Quadratic algebras related to elliptic curves

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Abstract

We construct quadratic finite-dimensional Poisson algebras and their quantum versions related to rank N and degree one vector bundles over elliptic curves with n marked points. The algebras are parameterized by the moduli of curves. For N=2 and n=1 they coincide with the Sklyanin algebras. We prove that the Poisson structure is compatible with the Lie-Poisson structure on the direct sum of n copies of sl(N). The derivation is based on the Poisson reduction from the canonical brackets on the affine space over the cotangent bundle to the groups of automorphisms of vector bundles.

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1 Introduction

In this article we construct quadratic Poisson algebras (the classical Sklyanin-Feigin-Odesskii algebras) based on the exchange relations with the Belavin-Drinfeld elliptic $\mathrm{sl}(N,\mathbb{C})$ r-matrix [2] and their quantum version related to the vertex elliptic matrix [1]. These algebras are parameterized by the moduli space of complex structures of elliptic curves with n marked points and the Planck constant living on curves in the quantum case. For SL_2 and $n = 1$ we come to the original Sklyanin algebra [8]. The constructed algebras are particular case of general construction [5], but in contrast of the generic case they are finitely generated. We describe explicitly the Poisson brackets between the generators and the corresponding quadratic relations in the quantum case in terms of quasi-periodic functions on the moduli space. In the classical case the Poisson algebras have a form of quadratic algebras on the direct product of n copies of $\mathrm{GL}(N,\mathbb{C})$ with a nontrivial mixing of the components. On the other hand, there exists the standard linear Lie-Poisson structure on direct sum $\oplus_{a=1}^{n}$ $\mathrm{Lie}(\mathrm{GL}(N,\mathbb{C}))$. We prove that the both Poisson structures are compatible.

The classical algebras define symmetries of the elliptic generalization of the Schlesinger and the Garnier systems [9, 4].

In Section 2 we derive the classical vertex r-matrix and the $\mathrm{GL}(N,\mathbb{C})$-valued Lax matrix with n simple poles from the canonical brackets on some generalization of the cotangent bundle of the $\mathrm{GL}(N,\mathbb{C})$ two-loop group by the Poisson reduction. In section 3 we present the explicit form of the brackets and prove that they are compatible with the Lie-Poisson brackets. The section 4 is devoted to the quantum case.

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2 Classical exchange relations from $\mathrm{GL}(N,\mathbb{C})$ two-loop group

2.1 Degree one vector bundles over elliptic curves

Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be an elliptic curve, with the modular parameter τ, $(\Im m\tau > 0)$. Consider a vector bundle E_N of a rank N over Σ_τ. It is described by its sections $s = (s_1(z,\bar z),\dots,s_N(z,\bar z))$ with monodromies

$$s^T(z+1,\bar z+1) = Qs^T(z,\bar z)\,,\quad s^T(z+\tau,\bar z+\bar\tau) = \tilde\Lambda s^T(z,\bar z)\,,$$

where

$$Q = \mathrm{diag}(1,\mathbf{e}_N,\dots,\mathbf{e}_N^{N-1})\,,\ \mathbf{e}_N = \exp\frac{2\pi\imath}{N}\,,\ \tilde\Lambda = \mathbf{e}_N^{-(z+\frac{\tau}{2})}\Lambda\,,\ \ \Lambda = (E_{j,j+1})\,,$$

where $E_{j,j+1}$ is a matrix with a unity on the $(j,j+1)$ place. Since $\det Q = \pm 1$ and $\det\tilde\Lambda = \pm\mathbf{e}_1^{-(z+\frac{\tau}{2})}$ the determinants of the transition matrices have the same quasi-periods as the Jacobi theta-functions. The theta-functions have simple poles in the fundamental domain Σ_τ. Thereby, the vector bundle E_N has degree one.

One can choose a holomorphic section $(\bar\partial s = 0)$ in the form

$$s(z) = (\theta\begin{bmatrix}\frac{1}{N}\\ 0\end{bmatrix}(z;N\tau),\dots,\theta\begin{bmatrix}1\\ 0\end{bmatrix}(z;N\tau))\,.$$

Define the transformations $s^T \to f(z,\bar z)s^T$ by smooth maps $f:\Sigma_\tau\to\mathrm{GL}(N,\mathbb{C})$ $(f\in\Omega^{(0,0)}_{C^\infty}(\Sigma_\tau,\mathrm{GL}(N,\mathbb{C})))$, with monodromies

$$f(z+1,\bar z+1) = Q^{-1}f(z,\bar z)Q\,,\quad f(z+\tau,\bar z+\bar\tau) = \tilde\Lambda^{-1}f(z,\bar z)\tilde\Lambda\,.$$

They preserve the degree of E_N and therefore generate the gauge group $\mathcal{G} = \{f(z,\bar z)\}$ of E_N.

In general, the operators

$$d_{\bar A} = \bar\partial + \bar A\,:\,\Omega^{(0,0)}(\Sigma_\tau,E_N)\to\Omega^{(0,1)}(\Sigma_\tau,E_N)$$

define a complex structure of E_N. A section is holomorphic if $d_{\bar A}(s^T) = 0$. Here we assume that $\bar A$ has the same monodromies as the sections of E_N

$$\bar A(z+1,\bar z+1) = Q^{-1}\bar A(z,\bar z)Q\,,\quad \bar A(z+\tau,\bar z+\bar\tau) = \tilde\Lambda^{-1}\bar A(z,\bar z)\tilde\Lambda\,. \tag{2.1}$$

Two complex structures, defined by $\bar A$ and $\bar A^f$, are called equivalent if they are related by the gauge transform

$$\bar A^f = f^{-1}\bar Af + f^{-1}\bar\partial f\,,\quad f\in\mathcal{G}\,. \tag{2.2}$$

The quotient of the space of generic connections $\mathcal{A} = \{\bar A\}$ with respect to the $\mathcal{G}$-action is the moduli space of holomorphic bundles $Bun(E_N) = \mathcal{A}/\mathcal{G}$.

Consider the two-loop group $LL(\mathrm{GL}(N,\mathbb{C}))$ represented by the space of sections $\{g(z,\bar z)\} = \Omega^{(0,0)}_{C^\infty}(\Sigma_\tau,\mathrm{GL}(N,\mathbb{C}))$ with the monodromies

$$g(z+1,\bar z+1) = Q^{-1}g(z,\bar z)Q\,,\quad g(z+\tau,\bar z+\bar\tau) = \tilde\Lambda^{-1}g(z,\bar z)\tilde\Lambda\,. \tag{2.3}$$

The two-loop group $LL(\mathrm{GL}(N,\mathbb{C}))$ with these quasi-periodicity conditions is the group $AutE_N$ of automorphisms of the degree 1 vector bundle E_N.

2.2 The Poisson structure on $\mathcal{R}$

The space $\mathcal{R} = \mathcal{A} \times \Omega^{(0,0)}_{C^\infty}(\Sigma_\tau, \mathrm{GL}(N,\mathbb{C})) = \{(\bar\partial + \bar A, g)\}$ can be endowed with the symplectic form

$$\omega = \int_{\Sigma_\tau} K\langle d(\bar A g^{-1}) \wedge dg\rangle + \frac{1}{2}\int_{\Sigma_\tau} K\langle g^{-1}dg \wedge \bar\partial(g^{-1}dg)\rangle\,, \tag{2.4}$$

where $\langle\,,\,\rangle$ is the trace in the vector representation, and K is a section of the canonical bundle over Σ_τ ($K \in \Omega^{(1,0)}(\Sigma_\tau)$). We choose $K = dz$. The space $\mathcal{R}$ is the affine space over the cotangent bundle to the two-loop group $T^*(LL(\mathrm{GL}(N,\mathbb{C})))$.

The transformations (2.2) along with

$$g \to f^{-1} g f \tag{2.5}$$

are canonical with respect to the symplectic form (2.4). The Hamiltonian vector fields V_ϵ, ($\epsilon \in Lie(\mathcal{G})$) on $\mathcal{R}$ ($V_\epsilon \omega = d\mu^*$) are generated by the Hamiltonian

$$\mu^*(\epsilon; \bar A, g) = \int_{\Sigma_\tau} K\langle \epsilon(g\bar A g^{-1} - \bar\partial g g^{-1} - \bar A)\rangle\,. \tag{2.6}$$

Remark 2.1 *Let $\Phi \in \Omega^{(1,0)}_{C^\infty}(\Sigma_\tau, End E_N)$ be the Higgs field and $g = \exp(\hbar K^{-1}\Phi)$, where $\hbar \in \mathbb{C}$. In the limit $\hbar \to 0$*

$$g \sim K^{-1}(Id + \hbar\Phi + \ldots)\,. \tag{2.7}$$

The form (2.4) in the first order becomes the canonical form on the Higgs bundle $\{(d_{\bar A}, \Phi)\}$. The symmetries defines the Hamiltonian $\mu^*(\epsilon; \bar A, \Phi) = \int_{\Sigma_\tau}\langle\epsilon(\bar\partial\Phi + [\bar A, \Phi])\rangle$. Thus, $\mathcal{R}$ is a deformation of the Higgs bundle.

The inversion of (2.4) defines the Poisson structure on $\mathcal{R}$. In terms of coordinates in the basis (B.4)
$\bar A = \sum_{\alpha\in\tilde{\mathbb{Z}}^{(2)}_N} \bar A_\alpha T_\alpha$ and $g = \sum_{a\in\mathbb{Z}^{(2)}_N} g_a T_a$ it takes the form

$$K\{\bar A_\alpha(z,\bar z), \bar A_\beta(w,\bar w)\} = C_{\alpha+\beta}\bar A_{\alpha+\beta}\delta(z-w, \bar z - \bar w) + \bar\partial\delta(z-w,\bar z-\bar w)\delta_{\alpha,-\beta}\,, \tag{2.8}$$

$$K\{g_a(z,\bar z), \bar A_\beta(w,\bar w)\} = \mathbf{e}_N^{(a\times\beta)} g_{a+\beta}(z,\bar z)\delta(z-w,\bar z-\bar w)\,, \tag{2.9}$$

$$\{g_a(z,\bar z), g_b(w,\bar w)\} = 0\,. \tag{2.10}$$

The brackets define the Poisson algebra $\mathcal{O}(\mathcal{R})$ with the symmetry group $\mathcal{G}$.

Define a Poisson subalgebra $\mathbf{P}_{\Sigma_\tau}$ of $\mathcal{O}(\mathcal{R})$. It satisfies the following conditions:

1. The connection $\bar A$ takes values in the subalgebra $\mathrm{sl}(N,\mathbb{C})$, while the field g is still takes value in $\mathrm{GL}(N,\mathbb{C})$;

2. $\mathbf{P}_{\Sigma_\tau}$ is generated by holomorphic functionals over $\mathcal{R}$ with the test functions vanishing at $z = 0$.

The subalgebra $\mathbf{P}_{\Sigma_\tau}$ has a center $\mathcal{Z}$ generated by $\det g(z,\bar z)$. The symmetry group $\mathcal{G}^s \subset \mathcal{G}$ of $\mathbf{P}_{\Sigma_\tau}$ is generated by the smooth maps $f : \Sigma_\tau \to \mathrm{SL}(N,\mathbb{C})$.

2.3 The Poisson reduction

Our goal is calculating the reduced Poisson structure with respect to the $\mathcal{G}^s$-action. The standard Poisson reduction $\mathbf{P}^{red}_{\Sigma_\tau}$ of $\mathbf{P}_{\Sigma_\tau}$ is described as follows. Let $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$ is the invariant Poisson subalgebra and

$$I^{\mathcal{G}^s} = \{\mu^*(\epsilon)F(\bar{A}, g) \,|\, F(\bar{A}, g) \in \mathbf{P}_{\Sigma_\tau}\}$$

is the ideal in $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$ generated by the functional $\mu^*(\epsilon)$ (2.6), where $\epsilon \in Lie(\mathcal{G}^s)$. The reduced Poisson algebra $\mathbf{P}^{red}_{\Sigma_\tau}$ is the factor algebra

$$\mathbf{P}^{red}_{\Sigma_\tau} = \mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}/I^{\mathcal{G}^s} := \mathbf{P}_{\Sigma_\tau}//\mathcal{G}^s\,. \tag{2.11}$$

In our construction we use another ideal in $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$. It will be defined below.

First, calculate the brackets in the invariant subalgebra $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$. Due to the monodromy conditions (2.1) the generic field $\bar{A}$ is gauge equivalent to the trivial $f^{-1}\bar{A}f + f^{-1}\bar{\partial}f = 0$. Therefore

$$\bar{A} = -\bar{\partial}f[\bar{A}]f^{-1}[\bar{A}]\,. \tag{2.12}$$

Again, the monodromies of the gauge matrices (2.3) prevent to have nontrivial residual gauge symmetries. Let $f[\bar{A}](z,\bar{z})$ be a solution of (2.12). Consider the transformation of g by solutions of (2.12)

$$L[\bar{A}, g](z,\bar{z}) = f[\bar{A}](z,\bar{z})g(z,\bar{z})f^{-1}[\bar{A}](z,\bar{z})\,. \tag{2.13}$$

The gauge invariant subalgebra $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$ is generated by the matrices L

$$\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau} = \{\Psi(\bar{A}, g) = \Psi(0, L)\}$$

Proposition 2.1 *The brackets on $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$ take the form of the classical exchange relations*

$$\{L_1(z,\bar{z}), L_2(w,\bar{w})\} = [r(z-w), L_1(z,\bar{z}) \otimes L_2(w,\bar{w})]\,, \tag{2.14}$$

where $L_1(z,\bar{z}) = L(z,\bar{z}) \otimes Id$, $L_2(w,\bar{w}) = Id \otimes L(w,\bar{w})$, and $r(z,w)$ is the classical Belavin-Drinfeld elliptic r-matrix [2].

Proof.
The calculation of brackets in $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$ is reduced to the calculation on shell ($\bar{A} = 0$, $f = Id$) of the Poisson brackets between the matrix elements of (2.13) by (2.8) - (2.10). In doing these calculations we need only the expression

$$r_{\alpha,\beta}(z,\bar{z}; z', \bar{z}') = \frac{\delta f_\alpha(z,\bar{z})}{\delta \bar{A}_\beta(z',\bar{z}')}|_{\bar{A}=0}\,. \tag{2.15}$$

The straightforward calculations of the brackets $\{L, L\}$ performed in [3] lead to the desired r-matrix form (2.14).

Let us find the r-matrix. Due to (2.12) r is the Green function of the operator $\bar{\partial}$

$$\bar{\partial}r_{\alpha,\beta}(z,\bar{z}; z', \bar{z}') = \delta_{\alpha+\beta,0}\delta(z-z', \bar{z}-\bar{z}')\,, \tag{2.16}$$

having the following quasi-periodicities

$$r(z+1, \bar{z}+1) = (Q^{-1} \otimes Id)\, r(z,\bar{z})\, (Q \otimes Id)\,,$$

$$r(z+\tau, \bar{z}+\bar{\tau}) = (\Lambda^{-1} \otimes Id)\, r(z,\bar{z})\, (\Lambda \otimes Id)\,.$$

It follows from (2.16) that $r_{\alpha,\beta}$ is a meromorphic and singular on the diagonal

$$\lim_{z'\to z} r_{\alpha,\beta}(z,z') = \frac{1}{z-z'} T_\alpha \otimes T_\beta \delta_{\alpha+\beta,0}\,. \tag{2.17}$$

Due to (A.9), (B.12), and (B.14)

$$r(z,w) = r(z-w) = \sum_\alpha \varphi_\alpha(z-w) T_\alpha \otimes T_{-\alpha}\,. \tag{2.18}$$

It is the Belavin-Drinfeld classical r-matrix [2]. This r-matrix satisfies the classical Yang-Baxter equation providing the Jacoby identity for the brackets (2.14). □

Remark 2.2 *In the limit (2.7) the only non-trivial brackets (2.9) assume the form*

$$\{\Phi_\alpha(z,\bar z), \bar A_\beta(w,\bar w)\} = \delta_{\alpha,-\beta}\delta(z-w,\bar z-\bar w)$$

and (2.14) is replaced by the linear brackets

$$\{L_1(z,\bar z), L_2(w,\bar w)\} = [r(z-w), L(z,\bar z)\otimes Id + Id\otimes Id L_2(w,\bar w)]\,.$$

Let us fix a divisor of non-coincident points on Σ_τ

$$D_n = (x_1,\ldots,x_n)\,,\;\; x_j \neq x_k\,,\;\; x_j \in \Sigma_\tau\,.$$

Define the subalgebra $Lie(D_n)(\mathcal{G}^s) \subset Lie(\mathcal{G}^s)$

$$Lie(D_n)(\mathcal{G}^s) = \{\varepsilon \in Lie(\mathcal{G}^s)\,|\,\varepsilon(x_j,\bar x_j) = 0\,,\;\; x_j \in D_n\}\,.$$

Consider the ideal $I(D_n)$ generated by the functional

$$\mu^*_{D_n}(\varepsilon;\bar A, g) = \mu^*_{D_n}(\varepsilon; L) = \int_{\Sigma_\tau} \langle \varepsilon \bar\partial L(z,\bar z)\rangle\,, \tag{2.19}$$

where ε belongs to $Lie(\mathcal{G}^s(D_n))$. Since $\mu^*_{D_n}$ depends only on L, $I(D_n) \subset \mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}$.

Consider the quotient Poisson algebra $\mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}/I(D_n)$

Proposition 2.2 *The reduced Poisson algebra*

$$\mathbf{P}^{red}_{\Sigma_\tau,D_n} = \mathbf{P}^{\mathcal{G}^s}_{\Sigma_\tau}/I(D_n)\,, \tag{2.20}$$

is finitely generated

$$\dim \mathbf{P}^{red}_{\Sigma_\tau,D_n} = nN^2\,.$$

*The matrix $L(z)$ in the classical exchange relations (*the Lax matrix*) takes the form*

$$L = S_0T_0 + \sum_{j=1}^n (S_0^j E_1(z-x_j)T_0 + \tilde L_j)\,,\quad \tilde L_j = \sum_\alpha S^j_\alpha \varphi_\alpha(z-x_j)T_\alpha\,, \tag{2.21}$$

where

$$\sum_{j=1}^n S_0^j = 0\,, \tag{2.22}$$

$\varphi_\alpha(z-x_j)$ are defined by (B.12), and $E_1(z-x_j)$ is the first Eisenstein series (A.2).

Proof
To prove it we analyze solutions of (2.19):

$$\mu^*_{D_n}(\varepsilon;L)=\int_{\Sigma_\tau}\langle\varepsilon\bar\partial L(z,\bar z)\rangle\ =0\,. \tag{2.23}$$

The solutions are meromorphic quasi-periodic maps having simple poles at the marked points. Let $L(z)=\sum_a L_a(z)T_a$ be the expansion of L in the basis T_a of $\mathrm{GL}(N,\mathbb{C})$. It follows from (2.3) that

$$L_a(z+1)=\mathbf{e}_N^{a_2}L_a(z)\,,\quad L_a(z+\tau)=\mathbf{e}_N^{-a_1}L_a(z)\,,\quad a=(a_1,a_2)\,.$$

The functions $\varphi_a(z-x_j)$ (B.12) have these monodromies (B.14) and simple poles at x_j. They form a n-dimensional basis in the space of quasi-periodic functions (B.24) with the poles at x_j. If $a=(0,0)$ then $L_0(z)$ is a double-periodic function with simple poles at x_j. The basis in this space is 1 and the Eisenstein functions $E_1(z-x_j)$ with vanishing sum of their residues. Thus, the space has dimension n. In this way we come to (2.21) and (2.22). $\square$

As we mentioned above $\det g$ generates the Casimir functionals in $\mathbf{P}^{red}_{\Sigma_\tau}$. Thereby, the brackets on $\mathbf{P}^{red}_{\Sigma_\tau,D_n}$ are degenerate. The function $\det L(z)$ is the generating function for the Casimir elements $C^\mu(j)$. Since $\det L(z)$ is a double periodic function it can be expanded in the basis of elliptic functions (A.6)

$$\det L(z)=C^0+\sum_j^n C^1(j)E_1(z-x_j)+C^2(j)E_2(z-x_j)+\ldots+C^N(j)E_N(z-x_j)\,. \tag{2.24}$$

Due to the condition

$$\sum_{j=1}^n C^1(j)=0\,, \tag{2.25}$$

the number of the independent Casimir is Nn. The generic symplectic leaf

$$\mathcal{R}^2_{n,N}=\mathbf{P}^{red}_{\Sigma_\tau,D_n}/\{(C^\mu(j)=C(j)^\mu_{(0)})\,,\ \mu=1,\ldots,N\,,j=1,\ldots,N\}\,.$$

has dimension

$$\dim(\mathcal{P}^2_{n,N})=nN(N-1)\,. \tag{2.26}$$

Note that it coincides with the sum of dimensions of n generic $\mathrm{GL}(N,\mathbb{C})$ coadjoint orbits.

3 The structure of the reduced Poisson space

3.1 Explicit form of quadratic brackets

Proposition 1.2 provides the reduced Poisson algebra $\mathbf{P}^{red}_{\Sigma_\tau,D_n}$ with the generators

$$\{S_0\,,\,(S^j_0\,,\mathbf{S}^j=\{S^j_\alpha\}\ ,j=1,\ldots,n)\,|\,\sum_{j=1}^n S^j_0=0\}\,. \tag{3.1}$$

The brackets between generators were calculated in [4].

Proposition 3.1 *The Poisson brackets on the space $\mathbb{C}^{nN^2}$ in terms of the generators (3.1) take the form*

$$\{S_0, S_0^j\}_2 = \{S_0^j, S_0^k\}_2 = \{S_\alpha^j, S_\alpha^k\}_2 = 0\,,^1 \tag{3.2}$$

$$\{S_0, S_\alpha^k\}_2 = \sum_{\gamma\neq\alpha} \mathbf{C}(\alpha,\gamma)\left(S_{\alpha-\gamma}^k S_\gamma^k E_2(\breve{\gamma}) - \sum_{j\neq k} S_{-\gamma}^j S_{\alpha+\gamma}^k f_\gamma(x_k - x_j)\right)\,, \tag{3.3}$$

$$\{S_\alpha^k, S_\beta^k\}_2 = \mathbf{C}(\alpha,\beta) S_0 S_{\alpha+\beta}^k + \sum_{\gamma\neq\alpha,-\beta} \mathbf{C}(\gamma,\alpha-\beta) S_{\alpha-\gamma}^k S_{\beta+\gamma}^k \mathbf{f}_{\alpha,\beta,\gamma} \tag{3.4}$$

$$+\mathbf{C}(\alpha,\beta) S_0^k S_{\alpha+\beta}^k (E_1(\breve{\alpha}+\breve{\beta}) - E_1(\breve{\alpha}) - E_1(\breve{\beta}))$$

$$-\mathbf{C}(\alpha,\beta)\sum_{j\neq k}[S_0^k S_{\alpha+\beta}^j \varphi_{\alpha+\beta}(x_k - x_j) - S_0^j S_{\alpha+\beta}^k E_1(x_k - x_j)]\}$$

$$-2\sum_{j\neq k} \mathbf{C}(\gamma,\alpha-\beta) S_{\alpha-\gamma}^k S_{\beta+\gamma}^k \varphi_{\beta+\gamma}(x_k - x_j)\}\,,$$

where $\mathbf{f}_{\alpha,\beta,\gamma}$, $E_2(\breve{\alpha})$, $E_1(\breve{\alpha})$ are defined by (B.23) and (B.10). For $j \neq k$

$$\{S_\alpha^j, S_\beta^k\}_2 = \sum_{\gamma\neq\alpha,-\beta} \mathbf{C}(\gamma,\alpha-\beta) S_{\alpha-\gamma}^j S_{\beta+\gamma}^k \varphi_\gamma(x_j - x_k) \tag{3.5}$$

$$-\mathbf{C}(\alpha,\beta)\left(S_0^j S_{\alpha+\beta}^k \varphi_\alpha(x_j - x_k) - S_0^k S_{\alpha+\beta}^j \varphi_{-\beta}(x_k - x_j)\right)\,,$$

and

$$\{S_0^j, S_\beta^k\}_2 = \begin{cases} 2\sum_\gamma \mathbf{C}(\gamma,-\beta) S_{-\gamma}^j S_{\beta+\gamma}^k \varphi_\gamma(x_k - x_j)\,, & j\neq k\,, \\ -2\sum_{m\neq k}\sum_\gamma \mathbf{C}(\gamma,-\beta) S_{-\gamma}^k S_{\beta+\gamma}^m \varphi_{\beta+\gamma}(x_k - x_m)\,, & j = k\,. \end{cases} \tag{3.6}$$

This algebra is an explicit particular form of general construction of quadratic Poisson algebras [5]. For $n = 1$ this algebra was calculated in [6] and for $n = 1$ and $N = 2$ it is the classical Sklyanin algebra [8].

3.2 Twisting bundles

We need another but equivalent form of this algebra on $\mathbb{C}^{nN^2}$. Consider the twisted bundle $E_N' = Aut(E_N)\otimes\mathcal{L}$, where $\mathcal{L}$ is a trivial line bundle over Σ_τ. The sections of E_N' are the sections of E_N multiplied by $\vartheta(z+\eta)/\vartheta(z)$, $(\eta \in \Sigma_\tau)$. Therefore, the transition functions of E_N' are

$$ad(Q) \text{ for } z \to z+1\,, \quad \exp(-2\pi\imath\eta)\cdot ad(\tilde{\Lambda}) \text{ for } z \to z+\tau\,.$$

It follows from (B.13), (B.15) that solutions of (2.19) with these monodromies and simple poles at the divisor

$$\tilde{D}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$$

is

$$L_{\tilde{D}_n}^\eta = \sum_{j=1}^n \left[\widetilde{S}_0^j \varphi_\eta(z - \widetilde{x}_j) T_0 + \sum_\alpha \widetilde{S}_\alpha^j \varphi_{\alpha,\eta}(z - \widetilde{x}_j) T_\alpha\right]\,, \tag{3.7}$$

[1]The subscript index $\{\,,\,\}_2$ means the quadratic brackets.

where $\varphi_\eta = \varphi_{0,\eta}$. The corresponding algebra $\tilde{\mathbf{P}}^{red}_{\Sigma_\tau,\tilde{D}_n}$ is defined, as above, by the classical exchange relations

$$\{L^\eta_{1,\tilde{D}_n}(z), L^\eta_{2,\tilde{D}_n}(w)\} = [r(z-w), L^\eta_{1,\tilde{D}_n}(z) \otimes L^\eta_{2,\tilde{D}_n}(w)] . \tag{3.8}$$

with the set of nN^2 generators

$$\tilde{\mathbf{S}}^j = \{\tilde{S}^j_a\} , \quad (a = (a_1, a_2) \in \mathbb{Z}^{(2)}_N , \; j = 1, \dots, n) .$$

The brackets between the generators can be extracted from (3.8) as before. We do not need their explicit form because we prove immediately the equivalence of these two algebras. The only thing we need in next Section is the brackets containing $\tilde{S}^k_0$ in the rhs (compare with (3.4) and (3.5))

$$\{\tilde{S}^k_\alpha, \tilde{S}^k_\beta\} = \sum_{\gamma \neq \alpha, -\beta} \mathbf{C}(\gamma, \alpha-\beta) \tilde{S}^k_{\alpha-\gamma} \tilde{S}^k_{\beta+\gamma} \tilde{\mathbf{f}}^\eta_{\alpha,\beta,\gamma} \tag{3.9}$$

$$+\mathbf{C}(\alpha,\beta) \tilde{S}^k_0 \tilde{S}^k_{\alpha+\beta} (E_1(\breve{\alpha} + \breve{\beta} + \eta) - E_1(\breve{\alpha}) - E_1(\breve{\beta}) - E_1(\eta))$$

$$-\mathbf{C}(\alpha,\beta) \sum_{j \neq k} [\tilde{S}^k_0 \tilde{S}^j_{\alpha+\beta} \varphi_{\alpha+\beta,\eta}(x_k - x_j) - \tilde{S}^j_0 \tilde{S}^k_{\alpha+\beta} \varphi_\eta(x_k - x_j)]\}$$

$$-2 \sum_{j \neq k} \mathbf{C}(\gamma, \alpha-\beta) \tilde{S}^k_{\alpha-\gamma} \tilde{S}^k_{\beta+\gamma} \varphi_{\beta+\gamma,\eta}(x_k - x_j)\} ,$$

where

$$\tilde{\mathbf{f}}^\eta_{\alpha,\beta,\gamma} = E_1(\breve{\gamma}) - E_1(\breve{\alpha} - \breve{\beta} - \breve{\gamma}) + E_1(\breve{\alpha} - \breve{\gamma} + \eta) - E_1(\breve{\beta} + \breve{\gamma} + \eta) , \tag{3.10}$$

$$\{\tilde{S}^j_\alpha, \tilde{S}^k_\beta\} = \sum_{\gamma \neq \alpha, -\beta} \mathbf{C}(\gamma, \alpha-\beta) \tilde{S}^j_{\alpha-\gamma} \tilde{S}^k_{\beta+\gamma} \varphi_{\gamma,\eta}(x_j - x_k) \tag{3.11}$$

$$-\mathbf{C}(\alpha,\beta) \left(\tilde{S}^j_0 \tilde{S}^k_{\alpha+\beta} \varphi_{\alpha,\eta}(x_j - x_k) - \tilde{S}^k_0 \tilde{S}^j_{\alpha+\beta} \varphi_{-\beta,\eta}(x_k - x_j) \right) .$$

To prove the equivalence we choose for simplicity $\tilde{x}_i = 0$ for some i.

Proposition 3.2 *Fix two indices $1 \leq i, k \leq n$ $(i \neq k)$. Define $x_k = -\eta$ and $x_j = \tilde{x}_j$ for $j \neq k$. Then Poisson algebras $\mathbf{P}^{red}_{\Sigma_\tau,D_n}$ and $\tilde{\mathbf{P}}^{red}_{\Sigma_\tau,\tilde{D}_n}$ are isomorphic. The corresponding canonical transformations are*

$$S_0 = \widetilde{S}^i_0 + \sum_{j \neq i}^n \frac{1}{\varphi_\eta(\widetilde{x}_j)} (E_1(\widetilde{x}_j) + E_1(\eta)) \widetilde{S}^j_0 ,$$

$$S^k_0 = -\sum_{j \neq i}^n \frac{\widetilde{S}^j_0}{\varphi_\eta(\widetilde{x}_j)} , \quad S^k_\alpha = \frac{\widetilde{S}^i_\alpha}{\varphi_\alpha(\eta)} + \sum_{j \neq i} \frac{\varphi_{-\alpha}(\widetilde{x}_j) \widetilde{S}^j_\alpha}{\varphi_\eta(\widetilde{x}_j) \varphi_\alpha(\eta)} , \tag{3.12}$$

$$S^{j \neq k}_0 = \frac{\widetilde{S}^j_0}{\varphi_\eta(\widetilde{x}_j)} , \quad S^{j \neq k}_\alpha = \frac{\widetilde{S}^j_\alpha}{\varphi_\eta(\widetilde{x}_j)} .$$

Proof
The Lax operator $L^\eta_{\tilde{D}_n}$ (3.7) after dividing on $\varphi_\eta(z)$ acquires the same monodromies as L (2.21). Consider the residues and the constant terms of these operators. First, we have:

$$L^\eta_{\tilde{D}_n} / \varphi_\eta(z) = \widetilde{S}^i_0 T_0 + \sum_{j \neq i}^n \left[\widetilde{S}^j_0 \frac{\varphi_\eta(z - \widetilde{x}_j)}{\varphi_\eta(z)} T_0 + \sum_\alpha \left(\widetilde{S}^j_\alpha \frac{\varphi_{\alpha,\eta}(z - \widetilde{x}_j)}{\varphi_\eta(z)} + \widetilde{S}^i_\alpha \frac{\varphi_{\alpha,\eta}(z)}{\varphi_\eta(z)} \right) T_\alpha \right] . \tag{3.13}$$

Applying (B.16), (B.17), and (B.18) we get

$$L^{\eta}_{\tilde{D}_n}/\varphi_\eta(z) = \left(\widetilde{S}^i_0 + \sum_{j\neq i}^{n} \frac{1}{\varphi_\eta(\widetilde{x}_j)}(E_1(\widetilde{x}_j) + E_1(\eta))\widetilde{S}^j_0\right) \cdot T_0$$

$$-E_1(z+\eta) \cdot \sum_{j\neq i}^{n} \frac{\widetilde{S}^j_0}{\varphi_\eta(\widetilde{x}_j)} \cdot T_0 + \sum_{j\neq i}^{n} E_1(z - \widetilde{x}_j) \cdot \frac{\widetilde{S}^j_0}{\varphi_\eta(\widetilde{x}_j)} \cdot T_0$$

$$+ \sum_{\alpha, j\neq i} \varphi_\alpha(z - \widetilde{x}_j) \frac{\widetilde{S}^j_\alpha}{\varphi_\eta(\widetilde{x}_j)} \cdot T_\alpha + \sum_{\alpha, j\neq i} \varphi_\alpha(z+\eta) \cdot \left(\frac{\varphi_{-\alpha}(\widetilde{x}_j)\widetilde{S}^j_\alpha}{\varphi_\eta(\widetilde{x}_j)\varphi_\alpha(\eta)} + \frac{\widetilde{S}^i_\alpha}{\varphi_\alpha(\eta)}\right) \cdot T_\alpha$$

Note that there is a new pole at $x_b = -\eta$. Comparing with (2.21) we come to (3.12). □

3.3 Bihamiltonian structure

Introduce on the space $\mathbb{C}^{nN^2}$ the linear (Lie-Poisson) brackets. To this end consider the direct sum of n copies of $\mathrm{gl}(N, \mathbb{C})$: $\mathfrak{g}^* = \mathrm{gl}(N, \mathbb{C}) \oplus \ldots \oplus \mathrm{gl}(N, \mathbb{C})$ with the brackets

$$\{S^j_\alpha, S^k_\beta\}_1 = C(\alpha, \beta) S^j_{\alpha+\beta} \delta^{jk} \,.^2 \tag{3.14}$$

Remark 3.1 *The Lie-Poisson brackets have the r-matrix form*

$$\{\tilde{L}_1(z), \tilde{L}_2(w)\} = [r(z-w), \tilde{L}_1(z) + \tilde{L}_2(w)]\,,$$

where r is same as for the quadratic brackets (2.18), and $\tilde{L} = \sum_{j=1}^n \tilde{L}_j(z)$ (2.21).

Two Poisson structures are called *compatible* (or, form *Poisson pair*) if their linear combinations are Poisson structures as well.

Proposition 3.3 *The linear (3.14) and quadratic (3.2) - (3.6) Poisson brackets on the space $\mathbb{C}^{nN^2}$ are compatible.*

Proof.
Choose a point $x_k \in \tilde{D}_n$ and replace the variable $\tilde{S}^k_0$ by $\tilde{S}^k_0 + \lambda$, where $\lambda \in \mathbb{C}$ is a number and therefore it Poisson commutes with all elements of the quadratic Poisson algebra. Substitute the new variable in (3.9) and (3.11). The change of variables does not spoil the Jacobi identity and therefore we come to the following Poisson structure

$$\{\tilde{S}, \tilde{S}\}_\lambda := \{\tilde{S}, \tilde{S}\}_2 + \lambda\{\tilde{S}, \tilde{S}\}_1\,.$$

Consider the linear brackets term.

$$\begin{array}{c} \{\tilde{S}^k_\alpha, \tilde{S}^k_\beta\}_1 = F_1 \tilde{S}^j_{\alpha+\beta} + F_2 \tilde{S}^k_{\alpha+\beta}, \\ \{\tilde{S}^k_\alpha, \tilde{S}^j_\beta\}_1 = G_2 \tilde{S}^j_{\alpha+\beta}, \\ \{\tilde{S}^j_\alpha, \tilde{S}^k_\beta\}_1 = \tilde{G}_2 \tilde{S}^j_{\alpha+\beta}, \\ \{\tilde{S}^j_\alpha, \tilde{S}^j_\beta\}_1 = H_2 \tilde{S}^j_{\alpha+\beta}\,, \end{array} \tag{3.15}$$

where up to the common multiplier $C(\alpha, \beta)$ the coefficients have the form

$$\begin{array}{c} F_1 = \varphi_{\alpha+\beta,\eta}(x_{kj}), \quad F_2 = -E_1(\breve{\alpha}) - E_1(\breve{\beta}) - E_1(\eta) + E_1(\breve{\alpha} + \breve{\beta} + \eta), \\ G_2 = -\varphi_\alpha(x_{kj}), \quad \tilde{G}_2 = -\varphi_\beta(x_{kj}), \\ H_2 = \varphi_{0,-\eta}(x_{kj})\,, \end{array} \tag{3.16}$$

where $x_{kj} = x_k - x_j$ The following Lemma completes the proof.

[2] The subscript index $_1$ means the linear brackets.

Lemma 3.1 *The linear Poisson algebra (3.15) is equivalent to the direct sum of Lie-Poisson algebras on* $\oplus_{l=1}^{n}\mathrm{gl}(N,\mathbb{C})$

$$\{t^j_\alpha, t^k_\beta\} = C(\alpha,\beta)t^j_{\alpha+\beta}\delta^{jk}\,. \tag{3.17}$$

Proof
Define

$$\tilde{S}^k_\alpha = a_\alpha t^k_\alpha + b_\alpha t^j_\alpha\,, \quad \tilde{S}^j_\alpha = H_2 t^j_\alpha$$

The brackets (3.15) are equivalent to (3.17) if

$$\begin{aligned} a_\alpha a_\beta &= a_{\alpha+\beta}F_2, \\ b_\alpha &= G_2, \\ b_\alpha b_\beta &= F_1 H_2 + b_{\alpha+\beta}F_2. \end{aligned} \tag{3.18}$$

Let us solve these equations. The solution of the first equation can be found from (A.22). It takes the form $a_\alpha = -\varphi_\alpha(\eta)$. Next prove that $b_\alpha = G_2 = -\varphi_\alpha(x_{kj})$ satisfies the last relation. With $b_\alpha = -\varphi_\alpha(x_{kj})$ it takes the form

$$\varphi_\alpha(x_{kj})\varphi_\beta(x_{kj}) = \varphi_{\alpha+\beta,\eta}(x_{kj})\varphi_{0,-\eta}(x_{kj}) +$$

$$+\varphi_{\alpha+\beta}(x_{kj})\Big(E_1(\breve{\alpha}) + E_1(\breve{\beta}) + E_1(\eta) - E_1(\breve{\alpha}+\breve{\beta}+\eta)\Big)\,.$$

It follows from (A.20) that

$$\varphi_{\alpha+\beta,\eta}(x_{kj})\varphi_{0,-\eta}(x_{kj}) = \varphi_{\alpha+\beta}(x_{kj})\Big(E_1(\breve{\alpha}+\breve{\beta}+\eta)) + E_1(-\eta) + E_1(x_{kj}) - E_1(\breve{\alpha}+\breve{\beta}+x_{kj})\Big)\,,$$

so the last relation in (3.18) is an identity and thereby we come from (3.15) to (3.17). □

4 Quantum algebra

4.1 General case

In this section we consider quantization of quadratic Poisson algebra for the case $n > 1$. Let us consider quantum R-matrix, having the following form:

$$R(z,w) = \sum_{a\in\mathbb{Z}_N^{(2)}} \varphi_a^\hbar(z-w) T_a \otimes T_{-a}\,, \tag{4.1}$$

where we put $\varphi_a^\hbar(z) \equiv \varphi_{\hbar,a}(z)$. Note, that in contrast with the classical r-matrix, there is an additional term

$$\varphi_0^\hbar(z-w)\sigma_0\otimes\sigma_0.$$

Quantum R-matrix satisfies the quantum Yang-Baxter equation:

$$R_{12}(z-w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z-w)\,. \tag{4.2}$$

The quantum Yang-Baxter equation allows us to define the associative algebra by the relation:

$$R(z-w)L_1^\hbar(z)L_2^\hbar(w) = L_2^\hbar(w)L_1^\hbar(z)R(z-w)\,, \tag{4.3}$$

The Lax operator in (4.3) has the following monodromies with respect to $\hbar$:

$$L^{\hbar+\tau}(z) = \mathbf{e}_N(-z)L^{\hbar}(z)\,, \quad L^{\hbar+1}(z) = L^{\hbar}(z)\,. \tag{4.4}$$

So, we have to suppose that the variables S depend on $\hbar$ and x_j. The new variables and the Lax operator in (4.3) takes the following form:

$$S^j_{new} = \hat{S}^j_0 \varphi^{\hbar}_0(x_j)\,,$$

$$L^{\hbar}(z) = \sum_{j=1}^{n} \left(\hat{S}^j_0 \varphi^{\hbar}_0(x_j) \varphi^{\hbar}_0(z-x_j) T_0 + \sum_{\alpha} \hat{S}^j_{\alpha} \varphi^{\hbar}_{\alpha}(x_j) \varphi^{\hbar}_{\alpha}(z-x_j) T_{\alpha} \right) = \tag{4.5}$$

$$= \sum_{j=1}^{n} \sum_{a \in \mathbb{Z}_N^{(2)}} \hat{S}^j_a \varphi^{\hbar}_a(x_j) \varphi^{\hbar}_a(z-x_j) T_a\,.$$

Proposition 5.1: *The relations in the associative algebra assume the form*

$$\sum_c f^{\hbar}(a,b,c) \cdot \hat{S}^j_{b+c} \hat{S}^j_{a-c} \varphi^{\hbar}_{b+c}(x_j) \varphi^{\hbar}_{a-c}(x_j) \mathbf{e}_N(+\frac{c\times(a-b)}{2}) \; +$$

$$+ \sum_c \sum_{k\neq j} \varphi^{\hbar}_{a-c}(x_j - x_k) \varphi^{\hbar}_{b+c}(x_j) \varphi^{\hbar}_{a-c}(x_k) \cdot \tag{4.6}$$

$$\cdot \left(\hat{S}^j_{b+c} \hat{S}^k_{a-c} \mathbf{e}_N(+\frac{c\times(a-b)}{2}) - \hat{S}^k_{a-c} \hat{S}^j_{b+c} \mathbf{e}_N(-\frac{c\times(a-b)}{2}) \right) = 0,$$

and:

$$\sum_c \varphi^{\hbar}_c(x_j - x_k) \varphi^{\hbar}_{b+c}(x_k) \varphi^{\hbar}_{a-c}(x_j) \cdot \left(\hat{S}^j_{a-c} \hat{S}^k_{b+c} \mathbf{e}_N(-\frac{c\times(a-b)}{2}) - \hat{S}^k_{b+c} \hat{S}^j_{a-c} \mathbf{e}_N(+\frac{c\times(a-b)}{2}) \right) = 0\,, \tag{4.7}$$

$$k \neq j\,,$$

where

$$f^{\hbar}(a,b,c) = E_1(c+\hbar) - E_1(a-b-c+\hbar) + E_1(a-c+\hbar) - E_1(b+c+\hbar)$$

and $a, b, c \in \mathbb{Z}_N^{(2)}$.

Proof.

Let us consider the certain matrix element $T_a \otimes T_b$. For this put (4.5) and (4.1) in (4.3), we get the following expressions:

$$\sum_{j,k} \sum_{c,a,b} \varphi^{\hbar}_c(z-w) \varphi^{\hbar}_a(z-x_j) \varphi^{\hbar}_b(w-x_k) \cdot \hat{S}^j_a \hat{S}^k_b \varphi^{\hbar}_b(x_k) \varphi^{\hbar}_a(x_j) \cdot T_c T_a \otimes T_{-c} T_b = \tag{4.8}$$

$$= \sum_{j,k} \sum_{c,a,b} \varphi^{\hbar}_c(z-w) \varphi^{\hbar}_a(z-x_j) \varphi^{\hbar}_b(w-x_k) \cdot \hat{S}^k_b \hat{S}^j_a \varphi^{\hbar}_b(x_k) \varphi^{\hbar}_a(x_j) \cdot T_a T_c \otimes T_b T_{-c}\,.$$

$$\sum_{j,k}\sum_{c,a,b}\varphi^\hbar_c(z-w)\varphi^\hbar_a(z-x_j)\varphi^\hbar_b(w-x_k)\cdot\hat S^j_a\hat S^k_b\varphi^\hbar_b(x_k)\varphi^\hbar_a(x_j)\mathbf{e}_N(-\frac{c\times(a-b)}{2})\cdot T_{c+a}\otimes T_{-c+b}= \quad (4.9)$$

$$=\sum_{j,k}\sum_{c,a,b}\varphi^\hbar_c(z-w)\varphi^\hbar_a(z-x_j)\varphi^\hbar_b(w-x_k)\cdot\hat S^k_b\hat S^j_a\varphi^\hbar_b(x_k)\varphi^\hbar_a(x_j)\mathbf{e}_N(+\frac{c\times(a-b)}{2})\cdot T_{a+c}\otimes T_{b-c}\,.$$

The functions of l.h.s and r.h.s. are equal because their poles and quasi-periods coincide. After changing the variables $a\to a-c,\ b\to b+c$, we get for the coefficients in front of the matrix element $T_a\otimes T_b$:

$$\sum_c\varphi^\hbar_c(z-w)\varphi^\hbar_{a-c}(z-x_j)\varphi^\hbar_{b+c}(w-x_k)\varphi^\hbar_{b+c}(x_k)\varphi^\hbar_{a-c}(x_j)\cdot \tag{4.10}$$
$$\cdot\left(\hat S^j_{a-c}\hat S^k_{b+c}\mathbf{e}_N(-\frac{c\times(a-b)}{2})-\hat S^k_{b+c}\hat S^j_{a-c}\mathbf{e}_N(+\frac{c\times(a-b)}{2})\right)=0\,.$$

We have to consider two types of these expressions:

$$k\neq j:\ \sum_c\varphi^\hbar_c(z-w)\varphi^\hbar_{a-c}(z-x_j)\varphi^\hbar_{b+c}(w-x_k)\varphi^\hbar_{b+c}(x_k)\varphi^\hbar_{a-c}(x_j)\cdot$$
$$\cdot\left(\hat S^j_{a-c}\hat S^k_{b+c}\mathbf{e}_N(-\frac{c\times(a-b)}{2})-\hat S^k_{b+c}\hat S^j_{a-c}\mathbf{e}_N(+\frac{c\times(a-b)}{2})\right)=0\,,$$
$$k=j: \tag{4.11}$$
$$\sum_c\left(\varphi^\hbar_c(z-w)\varphi^\hbar_{a-c}(z-x_j)\varphi^\hbar_{b+c}(w-x_j)-\varphi^\hbar_{a-b-c}(z-w)\varphi^\hbar_{a-c}(z-x_j)\varphi^\hbar_{b+c}(w-x_k)\right)\cdot$$
$$\cdot\hat S^j_{a-c}\hat S^k_{b+c}\varphi^\hbar_{b+c}(x_k)\varphi^\hbar_{a-c}(x_j)\mathbf{e}_N(-\frac{c\times(a-b)}{2})=0\,.$$

We get second expression after changing $c\to a-b-c$. Taking the limits $(z\to x_j,w\to x_j)$ and $(z\to x_j,w\to x_k)$, as it has been already done in section three, we get the coefficients which must be equal to zero. So we come to (4.6) and (4.7). $\square$

4.2 Quadratic algebra in $\mathrm{GL}(2,\mathbb{C})$ case

Let us consider the case $N=2$ in more detail. In this case quantum R-matrix take the following form:

$$R(z,w)=\sum_{a=0}^{3}\varphi^\hbar_a(z-w)\sigma_a\otimes\sigma_a, \tag{4.12}$$

where instead of T_a we use the basis of sigma-matrices.

Proposition 5.2: *The relations in the associative algebra assume the form*

$$[\hat S^j_\alpha,\hat S^j_\beta]_-=i\varepsilon_{\alpha\beta\gamma}c^1_1(j,j;\alpha,\beta,\gamma)[\hat S^j_\gamma,\hat S^j_0]_++ \tag{4.13}$$
$$+\sum_{k\neq j}i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\alpha}\left(\varphi^\hbar_\gamma(x_{jk})c^2_1(j,k;\alpha,\beta,\gamma)[\hat S^k_\gamma,\hat S^j_0]_+-\varphi^\hbar_0(x_{jk})c^3_1(j,k;\alpha,\beta,\gamma)[\hat S^j_\gamma,\hat S^k_0]_+\right)\,,$$

$$c_1^1(j,j;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)}\,,\quad c_1^2(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_k)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)}\,,\quad c_1^3(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)},$$

$$[\hat{S}_\alpha^j,\hat{S}_0^j]_-=i\varepsilon_{\alpha\beta\gamma}\frac{J_\beta-J_\gamma}{J_\alpha}c_2^1(j,j;\alpha,\beta,\gamma)[\hat{S}_\beta^j,\hat{S}_\gamma^j]_++ \tag{4.14}$$
$$+\sum_{k\neq j}i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\alpha J_\alpha}\cdot\Big(c_2^2(j,k;\alpha,\beta,\gamma)D(\alpha,\beta)[\hat{S}_\gamma^j,\hat{S}_\beta^k]_+-c_2^3(j,k;\alpha,\beta,\gamma)D(\alpha,\gamma)[\hat{S}_\gamma^k,\hat{S}_\beta^j]_+\Big)\,,$$

$$c_2^1(j,j;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_\beta^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)}\,,\quad c_2^2(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_\beta^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)}\,,\quad c_2^3(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_k)\varphi_\beta^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)}\,,$$
$$D(\alpha,\beta)=\left(k_\alpha(k_\alpha-k_\beta-ln'\varphi_\alpha^\hbar(x_{jk}))+ln'\frac{\varphi_\alpha^\hbar(x_{jk})}{\varphi_0^\hbar(x_{jk})}(\partial_z-E_1(\hbar))\right)\varphi_\beta^\hbar(x_{jk}),$$
$$ln'\varphi_u^\hbar(x)=\frac{\partial_u(\varphi_u^\hbar(x))}{\varphi_u^\hbar(x)},$$

and for $k\neq j$:

$$[\hat{S}_\alpha^j,\hat{S}_\beta^k]_-=i\varepsilon_{\alpha\beta\gamma}\tfrac{1}{k_\gamma}\Big(\varphi_\beta(x_{jk})c_3^1(j,k;\alpha,\beta,\gamma)[\hat{S}_\gamma^j,\hat{S}_0^k]_+-\varphi_\alpha(x_{jk})c_3^2(j,k;\alpha,\beta,\gamma)[\hat{S}_\gamma^k,\hat{S}_0^j]_+\Big)\,, \tag{4.15}$$

$$c_3^1(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_k)}\,,\quad c_3^2(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_k)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_k)},$$

$$[\hat{S}_\alpha^j,\hat{S}_0^k]_-=i\varepsilon_{\alpha\beta\gamma}\tfrac{1}{k_\alpha}\Big(\varphi_\beta(x_{jk})c_4^1(j,k;\alpha,\beta,\gamma)[\hat{S}_\gamma^j,\hat{S}_\beta^k]_+-\varphi_\alpha(x_{jk})c_4^2(j,k;\alpha,\beta,\gamma)[\hat{S}_\gamma^k,\hat{S}_\beta^j]_+\Big)\,, \tag{4.16}$$

$$c_4^1(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_j)\varphi_\beta^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_k)}\,,\quad c_4^2(j,k;\alpha,\beta,\gamma)=\frac{\varphi_\gamma^\hbar(x_k)\varphi_\beta^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_k)},$$

where

$$k_\gamma=E_1(\check{\gamma}+\hbar)-E_1(\check{\gamma})-E_1(\hbar),$$
$$J_\gamma=E_2(\check{\gamma}+\hbar)-E_2(\hbar)$$

(see Appendix B),

$$x_{jk}=x_j-x_k.$$

Proof.

Put (4.5) in (4.3) in the case $N = 2$, check the balance in front of two type fixed matrix elements $\sigma_\alpha \otimes \sigma_\beta$ and $\sigma_\alpha \otimes \sigma_0$ in left hand side (lhs) and right hand side (rhs). We fix these elements and compare the coefficients at the corresponding poles. We get the following expressions for brackets:

$$[\hat{S}^j_\alpha, \hat{S}^j_\beta]_- = i\varepsilon_{\alpha\beta\gamma} \cdot \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\hat{S}^j_\gamma, \hat{S}^j_0]_+ +$$

$$+\sum_{k\neq j} \frac{\varphi^\hbar_\gamma(x_{jk})}{f^\hbar(\alpha,\beta,0)} \cdot \frac{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\hat{S}^j_\alpha, \hat{S}^k_\beta]_- - \frac{\varphi^\hbar_\alpha(x_{jk})}{f^\hbar(\alpha,\beta,0)} \cdot \frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\hat{S}^k_\alpha, \hat{S}^j_\beta]_- + \tag{4.17}$$

$$+i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_0(x_{jk})}{f^\hbar(\alpha,\beta,0)} \cdot \frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\hat{S}^j_\gamma, \hat{S}^k_0]_+ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{f^\hbar(\alpha,\beta,0)} \cdot \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_0(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\hat{S}^k_\gamma, \hat{S}^j_0]_+ ,$$

$$[\hat{S}^j_\alpha, \hat{S}^j_0]_- = i\varepsilon_{\alpha\beta\gamma}\frac{J_\beta - J_\gamma}{J_\alpha} \cdot \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\hat{S}^j_\beta, \hat{S}^j_\gamma]_+ +$$

$$+\sum_{k\neq j} \frac{\varphi^\hbar_0(x_{jk})}{J_\alpha} \cdot \frac{\varphi^\hbar_0(x_k)}{\varphi^\hbar_0(x_j)}[\hat{S}^j_\alpha, \hat{S}^k_0]_- + \frac{\varphi^\hbar_\alpha(x_{jk})}{J_\alpha} \cdot \frac{\varphi^\hbar_\alpha(x_k)}{\varphi^\hbar_\alpha(x_j)}[\hat{S}^k_\alpha, \hat{S}^j_0]_- + \tag{4.18}$$

$$-i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{J_\alpha} \cdot \frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\hat{S}^j_\beta, \hat{S}^k_\gamma]_+ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{J_\alpha} \cdot \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_\beta(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\hat{S}^j_\gamma, \hat{S}^k_\beta]_+ ,$$

and for $k \neq j$:

$$[\hat{S}^j_\alpha, \hat{S}^k_\beta]_- = \frac{\varphi^\hbar_\gamma(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\hat{S}^k_\alpha, \hat{S}^j_\beta]_- + \tag{4.19}$$

$$+i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\alpha(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\hat{S}^k_\gamma, \hat{S}^j_0]_+ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_0(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\hat{S}^j_\gamma, \hat{S}^k_0]_+ ,$$

$$[\hat{S}^j_\alpha, \hat{S}^k_0]_- = \frac{\varphi^\hbar_\alpha(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\hat{S}^k_\alpha, \hat{S}^j_0]_- + \tag{4.20}$$

$$+i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\beta(x_j)\varphi^\hbar_\gamma(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\hat{S}^j_\beta, \hat{S}^k_\gamma]_+ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\beta(x_k)\varphi^\hbar_\gamma(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\hat{S}^k_\beta, \hat{S}^j_\gamma]_+ .$$

It is possible to express all commutators by the anti-commutators. In fact for the brackets $[\hat{S}^j_\alpha, \hat{S}^k_\beta]_-$, $[\hat{S}^j_\alpha, \hat{S}^k_0]_-$ we have two additional equations (permutation $j \leftrightarrow k$). Solving the system of six equations we get (4.13)-(4.16). $\square$

4.3 Quantum Determinant

In this section we prove for $\mathrm{GL}(2,\mathbb{C})$ that the quantum determinant generates central elements of the exchange algebra

$$R_{12}(z_1, z_2)\hat{L}_1(z_1)\hat{L}_2(z_2) = \hat{L}_2(z_2)\hat{L}_1(z_1)R_{12}(z_1, z_2) \tag{4.21}$$

for R and L defined in (4.12) and (4.5).

Let us start from the classical algebra (3.1)-(3.6). To prove in GL$(N,\mathbb{C})$ case that $\det L(z)$ generates the Casimir functions of the Poisson structure (3.1)-(3.6) consider each side of the equality

$$\{L_1(z)\dots L_N(z), L_{N+1}(w)\} = [L_1(z)\dots L_N(z)L_{N+1}(w), r_{1,N+1}(z,w) + \dots + r_{N,N+1}(z,w)]$$

as a linear operator acting on $\bigotimes_{i=1}^{N+1} V_i$, where $V_i \cong \mathbb{C}^N$ are vector spaces, $L_i \in \mathrm{End}V_i$ and $r_{ik} \in \mathrm{End}(V_i \otimes V_k)$. The determinant $\det L(z)$ obviously appears on the subspace $\left[\bigwedge_{i=1}^{N} V_i\right] \otimes V_{N+1}$. The r.h.s. on this subspace reduces to the following:

$$[\det L(z) \cdot L_{N+1}(w), Tr_1 r_{1,N+1}(z,w) + Tr_N r_{N,N+1}(z,w)].$$

Here traces Tr_i are taken over $\mathrm{End}V_i$ components. All of them vanish for the r-matrix (2.18). End of the proof for the classical case.

In quantum case the determinant is replaced by the quantum determinant:

$$\det_\hbar = \mathrm{tr}(P^- \hat{L}(z,\hbar) \otimes \hat{L}(z+2\hbar,\hbar))\,,$$

where P^- is the projection into skewsymmetric part of the tensor product:

$$P^- a \otimes b = \frac{1}{2}\left(a \otimes b - b \otimes a\right)\,.$$

Here we discuss only 2×2 case. The R-matrix

$$R_{12}(z,w) = \sum_{a=0}^{3} \varphi_a^\eta(z-w)\sigma_a \otimes \sigma_a$$

satisfies the following important condition:

$$R_{12}(z, z+2\hbar) = 4\frac{\vartheta'(0)}{\vartheta(2\hbar)}P^-,$$

and

$$P^- = \frac{1}{4}\left(1 \otimes 1 - \sum_{\alpha=1}^{3} \sigma_\alpha \otimes \sigma_\alpha\right)\,.$$

Consider the product $L_1(z_1)L_2(z_2)L_3(w) \in V^{\otimes 3}$.

It follows from the Yang-Baxter equation that

$$R_{12}R_{13}R_{23}\hat{L}_1\hat{L}_2\hat{L}_3 = \hat{L}_3\hat{L}_2\hat{L}_1R_{12}R_{13}R_{23}$$

Put $z_2 = z_1 + 2\hbar$. Then

$$P_{12}^- R_{13}R_{23}\hat{L}_1\hat{L}_2\hat{L}_3 = \hat{L}_3\hat{L}_2\hat{L}_1P_{12}^- R_{13}R_{23}$$

The next statement is the most important one:

$$P_{12}^- R_{13}R_{23} \sim P_{12}^- \otimes 1_3$$

It follows from direct calculations. For the simplicity one can use the following identity for $\alpha, \beta, \gamma \sim 1,2,3$ up to the cyclic permutations:

$$-\varphi_0^\hbar(x)\varphi_\gamma(x-2\hbar) + \varphi_\gamma^\hbar(x)\varphi_0(x-2\hbar) + \varphi_\beta^\hbar(x)\varphi_\alpha(x-2\hbar) + \varphi_\alpha^\hbar(x)\varphi_\beta(x-2\hbar) = 0 \qquad (4.22)$$

Using also a simple fact $Tr_{12}\left(P_{12}^-\hat{L}_1\hat{L}_2\right) = Tr_{12}\left(P_{12}^-\hat{L}_2\hat{L}_1\right)$ we come to the final result:

$$[Tr_{12}\left(P_{12}^-\hat{L}_1(z-2\hbar)\hat{L}_2(z)\right), \hat{L}_3(w)] = 0$$

4.4 Nonhomogeneous algebra and Reflection Equation

Consider the rank two case ($N = 2$) with four marked points $n = 4$. As an initial data we put the marked points on $z = 0$ and the half-periods of Σ_τ

$$x_0 = 0\,,\ x_1 = \frac{\tau}{2} = \omega_2\,,\ x_2 = \frac{1+\tau}{2} = \omega_1 + \omega_2\,,\ x_3 = \frac{1}{2} = \omega_1\,,$$

and assume that

$$S^j_\alpha = \delta^j_\alpha \tilde{\nu}_\alpha\,,\ (j = 1\,,2\,,3)\,, \tag{4.23}$$

while $S^0_\alpha = S_\alpha$ are arbitrary. This choice appears as a consequence of the reduction $L(z)L(-z) = 1 \times \det L(z)$.

Let R^- be the quantum vertex R-matrix, that arises in the XYZ model. We introduce also the matrix R^+

$$R^\pm(z, w) = \sum_{a=0}^{3} \varphi_a^{\frac{\hbar}{2}}(z \pm w)\sigma_a \otimes \sigma_a\,. \tag{4.24}$$

Define the quantum Lax operator

$$\hat{L}(z) = \hat{S}_0 \phi^\hbar(z)\sigma_0 + \sum_\alpha (\hat{S}_\alpha \varphi^\hbar_\alpha(z) + \tilde{\nu}_\alpha \varphi^\hbar_\alpha(z - \omega_\alpha))\sigma_\alpha\,. \tag{4.25}$$

Proposition 4.1 *The Lax operator satisfies the quantum reflection equation*

$$R^-(z, w)\hat{L}_1(z)R^+(z, w)\hat{L}_2(w) = \hat{L}_2(w)R^+(z, w)\hat{L}_1(z)R^-(z, w)\,, \tag{4.26}$$

if its components S_a generate the associative algebra with relations:

$$[\tilde{\nu}_\alpha, \tilde{\nu}_\beta] = 0\,,\ \ [\tilde{\nu}_\alpha, \hat{S}_a] = 0\,, \tag{4.27}$$

$$i[\hat{S}_0, \hat{S}_\alpha]_+ = [\hat{S}_\beta, \hat{S}_\gamma]\,, \tag{4.28}$$

$$[\hat{S}_\gamma, \hat{S}_0] = i\frac{K_\beta - K_\alpha}{K_\gamma}[\hat{S}_\alpha, \hat{S}_\beta]_+ - 2i\frac{1}{K_\gamma}(\tilde{\nu}_\alpha \rho_\alpha \hat{S}_\beta - \tilde{\nu}_\beta \rho_\beta \hat{S}_\alpha)\,, \tag{4.29}$$

where

$$K_\alpha = E_1(\hbar + \check{\alpha}) - E_1(\hbar) - E_1(\check{\alpha})\,,\quad \rho_\alpha = -\exp(-2\pi\imath\check{\alpha}\partial_\tau\check{\alpha})\phi(\check{\alpha} + \hbar, -\check{\alpha})\,.$$

The proof is based on the direct check. Details can be found in [7].

If all $\nu_\alpha = 0$ (4.27) – (4.29) the algebra coincides with the Sklyanin algebra. Therefore, the algebra (4.27) – (4.29) is a three parametric deformation of the Sklyanin algebra.

Two elements

$$C_1 = \hat{S}_0^2 + \sum_\alpha \hat{S}_\alpha^2\,,$$

$$C_2 = \sum_\alpha \hat{S}^2_\alpha K_\alpha(K_\alpha - K_\beta - K_\gamma) + 2\tilde{\nu}_\alpha \rho_\alpha K_\alpha \hat{S}_\alpha$$

belong to the center of the generalized Sklyanin algebra (4.27), (4.28). They are the coefficients of the expansion of the quantum determinant

$$\det_\hbar = \mathrm{tr}(P^- \hat{L}(z, \hbar) \otimes \hat{L}(z + 2\hbar, \hbar))\,.$$

5 Appendix

5.1 Appendix A. Elliptic functions.

We assume that $q = \exp 2\pi i\tau$, where τ is the modular parameter of the elliptic curve E_τ.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n \mathbf{e}(\frac{1}{2}n(n+1)\tau + nz) = \quad (\mathbf{e} = \exp 2\pi\imath) \tag{A.1}$$

The Eisenstein functions

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}, \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n). \tag{A.3}$$

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1. \tag{A.4}$$

Relation to the Weierstrass functions

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau)z, \quad \wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau). \tag{A.5}$$

The highest Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!}\partial^{(j-2)}E_2(z), \quad (j>2). \tag{A.6}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{A.7}$$

$$\phi(u,z) = \phi(z,u), \quad \phi(-u,-z) = -\phi(u,z). \tag{A.8}$$

It has a pole at $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \dots. \tag{A.9}$$

$$\partial_u\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(u)). \tag{A.10}$$

$$\partial_z\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(z)). \tag{A.11}$$

$$\lim_{z\to 0} \ln \partial_u\phi(u,z) = -E_2(u). \tag{A.12}$$

Heat equation

$$\partial_\tau\phi(u,w) - \frac{1}{2\pi i}\partial_u\partial_w\phi(u,w) = 0. \tag{A.13}$$

Quasi-periodicity

$$\vartheta(z+1) = -\vartheta(z)\,, \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}}e^{-2\pi i z}\vartheta(z)\,, \tag{A.14}$$

$$E_1(z+1) = E_1(z)\,, \quad E_1(z+\tau) = E_1(z) - 2\pi i\,, \tag{A.15}$$

$$E_2(z+1) = E_2(z)\,, \quad E_2(z+\tau) = E_2(z)\,, \tag{A.16}$$

$$\phi(u, z+1) = \phi(u, z)\,, \quad \phi(u, z+\tau) = e^{-2\pi \imath u}\phi(u, z)\,. \tag{A.17}$$

$$\partial_u \phi(u, z+1) = \partial_u \phi(u, z)\,, \quad \partial_u \phi(u, z+\tau) = e^{-2\pi \imath u}\partial_u \phi(u, z) - 2\pi\imath\phi(u, z)\,. \tag{A.18}$$

The Fay three-section formula:

$$\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1+u_2, z_1)\phi(u_2, z_2-z_1) - \phi(u_1+u_2, z_2)\phi(u_1, z_1-z_2) = 0\,. \tag{A.19}$$

From (A.11) and (A.19) we have:

$$\phi(u_1, z)\phi(u_2, z) = \phi(u_1+u_2, z)(E_1(u_1) + E_1(u_2) - E_1(u_1+u_2+z) + E_1(z))\,. \tag{A.20}$$

Particular cases of this formula are the functional equations

$$\phi(u, z)\partial_v \phi(v, z) - \phi(v, z)\partial_u \phi(u, z) = (E_2(v) - E_2(u))\phi(u+v, z)\,, \tag{A.21}$$

$$\begin{aligned} \phi(u, z_1)\phi(-u, z_2) &= \phi(u, z_1-z_2)(-E_1(z_1) + E_1(z_2) - E_1(u) + E_1(u+z_1-z_2)) = \\ &= \phi(u, z_1-z_2)(-E_1(z_1) + E_1(z_2) + \partial_u \phi(u, z_2-z_1))\,, \end{aligned} \tag{A.22}$$

$$\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u)\,. \tag{A.23}$$

$$\begin{aligned} \phi(v, z-w)\phi(u_1-v, z)\phi(u_2+v, w) - \phi(u_1-u_2-v, z-w)\phi(u_2+v, z)\phi(u_1-v, w) = \\ \phi(u_1, z)\phi(u_2, w)f(u_1, u_2, v)\,, \end{aligned} \tag{A.24}$$

where

$$\mathbf{f}(u_1, u_2, v) = E_1(v) - E_1(u_1-u_2-v) + E_1(u_1-v) - E_1(u_2+v)\,. \tag{A.25}$$

One can rewrite the last function as

$$\mathbf{f}(u_1, u_2, v) = -\frac{\vartheta'(0)\vartheta(u_1)\vartheta(u_2)\vartheta(u_2-u_1+2v)}{\vartheta(u_1-v)\vartheta(u_2+v)\vartheta(u_2-u_1+v)\vartheta(v)}\,. \tag{A.26}$$

Using (A.2), (A.4), (A.9) one can derive from (A.24) some important particular cases. One of them corresponding to $v = u_1$ (or $v = -u_2$), is the Fay identity (A.19). Another particular case comes from $u_1 = 0$ (or $u_2 = u$):

$$\begin{aligned} \phi(v, z-w)\phi(-v, z)\phi(u+v, w) - \phi(-u-v, z-w)\phi(u+v, z)\phi(-v, w) = \\ \phi(u_1, z)(E_2(u+v) - E_2(v))\,. \end{aligned} \tag{A.27}$$

If $u_2 \to -v$ then (A.24) in the first non-trivial order take the form for $u_1 = \alpha,\ u_2 = \beta$

$$\begin{aligned} \phi(-\beta, z-w)E_1(w)\phi(\alpha+\beta, z) - \phi(\alpha, z-w)E_1(z)\phi(\alpha+\beta, w) = \\ \phi(\alpha, z)\phi(\beta, w)(E_1(\alpha) + E_1(\beta) - E_1(\alpha+\beta))\,. \end{aligned} \tag{A.28}$$

5.2 Appendix B. Lie algebra $\mathrm{sl}(N,\mathbb{C})$ and elliptic functions

Introduce the notation

$$\mathbf{e}_N(z)=\exp(\frac{2\pi i}{N}z)$$

and two matrices

$$Q=\mathrm{diag}(\mathbf{e}_N(1),\dots,\mathbf{e}_N(m),\dots,1) \tag{B.1}$$

$$\Lambda=\delta_{j,j+1}\,,\quad(j=1,\dots,N\,,\ mod\,N)\,. \tag{B.2}$$

Let

$$\mathbb{Z}^{(2)}_N=(\mathbb{Z}/N\mathbb{Z}\oplus\mathbb{Z}/N\mathbb{Z})\,,\quad\tilde{\mathbb{Z}}^{(2)}_N)=\mathbb{Z}^{(2)}_N\setminus(0,0) \tag{B.3}$$

be the two-dimensional lattice of order N^2 and N^2-1 correspondingly. The matrices $Q^{a_1}\Lambda^{a_2}$, $a=(a_1,a_2)\in\mathbb{Z}^{(2)}_N$ generate a basis in the group $\mathrm{GL}(N,\mathbb{C})$, while $Q^{\alpha_1}\Lambda^{\alpha_2}$, $\alpha=(\alpha_1,\alpha_2)\in\tilde{\mathbb{Z}}^{(2)}_N$ generate a basis in the Lie algebra $\mathrm{sl}(N,\mathbb{C})$. More exactly, we introduce the following basis in $\mathrm{GL}(N,\mathbb{C})$. Consider the projective representation of $\mathbb{Z}^{(2)}_N$ in $\mathrm{GL}(N,\mathbb{C})$

$$a\to T_a=\frac{N}{2\pi i}\mathbf{e}_N(\frac{a_1a_2}{2})Q^{a_1}\Lambda^{a_2}\,, \tag{B.4}$$

$$T_aT_b=\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})T_{a+b}\,,\quad(a\times b=a_1b_2-a_2b_1)\,. \tag{B.5}$$

Here $\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})$ is a non-trivial two-cocycle in $H^2(\mathbb{Z}^{(2)}_N,\mathbb{Z}_{2N})$. The matrices T_α, $\alpha\in\tilde{\mathbb{Z}}^{(2)}_N$ generate a basis in $\mathrm{sl}(N,\mathbb{C})$. It follows from (B.5) that

$$[T_\alpha,T_\beta]=\mathbf{C}(\alpha,\beta)T_{\alpha+\beta}\,, \tag{B.6}$$

where $\mathbf{C}(\alpha,\beta)=\frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)$ are the structure constants of $\mathrm{sl}(N,\mathbb{C})$.

For $N=2$ the basis T_α is proportional to the basis of the Pauli matrices:

$$T_{(1,0)}=\frac{1}{\pi\imath}\sigma_3\,,\quad T_{(0,1)}=\frac{1}{\pi\imath}\sigma_1\,,\quad T_{(1,1)}=\frac{1}{\pi\imath}\sigma_2\,.$$

The Lie coalgebra $\mathfrak{g}^*=\mathrm{sl}(N,\mathbb{C})$ has the dual basis

$$\mathfrak{g}^*=\{\mathbf{S}=\sum_{\tilde{\mathbb{Z}}^{(2)}_N}S_\gamma t^\gamma\}\,,\quad t^\gamma=\frac{2\pi\imath}{N^2}T_{-\gamma}\,,\quad\langle T_\alpha t^\beta\rangle=\delta_\alpha^{-\beta}\,. \tag{B.7}$$

It follows from (B.6) that $\mathfrak{g}^*$ is a Poisson space with the linear brackets

$$\{S_\alpha,S_\beta\}=\mathbf{C}(\alpha,\beta)S_{\alpha+\beta}\,. \tag{B.8}$$

The coadjoint action in these basises takes the form

$$\mathrm{ad}^*_{T_\alpha}t^\beta=\mathbf{C}(\alpha,\beta)t^{\alpha+\beta}\,. \tag{B.9}$$

Let $\breve{\gamma}=\frac{\gamma_1+\gamma_2\tau}{N}$. Then introduce the following constants on $\tilde{\mathbb{Z}}^{(2)}$:

$$\vartheta(\breve{\gamma})=\vartheta(\frac{\gamma_1+\gamma_2\tau}{N})\,,\quad E_1(\breve{\gamma})=E_1(\frac{\gamma_1+\gamma_2\tau}{N})\,,\quad E_2(\breve{\gamma})=E_2(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.10}$$

$$\phi_\gamma(z)=\phi(\breve{\gamma},z)\,, \tag{B.11}$$

$$\varphi_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\phi_\gamma(z)\,, \tag{B.12}$$

$$\varphi_{\gamma,\eta}(z) = \mathbf{e}_N(\gamma_2 z)\phi(\eta + \frac{\gamma_1+\gamma_2\tau}{N}, z)\,. \tag{B.13}$$

They have the following quasi-periodicities

$$\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z)\,, \quad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z)\,, \tag{B.14}$$

$$\varphi_{\gamma,\eta}(z+1) = \mathbf{e}_N(\gamma_2)\varphi_{\gamma,\eta}(z)\,, \quad \varphi_{\gamma,\eta}(z+\tau) = \mathbf{e}_N(-\gamma_1-\eta)\varphi_{\gamma,\eta}(z)\,, \tag{B.15}$$

The important relations for these functions are

$$\frac{\varphi_\eta(z_1-z_2)}{\varphi_\eta(z_1)} = \frac{1}{\varphi_\eta(z_2)}(E_1(z_2)+E_1(\eta)+E_1(z_1-z_2)-E_1(z_1+\eta)), \tag{B.16}$$

$$\frac{\varphi_{\alpha,\eta}(z_1-z_2)}{\varphi_\eta(z_1)} = \frac{1}{\varphi_\eta(z_2)}\varphi_\alpha(z_1-z_2) + \frac{\varphi_{-\alpha}(z_2)}{\varphi_\eta(z_2)\varphi_\alpha(\eta)}\varphi_\alpha(z_1+\eta), \tag{B.17}$$

$$\frac{\varphi_{\alpha,\eta}(z)}{\varphi_\eta(z)} = \frac{\varphi_\alpha(z+\eta)}{\varphi_\alpha(\eta)}. \tag{B.18}$$

Another important relation in the case $N=2$ is

$$k_\gamma f^\hbar(\gamma,\alpha,0) = J_\gamma = E_2(\gamma+\hbar) - E_2(\hbar), \tag{B.19}$$

where

$$k_\gamma = E_1(\gamma+\hbar) - E_1(\gamma) - E_1(\hbar),$$

We give a short comment of this formula. From (A.24),(A.25) we have:

$$(E_1(\gamma+\hbar)-E_1(\gamma)-E_1(\hbar))(E_1(\alpha+\hbar)+E_1(-\beta+\hbar)-E_1(\gamma+\hbar)-E_1(\hbar)) = E_2(\gamma+\hbar)-E_2(\hbar), \tag{B.20}$$

where we suppose $\alpha-\beta=\gamma$. The function at r.h.s. and the function at l.h.s. have the coinciding poles ($\hbar=0, \hbar=-\gamma$) and zeroes ($\hbar=-\frac{1}{2}\gamma$), so we come to the equality of these functions.

Define the function

$$f_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\partial_u\phi(u,z)|_{u=\breve\gamma} = \varphi_\gamma(z)(E_1(\breve\gamma+z)-E_1(\breve\gamma))\,. \tag{B.21}$$

It follows from (A.10) that

$$f_\gamma(z) = \varphi_\gamma(z)(E_1(\breve\gamma+z)-E_1(\breve\gamma))\,. \tag{B.22}$$

$$\mathbf{f}_{\alpha,\beta,\gamma} = E_1(\breve\gamma) - E_1(\breve\alpha-\breve\beta-\breve\gamma) + E_1(\breve\alpha-\breve\gamma) - E_1(\breve\beta-\breve\gamma)\,. \tag{B.23}$$

(see (A.25))

It follows from (A.7) that

$$\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z)\,, \quad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z)\,. \tag{B.24}$$

$$f_\gamma(z+1) = \mathbf{e}_N(\gamma_2)f_\gamma(z)\,, \quad f_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)f_\gamma(z) - 2\pi\imath\varphi_\gamma(z)\,. \tag{B.25}$$

The modification of (A.24) is

$$\varphi_\gamma(z-x_j)\varphi_{-\gamma}(z-x_k) = \varphi_\gamma(x_k-x_j)(E_1(z-x_k)-E_1(z-x_j)) - f_\gamma(x_k-x_j)\,. \tag{B.26}$$

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