov*, Algebraic construction of contragradient quasi-Verma modules in positive characteristic. in: Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math. **40**, Math. Soc. Japan, Tokyo, 2004, pp. 27-68.

[Ba1] *E. Backelin*, The Hom-spaces between projective functors. Represent. Theory **5** (2001), 267-283.

[Ba2] *E. Backelin*, Koszul duality for parabolic and singular category $\mathcal{O}$. Represent. Theory **3** (1999), 139-152.

[BGS] *A. Beilinson, V. Ginzburg, W. Soergel*, Koszul duality patterns in representation theory. J. Amer. Math. Soc. **9** (1996), no. 2, 473-527.

[BG] *I. Bernstein, S. Gelfand*, Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245-285.

[BGG1] *I. Bernstein, I. Gelfand, S. Gelfand*, A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Prilozen. **10** (1976), no. 2, 1-8.

[BGG2] *I. Bernstein, I. Gelfand, S. Gelfand*, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. in: Lie Groups Represent., Proc. Summer Sch. Bolyai Janos Math. Soc., Budapest 1971 (1975), 21-64.

[BB] *A. Björner, F. Brenti*, Combinatorics of Coxeter groups. Graduate Texts in Mathematics, **231**. Springer, New York, 2005.

[Bo] *B. Boe*, A counterexample to the Gabber-Joseph conjecture. Contemp. Math. **139** (1992), 1-3.

[Ca] *K. Carlin*, Extensions of Verma modules. Trans. Amer. Math. Soc. **294** (1986), no. 1, 29-43.

[CPS] *E. Cline, B. Parshall, L. Scott*, Finite dimensional algebras and highest weight categories. J. Reine Angew. Math. **391** (1988), 85-99.

[CI] *D. Collingwood, R. Irving*, A decomposition theorem for certain self-dual modules in the category $\mathcal{O}$. Duke Math. J. **58** (1989), no. 1, 89-102.

[Di] *J. Dixmier*, Enveloping algebras. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, **11**, American Mathematical Society, Providence, RI, 1996.

[DM] *Yu. Drozd, V. Mazorchuk*, Koszul duality for extension algebras of standard modules, math.RT/0411528

[FKS] *I. Frenkel, M. Khovanov, C. Stroppel*, A categorification of finite-dimensional irreducible representations of quantum $sl(2)$ and their tensor products, math.QA/0511467

[GJ] *O. Gabber, A. Joseph*, Towards the Kazhdan-Lusztig conjecture. Ann. Sci. Éc. Norm. Supér., IV. Sér. **14** (1981), 261-302.

[Ha] *D. Happel*, Triangulated categories in the representation theory of finite dimensional algebras. London Mathematical Society Lecture Note Series, 119. Cambridge etc.: Cambridge University Press. 1988.

[Ir1] *R. Irving*, Projective modules in the category $\mathcal{O}_S$: Loewy series. Trans. Amer. Math. Soc. **291** (1985), no. 2, 733–754.

[Ir2] *R. Irving*, A filtered category $\mathcal{O}_S$ and applications. Mem. Amer. Math. Soc. **83** (1990).

[Ir3] *R. Irving*, Shuffled Verma modules and principal series modules over complex semisimple Lie algebras. J. London Math. Soc. (2) **48** (1993), no. 2, 263–277.

[IS] *R. Irving, B. Shelton*, Loewy series and simple projective modules in the category $\mathcal{O}_S$. Pacific J. Math. **132** (1988), no. 2, 319-342.

[Jo] *A. Joseph*, The Enright functor on the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Invent. Math. **67** (1982), no. 3, 423-445.

[KL] *D. Kazhdan, G. Lusztig*, Representations of Coxeter groups and Hecke algebras. Invent. Math. **53** (1979), no. 2, 165-184.

[KM] *O. Khomenko, V. Mazorchuk*, On Arkhipov's and Enright's functors. Math. Z. **249** (2005), no. 2, 357-386.

[Lu1] *G. Lusztig*, Cells in affine Weyl groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 255-287, Adv. Stud. Pure Math., **6**, North-Holland, Amsterdam, 1985.

[Lu2] *G. Lusztig*, Cells in affine Weyl groups. II. J. Algebra **109** (1987), no. 2, 536-548.

[Ma] *V. Mazorchuk*, Applications of the category of linear complexes of tilting modules associated with the category $\mathcal{O}$, math.RT/0501220

[MO1] *V. Mazorchuk, S. Ovsienko*, Finitistic dimension of properly stratified algebras. Adv. Math. **186** (2004), no. 1, 251-265.

[MO2] *V. Mazorchuk, S. Ovsienko*, A pairing in homology and the category of linear tilting complexes for a quasi-hereditary algebra, with an appendix by *C. Stroppel*, J. Math. Kyoto Univ. **45** (2005), 711-741.

[MOS] *V. Mazorchuk, S. Ovsienko, C. Stroppel*, Quadratic duals, Koszul dual functors, and applications, math.RT/0603475.

[MS1] *V. Mazorchuk, C. Stroppel*, Projective-injective modules, Serre functors and symmetric algebras, math.RT/0508119.

[MS2] *V. Mazorchuk, C. Stroppel*, Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. Trans. Amer. Math. Soc. **357** (2005), no. 7, 2939-2973.

[Ri] *C. Ringel*, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. Math. Z. **208** (1991), No.2, 209-224.

[RC] *A. Rocha-Caridi*, Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module. Trans. Amer. Math. Soc. **262** (1980), no. 2, 335-366.

[Sa] *B. Sagan*, The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, 2001.

[So1] *W. Soergel*, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3** (1990), no. 2, 421-445.

[So2] *W. Soergel*, Character formulas for tilting modules over Kac-Moody algebras. Represent. Theory **2** (1998), 432-448.

[St1] *C. Stroppel*, Category $\mathcal{O}$: gradings and translation functors, J. Algebra **268**, no. 1, 301-326.

[St2] *C. Stroppel*, Homomorphisms and extensions of principal series representations. J. Lie Theory **13** (2003), no. 1, 193-212.

[Ve] *D. Verma*, Möbius inversion for the Bruhat ordering on a Weyl group. Ann. Sci. École Norm. Sup. (4) **4** (1971), 393-398.

[Yu] *S. Yuzvinsky*, Linear representations of posets, their cohomology and a bilinear form. European J. Combin. **2** (1981), no. 4, 385-397.

[Zh] *B. Zhu*, On characteristic modules of graded quasi-hereditary algebras. Comm. Algebra **32** (2004), no. 8, 2919-2928.

Department of Mathematics, Uppsala University, SE-751 06, Uppsala, SWEDEN, e-mail: `mazor@math.uu.se`, web: http://www.math.uu.se/˜mazor/