On equivalences of derived categories of coherent sheaves on abelian varieties

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Introduction

Let A be an abelian variety and $\widehat{A}$ be the dual abelian variety. It was proved in [Mu1] that the derived categories of coherent sheaves $D^b_{coh}(A)$ and $D^b_{coh}(\widehat{A})$ are equivalent. This equivalence is called Fourier-Mukai transform. It is given with the help of the Poincare line bundle P_A on the product $A \times \widehat{A}$ by the formula

$$F \mapsto R^{\cdot} p_{2*}(P_A \otimes p_1^*(F)).$$

In the paper [Po1] this construction of Mukai is generalized in the following way. Consider two abelian varieties A and B . Let f be a homomorphism from $A \times \widehat{A}$ to $B \times \widehat{B}$ that is an isomorphism. It can be written as matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where α is a homomorphism from A to B , β is from $\widehat{A}$ to B and etc.. We say that f is symplectic iff the inverse for f has the following matrix form

$$f^{-1} = \begin{pmatrix} \widehat{\delta} & -\widehat{\beta} \\ -\widehat{\gamma} & \widehat{\alpha} \end{pmatrix}$$

It was proved in [Po1] that if the varieties $A \times \widehat{A}$ and $B \times \widehat{B}$ are symplectic isomorphic, where A and B are abelian varieties over algebraically closed field, then the derived categories of coherent sheaves $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent.

The aim of this paper is to give the proof for the inverse statement (see Th.2.7), i.e. if the derived categories $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent, then the varieties $A \times \widehat{A}$ and $B \times \widehat{B}$ are symplectic isomorphic. This theorem implies finiteness of the set of non-isomorphic abelian varieties, which have the derived categories of coherent sheaves equivalent to $D^b_{coh}(A)$ (Cor.2.8). The proof is based on the theorem from

[Or], which says that any exact equivalence between two derived categories of coherent sheaves on smooth projective varieties can be represented an object on the product.

Representing equivalences by objects on the product , we construct the map from the set of all exact equivalence between $D^b_{coh}(A)$ and $D^b_{coh}(B)$ to the set of symplectic isomorphism between $A\times\widehat{A}$ and $B\times\widehat{B}$. It is proved that this map is functorial (Prop.2.4). In particular, we get the homomorphism from the group of exact autoequivalences of $D^b_{coh}(A)$ to the group of the symplectic automorphisms of the variety $A\times\widehat{A}$. This homomorphism has the non-trivial kernel, which is isomorphic to the direct sum of $\mathbb{Z}$ and $(A\times\widehat{A})_k$, where the latter is the group of k -points for the variety $A\times\widehat{A}$. If the field k is algebraically closed and $\operatorname{char} k=0$, then we show that there is the short exact sequence

$$0\longrightarrow\mathbb{Z}\oplus(A\times\widehat{A})_k\longrightarrow\operatorname{Auteq}(D^b_{coh}(A))\longrightarrow Sp(A\times\widehat{A})\longrightarrow 1$$

Further, we consider the central extension of $Sp(A\times\widehat{A})$ with $\mathbb{Z}$ and write the explicit formula for 2-cocycle that gives this extension. It can be done, because we know that any equivalence between derived categories of coherent sheaves on abelian varieties is represented by the object, which is isomorphic to a sheaf up to translation functor (Prop.2.9).

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1 Preliminaries.

Let X and Y be some smooth projective varieties over a field k . Denote by $D^b_{coh}(X)$ and $D^b_{coh}(Y)$ the bounded derived categories of coherent sheaves on X and Y respectively. Recall that a derived category has the structure of a triangulated category.

For every object $E\in D^b_{coh}(Y\times X)$ we can define an exact functor Φ_E from $D^b_{coh}(X)$ to $D^b_{coh}(Y)$. Denote by p and π the projections of $D^b_{coh}(X\times Y)$ onto X and Y respectively:

$$\begin{array}{ccc} X\times Y & \xrightarrow{\pi} & Y \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

Then Φ_E is defined by the following formula:

$$\Phi_E(\bullet):=\pi_*(E\otimes p^*(\bullet)) \qquad (1)$$

(henceforth we abbreviate $f_*, f^*, \otimes$ and etc. instead of $R^\bullet f_*, L^\bullet f^*, \overset{L}{\otimes}$, because we consider only derived functors).

Moreover, the object $E \in D^b_{coh}(X \times Y)$ define another functor Ψ_E from $D^b_{coh}(Y)$ to $D^b_{coh}(X)$. This functor is given by the formula:

$$\Psi_E(\bullet) := p_*(E \otimes \pi^*(\bullet)) \tag{2}$$

It is known that the functor Φ_E has left and right adjoint functors Φ_E^* and $\Phi_E^!$ respectively, defined by the following formulas:

$$\begin{aligned} \Phi_E^* &= \Psi_{E^\vee \otimes \pi^* \omega_Y [dim Y]}, \\ \Phi_E^! &= \Psi_{E^\vee \otimes p^* \omega_X [dim X]}, \end{aligned} \tag{3}$$

where ω_X and ω_Y are the canonical sheaves on X and Y , and $E^\vee := \mathbf{R}^\bullet \mathcal{H}om(E, \mathcal{O}_{X \times Y})$.

Let X, Y, Z be smooth projective varieties and I, J, K objects of $D^b_{coh}(X \times Y)$, $D^b_{coh}(Y \times Z)$ and $D^b_{coh}(X \times Z)$, respectively. Consider the following diagram of projections:

$$\begin{tikzcd}
 & XYZ \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
XY \arrow[d, "\pi^1_{12}"'] \arrow[dr, "\pi^2_{12}"] & XZ \arrow[dl, "\pi^1_{13}"'] \arrow[dr, "\pi^3_{13}"] & YZ \arrow[dl, "\pi^2_{23}"'] \arrow[d, "\pi^3_{23}"] \\
X & Y & Z
\end{tikzcd}$$

The objects I, J, K give the triple of functors

$$\begin{aligned} \Phi_I :&\quad D^b_{coh}(X) \longrightarrow D^b_{coh}(Y), \\ \Phi_J :&\quad D^b_{coh}(Y) \longrightarrow D^b_{coh}(Z), \\ \Phi_K :&\quad D^b_{coh}(X) \longrightarrow D^b_{coh}(Z), \end{aligned}$$

defined by the formula (1), i.e.

$$\begin{aligned} \Phi_I &:= \pi^2_{12*}(I \otimes {\pi^1_{12}}^*(\cdot)), \\ \Phi_J &:= \pi^3_{23*}(J \otimes {\pi^2_{23}}^*(\cdot)), \\ \Phi_K &:= \pi^3_{13*}(K \otimes {\pi^1_{13}}^*(\cdot)). \end{aligned}$$

Let us consider the object $I \underset{Y}{\boxtimes} J \in D^b_{coh}(X \times Y \times Z)$. Henceforth, we will write $I \underset{Y}{\boxtimes} J$ instead of ${p_{23}}^* J \otimes {p_{12}}^* I$. The next proposition from [Mu1] gives us the composition law for exact fanctors between derived categories.

Proposition 1.1 *The composition* $\Phi_J \circ \Phi_I$ *is isomorphic to* Φ_K *with*

$$K = p_{13*}(I \underset{Y}{\boxtimes} J), \tag{4}$$

Thus, we associated to a smooth projective variety X the derived category of coherent sheaves $D^b_{coh}(X)$ and we attached to each object $\mathcal{E} \in D^b_{coh}(X \times Y)$ the functor $\Phi_{\mathcal{E}}$ from $D^b_{coh}(X)$ to $D^b_{coh}(Y)$.

There arise the following questions :

- When do two different smooth projective varieties have equivalent derived categories of coherent sheaves?
- What is the group of exact autoequivalences of the derived category of coherent sheaves for one X?

There are some previous results related to these questions. First, we can answer to these questions in the case of a variety with ample canonical or ample anticanonical sheaf.

Theorem 1.2 [BO2] *Let* X *be a smooth projective variety with ample canonical or anticanonical sheaf. If* $\mathcal{D} = D^b_{coh}(X)$ *is equivalent as a triangulated category to* $D^b_{coh}(X')$ *for some other smooth algebraic variety* X' *, then* X *is isomorphic to* X' *.*

Theorem 1.3 [BO2] *Let* X *be a smooth projective variety with ample canonical or anticanonical sheaf. Then the group of isomorphism classes of exact autoequivalences* $D^b_{coh}(X) \xrightarrow{\sim} D^b_{coh}(X)$ *is generated by the automorphisms of the variety, the twists by invertible sheaves and the translations.*

Moreover, we can add that in the hypothesis of Theorem 1.3 the group of autoequivalences $\mathrm{Aut}D^b_{coh}(X)$ is the semi-direct product of the normal subgroup $G_1 = \mathrm{Pic}X \oplus \mathbf{Z}$ and the subgroup $G_2 = \mathrm{Aut}X$ naturally acting on G_1 , i.e:

$$\mathrm{Auteq}D^b_{coh}(X) \cong \mathrm{Aut}X \ltimes (\mathrm{Pic}X \oplus \mathbf{Z})$$

Secondly, It is known that any equivalence can be represented by an object on the product.

Theorem 1.4 [Or] *Let* X *and* Y *be smooth projective varieties. Suppose* $F : D^b_{coh}(X) \xrightarrow{\sim} D^b_{coh}(Y)$ *is an exact equivalence. Then there exists a unique up to isomorphism object* $E \in D^b_{coh}(X \times Y)$ *such that the functors* F *and* Φ_E *are isomorphic.*

Further, having an object $\mathcal{E} \in D^b_{coh}(X \times Y)$ we need a good method to check fully faithfulness of the functor $\Phi_{\mathcal{E}} : D^b_{coh}(X) \longrightarrow D^b_{coh}(Y)$. It is not so simple to do. However, there is the useful criterion that will be needed for sequel.

Theorem 1.5 [BO1] *Let M and X be smooth algebraic varieties over algebraically closed field of the characteristic 0 and let $E \in D^b_{coh}(M \times X)$. Then the functor Φ_E is fully faithful, iff the following orthogonality conditions are verified:*

$$i) \quad \mathrm{Hom}^i_X(\Phi_E(\mathcal{O}_{t_1}),\ \Phi_E(\mathcal{O}_{t_2})) = 0 \qquad \textit{for every } i \textit{ and } t_1 \neq t_2.$$

$$ii) \quad \mathrm{Hom}^0_X(\Phi_E(\mathcal{O}_t),\ \Phi_E(\mathcal{O}_t)) = k,$$

$$\mathrm{Hom}^i_X(\Phi_E(\mathcal{O}_t),\ \Phi_E(\mathcal{O}_t)) = 0, \qquad \textit{for } i \notin [0, dim M].$$

Here t , t_1 , t_2 are points of M and $\mathcal{O}_{t_i}$ are corresponding skyscraper sheaves.

Now, let X , Y , Z and W be four smooth projective varieties . Suppose we have two morphisms f and g :

$$f : Y \times Z \longrightarrow Y \quad \text{and} \quad g : Y \times Z \longrightarrow Z$$

such that the following condition holds:

$(*)$ The morphisms (f, pr_2) and (pr_1, g) from $Y \times Z$ to itself are isomorphisms.

The morphisms f and g give us the morphisms

$$(id_X, f) : X \times Y \times Z \times W \longrightarrow X \times Y \quad \text{and} \quad (g, id_W) : X \times Y \times Z \times W \longrightarrow Z \times W,$$

defined in the obvious way.

Let $\mathcal{E}$ and $\mathcal{F}$ be objects of the categories $D^b_{coh}(X \times Y)$ and $D^b_{coh}(Z \times W)$ respectively. Using the morphisms (f, pr_2) and (pr_1, g) we define the object $\mathcal{H} \in D^b_{coh}((X \times Z) \times (Y \times W))$ by the rule:

$$\mathcal{H} := (id_X, f)^* \mathcal{E} \otimes (g, id_W)^* \mathcal{F}$$

Further, we can attach to the objects $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$ the following functors:

$$\begin{aligned} &\Phi_{\mathcal{E}} : D^b_{coh}(X) \longrightarrow D^b_{coh}(Y) \\ &\Phi_{\mathcal{F}} : D^b_{coh}(Z) \longrightarrow D^b_{coh}(W) \\ &\Phi_{\mathcal{H}} : D^b_{coh}(X \times Z) \longrightarrow D^b_{coh}(Y \times W) \end{aligned}$$

defined by formula (1).

Proposition 1.6 *In the above notation, if the functors $\Phi_{\mathcal{E}}$ and $\Phi_{\mathcal{F}}$ are full and faithful, then the functor $\Phi_{\mathcal{H}}$ is full and faithful too.*

Proof. To prove the proposition we must show that the functor

$$\Phi^*_{\mathcal{H}} \circ \Phi_{\mathcal{H}} : D^b_{coh}(X \times Z) \longrightarrow D^b_{coh}(X \times Z),$$

which is the composition of the functor $\Phi_{\mathcal{H}}$ with its left adjoint $\Phi^*_{\mathcal{H}}$, is isomorphic to the identity functor $id_{D^b_{coh}(X\times Z)}$.
Let us consider the following diagram:

$$\begin{tikzcd}
 & & & XZYWXZ \arrow[r, "{(g,1_W,g)}"] \arrow[d, "q_1"] \arrow[dr, phantom, "1"] & ZWZ \arrow[d, "q_3"] \\
 & XYXY \arrow[dr, phantom, "\star"] & & XZYXZ \arrow[ll, "{(1_X,f,1_X,f)}"'] \arrow[r, "{(g,g)}"] \arrow[dd, "q_2"] & ZZ \\
XYX \arrow[ur, "i_Y"] \arrow[dd, "q_4"'] \arrow[ddrr, phantom, "3"] & & XZYX \arrow[ll, "{(1_X,f,1_X)}"'] \arrow[ur, "j_Z"] \arrow[dd, "q_5"'] \arrow[dr, phantom, "2"] & & \\
 & & & XZXZ & \\
XX & & XZX \arrow[ll, "q_6"] \arrow[ur, "i_Z"'] & &
\end{tikzcd}$$

Here all q_i are projections, the morphisms i_Y, i_Z, j_Z are diagonal embeddings, i.e.

$$\begin{array}{llcl}
i_Y : & (x_1,y,x_2) & \mapsto & (x_1,y,x_2,y) \\
i_Z : & (x_1,z,x_2) & \mapsto & (x_1,z,x_2,z) \\
j_Z : & (x_1,z,y,x_2) & \mapsto & (x_1,z,y,x_2,z)
\end{array}$$

and the other morphisms are defined by the following rules:

$$\begin{array}{llcl}
(g,1_W,g): & (x_1,z_1,y,w,x_2,z_2) & \mapsto & (g(y,z_1),w,g(y,z_2)) \\
(g,g): & (x_1,z_1,y,x_2,z_2) & \mapsto & (g(y,z_1),g(y,z_2)) \\
(1_X,f,1_X,f): & (x_1,z_1,y,x_2,z_2) & \mapsto & (x_1,f(y,z_1),x_2,f(y,z_2)) \\
(1_X,f,1_X): & (x_1,y,x_2,z) & \mapsto & (x_1,f(y,z),x_2)
\end{array}$$

In this diagram the square marked with $\star$ is commutative; all other squares are Cartesian. This is clear for squares 1 and 2, and this follows from condition $(*)$ for square 3.

Let us consider the following object

$$T := \mathcal{H} \underset{Y\times W}{\boxtimes} \mathcal{H}^{\vee} \otimes p^*_{Y\times W}(\omega_{Y\times W})[dim(Y\times W)] \in D^b_{coh}((X\times Z)\times(Y\times W)\times(X\times Z))$$

To prove the lemma, it is sufficient to show that the object $q_{2*}q_{1*}(T)$ is isomorphic to the structure sheaf of the diagonal $\Delta_{X\times Z}\subset X\times Z$.

First, by the construction of $\mathcal{H}$, the object T is isomorphic to

$$(g,1_W,g)^*(\mathcal{F}\underset{W}{\boxtimes}\mathcal{F}^{\vee}\otimes p^*_W(\omega_W)[dimW])\otimes q_1^*((1_X,f,1_X,f)^*(\mathcal{E}\boxtimes\mathcal{E}^{\vee}[dimY])\otimes p^*_Y(\omega_Y))$$

Hence, by the projection formula, the object $q_{1*}(T)$ is isomorphic to

$$q_{1*}(g,1_W,g)^*(\mathcal{F}\underset{W}{\boxtimes}\mathcal{F}^{\vee}\otimes p^*_W(\omega_W)[dimW])\otimes(1_X,f,1_X,f)^*(\mathcal{E}\boxtimes\mathcal{E}^{\vee}[dimY])\otimes p^*_Y(\omega_Y)$$

Further, by the theorem of flat base change [H], we get the following isomorphism:

$$q_{1*}(g, 1_W, g)^*(\mathcal{F} \underset{W}{\boxtimes} \mathcal{F}^\vee \otimes p_W^*(\omega_W)[dimW]) \cong (g,g)^* q_{3*}(\mathcal{F} \underset{W}{\boxtimes} \mathcal{F}^\vee \otimes p_W^*(\omega_W)[dimW]).$$

By assumption, the functor $\Phi_{\mathcal{F}}$ is full and faithful. Consequently, the object $q_{3*}(\mathcal{F} \underset{W}{\boxtimes} \mathcal{F}^\vee \otimes p_W^*(\omega_W)[dimW])$ is isomorphic to $\mathcal{O}_{\Delta_Z}$. Besides, it immediately follows from condition $(*)$ that

$$(g,g)^*\mathcal{O}_{\Delta_Z} \cong j_{Z*}\mathcal{O}_{XZYX}$$

Therefore, we get the following isomorphism:

$$q_{1*}(T) \cong j_{Z*}\mathcal{O}_{XZYX} \otimes (1_X, f, 1_X, f)^*(\mathcal{E} \boxtimes \mathcal{E}^\vee[dimY]) \otimes p_Y^*(\omega_Y)$$

Further, using the projection formula again, we obtain

$$q_{1*}(T) \cong j_{Z*}(j_Z^*(1_X, f, 1_X, f)^*(\mathcal{E} \boxtimes \mathcal{E}^\vee[dimY]) \otimes p_Y^*(\omega_Y))$$

Since the square marked with $\star$ is commutative, we may write:

$$q_{1*}(T) \cong j_{Z*}((1_X, f, 1_X)^* i_Y^*(\mathcal{E} \boxtimes \mathcal{E}^\vee[dimY]) \otimes p_Y^*(\omega_Y)) \cong$$

$$j_{Z*}((1_X, f, 1_X)^*(\mathcal{E} \underset{Y}{\boxtimes} \mathcal{E}^\vee[dimY]) \otimes p_Y^*(\omega_Y))$$

Since the morphism $(f, pr_2) : Y \times Z \longrightarrow Y \times Z$ is an isomorphism, we have $(f, pr_2)^*\omega_{Y\times Z} \cong \omega_{Y \times Z}$. This implies that $f^*(\omega_Y) \otimes pr_2^*(\omega_Z) \cong pr_1^*(\omega_Y) \otimes pr_2^*(\omega_Z)$. Hence we get

$$f^*(\omega_Y) \cong pr_1^*(\omega_Y) \tag{5}$$

Therefore, we can write

$$q_{1*}(T) \cong j_{Z*}((1_X, f, 1_X)^*(\mathcal{E} \underset{Y}{\boxtimes} \mathcal{E}^\vee \otimes p_Y^*(\omega_Y)[dimY]))$$

Applying the functor q_{2*} to the object $q_{1*}(T)$ and using commutativity of diagram 2, we get

$$q_{2*}q_{1*}(T) \cong i_{Z*}q_{5*}(1_X, f, 1_X)^*(\mathcal{E} \underset{Y}{\boxtimes} \mathcal{E}^\vee \otimes p_Y^*(\omega_Y)[dimY])$$

Further, by the theorem of flat base change, we have

$$q_{2*}q_{1*}(T) \cong i_{Z*}q_6^* q_{4*}(\mathcal{E} \underset{Y}{\boxtimes} \mathcal{E}^\vee \otimes p_Y^*(\omega_Y)[dimY])$$

By assumption, the functor $\Phi_{\mathcal{E}}$ is full and faithful. This yields that

$$q_{4*}(\mathcal{E} \underset{Y}{\boxtimes} \mathcal{E}^\vee \otimes p_Y^*(\omega_Y)[dimY]) \cong \mathcal{O}_{\Delta_X}$$

Finally, we get

$$q_{2*}q_{1*}(T) \cong i_{Z*}q_6^*(\mathcal{O}_{\Delta_X}) \cong \mathcal{O}_{\Delta_{X\times Z}}$$

This completes the proof of this proposition. □

Corollary 1.7 *In the above notation, if the functors* $\Phi_{\mathcal{E}}$ *and* $\Phi_{\mathcal{F}}$ *are equivalences, then the functor* $\Phi_{\mathcal{H}}$ *is an equivalence too.*

Proof. By the proposition, the functor $\Phi_{\mathcal{H}}$ is full and faithful. Moreover, the left adjoint functor $\Phi^*_{\mathcal{H}}$ is coincides with

$$\Psi_{\mathcal{H}^\vee\otimes p^*_{Y\times W}(\omega_{Y\times W})}[dim Y\times W]$$

Using the equality (5) we obtain

$$\mathcal{H}^\vee\otimes p^*_{Y\times W}(\omega_{Y\times W})\cong(id_X,f)^*(\mathcal{E}^\vee\otimes p^*_Y(\omega_Y))\otimes(g,id_W)^*(\mathcal{F}^\vee\otimes p^*_W(\omega_W))$$

Therefore, by the proposition, the functor $\Phi^*_{\mathcal{H}}$ is full an faithful too. This yields that $\Phi_{\mathcal{H}}$ is an equivalence. □

Notice that $f=pr_1$ and $g=pr_2$ satisfy the condition (*). Consequently, we have the following corollary.

Corollary 1.8 *Suppose* $\mathcal{E}$ *and* $\mathcal{F}$ *are objects of* $D^b_{coh}(X\times Y)$ *and* $D^b_{coh}(Z\times W)$ *respectively such that the functors* $\Phi_{\mathcal{E}}$ *and* $\Phi_{\mathcal{F}}$ *are full and faithful. Then the functor*

$$\Phi_{\mathcal{H}}:D^b_{coh}(X\times Z)\longrightarrow D^b_{coh}(Y\times W),$$

where $\mathcal{H}:=\mathcal{E}\boxtimes\mathcal{F}$ *, is full and faithful too.*

2 Derived categories of abelian varieties.

Let A be an abelian variety. Denote by m_A the multiplication morphism $m_A: A\times A\longrightarrow A$. From now on, we write the group low in A additively. Moreover we will use the following notations:

– we denote by $d_A: A\times A\longrightarrow A$ the composition $m_A\circ(1_A,(-1)_A)$, i.e. $d_A: (a_1,a_2)\mapsto a_1-a_2$;

– for a closed point $a\in A$, by $T_a: A\otimes_k k(a)\longrightarrow A\otimes_k k(a)$ we denote the translation morphism $T_a(b)=a+b$;

– and the dual abelian variety of A will be denoted by $\widehat{A}:=\mathrm{Pic}^0 A$. Any closed k -point $a'\in\widehat{A}$ defines the line bundle $P_{a'}\in\mathrm{Pic}^0 A$ on A . Moreover, if we have some abelian varieties $A_1,...,A_m$ and a k -point $(a'_1,...,a'_m)\in\widehat{A}_1\times\cdots\times\widehat{A}_m$, then we will use the notation $P_{(a'_1,...,a'_k)}$ for the line bundle $P_{a'_1}\boxtimes\cdots\boxtimes P_{a'_k}$ on the product $A_1\times\cdots\times A_k$.

It was proved by Mukai in [Mu1] that the derived categories of coherent sheaves $D^b_{coh}(A)$ and $D^b_{coh}(\widehat{A})$ are equivalent. He considered the functor Φ_{P_A} , where P_A

is a Poincare line bundle on $A\times\widehat{A}$, and showed that this functor gives an equivalence between $D^b_{coh}(A)$ and $D^b_{coh}(\widehat{A})$.

Let us consider two abelian varieties A and B . By P_A and P_B denote Poincare line bundles on $A\times\widehat{A}$ and $B\times\widehat{B}$ respectively. Suppose the categories $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent as triangulated categories. By F denote a functor from $D^b_{coh}(A)$ to $D^b_{coh}(B)$ that gives some equivalence. By Theorem 1.4 , we can find an object $\mathcal{E}\in D^b_{coh}(A\times B)$ such that it represents this equivalence. In other words, there is some object $\mathcal{E}$ such that the functor F is isomorphic to the functor $\Phi_{\mathcal{E}}$ defined by formula (1).

Let us consider an object $\mathcal{E}\in D^b_{coh}(A\times B)$ and construct the objects $\mathcal{H}\in D^b_{coh}(A\times\widehat{A}\times A\times B)$ and $\mathcal{G}\in D^b_{coh}(A\times B\times B\times\widehat{B})$ by the following rule:

$$\begin{aligned}\mathcal{H}:=&\ \ p_{23}^*P_A\otimes(m_A,p_4)^*\mathcal{E}\\ \mathcal{G}:=&\ \ p_{24}^*P_B^{\vee}\otimes(p_1,d_B)^*\mathcal{E}^{\vee}\end{aligned}\tag{6}$$

Note that, by Proposition 1.6 , if $\Phi_{\mathcal{E}}$ is an equivalence, then the functors

$$\begin{aligned}\Phi_{\mathcal{H}}&:D^b_{coh}(A\times\widehat{A})\longrightarrow D^b_{coh}(A\times B)\\ \Phi_{\mathcal{G}}&:D^b_{coh}(A\times B)\longrightarrow D^b_{coh}(B\times\widehat{B})\end{aligned}$$

are equivalences too.

The composition of these functors $\Phi_{\mathcal{G}}\circ\Phi_{\mathcal{H}}[n]$ is a functor from $D^b_{coh}(A\times\widehat{A})$ to $D^b_{coh}(B\times\widehat{B})$. Denote by $\mathcal{J}\in D^b_{coh}(A\times\widehat{A}\times B\times\widehat{B})$ the object that represents this composition, i.e. $\Phi_{\mathcal{J}}\cong\Phi_{\mathcal{G}}\circ\Phi_{\mathcal{H}}[n]$. The object $\mathcal{J}$ can be constructed by the rule (4), i.e.

$$\mathcal{J}\cong p_{1256*}(\mathcal{H}\underset{A\times B}{\boxtimes}\mathcal{G})[n]\tag{7}$$

In this way, we attached to each object $\mathcal{E}\in D^b_{coh}(A\times B)$ the object $\mathcal{J}\in D^b_{coh}(A\times\widehat{A}\times B\times\widehat{B})$ so that if the functor $\Phi_{\mathcal{E}}$ is an equivalence, then the functor $\Phi_{\mathcal{J}}$ is an equivalence too. Now we would like to show that if the functor $\Phi_{\mathcal{E}}$ is an equivalence then the object $\mathcal{J}$ is an invertible sheaf on some subvariety that is a graf of some isomorphism between $A\times\widehat{A}$ and $B\times\widehat{B}$.

But, at first, we prove the following lemma that is needed for sequel. Let $\bar{X}:=X\times_k\bar{k}$ and denote by $\bar{\mathcal{F}}$ the inverse image of $\mathcal{F}$ under the morphism $\bar{X}\longrightarrow X$.

Lemma 2.1 *Let $\mathcal{F}$ be a coherent sheaf on a smooth variety X . Suppose there exist a closed subvariety $j:Z\hookrightarrow\bar{X}$ and an invertible sheaf $\mathcal{L}$ on Z such that $\bar{\mathcal{F}}\cong j_*\mathcal{L}$. Then there exist a closed subvariety $i:Y\hookrightarrow X$ and an invertible sheaf $\mathcal{M}$ on Y such that $\mathcal{F}\cong i_*\mathcal{M}$ and $j=\bar{i}$.*

Proof. The subvariety Y is the image of the composition $Z\xrightarrow{j}\bar{X}\longrightarrow X$. The question is local on X . Let us consider a closed point $y\in Y\subset X$. Denote by (A,m)

the local ring of X at the point y . Then, $\mathcal{F}_y$ is the module over A . We must prove that $\mathcal{F}_y \cong A/J$, i.e there exists a surjective homomorphism $A \longrightarrow \mathcal{F}_y \to 0$. By Nakayama's lemma, it is sufficient to show that $\mathcal{F}_y/m\mathcal{F}_y \cong A/mA$. Since A/mA is a field, we have $\mathcal{F}_y/m\mathcal{F}_y \cong (A/mA)^s$.

Now let us take a closed point $z \in Z \subset \bar{X}$ over the point y . And let (B, n) be the local ring of $\bar{X}$ at z . Further , we have $\bar{F}_z/m\bar{\mathcal{F}}_z \cong (B/mB)^s$. Hence, we get $\bar{F}_z/n\bar{\mathcal{F}}_z \cong (B/nB)^s$. This implies that $s = 1$. Thus, $\mathcal{F}_y \cong A/J$ for some ideal J and $\bar{\mathcal{F}}_z \cong B/JB = A/J \otimes_A B$. Lemma is proved. □

Remark It follows from the proof that the analogous statement is true not only for field extensions but for any surjective morphism.

The following lemma says us that the property to be full and faithful is stable under field extensions.

Lemma 2.2 *Let X and Y be smooth projective variety over field k and let $\mathcal{E}$ be an object of $D^b_{coh}(X \times Y)$. Then the functor $\Phi_{\mathcal{E}} : D^b_{coh}(X) \longrightarrow D^b_{coh}(Y)$ is full and faithful iff the functor $\Phi_{\bar{\mathcal{E}}} : D^b_{coh}(\bar{X}) \longrightarrow D^b_{coh}(\bar{Y})$ is full and faithful.*

Proof. $\Rightarrow$ Denote by $\Phi^*_{\mathcal{E}}$ the left adjoint functor for $\Phi_{\mathcal{E}}$. If the functor $\Phi_{\mathcal{E}}$ is full and faithful, then the composition $\Phi_{\mathcal{E}} \circ \Phi^*_{\mathcal{E}}$ is the identity functor $id_{D^b_{coh}(X)}$, which is represented by the structure sheaf of the diagonal $\mathcal{O}_{\Delta}$ on the product $X \times X$. Hence, by the theorem of flat base change, the composition $\Phi_{\bar{\mathcal{E}}} \circ \Phi^*_{\bar{\mathcal{E}}}$ is represented the structure sheaf $\mathcal{O}_{\bar{\Delta}}$, where $\bar{\Delta}$ is the diagonal in $\bar{X} \times \bar{X}$. Consequently, the functor $\Phi_{\bar{\mathcal{E}}}$ is full and faithful too.

$\Leftarrow$ Consider the composition $\Phi_{\mathcal{E}} \circ \Phi^*_{\mathcal{E}}$. It is represented by an object $\mathcal{J}$ on $X \times X$. There exists the canonical morphism $\phi : \mathcal{J} \longrightarrow \mathcal{O}_{\Delta}$. Since the functor $\Phi_{\bar{\mathcal{E}}}$ is full and faithful, the morphism $\bar{\phi} : \bar{\mathcal{J}} \longrightarrow \mathcal{O}_{\bar{\Delta}}$ is an isomorphism. This implies that ϕ is an isomorphism too. Thus, the functor $\Phi_{\mathcal{E}}$ is full and faithful. □

Proposition 2.3 *In the above notation, suppose that the functor $\Phi_{\mathcal{E}}$ is an equivalence. Then there exists an abelian subvariety $j : X \hookrightarrow A \times \widehat{A} \times B \times \widehat{B}$ such that the canonical homomorphisms $X \longrightarrow A \times \widehat{A}$ and $X \longrightarrow B \times \widehat{B}$ are isomorphisms, and $\mathcal{J} \cong j_* L$, where L is an invertible sheaf on X .*

Proof. By Lemmas 2.1 and 2.2 , we can assume that k is algebraically closed.

Denote by e and e' the identity points of $A \times \widehat{A}$ and $B \times \widehat{B}$ respectively. We know that $\Phi_{\mathcal{H}}(\mathcal{O}_e) \cong \mathcal{E}$. Moreover, the quasi-inverse for the functor $\Phi_{\mathcal{G}}[n]$ is the functor $\Psi_{\mathcal{G}^{\vee}}$ (see (3)) and $\Psi_{\mathcal{G}^{\vee}}(\mathcal{O}_{e'}) \cong \mathcal{E}$. Therefore, since $\Phi_{\mathcal{J}} \cong \Phi_{\mathcal{G}} \circ \Phi_{\mathcal{H}}[n]$ we get $\Phi_{\mathcal{J}}(\mathcal{O}_e) \cong \mathcal{O}_{e'}$. Hence we have

$$\mathcal{J} \otimes \mathcal{O}_{\{e\} \times (B \times \widehat{B})} \cong \mathcal{O}_{\{e\} \times \{e'\}}$$

This implies that there exists a Zariski neighborhood U of e such that the restriction $\mathcal{J}\mid_{U\times(B\times\widehat{B})}$ is a sheaf with the support which meets $\{e\}\times(B\times\widehat{B})$ only at the point $\{e\}\times\{e'\}$.

Denote by (R,m) and (R',m') the local rings of $A\times\widehat{A}$ and $B\times\widehat{B}$ at the points e and e' respectively. Then $(\tilde{R},\tilde{m}):=R\otimes_k R'$ is the local ring of the product at the point $\{e\}\times\{e'\}$. By J denote the stalk of the sheaf $\mathcal{J}\mid_{U\times(B\times\widehat{B})}$ at the point $\{e\}\times\{e'\}$. It is a module over ring $\tilde{R}$.

Further, we know that

$$(R/m\otimes_k R')\otimes_{\tilde{R}} J=k \quad\text{and}\quad \operatorname{Tor}_i^{\tilde{R}}(R/m\otimes_k R',J)=0 \quad\text{for all}\quad i>0$$

Consequently, we obtain $J/\tilde{m}J\cong\tilde{R}/\tilde{m}\tilde{R}$. By Nakayma's lemma , we have a surjective homomorphism $\tilde{R}\longrightarrow J\to 0$. Since $\tilde{R}$ is a flat algebra over R , we get

$$R/m\otimes_R J=k \quad\text{and}\quad \operatorname{Tor}_i^{R}(R/m,J)=0 \quad\text{for all}\quad i>0$$

This immediately yields that J is a free R -module of rang 1 and the composition $R\longrightarrow\tilde{R}\longrightarrow J$ is isomorphism. Consequently, there exist Zariski neighborhood V of e and subscheme $X(V)\subset V\times(B\times\widehat{B})$ such that the projection $X(V)\longrightarrow V$ is an isomorphism and $\mathcal{J}\mid_{V\times(B\times\widehat{B})}\cong\mathcal{O}_{X(V)}$.

Now, consider a closed point $(a,a')\in V$. We have

$$\Phi_{\mathcal{J}}(\mathcal{O}_{(a,a')})\cong\mathcal{O}_{(b,b')}$$

for some closed point $(b,b')\in B\times\widehat{B}$. Further, it follows from the construction of the objects $\mathcal{H}$ and $\mathcal{G}$ that

$$\Phi_{\mathcal{H}}(\mathcal{O}_{(a,a')})\cong T^*_{(a,0)}\mathcal{E}\otimes P_{(a',0)} \quad\text{and}\quad \Psi_{\mathcal{G}^\vee}(\mathcal{O}_{(b,b')})\cong T^*_{(0,-b)}\mathcal{E}\otimes P_{(0,b')}$$

Hence, we obtain

$$T^*_{(a,0)}\mathcal{E}\otimes P_{(a',0)}\cong T^*_{(0,-b)}\mathcal{E}\otimes P_{(0,b')}$$

Now take an arbitrary closed point $(a,a')\in A\times\widehat{A}$. We can represent it as a sum $(a,a')=(a_1,a_1')+(a_2,a_2')$ with $(a_1,a_1'),(a_2,a_2')\in V$. There exists the closed points (b_1,b_1') and (b_2,b_2') such that

$$T^*_{(a_i,0)}\mathcal{E}\otimes P_{(a_i',0)}\cong T^*_{(0,-b_i)}\mathcal{E}\otimes P_{(0,b_i')}$$

for $i=1,2$. Hence, we have

$$T^*_{(a,0)}\mathcal{E}\otimes P_{(a',0)}\cong T^*_{(0,-b)}\mathcal{E}\otimes P_{(0,b')}$$

for the point (b,b') that is the sum of (b_1,b_1') and (b_2,b_2') . Consequently, we get

$$\Phi_{\mathcal{J}}(\mathcal{O}_{(a,a')})\cong\mathcal{O}_{(b,b')}$$

for an arbitrary closed point (a,a') . Thus, repeating the procedure stated above, we can find for every closed point (a,a') a neighborhood W and a subscheme $X(W)\subset W\times(B\times\widehat{B})$ such that the projection $X(W)\longrightarrow W$ is an isomorphism and $\mathcal{J}\mid_{W\times(B\times\widehat{B})}\cong\mathcal{O}_{X(W)}$. Finally, there exist an abelian variety $j:X\hookrightarrow A\times\widehat{A}\times B\times\widehat{B}$ such that the canonical morphism $X\longrightarrow A\times\widehat{A}$ is an isomorphism and $\mathcal{J}\cong j_*L$, where L is an invertible sheaf on X . The subvariety X gives the homomorphism from $A\times\widehat{A}$ to $B\times\widehat{B}$, which induces an equivalence of derived categories of coherent sheaves. Hence, this homomorphism is an isomorphism. The proposition is proved. □

Remark. It follows from the proposition that if $\Phi_{\mathcal{E}}$ is an equivalence, then $\mathcal{E}$ gives the isomorphism $f_{\mathcal{E}}:A\times\widehat{A}\longrightarrow B\times\widehat{B}$ with the graf X . We see that $f_{\mathcal{E}}(a,a')=(b,b')$ iff $T^*_{(a,0)}\mathcal{E}\otimes P_{(a',0)}\cong T^*_{(0,-b)}\mathcal{E}\otimes P_{(0,b')}$.

For two abelian varieties A and B denote by $\mathcal{E}q(A,B)$ the set of all exact equivalences, up to isomorphism, from $D^b_{coh}(A)$ to $D^b_{coh}(B)$.

Let us introduce two categories $\mathfrak{A}$ and $\mathfrak{D}$, which are groupoids (i.e. all morphisms are isomorphisms). Objects of both categories are abelian varieties. Morphisms in $\mathfrak{A}$ are homomorphisms between abelian varieties, which are isomorphisms; morphisms in $\mathfrak{B}$ are exact equivalences between derived categories of coherent sheaves on abelian varieties, i.e.

$$\mathcal{M}or_{\mathfrak{A}}(A,\ B):=Iso(A,\ B)$$
$$\mathcal{M}or_{\mathfrak{D}}(A,\ B):=\mathcal{E}q(A,\ B)$$

By Proposition 2.3 , we have a map from the set $\mathcal{E}q(A,B)$ to the set $Iso(A\times\widehat{A},\ B\times\widehat{B})$, which attaches to each equivalence $\Phi_{\mathcal{E}}$ the isomorphism $f_{\mathcal{E}}$. Consider the map $F:\mathfrak{D}\longrightarrow\mathfrak{A}$ that takes an abelian variety A to $A\times\widehat{A}$ and acts on morphisms as above, i.e

$$F(\Phi_{\mathcal{E}})=f_{\mathcal{E}}$$

Proposition 2.4 *In the above notation, the map* $F:\mathfrak{D}\longrightarrow\mathfrak{A}$ *is a functor.*

Proof. We can assume that field k is algebraically closed. Consider three abelian variety A,B,C . Let $\mathcal{E}$ and $\mathcal{F}$ be objects from $D^b_{coh}(A\times B)$ and $D^b_{coh}(B\times C)$ respectively such that the functors

$$\Phi_{\mathcal{E}}:D^b_{coh}(A)\longrightarrow D^b_{coh}(B)$$
$$\Phi_{\mathcal{F}}:D^b_{coh}(B)\longrightarrow D^b_{coh}(C)$$

are equivalence. By $\mathcal{I}$ denote the object $p_{13*}(\mathcal{E} \underset{B}{\boxtimes} \mathcal{F})$ in $D^b_{coh}(A \times C)$.

We must show that

$$f_{\mathcal{I}} = f_{\mathcal{F}} \circ f_{\mathcal{E}}$$

It is sufficient to check for closed point. We know that $f_{\mathcal{E}}(a,a') = (b,b')$ for $(a,a') \in A \times \widehat{A}$ and $(b,b') \in B \times \widehat{B}$ iff $T^*_{(a,0))}\mathcal{E} \otimes P_{(a',0)} \cong T^*_{(0,-b)}\mathcal{E} \otimes P_{(0,b')}$.

Now fix point $(a,a') \in A \times \widehat{A}$. Let $f_{\mathcal{E}}(a,a') = (b,b')$ and $f_{\mathcal{F}}(b,b') = (c,c')$. We must check that $f_{\mathcal{I}}(a,a') = (c,c')$.

First, we have the following sequence of isomorphisms:

$$T^*_{(a,0)}\mathcal{I} \otimes P_{(a',0)} \cong p_{13*}((T^*_{(a,0)}\mathcal{E} \otimes P_{(a',0)}) \underset{B}{\boxtimes} \mathcal{F}) \cong p_{13*}((T^*_{(0,-b)}\mathcal{E} \otimes P_{(0,b')}) \underset{B}{\boxtimes} \mathcal{F})$$

Further, it is clear that for any object $\mathcal{S} \in D^b_{coh}(A \times B \times C)$ we have

$$p_{13*}(T^*_b \mathcal{S}) \cong p_{13*}(\mathcal{S})$$

Consequently, we obtain

$$T^*_{(a,0)}\mathcal{I} \otimes P_{(a',0)} \cong p_{13*}(\mathcal{E} \underset{B}{\boxtimes} (T^*_{(b,0)}\mathcal{F} \otimes P_{(b',0)})) \cong$$

$$p_{13*}(\mathcal{E} \underset{B}{\boxtimes} (T^*_{(0,-c)}\mathcal{F} \otimes P_{(0,c')})) \cong T^*_{(0,-c)}\mathcal{I} \otimes P_{(0,c')}$$

This yields $f_{\mathcal{I}}(a,a') = (c,c')$ and the proposition is proved. □

Thus, we have got the functor $F : \mathfrak{D} \longrightarrow \mathfrak{A}$. Now we would like to describe it. First, not every element from $Iso(A \times \widehat{A},\ B \times \widehat{B})$ can be realized as $f_{\mathcal{E}}$ for some $\mathcal{E}$.

Let us consider a morphism $f : A \times \widehat{A} \longrightarrow B \times \widehat{B}$. It can be written as matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where α is the morphism from A to B , β is from $\widehat{A}$ to B and etc. Any morphism f determines two morphisms $\widehat{f}$ and $\tilde{f}$ from $B \times \widehat{B}$ to $A \times \widehat{A}$. They have the following matrix forms:

$$\widehat{f} = \begin{pmatrix} \widehat{\delta} & \widehat{\beta} \\ \widehat{\gamma} & \widehat{\alpha} \end{pmatrix}; \qquad \tilde{f} = \begin{pmatrix} \widehat{\delta} & -\widehat{\beta} \\ -\widehat{\gamma} & \widehat{\alpha} \end{pmatrix}$$

Let us introduce the set $Sp(A \times \widehat{A},\ B \times \widehat{B})$. It is a subset of $Iso(A \times \widehat{A},\ B \times \widehat{B})$ consisting of all f such that $\tilde{f}$ coincides with inverse for f , i.e.

$$Sp(A \times \widehat{A},\ B \times \widehat{B}) := \left\{ f \in Iso(A \times \widehat{A},\ B \times \widehat{B}) \;\middle|\; \tilde{f} = f^{-1} \right\}$$

(Note that $Sp(A \times \widehat{A})$ is a subgroup of $Aut(A \times \widehat{A})$).

Definition 2.5 *We say that a morphism* $f : A\times\widehat{A}\longrightarrow B\times\widehat{B}$ *is* symplectic *if it belongs to* $Sp(A\times\widehat{A},\ B\times\widehat{B})$.

Proposition 2.6 *For any equivalence* $\Phi_{\mathcal{E}} : D^b_{coh}(A)\longrightarrow D^b_{coh}(B)$ *the morphism* $f_{\mathcal{E}}$ *is symplectic.*

Proof. First, going over to the algebraic closure, we can assume that field k is algebraically closed. To check the equality $\tilde{f}=f^{-1}$ it is sufficient to consider the closed points. Let f takes $(a,a')\in A\times\widehat{A}$ to $(b,b')\in B\times\widehat{B}$. We must show that $\widehat{f}(b,-b')=(a,-a')$.

The isomorphism $f_{\mathcal{E}}$ is given by the abelian subvariety $X\hookrightarrow A\times\widehat{A}\times B\times\widehat{B}$. Consequently, we have to check that $P_{(-a',a,b',-b)}\otimes\mathcal{O}_X\cong\mathcal{O}_X$. It is equivalent to show that $P_{(-a',a,b',-b)}\otimes\mathcal{J}\cong\mathcal{J}$.

From the beginning let us consider the object $\mathcal{H}=p_{23}^*P_A\otimes(m_A,p_4)^*\mathcal{E}$ from $D^b_{coh}(A\times\widehat{A}\times A\times B)$ (see (6)). We have the following sequence of isomorphisms:

$$
\begin{aligned}
&P_{(-a',a,0,0)}\otimes\mathcal{H}\cong p_{23}^*(P_A\otimes P_{(a,0)})\otimes(m_A,p_4)^*\mathcal{E}\otimes P_{(-a',0,0,0)}\cong\\
&p_{23}^*(T^*_{(0,a)}P_A)\otimes(m_A,p_4)^*\mathcal{E}\otimes P_{(-a',0,0,0)}\cong\\
&T^*_{(0,0,a,0)}(p_{23}^*P_A\otimes T^*_{(0,0,-a,0)}((m_A,p_4)^*\mathcal{E})\otimes P_{(-a',0,0,0)})\cong\\
&T^*_{(0,0,a,0)}(p_{23}^*P_A\otimes(m_A,p_4)^*(T^*_{(-a,0)}\mathcal{E}\otimes P_{(-a',0)})\otimes P_{(0,0,a',0)})\cong\\
&T^*_{(0,0,a,0)}(p_{23}^*P_A\otimes(m_A,p_4)^*(T^*_{(0,b)}\mathcal{E}\otimes P_{(0,-b')})\otimes P_{(0,0,a',0)})\cong\\
&T^*_{(0,0,a,b)}(p_{23}^*P_A\otimes(m_A,p_4)^*(\mathcal{E}))\otimes P_{(0,0,a',-b')}\cong T^*_{(0,0,a,b)}\mathcal{H}\otimes P_{(0,0,a',-b')}
\end{aligned}
$$

Further, we have the similar sequence of isomorphisms for the object $\mathcal{G}:=p_{24}^*P_B^{-1}\otimes(p_1,d_B)^*\mathcal{E}^{\vee}$ from $D^b_{coh}(A\times B\times B\times\widehat{B})$:

$$
\begin{aligned}
&\mathcal{G}\otimes P_{(0,0,b',-b)}\cong p_{24}^*(P_B^{-1}\otimes P_{(0,-b)})\otimes(p_1,d_B)^*\mathcal{E}^{\vee}\otimes P_{(0,0,b',0)}\cong\\
&p_{24}^*(T^*_{(b,0)}P_B^{-1})\otimes(p_1,d_B)^*\mathcal{E}^{\vee}\otimes P_{(0,0,b',0)}\cong\\
&T^*_{(0,b,0,0)}(p_{24}^*P_B^{-1}\otimes T^*_{(0,-b,0,0)}((p_1,d_B)^*\mathcal{E}^{\vee})\otimes P_{(0,0,b',0)})\cong\\
&T^*_{(0,b,0,0)}(p_{24}^*P_B^{-1}\otimes(p_1,d_B)^*(T^*_{(0,-b)}\mathcal{E}^{\vee}\otimes P_{(0,-b')})\otimes P_{(0,b',0,0)})\cong\\
&T^*_{(0,b,0,0)}(p_{24}^*P_B^{-1}\otimes(p_1,d_B)^*(T^*_{(a,0)}\mathcal{E}^{\vee}\otimes P_{(-a',0)})\otimes P_{(0,b',0,0)})\cong\\
&T^*_{(a,b,0,0)}(p_{24}^*P_B^{-1}\otimes(p_1,d_B)^*(\mathcal{E}^{\vee}))\otimes P_{(-a',b',0,0)}\cong T^*_{(a,b,0,0)}\mathcal{G}\otimes P_{(-a',b',0,0)}
\end{aligned}
$$

Finally, we know that the object $\mathcal{J}$ is isomorphic to $p_{1256*}(\mathcal{H}\underset{A\times B}{\boxtimes}\mathcal{G})$. Hence, combining the previous calculations, we get

$$
P_{(-a',a,b',-b)}\otimes\mathcal{J}\cong p_{1256*}(T^*_{(0,0,a,b,0,0)}(\mathcal{H}\underset{A\times B}{\boxtimes}\mathcal{G}))\cong\mathcal{J}.
$$

This proves the proposition. □

Propositions 2.3 and 2.6 implies the following theorem

Theorem 2.7 *Let A and B be two abelian varieties. If the derived categories of coherent sheaves $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent as triangulated categories, then $A \times \widehat{A}$ and $B \times \widehat{B}$ are symplectic isomorphic.*

The inverse statement was prove in [Po1] for algebraically closed field. In the next chapter we will give a little different proof of it.

Corollary 2.8 *For each abelian variety A there are only finite number of non-isomorphic abelian varieties , which have the derived categories of coherent sheaves equivalent to $D^b_{coh}(A)$ (as triangulated categories).*

Proof. It was proved in [LOZ] that for each abelian variety Z there are, up to isomorphism, only finitely many abelian varieties that admit an embedding in X as an abelian subvarieties. This statement for $Z = A \times \widehat{A}$ and Theorem 2.7 imply our statement. □

Further, it follows from Propositions 2.4 and 2.6 that we have homomorphism from the group of exact equivalence $\mathrm{Auteq}(D^b_{coh}(A))$ to the group of symplectic automorphisms $Sp(A\times \widehat{A})$. Before to describe the kernel and the image of this homomorphism we show that if the functor $\Phi_{\mathcal{E}}$ is an equivalence then $\mathcal{E}$ have to be a sheaf, up to translation. Note that this fact is not valid in general, i.e. there are examples of equivalences, which are represented by complexes, having more than one cohomology.

Proposition 2.9 *Let $\mathcal{E}$ be an object in $D^b_{coh}(A \times B)$ such that the functor $\Phi_{\mathcal{E}}$: $D^b_{coh}(A) \xrightarrow{\sim} D^b_{coh}(B)$ is an exact equivalence. Then $\mathcal{E}$ has only one cohomology, i.e. it is isomorphic to a sheaf up to translation.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc}
(A \times \widehat{A}) \times (A \times B) \times (B \times \widehat{B}) & \xrightarrow{p} & (A \times \widehat{A}) \times (B \times \widehat{B}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\
A \times B \times A \times B & \xrightarrow{g} & A \times B
\end{array}$$

Here p and q are projections and the morphisms f and g are defined by the following rules:

$$\begin{array}{llll}
f: & (a_1, a', a_2, b_1, b_2, b') & \mapsto & (a_1 + a_2, b_1, a_2, b_1 - b_2) \\
g: & (a_1, b_1, a_2, b_2) & \mapsto & (a_1 - a_2, b_1 - b_2).
\end{array}$$

Now take the object

$$\mathcal{K} := (\mathcal{H} \underset{A\times B}{\boxtimes} \mathcal{G}) \otimes p_{56}^* P_B[2n],$$

which belongs to the derived category of coherent sheaves on $(A\times \widehat{A})\times (A\times B)\times (B\times \widehat{B})$. It follows from definition of $\mathcal{H}$ and $\mathcal{G}$ (see (6)) that

$$f_*\mathcal{K}\cong f_*(p_{23}^*P_A\otimes p_{46}^*P_B^{-1}\otimes p_{56}^*P_B)\otimes (\mathcal{E}\boxtimes \mathcal{E}^{\vee})[2n]\cong \mathcal{O}_M\otimes (\mathcal{E}\boxtimes \mathcal{E}^{\vee}).$$

It is easily to see that M is $A\times B\times\{0\}\times\{0\}$ as abelian subvariety of $A\times B\times A\times B$. Therefore, we get

$$g_*f_*\mathcal{K}\cong \mathcal{E}\otimes (\mathcal{E}^{\vee}\mid_{(0,0)}),$$

where $\mathcal{E}^{\vee}\mid_{(0,0)}$ is a complex of vector spaces that is the inverse image of $\mathcal{E}^{\vee}$ under the closed embedding of the point $(0,0)$ into $A\times B$.

On the other side, it follows from Proposition 2.3 that

$$p_*\mathcal{K}\cong j_*\mathcal{L}$$

where j is an embedding of the abelian variety X into $A\times \widehat{A}\times B\times \widehat{B}$ that gives the isomorphism $f_{\mathcal{E}}$ between $A\times \widehat{A}$ and $B\times \widehat{B}$, and $\mathcal{L}$ is an invertible sheaf on X . Denoting by q' the restriction q on X , we obtain the isomorphism

$$g_*f_*\mathcal{K}\cong q'_*\mathcal{L}$$

We know that q' is a morphism of abelian varieties. Let us denote by d the dimension of $Ker(q')$. Then $\dim Im(q')=2n-d$ and the cohomology sheaves $H^j(q'_*\mathcal{L})$ are trivial for $j\notin[0,d]$.

Now we assume that the last right non-trivial cohomology of $\mathcal{E}$ is $H^0(\mathcal{E})$. Let $H^{-i}(\mathcal{E})$ with $i\geq 0$ be the last left non-trivial cohomology.

Supports of all cohomology sheaves of the object $\mathcal{E}$ are subsets of $Im(q')$. This implies that $\operatorname{codim}\operatorname{Supp}H^{-i}(\mathcal{E})\geq d$. Consequently, all cohomologies of the object $(H^{-i}(\mathcal{E}))^{\vee}[-i]$, which have index less than $i+d$, are trivial.

Further, we have the canonical homomorphism $H^{-i}(\mathcal{E})[i]\longrightarrow \mathcal{E}$. Hence, there exists a non-trivial homomorphism

$$\mathcal{E}^{\vee}\longrightarrow (H^{-i}(\mathcal{E}))^{\vee}[-i]$$

We know that all non-trivial cohomology of the second object lie in $[i+d,\infty]$. Hence, there exists $j\geq i+d$ such that $H^j(\mathcal{E}^{\vee})\neq 0$. Replacing $\mathcal{E}$ with $T^*_{(a,b)}\mathcal{E}$, if it is necessary, we can assume that the point $(0,0)$ is contained in $\operatorname{Supp}H^j(\mathcal{E}^{\vee})$. Consequently, the object

$$q'_*\mathcal{L}\cong g_*f_*\mathcal{K}\cong \mathcal{E}^{\vee}\mid_{(0,0)}\otimes\mathcal{E} \tag{8}$$

has non-trivial cohomology for the same $j \geq i+d$. Hence , we get $i=0$ and the object $\mathcal{E}$ has only one non-trivial cohomology $H^0(\mathcal{E})$. Thus, it is isomorphic to a sheaf. □

Let us put $B = A$. Suppose $\mathcal{E}$ is a sheaf on $A \times A$ such that $\Phi_{\mathcal{E}}$ is an equivalence and $f_{\mathcal{E}}$ is the identity, i.e. the graf X is the diagonal of $(A\times \widehat{A})\times(A\times \widehat{A})$. By $\Delta : A \longrightarrow A \times A$ denote the diagonal embedding. It follows from (8) that there exists an object W on A such that $\Delta_*(W) \cong \mathcal{E}^{\vee}\mid_{(0,0)} \otimes \mathcal{E}$.

First, assume that the point $(0,0)$ belongs to $\operatorname{Supp}\mathcal{E}$. This implies that $\mathcal{E}^{\vee}\mid_{(0,0)}$ is non-trivial complex of vector spaces. Hence, there exists some sheaf E on A such that $\Delta_*(E) \cong \mathcal{E}$. Therefore, we have $\Phi_{\mathcal{E}}(\cdot) \cong E \otimes (\cdot)$. Since $\Phi_{\mathcal{E}}$ is an equivalence, E is an invertible sheaf. It is easily to check that $E \in \operatorname{Pic}^0(A)$, because $f_{\mathcal{E}}$ is the identity.

If now the point $(0,0)$ does not belong to $\operatorname{Supp}\mathcal{E}$, we take a sheaf $\mathcal{E}' := T^*_{(a_1,a_2)}\mathcal{E}$ such that its support contains $(0,0)$. It is clear that $f_{\mathcal{E}'} = f_{\mathcal{E}}$. As above, we get $\mathcal{E}' \cong \Delta_*(E')$ with $E' \in \operatorname{Pic}^0(A)$. Hence, $\mathcal{E} \cong T^*_{(a_1-a_2,0)}\Delta_*(E')$. Thus, we have proved the following corollary.

Corollary 2.10 *Let A be an abelian variety. Then the kernel of the homomorphism $\operatorname{Auteq}(D^b_{coh}(A)) \longrightarrow Sp(A\times\widehat{A})$ is isomorphic to the group $\mathbb{Z}\oplus(A\times\widehat{A})_k$, where $(A\times\widehat{A})_k$ denotes the group of k -points of the abelian variety $A\times\widehat{A}$.*

3 Semi-homogeneous vector bundles.

In the previous chapter we showed that the morphism $f_{\mathcal{E}}$ is symplectic. Now, using the technique of [Mu2] and results from [BO1], we are going to prove that every symplectic morphism $f: A\times\widehat{A} \longrightarrow B\times\widehat{B}$ can be realized the same way. This fact can be deduced from [Po1] and [Po2]. It was proved in the paper [Po1] that if the variety $A\times\hat{A}$ and $B\times\hat{B}$ are symplectically isomorphic, then the derived categories $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent. We will give a little different proof for this statement.

Henceforth, we assume that k is an algebraically closed field and $\operatorname{char}(k)=0$.

Definition 3.1 *A vector bundle on an abelian variety Z is* semi-homogeneous *if for every $z \in Z$, there exists a line bundle L on Z such that $T^*_z(E)\cong E\otimes L$.* (Note that L has to belong to $\operatorname{Pic}^0(Z)$).

The following theorem was proved in [Mu2].

Proposition 3.2 ([Mu2], Th.5.8.) *Let E be a simple vector bundle on Z . Then* *the following conditions are equivalent:*

(1) $\dim \mathrm{H}^j(Z, \mathcal{E}nd(E)) = \binom{n}{j}$ *for all* $j = 0, ..., n,$

(2) E *is semi-homogeneous,*

(3) $\mathcal{E}nd(E)$ *is a homogeneous vector bundle,*

(4) *there exist an isogeny* $\pi : Y \longrightarrow Z$ *and a line bundle* L *on* Y *such that* $E \cong \pi_*(L)$

Let E be a vector bundle on Z . We denote by $\mu(E)$ the equivalence class of $\frac{det(E)}{r(E)}$ in $\mathrm{NS}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$.

It is well-known that any line bundle L on Z gives the homomorphism $\phi_L : Z \longrightarrow \widehat{Z}$ and this mapping induces the canonical injection of $\mathrm{NS}(Z)$ into $\mathrm{Hom}(Z, \widehat{Z})$ (see, for example, [Mum]). Further, every element $\mu = \frac{[L]}{l} \in \mathrm{NS}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ defines the correspondence $\Phi_\mu \subset Z \times \widehat{Z}$. It is clear that $\Phi_\mu = \mathrm{Im}\Big[Z \stackrel{(l,\phi_L)}{\longrightarrow} Z \times \widehat{Z}\Big]$. Denote by q_1 and q_2 the projections of Φ_μ onto Z and $\widehat{Z}$ respectively.

The following proposition was proved in [Mu2].

Proposition 3.3 ([Mu2], Th.7.10.) *Let* $\mu = \frac{[L]}{l}$ *, where* $[L]$ *is the equivalence class in* $\mathrm{NS}(Z)$ *of a line bundle* L *and* l *is a positive integer.*

(1) *There exists a simple semi-homogeneous vector bundle* E *with* $\mu(E) = \mu$

(2) *Every simple semi-homogeneous vector bundle* E' *with* $\mu(E') = \mu$ *is isomorphic to* $E \otimes M$ *for some line bundle* M *in* $\mathrm{Pic}^0(Z)$

(3) $r(E)^2 = \deg(q_1)$, *and* $\chi(E)^2 = \deg(q_2)$

It is easy to show that for any point $(z, z') \in \Phi_\mu$ we have isomorphism $T_z^*(E) \cong E \otimes P_{z'}$. Actually, let $l = r(E)$ and $L = det(E)$. We know that $(z, z') = (lx, \phi_L(x))$ for some point $x \in Z$. Moreover, since E is semi-homogeneous, we have $T_x^*(E) \cong E \otimes M$ for some line bundle M in $\mathrm{Pic}^0(Z)$. Taking determinants, we obtain $T_x^*(L) \cong L \otimes M^{\otimes l}$. It means that $P_{\phi_L(x)} = M^{\otimes l}$. On the other side, we have $T_{lx}^*(E) \cong E \otimes M^{\otimes l}$. Since $(z, z') = (lx, \phi_L(x))$, we get $T_z^*(E) \cong E \otimes P_{z'}$.

Now let us introduce a closed subgroup $\Sigma^0(E) \subset \widehat{Z}$ defined by the following rule

$$\Sigma^0(E) := \{z' \in \widehat{Z} \mid E \otimes P_{z'} \cong E\}$$

If a bundle E is simple semi-homogeneous, then by Proposition 3.2 , there exist an isogeny $\pi : Y \longrightarrow Z$ and a line bundle M on Y such that $E \cong \pi_*(M)$. It is clear that $\pi^*(E) \cong M_1 \oplus M_2 \oplus \cdots \oplus M_r$ for some line bundles M_i on Y with $M_1 = M$. If $z' \in \Sigma^0(E)$ then $\pi^*(P_{z'}) \otimes M \cong M_i$ for $i = 1, .., r$. This yields that $\mathrm{ord}\Sigma^0(E) \le r^2$. On the other side, we have $q_{2*}(Ker(q_1)) \subset \Sigma^0(E)$. Hence, we obtain $\mathrm{ord}\Sigma^0(E) = r^2$ and $q_{2*}(Ker(q_1)) = \Sigma^0(E)$.

Further, let us suppose that $T_z^*(E) \cong E \otimes P_{z'}$ for some point $(z,z') \in Z \times \widehat{Z}$. Since the homomorphism $\Phi_\mu \longrightarrow Z$ is an isogeny, there exists a point $z^{''} \in \widehat{Z}$ such that $(z,z^{''}) \in \Phi_\mu$. Hence, we obtain $E \otimes P_{(z^{'}-z^{''})} \cong E$. Consequently, $(z^{'} - z^{''}) \in \Sigma^0(E)$. But we have proved that $q_{2*}(Ker(q_1)) = \Sigma^0(E)$. This immediately implies that $(0, z^{'} - z^{''}) \in \Phi_\mu$. Therefore, we get $(z,z') \in \Phi_\mu$. Thus, we have proved

Lemma 3.4 *In the above notation, if E is a simple semi-homogeneous vector bundle on Z . Then $T_z^*(E) \cong E \otimes P_{z'}$ iff $(z,z') \in \Phi_\mu$.*

Now we are ready to prove that if $f \in Sp(A \times \widehat{A},\ B \times \widehat{B})$ then there exists a sheaf $\mathcal{E}$ on $A \times B$ such that $f_{\mathcal{E}} = f$.

Let us fix an symplectic isomorphism $f : A \times \widehat{A} \longrightarrow B \times \widehat{B}$. Let X be the graf of f . Isomorphism f can be written as matrix

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

We can assume that $\beta : \widehat{A} \longrightarrow B$ is an isogeny, because every f can be represented as a product of $f_1 \in Sp(A \times \widehat{A},\ B \times \widehat{B})$ and $f_2 \in Sp(A \times \widehat{A})$ such that β_1 and β_2 are isogenies.

Since β is an isogeny, we can attach to f the element $g \in \mathrm{Hom}(A \times \widehat{A},\ B \times \widehat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ that has the following matrix form:

$$g = \begin{pmatrix} \beta^{-1}\alpha & -\beta \\ -\widehat{\beta}^{-1} & \delta\beta^{-1} \end{pmatrix}$$

As f is symplectic, it can easily be checked that $\widehat{g} = g$. There exists the canonical injection of $\mathrm{NS}(A \times B) \otimes_{\mathbb{Z}} \mathbb{Q}$ into $\mathrm{Hom}(A \times \widehat{A},\ B \times \widehat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and it is known that an element g belongs to the image of this injection iff $\widehat{g} = g$ (see [Mum]). This yields that there exists $\mu = \frac{[L]}{l} \in \mathrm{NS}(A \times B)$ such that Φ_μ coincides with the graf of the correspondence g . By Proposition 3.3 , we can find a simple semi-homogeneous bundle $\mathcal{E}$ on $A \times B$ with $\mu(\mathcal{E}) = \mu$.

We affirm that $\Phi_{\mathcal{E}}$ is an equivalence and $f_{\mathcal{E}} = f$. Before to prove it let us compare X and Φ_μ , which are the grafs of f and g . If a point (a, a', b, b') is contained in X , then

$$\begin{array}{rcl} b & = & \alpha(a) + \beta(a') \\ b' & = & \gamma(a) + \delta(a') \end{array} \quad \text{and, consequently} \quad \begin{array}{rcrcl} a' & = & -\beta^{-1}\alpha(a) & + & \beta^{-1}(b) \\ b' & = & (\gamma - \delta\beta^{-1}\alpha)(a) & + & \delta\beta^{-1}(b) \end{array}$$

Since f is symplectic, we have the equality $(\gamma - \delta\beta^{-1}\alpha) = \widehat{\beta}^{-1}$. This follows that (a, a', b, b') belongs to X iff $(a, -a', b, b')$ belongs to Φ_μ . Consequently, $\Phi_\mu = (1_A, -1_{\widehat{A}}, 1_B, 1_{\widehat{B}})X$.

Proposition 3.5 *In the above notation, the functor* $\Phi_{\mathcal{E}} : D^b_{coh}(A) \longrightarrow D^b_{coh}(B)$ *is an equivalence.*

Proof. Let us denote by $\mathcal{E}_a$ the restriction of the bundle $\mathcal{E}$ on $\{a\} \times B$. By Theorem 1.5 , to prove that $\Phi_{\mathcal{E}}$ is full and faithful we must check that $\mathcal{E}_a$ are simple and they are orthogonal to each other for different points.

First, notice that the rank of the bundle $\mathcal{E}$ is equal to $\sqrt{\deg(\beta)}$. This follows from Proposition 3.3 and from the fact that the degree of the homomorphism $X \longrightarrow A \times B$ is equal to $\deg(\beta)$.

It is clear that $\mathcal{E}_a$ is semi-homogeneous (see (4) of Prop.3.2) and $\mu(\mathcal{E}_a) = \delta\beta^{-1} \in \mathrm{NS}(B) \otimes \mathbb{Q} \subset \mathrm{Hom}(B, \widehat{B}) \otimes \mathbb{Q}$. By ν denote $\delta\beta^{-1}$ as the element of $\mathrm{NS}(B) \otimes \mathbb{Q}$. By Proposition 3.3 , there exists a simple semi-homogeneous vector bundle F on B with $\mu(F) = \nu$. It is easily to see that $\Phi_\nu = \mathrm{Im}\Big[\widehat{A} \stackrel{(\beta,\delta)}{\longrightarrow} B \times \widehat{B}\Big]$. Since f is an isomorphism, the homomorphism $\widehat{A} \stackrel{(\beta,\delta)}{\longrightarrow} B \times \widehat{B}$ is an embedding. Hence, we obtain $r(F) = \sqrt{\deg(\beta)} = r(\mathcal{E}_a)$.(Moreover, we can see that the canonical projection $\Sigma^0(\mathcal{E}) \longrightarrow \Sigma^0(\mathcal{E}_a)$ is an isomorphism.) The bundles F and $\mathcal{E}_a$ are semi-homogeneous and they have the same slope $\mu \in \mathrm{NS}(B)$ and the same rank. Moreover, F is simple. Hence, $\mathcal{E}_a$ is simple too (see [Mu2] 6.18).

Further, since $\mathcal{E}$ is semi-homogeneous, we have $\mathcal{E}_a \cong \mathcal{E}_0 \otimes M$ for some $M \in \mathrm{Pic}^0(B)$. It follows from condition (4) of Proposition 3.2 and the projection formula that either there exists non-trivial morphism $\mathcal{E}_0 \longrightarrow \mathcal{E}_0 \otimes M$ or they are orthogonal, i.e.

$$\mathrm{Hom}^i(\mathcal{E}_0,\, \mathcal{E}_0 \otimes M) = 0, \quad \text{and} \quad \mathrm{Hom}^i(\mathcal{E}_0 \otimes M,\, \mathcal{E}_0) = 0 \quad \text{for all } i .$$

But, it is known that a simple semi-homogeneous vector bundle is stable (see [Mu2] 6.15.). Hence, we have either $\mathcal{E}_0$ and $\mathcal{E}_0 \otimes M$ are isomorphic or they are orthogonal.

Now let us assume that $\mathcal{E}_0$ is isomorphic to $\mathcal{E}_a$ for some point $a \in A$. Since $\mathcal{E}$ is semi-homogeneous, we have $T^*_{(a,0)}\mathcal{E} \cong \mathcal{E} \otimes P_{(a',b')}$ for some point $(a',b') \in \widehat{A} \times \widehat{B}$. This implies

$$\mathcal{E}_a \otimes P_{b'} \cong \mathcal{E}_0 \cong \mathcal{E}_a$$

Hence, $P_{b'} \in \Sigma^0(\mathcal{E}_a)$. As the morphism $\Sigma^0(\mathcal{E}) \longrightarrow \Sigma^0(\mathcal{E}_a)$ is an isomorphism, there exists $a'' \in \widehat{A}$ such that $\mathcal{E} \otimes P_{(a'',b')} \cong \mathcal{E}$. Therefore, we obtain $T^*_{(a,0)}\mathcal{E} \cong \mathcal{E} \otimes P_{(a'-a'',0)}$. Hence, by Lemma 3.4 , we obtain $(a, a'-a'', 0, 0) \in \Phi_\mu$. On the other hand, as f is an isomorphism, the projection $\Phi_\mu \longrightarrow A \times \widehat{A}$ is an isomorphism too. This immediately implies $a = 0$. Thus, $\mathcal{E}_{a_1}$ and $\mathcal{E}_{a_2}$ are orthogonal for any two different points a_1 and a_2. By Theorem 1.5 , the functor $\Phi_{\mathcal{E}} : D^b_{coh}(A) \longrightarrow D^b_{coh}(B)$ is full and faithful. By the same reason the adjoint functor $\Psi_{\mathcal{E}^\vee}$ is full and faithful too. This yields that $\Phi_{\mathcal{E}}$ is an equivalence. The proposition is proved. $\square$

By Proposition 2.3 , the equivalence $\Phi_{\mathcal{E}}$ gives the isomorphism $f_{\mathcal{E}} : A \times \widehat{A} \longrightarrow B \times \widehat{B}$ and $f_{\mathcal{E}}(a, a') = (b, b')$ iff $T^*_{(a,b)}\mathcal{E} \cong \mathcal{E} \otimes P_{(-a',b')}$. Hence, if Y is the graf of $f_{\mathcal{E}}$, then $Y = (1_A, -1_{\widehat{A}}, 1_B, 1_{\widehat{B}})\Phi_\mu$ for $\mu = \mu(\mathcal{E})$. Therefore, the graf $f_{\mathcal{E}}$ coincides with the graf of f , i.e $f_{\mathcal{E}} = f$.

Thus, we have proved the following theorems

Theorem 3.6 *Let A and B be abelian varieties over algebraically closed field k with $\mathrm{char}(k) = 0$. Then the derived categories of coherent sheaves $D^b_{coh}(A)$ and $D^b_{coh}(B)$ are equivalent as triangulated categories iff there exists a symplectic isomorphism $f : A \times \widehat{A} \xrightarrow{\sim} B \times \widehat{B}$*

Theorem 3.7 *Let A be an abelian variety over algebraically closed field k with $\mathrm{char} k = 0$. Then the group of exact autoequivalence, up to isomorphism, $\mathrm{Auteq}(D^b_{coh}(A))$ can be included into the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \oplus (A \times \widehat{A})_k \longrightarrow \mathrm{Auteq}(D^b_{coh}(A)) \longrightarrow Sp(A \times \widehat{A}) \longrightarrow 1$$

Let us describe the group $\mathrm{Auteq}(D^b_{coh}(A))$ more precisely. It has the normal subgroup $(A \times \hat{A})_k$, which consists of the functors $T^*_a(\cdot) \otimes P_{a'}$ with $(a, a') \in A \times \hat{A}$. The quotient group is a central extension of $Sp(A \times \hat{A})$ with $\mathbb{Z}$. We denote this central extension by $\widetilde{Sp}(A \times \hat{A})$. We have the short exact sequences:

$$0 \longrightarrow (A \times \hat{A})_k \longrightarrow \mathrm{Auteq}(D^b_{coh}(A)) \longrightarrow \widetilde{Sp}(A \times \hat{A}) \longrightarrow 1 \tag{9}$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{Sp}(A \times \hat{A}) \longrightarrow Sp(A \times \hat{A}) \longrightarrow 1 \tag{10}$$

To describe the central extension (10) it is sufficient to give a 2-cocycle $\lambda(g_1, g_2) \in \mathbb{Z}$, where g_1 and g_2 are elements of $Sp(A \times \hat{A})$.

By Proposition 2.9 , if $\Phi_{\mathcal{E}}$ is an equivalence, then $\mathcal{E} \in D^b_{coh}(A \times A)$ is isomorphic to $E[k]$ for some sheaf E on $A \times A$. Let E and F be some sheaf on $A \times A$ such that Φ_E and Φ_F are equivalence. The composition $\Phi_F \circ \Phi_E$ is represented by the object $G[k]$, where G is a sheaf. Let us put $\lambda(f_F, f_E) = k$. It is obvious that λ is 2-cocycle on the group $Sp(A \times \hat{A})$.

We can calculate λ explicitly. Suppose $k = \mathbb{C}$. Then $A \cong \mathbb{C}^n/\mathbb{Z}^{2n}$ and we can associated with any line bundle L on A the Hermitian form $H(L)$ on $\mathbb{C}^n$ (see [Mum]). Denote by $p(H)$ the number of positive eigenvalues of H . Thus, we get the function $p : \mathrm{NS}(A) \to \mathbb{Z}$. We can extend it to $\mathrm{NS}(A) \otimes \mathbb{R}$ by the rule: $p(r[L]) = p(H(L))$ for $r \in \mathbb{R}_+$. Let us take two elements of $Sp(A \times \hat{A})$:

$$g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \tag{11}$$

such that β_1 and β_2 are isogenies. This means that there exist the inverses $\beta_1^{-1}, \beta_2^{-1} \in \mathrm{Hom}(A,\, \hat{A}) \otimes \mathbb{Q}$. Further, let us write:

$$g_1 g_2 = \begin{pmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{pmatrix}$$

Now consider the element $\beta_1^{-1}\beta_3\beta_2^{-1} = \beta_1^{-1}\alpha_1 + \delta_2\beta_2^{-1}$ of $\mathrm{Hom}(A,\, \hat{A})\otimes\mathbb{Q}$. It belongs to $\mathrm{NS}(A) \otimes \mathbb{Q}$, hence it can be written as $\frac{[L]}{l}$ for some line bundle L on A and natural integer l . It is not hard to prove that

$$\lambda(g_1, g_2) = p(\beta_1^{-1}\beta_3\beta_2^{-1}) - n.$$

Thus, we have given the formula for $\lambda(g_1, g_2)$, when β_1 and β_2 are invertible. Since λ is a cocycle, it is uniquely determined by this formula. Moreover, we see that our cocycle can be extend onto the group $Sp(A \times \hat{A},\, \mathbb{R}) \subset \mathrm{End}(A \times \hat{A}) \otimes \mathbb{R}$.

Now suppose $A = E^n$, where E is an elliptic curve without complex multiplication, then the group $Sp(A \times \hat{A})$ is isomorphic to $\mathrm{Sp}_{2n}(\mathbb{Z})$. Consider the real symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{R})$. It is known that the Maslov index defines the $\mathbb{Z}$ -valued 2-cocycle on the symplectic group by the rule:

$$\mu_l(g_1, g_2) = \tau(l, g_1 l, g_1 g_2 l),$$

where l is some Lagrangian plane, and τ is the Maslov index (see [LV]). There is the formula for the cocycle μ . If g_1 and g_2 are written as matrices like in (11), then

$$\mu_l(g_1, g_2) = sign(\beta_1^{-1}\beta_3\beta_2^{-1}),$$

where the right side denotes the signature of symmetric matrix $\beta_1^{-1}\beta_3\beta_2^{-1}$. We can compare the cocycles λ and μ . It is easy to check that cocycle $2\lambda - \mu$ is trivial as element of the second cohomology , i.e it gives the splitting extension. The symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{R})$ has the universal covering group $\widetilde{G}$. It is known that the central extension $\widetilde{G}_\mu$, which is given by the cocycle μ , is included in the exact sequence $1 \to \widetilde{G} \to \widetilde{G}_\mu \to \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$. Hence, we have the exact sequence $1 \to \widetilde{G} \to \widetilde{G}_\lambda \to \mathbb{Z}/2\mathbb{Z} \to 0$ for $\widetilde{G}_\lambda$. It is easily to show that this sequence splits. Therefore, $\widetilde{G}_\lambda$ is isomorphic to $\widetilde{G} \times \mathbb{Z}/2\mathbb{Z}$.

Example. Let A be an abelian variety such that $\mathrm{End}(A) = \mathbb{Z}$. Then the Neron-Severy group $\mathrm{NS}(A)$ is isomorphic to $\mathbb{Z}$. Let L and M be a generators of $\mathrm{NS}(A)$ and $\mathrm{NS}(\hat{A})$ respectively such that $\phi_M \circ \phi_L = N \cdot id_A$ with $N > 0$. In this case the group $Sp(A \times \hat{A})$ is isomorphic to congruenz-subgroup $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$. Further, if B is an abelian variety such that $B \times \hat{B}$ is isomorphic to $A \times \hat{A}$, then B can be represented as the image of some morphism $A \stackrel{k\cdot id, m\phi_L}{\longrightarrow} A\times\hat{A}$. We can assume that $g.c.d.(k,m)=1$. Denote by ψ this morphism from A to B . The kernel of ψ is equal to $Ker(m\phi_L)\cap A_k$. Since $g.c.d.(k,m)=1$, we get $Ker(\psi)=Ker(\phi_L)\cap A_k$. Besides, we know that $Ker(\phi)\subset A_N$. Hence, we can assume that k is a divisor of N . Thus, any k , which is a divisor of N , gives us the abelian variety B , where $B:=A/(Ker(\phi_L)\cap A_k)$. It is clear that two different divisors determine non-isomorphic abelian varieties. Moreover, it is easily to prove that the embedding B into $A\times\hat{A}$ is split iff $g.c.d.(k,\frac{N}{k})=1$. Therefore, the number of abelian varieties B such that $D^b_{coh}(B)\simeq D^b_{coh}(A)$ is equal to 2^s , where s is the number of the prime divisors of N .

Let us assume, in addition, that A is principal polarized. So we have $N=1$ and if $D^b_{coh}(A)\simeq D^b_{coh}(B)$ for an abelian variety B , then $B\cong A$. Moreover, the group $Sp(A\times\hat{A})$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. Further, it was proved in [Po2] that the sequence (9) splits for principal polarized abelian variety. Finally, we get the description of $\mathrm{Auteq}(D^b_{coh}(A))$ as semi-direct product of normal subgroup $(A\times\hat{A})_k$ and the group $\widetilde{Sp}(A\times\hat{A})$, i.e.

$$\mathrm{Auteq}(D^b_{coh}(A))\cong \widetilde{Sp}(A\times\hat{A})\ltimes(A\times\hat{A})_k$$

and $\widetilde{Sp}(A\times\hat{A})$ is the central extension of $\mathrm{SL}(2,\mathbb{Z})$, which is determined by cocycle λ . We can include it in the following exact sequence:

$$1\longrightarrow B_3\longrightarrow \widetilde{Sp}(A\times\hat{A})\longrightarrow \mathbb{Z}/2n\mathbb{Z},$$

where B_3 is the braid group of 3 threads. (B_3 is the central extension of $\mathrm{SL}(2,\mathbb{Z})$, which is induced by the universal covering of $\mathrm{SL}(2,\mathbb{R})$) .

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