Relations between the correlators

of the topological sigma–model coupled to gravity

M. Kontsevich, Yu. Manin

Abstract

We prove a new recursive relation between the correlators $\langle\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{g,\beta}$, which together with known relations allows one to express all of them through the full system of Gromov–Witten invariants in the sense of Kontsevich–Manin and the intersection indices of tautological classes on $\overline{M}_{g,n}$, effectively calculable in view of earlier results due to Mumford, Kontsevich, Getzler, and Faber. This relation shows that a linear change of coordinates of the big phase space transforms the potential with gravitational descendants to another function defined completely in terms of the Gromov–Witten correspondence and the intersection theory on $V^n\times\overline{M}_{g,n}$. We then extend the formalism of gravitational descendants from quantum cohomology to more general Frobenius manifolds and Cohomological Field Theories.

0. Introduction.

This note furnishes a list of relations between the correlators

$$\langle\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{g,\beta}$$

where $\gamma_i\in H^*(V)$, V a smooth projective manifold, which allows one to express all of them through the following data:

(i) *The (full) quantum cohomology of* V in the sense of [KM], consisting of the maps $I^V_{g,n,\beta}\,:\,H^*(V^n)\to H^*(\overline{M}_{g,n})$. Notice that quantum cohomology directly contains all the stable range correlators with $d_i=0$ (cap $I^V_{g,n}$ with the fundamental class of the moduli space), but generally for $g>0$ cannot be reconstructed from these correlators only.

(ii) *The intersection indices of tautological classes on* $\overline{M}_{g,n}$, effectively calculable in view of the known results of Mumford, Kontsevich, Getzler, and Faber (cf. [F]).

Our main trick consists in introducing the generalized correlators which we denote $\langle\tau_{d_1,e_i}\gamma_1\dots\tau_{d_n,e_n}\gamma_n\rangle_{g,\beta}$ and in deriving for them a general recursion relation. This is the content of the Theorem 1.2 which is the central result of §1. In the remaining part of §1 we collect some further (and well known) recursion formulas for the reader's convenience: cf. [W1], [W2], [DijW], [DijVV], [EHX1], [EHX2]. Taken together, they provide transparent computation algorithms.

In §2 we apply these formulas to the comparison of two generating functions: the standard one involving the gravitational descendants ($e_i=0$), and the new one, involving the modified gravitational descendants which in the notation above correspond to $d_i=0$. We prove that the two functions are related by an invertible linear transformation T of the big phase space, common for all genera, and defined entirely in terms of two–point genus zero correlators with descendants at one point. A further calculation shows that T is also entirely determined by the three–point primary correlators. This might shed some light to the problem of Virasoro constraints, cf. [EHX1], [EHX2]. In fact, any system of differential equations for any generating function of the correlators with descendants equally well serves for the modified correlators after the coordinate change defined by T. On the other hand, the transition to the modified correlators almost decouples the a–indices from the d–indices: cf. formula (27) below.

Finally, in §3 we extend the new formalism of the gravitational descendants from quantum cohomology to the more general Frobenius manifolds and Cohomological Field Theories.

§1. Generalized correlators

1.1. The setting. The mathematical definition of the conventional correlators in the notation of [BM] is

$$\langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n\rangle_{g,\beta} :=$$

$$\int_{J_{g,n}(V,\beta)} c_1(L_{1;g,n}(V,\beta))^{d_1} \cup ev_1^*(\gamma_1) \cup \dots \cup c_1(L_{n;g,n}(V,\beta))^{d_n} \cup ev_n^*(\gamma_n). \qquad (1)$$

where $J_{g,n}(V,\beta) \in A_*(\overline{M}_{g,n}(V,\beta))$ is the virtual fundamental class, the line bundle $L_i = L_{i;g,n}(V,\beta)$, $i = 1,\dots,n$ has the geometric fiber $T^*_{x_i}C$ at the point $[(C, x_1, \dots, x_n, f : C \to V)]$, and ev_i sends this point to $f(x_i)$. Recall also that β varies in the semigroup of the effective algebraic classes of $H_2(V,\mathbf{Z})/(tors)$.

Put $\psi_i := c_1(L_{i;g,n}(V,\beta))$.

In the stable range $2g-2-n > 0$ we have the absolute stabilization map $st : \overline{M}_{g,n}(V,\beta) \to \overline{M}_{g,n}$, and the respective bundles L_i on $\overline{M}_{g,n}$. Put $\phi_i := st^*(c_1(L_i))$.

Our generalized correlators, by definition, are:

$$\langle \tau_{d_1,e_1}\gamma_1 \dots \tau_{d_n,e_n}\gamma_n\rangle_{g,\beta} :=$$

$$\int_{J_{g,n}(V,\beta)} \psi_1^{d_1}\phi_1^{e_1} \cup ev_1^*(\gamma_1) \cup \dots \cup \psi_n^{d_n}\phi_n^{e_n} \cup ev_n^*(\gamma_n). \qquad (2)$$

Since $\overline{M}_{0,2}(V,0) = \emptyset$, we have

$$\langle \tau_{d_1}\gamma_1\, \tau_{d_2}\gamma_2\rangle_{0,0} = 0. \qquad (3)$$

Furthermore, in the stable range we have

$$\langle \prod_{i=1}^n \tau_{d_i,0}\gamma_i\rangle_{g,\beta} = \langle \prod_{i=1}^n \tau_{d_i}\gamma_i\rangle_{g,\beta}$$

1.2. Theorem. *If $2g-2+n > 0$, then for any j with $d_j \geq 1$ we have*

$$\langle \prod_{i=1}^n \tau_{d_i,e_i}\gamma_i\rangle_{g,\beta} = \langle \prod_{i=1}^n \tau_{d_i-\delta_{ij},e_i+\delta_{ij}}\gamma_i\rangle_{g,\beta}$$

$$+ \sum_{a,\,\beta_1+\beta_2=\beta} \pm \langle \tau_{d_j-1}\gamma_j\, \tau_0\Delta^a\rangle_{0,\beta_1} \langle \tau_{0,e_j}\Delta_a \prod_{i:\,i\neq j} \tau_{d_i,e_i}\gamma_i\rangle_{g,\beta_2}. \qquad (4)$$

Here (Δ_a), (Δ^a) are Poincaré dual bases of $H^*(V)$, and the sign arises from permuting γ_j with γ_i for all $i < j$.

1.3. Corollary. *For $g=0$, $n=3$, $d_1\ge 1$ we have:*

$$\langle \tau_{d_1}\gamma_1\tau_{d_2}\gamma_2\tau_{d_3}\gamma_3\rangle_{0,\beta} = \sum_{a,\,\beta_1+\beta_2=\beta} \langle \tau_{d_1-1}\gamma_1\,\tau_0\Delta^a\rangle_{0,\beta_1}\langle\tau_0\Delta_a\,\tau_{d_2}\gamma_2\,\tau_{d_3}\gamma_3\rangle_{0,\beta_2}. \quad (4a)$$

In fact, $\phi_i=0$ here, so one should put $e_i=0$ in (4), and the first summand will vanish.

This is a well known identity.

Sketch of proof. Consider the morphism of universal curves $\widetilde{st}: C_{g,n}(V,\beta)\to C_{g,n}$ covering st. It induces the morphism of relative 1–forms sheaves $\omega\to\omega(V,\beta)$, at least at the complement of singular points of the fiber. Restricting the latter to the j–th section ($j\in S$ being fixed), we get the morphism $st^*(L_{j;g,n})\to L_{j;g,n}(V,\beta)$ on $\overline{M}_{g,n}(V,\beta)$. It is a local isomorphism everywhere except for the points in this stack over which the j–th section lies on the component of fiber which gets contracted by $\widetilde{st}$. These points constitute the union of boundary strata $\overline{M}(V,\sigma(\beta_1,\beta_2))$ where $\sigma(\beta_1,\beta_2)$ is one–edge, two–vertex n–graph with one vertex of genus 0, class β_1, with tail j, and another of genus g, class β_2, with tails $\ne j$. Naively, one would expect that all these boundaries are divisors, and over them sections of $st^*(L_{j;g,n})$ have an extra zero of the first order. Hence in (2) we could replace one factor ψ_j by $\phi_j+\sum_{\beta_1+\beta_2=\beta}[\overline{M}(V,\sigma(\beta_1,\beta_2))]$. Then the restriction to the boundary would give (4). A more precise reasoning uses the pullback property of the virtual fundamental classes $J(V,\sigma)$ similar to Lemma 10 of [B]. The details will treated in [M2].

Clearly, these relations allow us to reduce all the generalized (in particular, the conventional ones) correlators to those with $\beta=0$, to the conventional ones in the unstable range and to the generalized ones with all $d_i=0$ in the stable range. Using (2) and the projection formula, one can rewrite the latter in the form

$$\langle \tau_{0,e_1}\gamma_1\dots\tau_{0,e_n}\gamma_n\rangle_{g,\beta} :=$$

$$\int_{I_{g,n}(V,\beta)} c_1(pr_2^*(L_1))^{e_1}\cup pr_1^*(\gamma_1)\cup\dots\cup c_1(pr_2^*(L_n))^{e_n}\cup pr_1^*(\gamma_n).$$

where this time the integration refers to $V^n\times\overline{M}_{g,n}$, $I=(ev,st)_*J$ is the Gromov–Witten correspondence, and pr_i are the two projections. Hence the correlators in the stable range with $d_i=0$ are calculable if we know the full (not just top) Gromov–Witten invariants. We will call the expressions above *the modified correlators.*

Notice that for $\beta=0$ we have $\psi_i=\phi_i$, hence $\tau_{d,e}=\tau_{d+e}$, so that (4) gives no new information and is tautologically true because of (3). So we will remind what happens in the case $\beta=0$, $\dim V>0$ separately.

1.4. The mapping to a point case. Recall that $\overline{M}_{g,n}(V,0)$ is canonically isomorphic to $\overline{M}_{g,n}\times V$, and with this identification,

$$[\overline{M}_{g,n}(V,0)]^{virt} = J_{g,n}(V,0) = c_G(\mathcal{E}\boxtimes\mathcal{T}_V)\cap[\overline{M}_{g,n}\times V], \quad (5)$$

where $\mathcal{E}=R^1\pi_*\mathcal{O}_C$, $\pi: C\to\overline{M}_{g,n}$ is the universal curve, and $G=g\dim V$. Consider the Chern classes and Chern roots of $\mathcal{E}$ and $\mathcal{T}_V$:

$$c_t(\mathcal{E}) = \prod_{i=1}^{g}(1+a_it) = \sum_{i=0}^{g}(-1)^i\lambda_{i;g,n}t^i,$$

where λ_i are Mumford's tautological classes defined as Chern classes of $\pi_*(\omega_\pi)$,

$$c_t(\mathcal{T}_V) = \prod_{j=1}^{\delta}(1+v_j t) = \sum_{j=0}^{\delta} c_j(V)t^j,\ \delta = \dim V.$$

Then we get

$$c_G(\mathcal{E}\boxtimes\mathcal{T}_V) = \prod_{i=1}^{g}\prod_{j=1}^{\delta}(a_i\boxtimes 1 + 1\boxtimes v_j) = \prod_{j=1}^{\delta}\sum_{i=0}^{g}(-1)^i\lambda_{i;g,n}\boxtimes v_j^{g-i}$$

$$= \sum_{(i_1,\dots,i_\delta)}(-1)^{i_1+\dots+i_\delta}\lambda_{i_1;g,n}\dots\lambda_{i_\delta;g,n}\boxtimes v_1^{g-i_1}\dots v_\delta^{g-i_\delta}$$

$$= (-1)^G\sum_{0\le i_1\le\dots\le i_\delta\le g}\lambda_{i_1;g,n}\dots\lambda_{i_\delta;g,n}\boxtimes m_{g-i_1,\dots,g-i_\delta}(c_0(V),\dots,c_\delta(V)). \tag{6}$$

Here $m_{g-i_1,\dots,g-i_\delta}$ is the symmetric function obtained by symmetrization of the obvious monomial in $-v_j$ and expressed via the Chern classes of V.

Furthermore, $L_{i;g,n}(V,0)$ is the lift of $L_{i;g,n}$ wrt the projection $\overline{M}_{g,n}\times V\to\overline{M}_{g,n}$ and ev_i is the projection $\overline{M}_{g,n}\times V\to V$. Hence we get

$$\langle\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{g,0} =$$

$$= (-1)^G\sum_{0\le i_1\le\dots\le i_\delta\le g}\left(\int_{\overline{M}_{g,n}}\lambda_{i_1;g,n}\dots\lambda_{i_\delta;g,n}\psi_{1;g,n}^{d_1}\dots\psi_{n;g,n}^{d_n}\right.$$

$$\left.\times\int_V m_{g-i_1,\dots,g-i_\delta}(c_0(V),\dots,c_\delta(V))\gamma_1\dots\gamma_n\right), \tag{7}$$

where $\psi_{i;g,n} = c_1(L_{i;g,n})$.

The generalized correlators give nothing new: $\tau_{d,e} = \tau_{d+e}$.

Most of the correlators (7) vanish for dimensional reasons. Here is the list of those that may remain.

1.4.1. Proposition. *The correlators (7) identically vanish except for the following cases.*

a) $g=0,\ n\ge 3,\ \sum d_i = n-3,\ \sum|\gamma_i| = 2\delta,$ *where* $\gamma\in H^{|\gamma|}(V),\ \delta = \mathrm{dim}V:$

$$\langle\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{0,0} = \frac{(d_1+\dots d_n)!}{d_1!\dots d_n!}\int_V\gamma_1\dots\gamma_n. \tag{8}$$

b) $g=1,\ n\ge 1,\ \sum d_i = n$ *(resp.* $n-1$*)*, $\sum|\gamma_i| = 0,$ *(resp. 2):*

$$\langle\tau_{d_1}1\dots\tau_{d_n}1\rangle_{1,0} = \deg c_\delta(V)\int_{\overline{M}_{1,n}}\psi_{1;1,n}^{d_1}\dots\psi_{n;1,n}^{d_n}, \tag{9}$$

$$\langle\tau_{d_1}\gamma\,\tau_{d_2}1\dots\tau_{d_n}1\rangle_{1,0} = -(c_{\delta-1}(V),\gamma)\int_{\overline{M}_{1,n}}\lambda_{1,1,n}\psi_{1;1,n}^{d_1}\dots\psi_{n;1,n}^{d_n} \tag{10}$$

for $|\gamma| = 2$.

c) $g \geq 2$, $n \geq 0$, $\sum |\gamma_i|/2 \leq \delta \leq 3, \sum (d_i + |\gamma_i|/2) = (g-1)(3-\delta) + n$.

In particular, the $g \geq 2$, $\beta = 0$ *correlators vanish for* $\dim V \geq 4$.

Proof. First of all, $\mathcal{E} = \mathcal{E}_{g,n}$ is lifted from $\overline{M}_{\geq 2,0}$, $\overline{M}_{1,1}$ or $\overline{M}_{0,3}$.

For $g = 0$, $\mathcal{E}$ is the zero bundle, and $J_{0,n}(V,0) = [\overline{M}_{0,n} \times V]$. Formula (8) follows from this and from the known expression for $g = 0$, $V =$ *a point* correlators:

$$\int_{\overline{M}_{0,n}} \psi_{1;0,n}^{d_1} \dots \psi_{n;0,n}^{d_n} = \frac{(d_1 + \dots + d_n)!}{d_1! \dots d_n!}. \tag{11}$$

For $g = 1$, (6) becomes

$$c_\delta(\mathcal{E} \boxtimes \mathcal{T}_V) = c_\delta(V) \boxtimes 1 - c_{\delta-1}(V) \boxtimes \lambda_{1,1,n}$$

from which (9) and (10) follow.

Finally, for $g \geq 2$ one sees that the virtual fundamental class can be non–zero only if the virtual dimension for $n = 0$ is non–negative, which means that $\dim V \leq 3$. The remaining inequalities follow from the dimension matching.

One can further specialize (7) and write formulas similar to (8)–(10) separately for curves, surfaces and threefolds, $g \geq 2$.

1.5. Unstable range case. If $2g - 2 + n \leq 0$, we cannot use the absolute stabilization morphism as in 1.2 and 1.3 because $\overline{M}_{g,n}$ is empty, whereas for $\beta \neq 0$, the stack $\overline{M}_{g,n}(V,\beta)$ may well be non–empty. Always assuming this (otherwise the relevant correlators vanish), we will use instead the forgetful morphism $\overline{M}_{g,n+1}(V,\beta) \to \overline{M}_{g,n}(V,\beta)$ to produce recursion.

1.5.1. Proposition. *All the unstable range correlators can be calculated through the genus zero and one primary* ($d_i = 0$) *stable range correlators, and the* $\beta = 0$ *correlators.*

Proof. We will be considering the cases $(g,n) = (0,2), (0,1), (0,0), (1,0)$ in this order, reducing each in turn to the previously treated ones.

1.5.2. Lemma. *Let* γ_0 *be a divisor class on* V *or more generally, a class in* $H^2(V)$. *Then we have*

$$\langle \gamma_0\, \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n \rangle_{g,\beta} = (\gamma_0, \beta)\, \langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n \rangle_{g,\beta}$$

$$+ \sum_{k:\, d_k \geq 1} \langle \tau_{d_1}\gamma_1 \dots \tau_{d_k - 1}(\gamma_0 \cup \gamma_k) \dots \tau_{d_n}\gamma_n \rangle_{g,\beta} \tag{12}$$

(We omit sometimes τ_0 in notation).

This is a generalization of the Divisor Axiom in [KM] following from the properties of $J(V,\beta)$.

To treat the two–point correlators with, say $d_1 > 0$, we first use (12) and write for some γ_0 with $(\gamma_0, \beta) \neq 0$:

$$\langle \tau_{d_1}\gamma_1\, \tau_{d_2}\gamma_2 \rangle_{0,\beta} =$$

$$\frac{1}{(\gamma_0,\beta)}\left(\langle\gamma_0\,\tau_{d_1}\gamma_1\,\tau_{d_2}\gamma_2\,\rangle_{0,\beta}-\langle\tau_{d_1-1}(\gamma_0\,\cup\,\gamma_1)\,\tau_{d_2}\gamma_2\rangle_{0,\beta}-\langle\tau_{d_1}\gamma_1\,\tau_{d_2-1}(\gamma_0\,\cup\,\gamma_2)\rangle_{0,\beta}\right). \tag{13}$$

The last two terms in (13) contain only two–point correlators with smaller sum d_1+d_2-1. To the first term we apply the Corollary 1.3:

$$\langle\gamma_0\,\tau_{d_1}\gamma_1\,\tau_{d_2}\gamma_2\rangle_{0,\beta}=\sum_{a,\,\beta_1+\beta_2=\beta}\langle\tau_{d_1-1}\gamma_1\,\Delta_a\rangle_{0,\beta_1}\langle\Delta^a\,\gamma_0\,\tau_{d_2}\gamma_2\rangle_{0,\beta_2}. \tag{14}$$

The right hand side contains only two–point correlators with smaller sum d_1-1 and three–point correlators with maximum one $\tau_d, d\neq 0$. If necessary, we can again apply (14) to the three–point correlators there, again reducing the order of the gravitational descendants involved.

Iterating this procedure, we will arrive to the expressions containing only primary correlators. Finally, the two–point primary correlators can be reduced to the three–point stable range ones:

$$\langle\gamma_1\gamma_2\rangle_{0,\beta}=\frac{1}{(\gamma_0,\beta)}\langle\gamma_0\gamma_1\gamma_2\rangle_{0,\beta}. \tag{15}$$

For later use, we register the following explicit reduction of some two–point correlators to the three–point ones following from (13):

$$\langle\tau_d\gamma_1\,\tau_0\gamma_2\rangle_{0,\beta}=\sum_{j=1}^{d+1}(-1)^{j+1}(\gamma_0,\beta)^{-j}\langle\gamma_0\,\tau_{d+1-j}\gamma_1\,\tau_0(\gamma_0^{j-1}\,\cup\,\gamma_2)\rangle_{0,\beta}. \tag{15a}$$

Clearly, one can invoke (12) in the same way in order to calculate the one–point and zero–point correlators. Alternatively, one can exploit the following identity, called *the dilaton equation*:

1.5.3. Lemma. *We have*

$$\langle\tau_1 1\,\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{g,\beta}=(2g-2+n)\,\langle\tau_{d_1}\gamma_1\dots\tau_{d_n}\gamma_n\rangle_{g,\beta}$$

This again follows from the axioms for $J(V,\beta)$ stated in [BM] and proved in [B].

1.6. Correlators for zero–dimensional V. This case is covered by the Witten–Kontsevich theory and additional relations summarized in [F].

§2. Generating functions on the big phase space

2.1. The big phase space. The conventional gravitational potential is a generating series for the correlators (1) considered as a formal function on the extended phase (super)space $\oplus_{d=0}^{\infty} H^*(V)[d]$. The d–th copy of $H^*(V)$ accomodates $\tau_d\gamma$'s. Thus the symbol τ_d acquires an independent meaning as the linear operator identifying $H^*(V) = H^*(V)[0]$ with $H^*(V)[d]$ or even shifting each $H^*(V)[e]$ to $H^*(V)[e+d]$ so that we can write $\tau_d = \tau_1^d$.

For convenience choose a basis $\{\Delta_a \,|\, a = 0, \dots r\}$ of $H^*(V, \mathbf{C})$. Denote by $\{x_{d,a}\}$ the dual coordinates to $\{\tau_d\Delta_a\}$ and by $\Gamma = \sum_{a,d} x_{d,a}\tau_d\Delta_a$ the generic even element of the extended phase superspace. As usual, $x_{d,a}$ has the same $\mathbf{Z}_2$–parity as Δ_a, and the odd coordinates anticommute. The formal functions we will be considering are formal series in weighted variables, where the weight of $x_{d,a}$ is d.

We need the universal character $B(V) \to \Lambda : \beta \mapsto q^\beta$ with values in the Novikov ring Λ which is the completed semigroup ring of $B(V)$ eventually localized with respect to the multiplicative system q^β. It is topologically spanned by the monomials $q^\beta = q_1^{\beta_1} \dots q_m^{b_m}$ where $\beta = (b_1, \dots, b_m)$ in a basis of the numerical class group of 1–cycles, and $(q_1, \dots, q_m)$ are independent formal variables. We will not need the genus expansion parameter because our main formula does not mix genera.

We now put formally

$$F_g(x) = \sum_\beta q^\beta \langle e^\Gamma \rangle_{g,\beta} = \sum_\beta q^\beta \sum_n \frac{\langle \Gamma^{\otimes n} \rangle_{g,\beta}}{n!}$$

$$= \sum_{n,(a_1,d_1),\dots,(a_n,d_n)} \epsilon(a) \, \frac{x_{d_1,a_1} \dots x_{d_n,a_n}}{n!} \sum_\beta q^\beta \langle \tau_{d_1}\Delta_{a_1} \dots \tau_{d_n}\Delta_{a_n} \rangle_{g,\beta} \tag{16}$$

where ϵ is the standard sign in superalgebra. We define $F_g^{st}(x)$ by the same formula in which the last summation is restricted to the stable range of (g, n) that is, $n \geq 3$ for $g = 0$ and $n \geq 1$ for $g = 1$.

We will introduce the generating function $G_g(x)$ for modified correlators by the same formula as F_g^{st} in which every τ_d in the stable range correlators is replaced by $\tau_{0,d}$:

$$G_g(x) = \sum_{n,(a_1,d_1),\dots(a_n,d_n)} \epsilon(a) \, \frac{x_{d_1,a_1} \dots x_{d_n,a_n}}{n!} \sum_\beta q^\beta \langle \tau_{0,d_1}\Delta_{a_1} \dots \tau_{0,d_n}\Delta_{a_n} \rangle_{g,\beta}. \tag{17}$$

We will prove that the two functions are connected by a linear change of coordinates of the big phase space.

2.2. Theorem. *We have for all $g \geq 0$*

$$F_g^{st}(x) = G_g(y) \tag{18}$$

where

$$y_{c,b} = x_{c,b} + \sum_{(a,d),d\geq c+1} \sum_\beta q^\beta x_{d,a} \langle \tau_{d-c-1}\Delta_a \, \tau_0\Delta^b \rangle_{0,\beta}. \tag{19}$$

Proof. For $d \geq 1$, define the linear operators

$$U_d : H^*(V, \Lambda) \to H^*(V, \Lambda)$$

by the formula

$$U_d(\gamma) := \sum_{a,\beta} q^\beta \langle \tau_{d-1}\gamma\, \tau_0 \Delta_a \rangle_{0,\beta} \Delta^a \tag{20}$$

and put $U_0(\gamma) = \gamma$.

The formula (4) means that in the stable range and for $d \geq 1$ the correlator of any element of the form

$$\tau_{d,e}\gamma - \tau_{d-1,e+1}\gamma - \tau_{0,e}(U_d(\gamma))$$

with any product of others $\tau_{d_i}\gamma_i$ vanishes; the same is true for $d = 0$ by the definition of U_0. Hence by induction, in any stable range correlator we can replace any expression $\tau_{d,0}\gamma$ by $\sum_{j=0}^{d} \tau_{0,j}(U_{d-j}(\gamma))$ without changing the value of the correlator. In particular,

$$F_g^{st}(x) = \sum_{n,\beta} \frac{q^\beta}{n!} \langle \prod_{i=1}^{n} \sum_{a_i,d_i} x_{d_i,a_i} \tau_{d_i} \Delta_{a_i} \rangle_{g,\beta}$$

$$= \sum_{n,\beta} \frac{q^\beta}{n!} \langle \prod_{i=1}^{n} \sum_{a_i,d_i} x_{d_i,a_i} \sum_{j_i=0}^{d_i} \tau_{0,j_i}(U_{d_i-j_i}(\Delta_{a_i})) \rangle_{g,\beta}$$

$$= \sum_{n,\beta} \frac{q^\beta}{n!} \langle \prod_{i=1}^{n} \sum_{c_i,b_i} y_{c_i,b_i} \tau_{0,c_i} \Delta_{b_i} \rangle_{g,\beta} = G_g(y).$$

To obtain the last equality, use (20) in order to represent each sum in the correlator product as a linear combination of terms $\tau_{0,c}\Delta_b$. The straightforward calculation of coefficients furnishes (19).

Remark. The operator T defined by $y = T(x)$ is a linear transformation of the big phase space with coefficients in Λ defined entirely in terms of genus zero two–point correlators. It is invertible, because (19) shows that it is the sum of identity and the operator which strictly raises the gravitational weight c. Hence we may define the corrected version of $G_g(x)$ by $\widetilde{G}_g(x) := F_g(T^{-1}(x))$. Equivalently, we can extend the modified correlators to the unstable range keeping the natural functional equations.

One can also use these formulas in order to give independent meaning to the symbols $\tau_{0,d}$ as linear operators on the infinite sum of the Λ–modules $H^*(V, \Lambda)[d]$.

2.3. Expressing T through the three–point primary correlators. Formulas (16) and (19) make the following definition natural:

$$\langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n \rangle_g := \sum_\beta q^\beta \langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n \rangle_{g,\beta}. \tag{21}$$

We will write simply $\langle \dots \rangle$ when $g = 0$. These correlators are Λ–polylinear functions on the Λ–module $\oplus_{d\geq 0} H^*(V, \Lambda)[d]$. Setting in (14) $d_2 = 0$, multiplying by q^β and summing, we obtain:

$$\langle \gamma_0\, \tau_d\gamma_1\, \gamma_2 \rangle = \sum_a \langle \tau_{d-1}\gamma_1\, \Delta_a \rangle \langle \Delta^a\, \gamma_0\, \gamma_2 \rangle \tag{22}$$

Put

$$\gamma_0 \cdot \gamma_2 := \sum_a \Delta_a \langle \Delta^a \gamma_0 \, \gamma_2 \rangle \tag{23}$$

(this is essentially the product in "small" quantum cohomology where the structure constants are the third derivatives of the genus zero potential restricted to H^2).

Then we can rewrite (22) as

$$\langle \gamma_0 \, \tau_d \gamma_1 \, \gamma_2 \rangle = \langle \tau_{d-1} \gamma_1 \, \gamma_0 \cdot \gamma_2 \rangle. \tag{24}$$

Now let l be any linear function on $H_2(V, \Lambda)$. It defines the derivation $\partial_l : \Lambda \to \Lambda$, $\partial_l q^\beta := l(\beta)\, q^\beta$. We extend it to formal series over Λ coefficientwise. If γ_0 is an ample divisor class considered as a linear function on H_2, we write ∂_{γ_0} for this derivation. Turning now to the equation (15a), multiply it by q^β and sum over all β. The left hand side of (15a) vanishes for $\beta = 0$, and the right hand side does not make sense, so we get:

$$\langle \tau_d \gamma_1 \, \gamma_2 \rangle =$$

$$\sum_{j=1}^{d+1} (-1)^{j+1} \partial_{\gamma_0}^{-j} [\langle \gamma_0 \, \tau_{d+1-j} \gamma_1 \, \tau_0 (\gamma_0^{j-1} \cup \gamma_2) \rangle - \langle \gamma_0 \, \tau_{d+1-j} \gamma_1 \, \tau_0 (\gamma_0^{j-1} \cup \gamma_2) \rangle_{0,0}].$$

To interpret this, notice that since $(\gamma_0, \beta) \neq 0$ for all algebraic effective non–zero 2–homology classes on V, $\partial_{\gamma_0}^{-1} F$ makes sense for any series F whose coefficients are correlators not involving the $\beta = 0$ ones. As the result of this "integration" we take the series again not involving the $\beta = 0$ terms.

Actually, in view of (8), the $\beta = 0$ terms vanish unless $j = d + 1$. Separating this summand and replacing the remaining triple correlators with the help of (24), we get the following result.

2.3.1. Proposition. *The matrix coefficients of T can be expressed inductively through the triple primary correlators, that is, Gromov–Witten invariants, of genus zero: for $d \geq 1$*

$$\langle \tau_d \gamma_1 \, \gamma_2 \rangle = \sum_{j=1}^{d} (-1)^{j+1} \partial_{\gamma_0}^{-j} \langle \tau_{d-j} \gamma_1 \, \gamma_0 \cdot (\gamma_0^{j-1} \cup \gamma_2) \rangle +$$

$$(-1)^d \partial_{\gamma_0}^{-(d+1)} [\langle \gamma_0 \, \gamma_1 \, \gamma_0^d \cup \gamma_2 \rangle - \langle \gamma_0 \, \gamma_1 \, \gamma_0^d \cup \gamma_2 \rangle_{0,0}]. \tag{25}$$

§3. Coupling of Frobenius manifolds and Cohomological Field Theories to topological gravity

3.1. Coupling of Frobenius manifolds to topological gravity. The restriction $\Phi(x)$ to the small phase space ($x_{d,a} = 0$ for $d > 0$) of the genus zero potential $F_0(x)$ from (16) satisfies the so called Associativity Equations and defines on $H^*(V, \Lambda)$ the structure of the formal Frobenius manifold, or the tree level quantum cohomology of V. The notion of Frobenius manifold was axiomatized and studied by B. Dubrovin in [D]. There are many interesting examples which do not come from quantum cohomology. In the sec. 6 of [D] Dubrovin sets to reconstruct the whole potential with gravitational descendants from its small phase space part. Our previous discussion shows how one can do it for quantum cohomology potentials. In this subsection we show how to do this for a wider class of formal Frobenius manifolds which are not supposed to come from quantum cohomology. Our approach considerably differs from that of [D]. It would be important to relate it to the integrable hierarchies as in [D].

We will divide our discussion into two steps.

First, we will introduce the modified potential with gravitational descendants which reduces to $G_0(x)$ in the quantum cohomology case.

Second, we will discuss the additional conditions needed to define the analog of the linear transformation T and the conventional potential with gravitational descendants $F_0(x) := G_0(T(x))$.

3.1.1. The big phase space and the modified potential. We will use the formalism of Frobenius manifolds as it was presented in [M1].

Let Λ be a $\mathbf{Q}$–algebra (playing role of the Novikov ring), H a free $\mathbf{Z}_2$–graded Λ–module of finite rank (in the quantum cohomology case $H = H^*(V, \Lambda)$), η a symmetric non–degenerate pairing on H replacing the Poincaré form. To keep intact as much notation as possible, we introduce formally *the big phase space* as linear infinite dimensional formal supermanifold $\oplus_{d\geq 0} H[d]$ with basis $\tau_d\Delta_a$ and coordinates $x_{d,a}$ as in sec. 6 above. Put $x_a = x_{0,a}$, $x = \{x_a\}$. By definition, *a Frobenius potential* on (H, η) is a formal series $\Phi(x) \in \Lambda[[x]]$ whose third derivatives $\Phi_{ab}{}^{c}$ (with one index raised by η) form the structure constants of the commutative, associative $\Lambda[[x]]$–module spanned by $\partial_a := \partial/\partial x_a$. Finally, any such triple $M = (H, \eta, \Phi)$ is called *a formal Frobenius manifold* (over Λ).

The primary correlators of M are by definition the symmetric polylinear functions $H^{\otimes n} \to \Lambda$, $n \geq 3$, whose values on the tensor products of $\tau_0\Delta_a$ are essentially the coefficients of Φ written as in (16):

$$\Phi(x) = \sum_{n, a_1, \dots, a_n} \epsilon(a) \, \frac{x_{a_1} \dots x_{a_n}}{n!} \, \langle \tau_0\Delta_{a_1} \dots \tau_0\Delta_{a_n} \rangle. \tag{26}$$

In the case of quantum cohomology this agrees with our notation (21). Notice that the Associativity Equations do not constrain the terms of Φ of degree ≤ 2. In this subsection we will use only correlators with ≥ 3 arguments.

In order to extend the potential Φ to a formal function on the big phase space which in the quantum cohomology case will coincide with G_0, we will use the Second Reconstruction Theorem of [KM], proved in [KMK] and [M1]:

3.1.2. Proposition. *For any Frobenius manifold M as above, there exists a unique sequence of Λ–linear maps $I_n^M : H^{\otimes n} \to H^*(\overline{M}_{0,n}, \Lambda)$, $n \geq 3$, satisfying the folowing properties:*

(i) S_n–invariance and compatibility with restriction to boundary divisors (cf. [KM] or [M1], p. 101).

(ii) The top degree term of I_n^M capped with the fundamental class is the correlator of M with n arguments.

Moreover, in the quantum cohomology case

$$I_n^M = \sum_\beta q^\beta I_{0,n,\beta}^V$$

where $I_{0,n,\beta}^V$ are the genus zero Gromov–Witten invariants discussed in [KM].

We now define *the modified M–correlators* with gravitational descendants by

$$\langle \tau_{0,d_1}\Delta_{a_1} \dots \tau_{0,d_n}\Delta_{a_n} \rangle := \int_{[\overline{M}_{0,n}]} I_n^M(\tau_0\Delta_{a_1} \otimes \dots \otimes \tau_0\Delta_{a_n})\phi_1^{d_1} \dots \phi_n^{d_n} \qquad (27)$$

where ϕ_i are defined in 1.1. Finally put

$$G_0^M(x) = \sum_{n,(a_1,d_1),\dots(a_n,d_n)} \epsilon(a)\, \frac{x_{d_1,a_1} \dots x_{d_n,a_n}}{n!} \langle \tau_{0,d_1}\Delta_{a_1} \dots \tau_{0,d_n}\Delta_{a_n} \rangle \qquad (28)$$

where this time x denotes coordinates on the big phase space. Clearly, if M is quantum cohomology, we have reproduced (17).

The expressions (27) are universal polynomials in the coefficients of Φ and η^{ab} depending only on the superrank of H and (a_i, d_i). They can be calculated using some results of [Ka].

To explain this, recall that $H_*(\overline{M}_{0,n})$ is spanned by the classes of the boundary strata $\overline{M}_{0,\tau}$ indexed by trees whose tails are labelled by $\{1, \dots, n\}$. Any cohomology class is uniquely defined by its values on these classes. For I_n^M these values are given in [KMK], (0.7). For $\phi_1^{d_1} \dots \phi_n^{d_n}$ they are products of multinomial coefficients over all vertices of τ: put on each flag d_i if this is a tail with label i, 1 otherwise, and divide the factorial of the sum of labels at each vertex by the product of factorials of labels.

It remains to calculate the cup product of the described classes. This problem was solved in [Ka]. Admittedly, the explicit formula is rather complicated.

3.1.3. Higher genus case. If $I_n^M = I_{0n}^M$ is extended to a Cohomological Field Theory I_{gn}^M, as defined in [KM], one can use the evident version of formula (26) in order to define the modified correlators and functions $G_g(x)$ of any genus. However, unlike the genus zero case, a CohFT cannot be reconstructed only from its primary correlators.

3.2. The operator T on the big phase space. If we want to extrapolate the construction of T from the case of quantum cohomology to more general Frobenius manifolds, we encounter several difficulties. The basic problem is that the inductive formula (25) for the coefficients of T involves some additional structures, not required in the general definition of fromal Frobenius manifolds. Namely, we need submodules H_2 and H^2 in H, a semigroup in H_2 with indecomposable zero accomodating β, the ring Λ with derivatives ∂_{γ_0}. All of these structures must satisfy several conditions, ensuring in particular the independence of the right hand side of (25) from the choice of γ_0.

The following seems to be the most straightforward way to describe the additional restrictions starting with the more conventional data on $M = (H, \eta, \Phi)$.

(i) Assume that M is endowed with the flat identity e and an Euler vector field E, such that $\mathrm{ad}\, E$ is semisimple on H. Assume that the spectrum $D, (d_a)$ belongs to Λ (see [M1], Ch.1, §2 for precise definitions).

(ii) Denote by $H^2 \subset H$ the submodule of H coresponding to the zero eigenvalue of $\mathrm{ad}\, E$. Assume that it is a free direct submodule. Denote by $H_2 \subset H$ the submodule of H coresponding to the eigenvalue $-D$ of $\mathrm{ad}\, E$. Assume that it is a free direct submodule, and that η makes H_2 strict dual to H^2.

(iii) Assume that a semigroup $B \subset H_2$ with indecomposable zero and finite decomposition is given such that $\Phi(x)$ can be expanded into a formal Fourier series with respect to the part of the coordinates dual to a basis of H_2, with coefficients vanishing outside B. Denote by Ψ the part corresponding to $\beta \neq 0$. Assume finally that $\Phi = \Psi + c$ where c is a cubic form, $E\Psi = (D + d_0)\Psi$ (without additional terms of degree ≤ 2, cf. [M1], Ch.1, (2.7)) and $E_1 c = (D + d_0)c$ where E_1 is the projection of E to the orthogonal complement to H^2.

These structures allow us to imitate the constructions of §2, starting with β–decomposition of the primary correlators, and to define T via (25). For more details, see [M3].

Notice that the cup product on H and the $\langle \dots \rangle_{0,0}$ correlators are defined using the constant terms of the relevant Fourier decomposition. The independence of (25) from the choice of γ_0 follows from the postulated properties.

Acknowledgements. One of the authors (Yu.M.) is thankful to Ezra Getzler for stimulating discussions which prompted him to focus on this part of a larger project [M2], and to Kai Berend for his explanations about the proof of Lemma 10 in [B] and the restriction formula for virtual fundamental classes.

References

[B] K. Behrend. *Gromov–Witten invariants in algebraic geometry.* Inv. Math., 127 (1997), 601–617.

[BF] K. Behrend, B. Fantechi. *The intrinsic normal cone.* Inv. Math., 128 (1997), 45–88.

[BM] K. Behrend, Yu. Manin. *Stacks of stable maps and Gromov–Witten invariants.* Duke Math. J., 85:1 (1996), 1–60.

[DijVV] R. Dijkgraaf, H. Verlinde, E. Verlinde. *Loop equations and Virasoro constraints in non–perturbative two–dimensional quantum gravity.* Nucl. Phys. B348 (1991), 435–456.

[DijW] R. Dijkgraaf, E. Witten. *Mean field theory, topological field theory, and multimatrix models.* Nucl. Phys. B342 (1990), 486–522.

[D] B. Dubrovin. *Geometry of 2D topological field theories.* In: Springer LNM, 1620 (1996), 120–348

[EHX1] T. Eguchi, K. Hori, Ch.–Sh. Xiong. *Gravitational quantum cohomology.* Preprint UT–753, hep–th/9605225.

[EHX2] T. Eguchi, K. Hori, Ch.–Sh. Xiong. *Quantum cohomology and Virasoro algebra.* Preprint UT–769, hep–th/9703086.

[F] C. Faber. *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians.* Preprint, 1997.

[Ka] R. Kaufmann. *The intersection form in $H^*(\overline{M}_{0n})$ and the explicit Künneth formula in quantum cohomology.* Int. Math. Res. Notices, 19 (1996), 929–952.

[KM] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Comm. Math. Phys., 164:3 (1994), 525–562.

[KMK] M. Kontsevich, Yu. Manin (with Appendix by R. Kaufmann). *Quantum cohomology of a product.* Inv. Math., 124 (1996), f. 1–3, 313–340.

[M1] Yu. Manin *Frobenius manifolds, quantum cohomology, and moduli spaces (Chapters I, II, III).* Preprint MPI 96–113, 1996.

[M2] Yu. Manin. *Algebraic geometric introduction to the gravitational quantum cohomology* (in preparation).

[M3] Yu. Manin. *Frobenius manifolds of quantum cohomology type and tensor products* (in preparation)

[W1] E. Witten. *On the structure of the topological phase of two–dimensional gravity.* Nucl. Phys. B340 (1990), 281–332.

[W2] E. Witten. *Two–dimensional gravity and intersection theory on moduli space.* Surveys in Diff. Geometry, 1 (1991), 243–310.