Yang-Mills theory and a superquadric.

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November 16, 2004

Abstract

We construct a supermanifold ST which turns to be an open subset of a superquadric $Q(5|6) \subset \mathbf{P}^{3|3} \times \mathbf{P}^{3|3}$. The Dolbeault algebra $\Omega^{0\bullet}(ST)$ is quasiisomorphic to N=3, D=4 YM algebra in Batalin-Vilkovisky formulation. We construct a $\bar{\partial}$-closed functional $tr : \Omega^{0\bullet}(ST) \to \mathbb{C}$. We conjecture that Chern-Simons theory associated with a triple $(\Omega^{0\bullet}(ST) \otimes Mat_n, \bar{\partial}, tr \otimes tr_{Mat_n})$ is equivalent to N=3, D=4 YM theory with gauge group U_n in euclidean signature.

1 Introduction

It is well known since long ago [1] that it is possible to encode solutions $N = 3$ supersymmetric YM-equation in dimension $D = 4$ by complex structures on a vector bundle defined over an open subset U in a superquadric $Q(5|6)$ (twistor superspace). Symbol $n|m$ denotes dimension of a supermanifold. The quadric $Q(5|6) \subset \mathbf{P}^{3|3} \times \mathbf{P}^{3|3}$ is defined by equation

$$\sum_{i=0}^{3} x_i x^i + \sum_{i=1}^{3} \psi_i \psi^i = 0, \tag{1}$$

in homogeneous coordinates

$$x_0, x_1, x_2, x_3, x^0, x^1, x^2, x^3, \psi_1, \psi_2, \psi_3, \psi^1, \psi^2, \psi^3 \tag{2}$$

on $\mathbf{P}^{3|3} \times \mathbf{P}^{3|3}$. The quadric a is complex supermanifold. It makes sense therefore to talk about differential (p, q)-forms $\Omega^{p,q}(Q(5|6))$.

It was suggested in [2] that it would be nice to use geometric methods of [3] to define the Lagrangian of N=3 D=4 YM theory in twistor formulation in a way similar to holomorphic Chern-Simons. If a is $(0, 1)$ matrix-valued form then the Lagrangian should be

$$CS(a) = tr_{Mat_n} \int (\frac{1}{2} a\bar{\partial}a + \frac{1}{6}a^3)Vol \tag{3}$$

where Vol is some integral form.

The main result of the present note is that we construct integral form Vol and give a definition of the domain of integration Z.

The following definition will be useful

Definition 1 *Suppose we are given a differential graded algebra (A, d) with a d-closed trace functional tr. We can consider A as a space of fields in some field theory with Lagrangian defined by the formula*

$$CS(a) = tr(\frac{1}{2}ad(a) + \frac{1}{6}a^3) \tag{4}$$

We call it a Chern-Simons theory associated with dga (A, d, tr)

Thus the matrix-valued Dolbeault complex $(\Omega^{0,\bullet}(U) \otimes Mat_n, \bar{\partial})$ with trace defined by the formula $tr(a) = tr_{Mat_n} \int_Z aVol$ would give an example of such algebra, had we the domain Z and a supermeasure Vol.

We construct measure Vol and a domain of integration Z in several steps.

Step 1. We define a compact supermanifold $\widetilde{\Pi F}$ and construct an integral form μ on it in a spirit of [3]. Let A_{rd} be a Dolbeault complex of $\widetilde{\Pi F}$. Integration of an element $a \in A$ against μ over $\widetilde{\Pi F}$ defines a $\bar{\partial}$-closed trace functional on A_{rd}. The Chern-Simons theory associated with dga $(A_{rd} \otimes Mat_n, \bar{\partial}, tr \otimes tr_{Mat_n})$ turns to be equivalent to $N = 3, D = 4$ Yang-Mills theory with gauge group U_n reduced to a point.

Step 2. From the algebra A_{rd} we reconstruct a differential algebra A. The algebra $A \otimes Mat_n$ conjecturally encodes full $N = 3, D = 4$ Yang-Mills theory with gauge group U_n in a sense of definition (1). If we put aside the differential d, A is equal to $A_{rd} \otimes C^{an}(\mathbb{R}^4)$($an$ stands for analytic). The integral form we are looking for is equal to $Vol = \mu dx_1 dx_2 dx_3 dx_4$, where $\mathbb{R}^4$ is equipped with coordinates x_1, x_2, x_3, x_4.

Step 3.

The domain of integration Z is not a complex submanifold of $Q(5|6)$. Instead, metaphorically speaking, it is as much real as it could be. Still the manifold Z carries some traces of complex structure of ambient $Q(5|6)$. It has a foliation $\mathcal{F}$ with complex leaves. This enables us to define an algebra B of Dolbeault forms $\Omega^{0\bullet}_{\mathcal{F}}(Z)$ defined along leaves of $\mathcal{F}$ (such beasts are discussed in the Appendix (5.2)). If one impose some real analyticity property on elements of $\Omega^{0\bullet}_{\mathcal{F}}(Z)$ the algebra becomes isomorphic to A from Step 2 and acquires a trace functional which we denote by tr_Z. The key role in our construction is played by an open subset $U = ST \subset Q(5|6)$-an orbit of super Poincare group $SP \subset GL(4|3)$. The domain of integration Z is some subset of ST. The projection $\pi : \Omega^{0\bullet}(ST) \to \Omega^{0\bullet}_{\mathcal{F}}(Z)$ could be used to pull the trace tr_Z back to $\Omega^{0\bullet}(ST)$. The projection π becomes a quasiisomorphism after appropriate completion of $\Omega^{0\bullet}(ST)$. We can use a triple $(\Omega^{0\bullet}(ST), \bar{\partial}, tr_Z)$ to define a Chern-Simons theory, which conjecturally should be equivalent to N=3,D=4 YM theory in euclidean signature.

It looks like the space ST is a maximal subset of $Q(5|6)$ such that solutions of SYM corresponding to holomorphic bundles on U are dense in a space of all solutions. Some computations supporting this statement are done in section (4.3).

Few words about the structure of the note are in order. In the section (2) we introduce basic definitions and setup notations.

In sections (3.1-3.3) we give a geometric twistor like description of $N = 3$ $D = 4$ YM theory reduced to a point(Step 1). In Section (4) we do Step 2, section (4.2) we do some preparatory work for Step 3. In section (4.4) we do Step 3. The main propositions are (16 and 24).

The exposition is rather sketchy. Not all the proofs in this note are spelled out. We hope to write a more expanded version elsewhere.

2 Definitions and notations

2.1 Group $GL(4|3)$ and its homogeneous space $Q(5|6)$.

In this section we examine the manifold $Q(5|6)$ as a homogeneous space. It will allow us to exhibit a global holomorphic super volume form *vol*- a section of holomorphic Berezinian line bundle. This volume form will be one of the ingredients of the supermeasure Vol.

Denote $Q(5|6)$ a supermanifold which a submanifold of $\mathbf{P}^{3|3} \times \mathbf{P}^{3|3}$ defined by equation (1). The action of $GL(4|3)$ preserves the equation (1). Hence $Q(5|6)$ is $GL(4|3)$-space. In fact it is a homogeneous space .

Let $\mathbb{C}^{4|3}$ be a complex linear super space, on which $GL(4|3)$ acts. We can identify the quadric $Q(5|6)$ with the space of partial flags as it is done in purely even case (see [5] for example). Any flag

$$F_1 \subset F_2 \subset \mathbb{C}^{4|3} \tag{5}$$

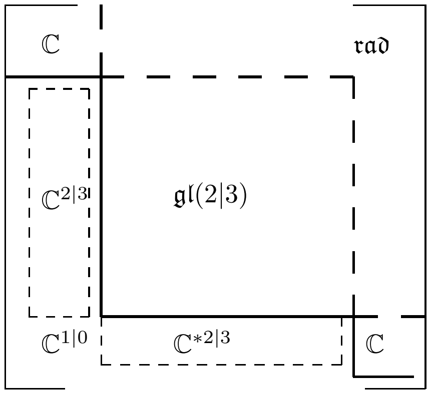
where $F_1 \cong \mathbb{C}^{1|0}$ and $F_2 \cong \mathbb{C}^{3|3}$ can be interpreted as a pair of points $F_1 \in \mathbf{P}^{3|3}$, $F_2 \in \mathbf{P}^{*3|3}$. The $*$ sign means a dual space. The condition (5) is equivalent to (1).

Let us choose a standard basis $e_1, \ldots, e_7$ of $\mathbb{C}^{4|3}$ with the parity ε of elements is $\varepsilon(e_1) = \varepsilon(e_2) = \varepsilon(e_3) = \varepsilon(e_7) = 1$, $\varepsilon(e_4) = \varepsilon(e_5) = \varepsilon(e_6) = -1$. In this notations a standard flag F

$$\begin{aligned} &F_1 = \mathrm{span} < e_7 > \\ &F_2 = \mathrm{span} < e_2, \ldots e_7 > \end{aligned} \tag{6}$$

defines a point in the space $Q(5|6)$.

It is easy to compute a shape of the matrix of the stabilizer of F. The following picture is useful



(7)

The stabilizer P with a Lie algebra $\mathfrak{p}$ is formed by matrices with zero entries below thick solid line in picture (7). The stabilizer has a form $P = \mathbb{C}^* \times \mathbb{C}^* \times GL(2|3) \ltimes Rad$, where Rad is the radical of the parabolic subgroup P.

Denote $i : GL(4|3) \to GL(4|3)/P$ the projection. The tangent space to a point $i(1)$ in homogeneous space $GL(4|3)/P$ can be identified with $\mathfrak{gl}(4|3)/\mathfrak{p}$. A Lie algebra $\mathfrak{m}$ is formed by elements with nonzero entries below thick solid line isomorphically projects into $\mathfrak{gl}(4|3)/\mathfrak{p}$. However this map is not a map of P representations. The quotient space is a representation of P which is isotropy representation of P at the tangent space at the point $i(1) \in GL(4|3)/P$.

The space $\mathfrak{m}$ decomposes into a sum $\mathbb{C}^{2|3} + \mathbb{C}^{*2|3} + \mathbb{C}^{1|0}$ of irreducible $GL(2|3)$ representations

The reader can easily identify these spaces with matrix block in the picture (7).

The elements $a_{21}, a_{31}, \alpha_{41}, \alpha_{51}, \alpha_{61}, a_{71}, a_{72}, a_{73}, \alpha_{74}, \alpha_{75}, \alpha_{76}$ stand for coordinates on the linear space $\mathfrak{m}$. The subscript (i, j) indicates on what matrix entry it takes nonzero value.

Proposition 2 *An element*

$$vol = da_{21} \wedge da_{31} \wedge d\alpha_{41} \wedge d\alpha_{51} \wedge d\alpha_{61} \wedge da_{71} \wedge da_{72} \wedge da_{73} \wedge d\alpha_{74} \wedge d\alpha_{75} \wedge d\alpha_{76} \quad (8)$$

is an element of $Ber(\mathfrak{gl}(4|3)/\mathfrak{p})^*$. *It is invariant with respect to the action of* P.

Proof. The proof is given in subsection (4.5). ■

As a corollary of proposition (8) we get

Proposition 3 *The element vol can be extended to* $GL(4|3)$*-invariant section vol of* $Ber_{\mathbb{C}} Q(5|6)$.

2.2 Group SP and its homogeneous space ST.

In this section we introduce a subset $ST = U \subset Q(5|6)$ which is invariant with respect to super-Poincare group SP. We make a treatment of ST as a homogeneous SP-space. The space ST will play an important role in construction of the domain $X \subset Q(5|6)$ over which the Lagrangian density will be integrated.

Introduce notations: W is a two-dimensional vector space, $W =< w^+, w^- >$. We shall have two copies of W which we denote W_l, W_r. Let T be 3-dimensional vector space, T^*-its dual and $V = W_l \otimes W_r$.

The group $GL(4|3)$ acts on $W_l + W_r + \Pi T \cong \mathbb{C}^{4|3}$ via its tautological representation. The sign Π denotes a parity change.

Define a super Lie algebra $\mathfrak{susy}_0(3) = \mathfrak{susy}_0 = V$, $\mathfrak{susy}_1(3) = \mathfrak{susy}_1 = W_l \otimes T + W_r \otimes T^*$. The nontrivial bracket on $\mathfrak{susy}_1$ defined by the formula: $\{w_l \otimes t, w_r \otimes t^*\} = w_l \otimes w_r(t, t^*)$, where (t, t^*) is a contraction of vector and covector.

The group $SL(W_l) \times SL(W_r) \times GL(T) = SL_2^l \times SL_2^r \times GL_3$ is acting on $\mathfrak{susy}$ in an obvious manner. Denote

$$SP = (SL_2^l \times SL_2^r) \ltimes \mathfrak{susy} \tag{9}$$

Similarly define

$$GrSP = (SL_2^l \times SL_2^r) \ltimes (V + W_l \otimes T + W_r \otimes T^*) \tag{10}$$

with the factor in the second bracket equipped with trivial Lie operation. We can also define a crossproduct $GL_3 \ltimes SP$ and $GL_3 \ltimes GrSP$ with GL_3 acting nontrivially only of the spaces T, T^*.

The group $GL_3 \ltimes SP$ embeds into $GL(4|3)$ by the rule:

$$\left[\begin{array}{c|c|c} SL_2 & W_r \otimes T & V \\ \hline 0 & GL_3 & W_l \otimes T^* \\ \hline 0 & 0 & SL_2 \end{array}\right] \tag{11}$$

It implies that the group SP acts on the superquadric $Q(5|6)$.

Definition 4 *Denote by ST the SP orbit of the standard flag (6).*

Definition 5 *For a supermanifold M denote M_{red} the underlying ordinary manifold.*

It is easy to see that the group $(SO(4) \ltimes V) \times GL_3$ is a parabolic subgroup of $GL(4|3)_{red}$. It means that the action of the former group on a generic point sweep a union of Schubert cells in $Q(5|6)_{red}$. The precise $GL(4|3)$-intrinsic description of the ST is unknown, except it is a dense in $Q(5|6)$.

Define $C^i = \{\begin{pmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{pmatrix} \in SL_2^i\}$ and $B^i = \{\begin{pmatrix} z_i & a_i \\ 0 & z_i^{-1} \end{pmatrix} \in SL_2^i\}$; $i = l, r$. The linear spaces $< w_i^- >; i = l, r$ are invariant with respect to B^i.

Definition 6 *A set $< w_l^- \otimes w_r^-, w_l^- \otimes T, w_r^- \otimes T^* >$ spans a subalgebra A in $\mathfrak{susy}$ which is invariant with respect to the action of $B^l \times B^r$.Denote by H the Lie group $(B^l \times B^r) \ltimes A$ with Lie algebra $\mathfrak{h}$.*

Proposition 7 *The stabilizer of the flag (6) is equal to $(B^l \times B^r \times GL_3) \ltimes A$.*

Proof. It becomes obvious if superimpose pictures (7 and 11). ■

The restriction of the volume form vol from (8) defines SP-invariant section of $vol_{ST} \in Ber_{\mathbb{C}} ST$

Definition 8 *Extend the Lie group SP to SP_G by adding "ghosts" for translations. We get*

$$SP_G = SP \ltimes \Pi V \tag{12}$$

The action of SP on ΠV factors through $SL_2^l \times SL_2^r$-factor. The Lie algebra of SP_G carries a differential d_G which maps an element $v \in \Pi V$ to the same element in V

3 Reduced theory

As a preliminary step in construction of the supermeasure we introduce a manifold $\widetilde{\Pi F}$ and an integral form on it. The form will define a trace tr_D on Dolbeault complex of this manifold. The CS theory constructed by $(\Omega^{0\bullet}(\widetilde{\Pi F}) \otimes Mat_n, \bar{\partial}, tr_D \otimes tr_{Mat_n})$ will become conjecturally equivalent to D=4 N=3 YM theory with gauge group U_n reduced to a point.

3.1 Manifold ΠF

The manifold $\widetilde{\Pi F}$ is a deformation of a more simple ΠF. In this section we give relevant definitions concerning ΠF.

Denote $X = \mathbf{P}^1 \times \mathbf{P}^1$. It has two projections $p_i : X \to \mathbf{P}^1, i = l, r$. Denote $\mathcal{O}(1)$ the dual to the Hopf line bundle over $\mathbf{P}^1$. The Picard group of X is $\mathbb{Z}+\mathbb{Z}$. It is generated by the classes of line bundles $\pi_l^*\mathcal{O}(1) = \mathcal{L}_l, \quad \pi_r^*\mathcal{O}(1) = \mathcal{L}_r$ which can serve as coordinates in $Pic(X)$. A line bundle $\mathcal{L}_l^{\otimes a} \otimes \mathcal{L}_r^{\otimes b}$ from $Pic(X)$ with coordinates (a, b) will be denoted as $\mathcal{O}(a, b)$.

Convention For a holomorphic (super)manifold Y and a vector bundle G on Y we denote $H^\bullet(Y, G)$ the cohomology of Y with coefficients in G. It can be computed as cohomology of Dolbeault complex $\Omega^{0,\bullet}(Y) \otimes G$. It is tacitly assumed that the omitted argument Y in $\Omega^{0,\bullet}(Y) \otimes G$ implies $Y = X = \mathbf{P}^1 \times \mathbf{P}^1$. If G-argument is missing we assume that $G = \mathcal{O}$.

Denote $\mathrm{Sym}V, \Lambda V$ symmetric and exterior algebras of a vector space (bundle).

Denote $\Theta = \mathcal{O}(1,1)$. We construct a vector bundle F over X. It is a direct sum of line bundles

$$\begin{aligned} F &= T \otimes \mathcal{L}_l + T^* \otimes \mathcal{L}_r + \Theta^* \\ H &= T \otimes \mathcal{L}_l + T^* \otimes \mathcal{L}_r. \end{aligned} \tag{13}$$

The supermanifold ΠF is the total space of vector bundle F with changed parity fibers. The manifold ΠF is closely related to ST and the relation will be examined in the following sections.

3.2 Properties of ΠF.

Aiming to define the manifold $\widetilde{\Pi F}$ which is a deformation of ΠF we exhibit in this section the deformation cocycle, which defines an infinitesimal deformation of the complex structure on ΠF. We extend it to actual deformation of ΠF. The manifold $\widetilde{\Pi F}$ is not homogeneous, which requires an extra work for definition of the integral form. We construct the form in the present section.

The manifold ΠF is a complex supermanifold. One can define a Dolbeault complex $(\Omega^{0,\bullet}(\Pi F), \bar{\partial})$. This is a differential graded algebra. The manifold ΠF is split, therefore the algebra $(\Omega^{0,\bullet}(\Pi F), \bar{\partial})$ contains as a differential subalgebra the Dolbeault complex $\Omega^{0,\bullet}\Lambda F^*$ of vector bundle ΛF^*.

Proposition 9 *Differential algebras $\Omega^{0,\bullet}(\Pi F)$ and $\Omega^{0,\bullet}\Lambda F^*$ are quasiisomorphic.*

The canonical class $\mathcal{K}_X$ is equal to $\mathcal{O}(-2,-2)$.There is a nontrivial cohomology class-"fundamental" class: $\alpha \in H^2(X, \mathcal{O}(-2,-2)) \overset{id}{\subset} H^2(X, \mathcal{O}(-2,-2) \otimes T \otimes T^*) \subset H^2(X, \Lambda^2(H^*) \otimes \Theta^*) \subset H^2(X, \Lambda F^* \otimes \Theta^*)$. We interpret $\Lambda(F^*) \otimes \Theta^*$ as a sheaf of local holomorphic differentiations of ΠF in direction of Θ^*.

A representative $\alpha = \alpha_{ij} d\bar{z}_i d\bar{z}_j \frac{\partial}{\partial\theta}$ of the class $[\alpha]$ can be extended to differentiation of $\Omega^{0,\bullet}(\Pi F)$. The properties of $D = \bar{\partial} + \alpha$:

1) differentiation of $\Omega^{0,\bullet}(\Pi F)$,

2) $D^2 = 0$

are corollaries of $\bar{\partial}$-cocycle equation for α. The operator D defines a new "holomorphic" structure on ΠF [1]. This new complex manifold will be denoted by $\widetilde{\Pi F}$.

The manifold ΠF is Calabi-Yau. By this we mean that $Ber_{\mathbb{C}}$ is trivial. Indeed the determinant line bundle of F is equal to $det(T \otimes \mathcal{L}_l) \otimes det(T^* \otimes \mathcal{L}_r) \otimes det(\mathcal{O}(-1,-1)) = \mathcal{O}(3,0) \otimes \mathcal{O}(0,3) \otimes \mathcal{O}(-1,-1) = \mathcal{O}(2,2)$; $\quad Ber_{\mathbb{C}}\Pi F = \mathcal{K}_X \otimes detF = \mathcal{O}$-is trivial.

It implies that the bundle $Ber_{\mathbb{C}}\Pi F$ admits a nonvanishing section.

This section is $SO(4)$-invariant. The action of $\mathbb{C}+\mathbb{C}$-the unipotent subgroup of parabolic $B \subset SO(4)$ (for which $X = SO(4)/B$) on large Schubert cell of X is by translations (the unipotent subgroup is commutative and by definition the action is free). Hence the section of $Ber_{\mathbb{C}}\Pi F$ is

$$vol_{\Pi F} = da_l \wedge da_r \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\tilde{\alpha}_1 \wedge d\tilde{\alpha}_2 \wedge d\tilde{\alpha}_3 \wedge d\theta,$$

where $\alpha_1, \ldots, \tilde{\alpha}_3, \theta$ are $\mathbb{C}+\mathbb{C}$-invariant coordinates on the odd fiber.

On manifold ΠF we can construct a global holomorphic integral 2-form the way explained in (5.1). In our case it is equal to $c_{\Pi F} = \alpha_1 \ldots \tilde{\alpha}_3 \theta d\alpha_1 \wedge \cdots \wedge d\tilde{\alpha}_3 \wedge d\theta$.

Proposition 10 *The form* $\mu = vol_{\Pi F} \otimes \bar{c}_{\Pi F}$ *is* D *closed nontrivial integral* $(0,-2)$*-form on* ΠF.

It defines a $\bar{\partial} + \alpha$-closed trace on $\Omega^{0\bullet}(\Pi F) \otimes Mat_n$ by the formula

$$tr_\mu(a) = tr \int_{\Pi F} a\mu$$

The algebra of cohomology of $(\Omega^{0,\bullet}(\Pi F), \bar{\partial})$ coincides with an A_∞-algebra $C = H^\bullet(X, \Lambda(H^*) \otimes \Lambda(\Theta))$ which we will describe now. The differential α will induce

[1] This definition is not quite conventional one, because usually the deformation cocycle is $\alpha \in \Omega^{0,1}(\mathcal{T})$ ($\mathcal{T}$ is a holomorphic tangent bundle), where as in our case $\alpha \in \Omega^{0,2}(\mathcal{T})$. We however continue to use a traditional wording and call it a deformation of complex structure, though a more precise term would be deformation of the Dolbeault algebra $\Omega^{0,\bullet}$. This algebra in our approach becomes a substitute for the underlying manifold. Instead of deforming the manifold we are deforming the differential $\bar{\partial} \to \bar{\partial} + \alpha$

a differential $[\alpha]$ on C. The ghost grading of the group $H^i(X, \Lambda^k(H^*) \otimes \Lambda^s(\Theta))$ is equal to $i+s$, the additional grading is equal to $k+2s$(preserved by $\bar{\partial}$ and α). We used somewhat perverted ghost grading, which differs from conventional by one. In particular the ghost grading of the gauge and spinor fields is equal to one. In the table below you will find the field content :

gh	deg		gh	deg	
0	0	$H^0(X,\Lambda^0(H^*)) = \mathbb{C}$	1	2	$H^0(X,\Lambda^0(H^*)\otimes\Theta) = V$
1	2	$H^1(X,\Lambda^2(H^*)) = \Lambda^2(T) + \Lambda^2(T^*)$	1	3	$H^0(X,\Lambda^1(H^*)\otimes\Theta) = W_l\otimes T + W_r\otimes T^*$
1	3	$H^1(X,\Lambda^3(H^*)) = W_l + W_r$	1	4	$H^0(X,\Lambda^2(H^*)\otimes\Theta) = T\otimes T^*$
2	4	$H^2(X,\Lambda^4(H^*)) = T\otimes T^*$	2	5	$H^1(X,\Lambda^3(H^*)\otimes\Theta) = W_l + W_r$
2	5	$H^2(X,\Lambda^5(H^*)) = W_l\otimes\Lambda^2(T) + W_r\otimes\Lambda^2(T^*)$	2	6	$H^1(X,\Lambda^4(H^*)\otimes\Theta) = T + T^*$
2	6	$H^2(X,\Lambda^6(H^*)) = V$	3	8	$H^2(X,\Lambda^6(H^*)\otimes\Theta) = \mathbb{C}$

(14)

The groups $H^0(X, \Lambda^2(H^*) \otimes \Theta)$ and $H^2(X, \Lambda^4(H^*))$ are contracting pairs, are killed by the differential $[\alpha]$ and should be considered as auxiliary fields. It is possible to reconstruct all operations in this algebra.

Proposition 11 *The group SP_G acts on $\widetilde{\Pi F}$. The Lie derivatives of μ with respect to generators of Lie algebra of SP_G are $\bar{\partial} + \alpha$-exact.*

Proof. The proof is given in subsection (4.5). ■

Proposition 12 *The CS-theory constructed by differential graded algebra with a trace $(\Omega^{0,\bullet}(\widetilde{\Pi F}) \otimes Mat_n, \bar{\partial} + \alpha, tr_\mu)$*

is equivalent IKKT.

Proof. One can adopt a technique of [4] to prove this statement. ■

3.3 Relation of $\widetilde{\Pi F}$ and ST

It might seem that the manifold $\widetilde{\Pi F}$ has appeared from nowhere. In this section we demonstrate an intimate relation between $\widetilde{\Pi F}$ and ST.

The subgroup of translations $V \subset SP$ acts on ST.

Consider a tensor product $\Omega^{0,\bullet}(ST) \otimes \Lambda(V)$ with a differential equal to $\bar{\partial} + d_{Vgh}$. If we choose a basis x_{ij} in V then $d_{Vgh} = \sum \tilde{x}^{ij} L_{x_{ij}}$, where $\tilde{x}^{ij}$ stands for the basis in the space of V-"ghosts", $L_{x_{ij}}$ for Lie derivative along the vector field on ST corresponding to a generator $x_{ij} \in V$.

Proposition 13 *There is a homomorphism of differential graded algebras* $\Omega^{0,\bullet}(ST) \otimes \Lambda(V), \bar{\partial} + d_{Vgh}) \to (\Omega^{0,\bullet}(\widetilde{\Pi F}), D)$ *which is a quasiisomorphism.*

One can interpret this result as a statement that $\widetilde{\Pi F}$ is a partial minimal model of $\Omega^{0,\bullet}(ST) \otimes \Lambda(V)$.

Since the action of V on ST is not free one can say that this is an intelligent way of reducing the theory to a point.

4 Nonreduced theory

This is the main section in the note. The integral form in its final shape is constructed in this section.

4.1 Construction of the integral form.

The integral form which will be used in construction of the Lagrangian will appear naturally in the process of recovering ST from $\widetilde{\Pi F}$. One may think it reverses reduction made in proposition (13) in which the algebra gets smaller. As a byproduct of such thing we get an algebra A with a trace. It will have a desired relation to SYM. We postpone however the discussion of the relation of this algebra to the twistor manifold $Q(5|6)$ till the section (4.4).

We will build a "cross-product" of $\Omega^{0,\bullet}(\widetilde{\Pi F})$ and $Hol(V)$-holomorphic functions on V. The linear space V has coordinates $x_{ij}, 1 \leq i, j \leq 2$. Define holomorphic differentiations of $\Omega^{0,\bullet}(\widetilde{\Pi F})\hat{\otimes}Hol(V)$ as follows.

There is an $SO(4)$ equivariant identification $V \cong H^0(\mathcal{O}(1,1))$. The line bundle $\mathcal{O}(1,1)$ is generated by its global sections. We have a short exact sequence:

$$0 \to M \to V \xrightarrow{m} \Theta = \mathcal{O}(1,1) \to 0 \qquad (15)$$

where V is considered as a trivial vector bundle with fiber V. **This sequence is defining for M.** The differentiation d_{DR} of $\Omega^{0,\bullet}(\widetilde{\Pi F})\hat{\otimes}Hol(V)$ is equal

$d_{DR}(x_{ij}) = m(x_{ij})$ and zero on the rest of the generators.

Recall that the Dolbeault differential D in $\Omega^{0,\bullet}(\widetilde{\Pi F})$ is equal to $\bar{\partial} + \alpha$. It becomes clear from explicit computation of cohomology that $\alpha d_{DR}(x_{ij})$ is $\bar{\partial}$-exact.

Choose β_{ij} such that $\bar{\partial}\beta_{ij} = \alpha d_{DR}(x_{ij})$. Define a holomorphic differentiation β of $\Omega^{0,\bullet}(\widetilde{\Pi F})\hat{\otimes} Hol(V)$ by setting $\beta(x_{ij}) = -\beta_{ij}$ and zero on the rest of the generators.

It follows from our construction that $D_{ext} = D + d_{DR} + \beta$ satisfies $D^2_{ext} = 0$. Denote $D' = d_{DR} + \beta$.

One can define an integral form on manifold $\widetilde{\Pi F} \times V$:

$$Vol = \mu \otimes \bigotimes_{i,j=1}^{2} dx_{ij}. \tag{16}$$

Proposition 14 *Vol is invariant with respect to D, D' and therefore with respect to D_{ext}.*

Proposition 15 *The algebra $(\Omega^{0,\bullet}(\widetilde{\Pi F}) \otimes Hol(U) \otimes Mat_n, D_{ext})$ is quasiisomorphic to $(\Omega^{0,\bullet}(ST), \bar{\partial})$.*

To use Vol for integration one needs to choose a real four-dimensional cycle in $V = \mathbb{C}^4$. The linear space V carries a bilinear form $(.,.)$. Suppose that the restriction of $(.,.)$ on the cycle Σ is positive definite. We call such cycle riemannian.

Conjecture 16 *Suppose U is a small neighborhood of a riemannian cycle $\Sigma \subset \mathbb{C}^4$ such that Σ is a real linear subspace in V. Denote $Z = \Pi F \times \Sigma$. The algebra $\Omega^{0,\bullet}(\widetilde{\Pi F}) \otimes Hol(U) \otimes Mat_n$ is equipped with a trace*

$$Tr_Z(a) = tr \int_Z aVol \tag{17}$$

The CS theory constructed by this data and matrix algebra Mat_n is equivalent to N=3 SYM in dimension four build by euclidean metric and a gauge group U_n. The equivalence should hold also on quantum level.

4.2 Construction of ST via deformation.

The manifold ST is a homogeneous space of the Poincare group SP. We however are forced to drop this point of view constructing the domain of integration Z. Instead we define ST as a deformation of a more simple manifold $GrST$, which is not SP homogeneous . It is easy to define Z in terms of $GrST$ and then lift Z to ST. This motivates constructions present in this section.

Consider a manifold $TotM^*$ which is a total space of the vector bundle M^*. It has a projection $p : TotM^* \to X$. The total space of the pullback ΠH on $TotM^*$ via p we denote by $GrST$.

Lemma 17 *The space of global sections of M^* is isomorphic to V.*

The space $TotM^*$ is homogeneous space of $SO(4) \ltimes V$

, hence the space $GrST$ is a homogeneous space of $GrSP$.

Proposition 18 *There is a nontrivial $SO(4)$-invariant cohomology class $[\beta] \in H^1(M^*(-1,-1))$.*

Proof. The proof is given in subsection (4.5). ■

Proposition 19 *The representative $\beta \in \Omega^{0,1}M^*(-1,-1)$ defines a nontrivial deformation of complex structure on $GrST$. The element β can be chosen to be $SO(4,\mathbb{R})$-invariant.*

Proof. The proof is given in subsection (4.5). ■

It is easy to see that β commutes with action of V on $GrST$.

Proposition 20 *A deformation of the manifold $GrST$ defined by β is isomorphic to ST.*

Proof. The proof is given in subsection (4.5). ■

The algebra $\Omega^{0,\bullet}\Lambda(H^*) \otimes \mathrm{Sym}M$ is a subalgebra in $\Omega^{0,\bullet}(GrST)$.

Proposition 21 *The image of embedding $i : \Omega^{0,\bullet}\Lambda(H^*)\otimes\mathrm{Sym}M \to \Omega^{0,\bullet}(GrST)$ is invariant with respect to differential β. By proposition (20)* $(\Omega^{0,\bullet}(GrST), \bar{\partial}+$

$\beta) \cong (\Omega^{0,\bullet}(ST), \bar{\partial})$. *We can modify operator* $\bar{\partial}$ *on* $\Omega^{0,\bullet}\Lambda(H^*)\otimes\mathrm{Sym}M$ *by adding* β. *With such modification* $i : (\Omega^{0,\bullet}\Lambda(H^*) \otimes \widehat{\mathrm{Sym}}M, \bar{\partial} + \beta) \to (\Omega^{0,\bullet}(ST), \bar{\partial})$ *is a quasiisomorphism, where* $\widehat{\mathrm{Sym}}M$ *is an appropriate completion of* $\mathrm{Sym}M$.

Since the bundle M is defined via embedding in short exact sequence (15) we have a map $i' : \Omega^{0,\bullet}\Lambda(H^*) \otimes \mathrm{Sym}M \to \Omega^{0,\bullet}\Lambda(F)\hat{\otimes}Hol(V)$, which can be extended to a completion $\Omega^{0,\bullet}\Lambda(H^*)\otimes\widehat{\mathrm{Sym}}M$. The image of i' is invariant with respect to D and D'. It is easy to see that α and d_{DR} act trivially on $\mathrm{Im}i'$.

Proposition 22 *The homomorphism* i' *commutes with differential* $\bar{\partial} + \beta$ *and is a quasiisomorphism of algebras.*

4.3 Computation of some cohomology.

In this section we give arguments in support that the open subset $ST \subset Q(5|6)$ has indeed some relation to D=4 N=3 SYM. It is not apriory entirely obvious, because standard theorems about twistor transform assert only the existence of some subset of $Q(5|6)$. We show that the first cohomology of the structure sheaf on ST is isomorphic to the space of solutions of the linearized SYM equation.

In the quest for a good subset $U \subset Q(5|6)$, holomorphic bundles on which parametrize solutions N=3 D=4 SYM equation we encounter with the following dilemma: if we take the subspace U too large , for example $U = Q(5|6)$, then the algebra $\Omega^{0,\bullet}(U)$ will have no trace functional on it. On the other hand if we take U two small Zarissky open subset of $Q(5|6)$ then $H^1(U, \mathcal{O}) = 0$ and we loose interpretation of $H^1(U, \mathcal{O})$ as space of solutions of complexified Maxwell equation. The goal of this section is to show that if we would like to have a true action (opposite to infinitesimal action) of SP on U, then $U = ST$ is the right choice.

The n-th symmetric power of the short exact sequence (15) give rise to a short exact sequence

$$0 \to \mathrm{Sym}^n(M) \to \mathrm{Sym}^n(V) \to \mathrm{Sym}^{n-1}(V) \otimes \Theta \to 0 \qquad (18)$$

Using long exact sequence arguments and already computed groups (14) we infer that $H^i(\Lambda^j(H^*)\otimes \mathrm{Sym}^k(M))$ can be computed from exact sequences:

$$
\begin{aligned}
&0 \to \mathrm{Sym}^n(V) \to V\otimes \mathrm{Sym}^{n-1}(V) \to H^1(\mathrm{Sym}^n(M)) \to 0\\
&(W_l\otimes T + W_r\otimes T^*)\otimes \mathrm{Sym}^{n-1}(V) = H^1(\Lambda^1(H^*)\otimes \mathrm{Sym}^n(M))\\
&0 \to T\otimes T^*\otimes \mathrm{Sym}^{n-1}(V) \to H^1(\Lambda^2(H^*)\otimes \mathrm{Sym}^{n-1}(M)) \to (\Lambda^2(T)+\Lambda^2(T^*))\otimes \mathrm{Sym}^n(V) \to 0\\
&0 \to H^1(\Lambda^3(H^*)\otimes \mathrm{Sym}^n(M)) \to (W_l+W_r)\otimes \mathrm{Sym}^n(V) \to (W_l+W_r)\otimes \mathrm{Sym}^{n-1}(V) \to 0\\
&0 \to (T+T^*)\otimes \mathrm{Sym}^{n-1}(V) \to H^2(\Lambda^4(H^*)\otimes \mathrm{Sym}^n(M)) \to T\otimes T^*\otimes \mathrm{Sym}^n(V) \to 0\\
&H^2(\Lambda^5(H^*)\otimes \mathrm{Sym}^n(M)) = (W_l\otimes \Lambda^2(T) + W_r\otimes \Lambda^2(T^*))\otimes \mathrm{Sym}^n(V)\\
&0 \to H^2(\Lambda^6(H^*)\otimes \mathrm{Sym}^n(M)) \to V\otimes \mathrm{Sym}^n(V) \to \mathrm{Sym}^{n-1}(V) \to 0
\end{aligned}
\tag{19}
$$

The groups $H^i(\Lambda^j(H^*)\otimes \mathrm{Sym}^k(M))$ can be interpreted as cohomology of the structure sheaf $\mathcal{O}$ of the supermanifold $GrST$ with no differential β taken into consideration. As the reader can see, for example, $H^1(\mathrm{Sym}(M)) = V\otimes \mathrm{Sym}(V)/\mathrm{Sym}(V)$ parametrizes classes of polynomial connections in trivial line bundle over V. If we take the differential β into consideration, then it will define the missing part of the field equations. For example in case of $H^1(\mathrm{Sym}(M))$ it defines a map $\mathrm{MX} : H^1(\mathrm{Sym}(M)) \to H^1(\Lambda^6(H^*)\otimes\mathrm{Sym}(M))$. This map is transgression in some spectral sequence and is defined as follows. A representative x of $[x]\in H^1(\mathrm{Sym}(M))$, can be extended to a chain of elements $x = x_0, x_2, x_4$, $x_i \in \Omega^{0.1}\Lambda^{2i}(H^*)\otimes \mathrm{Sym}(M)$. They satisfy $\beta x_i = \bar\partial x_{i+2}$ and $\mathrm{MX}([x]) = [\beta x_4]$

The spectral sequence we mentioned in the previous paragraph has the following construction. First we need to formulate a proposition.

Proposition 23 *The cohomology groups $H^*(ST,\mathcal{O})$, where $\mathcal{O}$ is the structure sheaf on ST vanish in all dimensions, except dimensions zero, where it is equal to $\mathbb{C}$ and dimensions one, where it coincides with a space analytic solution of supersymmetric Maxwell equation on V.*

This theorem can be traced, perhaps in a different form, to the beginnings of twistor methods in YM-theory.

To compute the groups $H^*(ST, \mathcal{O})$ one needs to compute cohomology of Dolbeault complex $\Omega^{0,\bullet}(ST)$, which is the same (as we know from proposition (21)) as cohomology of $\Omega^{0,\bullet}\Lambda H^* \otimes \mathrm{Sym}M$. One can filter $\Omega^{0,\bullet}\Lambda H^* \otimes \mathrm{Sym}M$ as $F^s = \bigoplus_{k\geq s} \Omega^{0,\bullet}\Lambda^k H^* \otimes \mathrm{Sym}M$. Then $E_1^{i,j} = H^i(\Lambda^j H^* \otimes \mathrm{Sym}M)$ and one of the higher differentials in this sequence coincide with MX. In fact this argument can serve as an alternative proof of proposition (23).

4.4 A small variation of the previous themes.

In this section we give a geometric interpretation of the algebra $A = \Omega^{0\bullet}(\widetilde{\Pi F}) \otimes Hol(V)$ in terms of the submanifold $X \subset Q(5|6)$. It will provide a link between construction of section (4.1) and the twistor space $Q(5|6)$.

The linear space Σ is the same as in proposition (16). Choose a linear basis $e_1, \dots, e_4 \in \Sigma$. Define $\frac{\partial}{\partial x_i}$ the differentiations in the direction of e_i. Restrict a map m from short exact sequence (15) on $\Sigma \subset V$, then $m(e_i)$ is a set of holomorphic sections of Θ. Denote $\frac{\partial}{\partial \bar{z}_i}$, the projection of $\frac{\partial}{\partial x_i} \in \mathcal{T}(\Sigma) \otimes \mathbb{C}$ into $\overline{\mathcal{T}}_{\mathbb{C}}(\Sigma)$. For any point $x \in X$ we have a subbundle $\overline{\mathcal{H}}_x \subset \overline{\mathcal{T}}_{\mathbb{C}}(\Sigma)$ panned by

$$\sum_{i=1}^{4} m(e_i)_x \frac{\partial}{\partial \bar{z}_i}, \tag{20}$$

which sweep a foliation of $X \times \Sigma$ in a sense of subsection (5.2). A key ingredient of this construction is $SO(4, \mathbb{R})$-invariant metric.

A projection

$$X \times \Sigma \to \Sigma \tag{21}$$

has holomorphic fibers. We can define $\overline{\mathcal{G}}$ as $\overline{\mathcal{T}}_{\mathbb{C}}(X)$-antiholomorphic tangent bundle to the fibers of projection (21).

The foliations $\overline{\mathcal{G}}$ and $\overline{\mathcal{H}}$ commute in a sense of subsection (5.2) , simply because $m(e_i)$ are holomorphic sections. $\overline{\mathcal{G}}$ and $\overline{\mathcal{H}}$ are transversal. Define $\overline{\mathcal{F}} = \overline{\mathcal{G}} + \overline{\mathcal{H}}$. According to subsection (5.2) $\Omega_{\overline{\mathcal{F}}} = \Omega_{\overline{\mathcal{H}}} \otimes \Omega_{\overline{\mathcal{G}}}$ and we can identify Θ and $\Omega^1_{\overline{\mathcal{H}}}$

If one pulls back the vector bundle H^* on $X \times \Sigma$ it becomes flat along leaves of $\overline{\mathcal{F}}$ (see Appendix for discussion).

We need to make a connection of $X \times \Sigma$ with the open orbit $ST \subset Q(5|6)$. We know from proposition (20) that manifold ST is a deformation of a more simple complex manifold $GrST$.

We can identify $GrST$ with the total space $Tot(\Pi H \oplus M^*)$, the symbol $\oplus$ denotes a direct sum of vector bundles. There is a projection $p : \Pi H \oplus V \rightarrow \Pi H \oplus M^*$ with one-dimensional fiber.

Then the map p restricted to $\Pi H \times \Sigma$ is an embedding. The signature of the restriction of $(.,.)$ is important in our construction, because projection p shrinks all isotropic directions.

In case of euclidean signature the structure holomorphic fibration comes as a free bonus with such embedding: the two-dimensional plain tangent to foliation is a maximal complex subspace which contains in Σ when it is mapped to a complex 3-dimensional fiber of M^*.

The restriction map $\Omega^{0,\bullet}(GrST) \rightarrow \Omega^{\bullet}_{\overline{\mathcal{F}}}(\Pi H \times \Sigma)$ commutes with differential $\bar{\partial}$. The differential β also can be pushed to $\Omega^{\bullet}_{\overline{\mathcal{F}}}(\Pi H \times \Sigma)$, which together with $\bar{\partial}$ define a deformed differential D. Denote $\Omega^{0,\bullet}_{\mathcal{F}}(Z)$ the algebra $\Omega^{\bullet}_{\overline{\mathcal{F}}}(\Pi H \times \Sigma)$ with deformed differential D. The global integral form $Vol = vol_{GrST} \otimes \bar{c}_{\Pi H}$ can be restricted to $\Pi H \times \Sigma$.

it becomes a section of $\Omega^{-3}_{int\ \overline{\mathcal{F}}}$.

Proposition 24 *The manifold $\Pi H \times \Sigma$ is equipped with a volume form $Vol \in \Omega^{-3}_{int\ \mathcal{F}}$. It is D-closed. The integral form defines a trace functional on $\Omega^{\bullet}_{\overline{\mathcal{F}}}(\Pi H \times \Sigma)$.*

Conjecture 25 *The CS-action associated with such functional is equivalent to $N = 3$, $D = 4$ YM theory in euclidean signature.*

This as a more ST-intrinsic construction of the manifold $\Pi F \times \Sigma$, which first appeared in conjecture (16). The point is that we interpreted a variable θ as one-form defined along one-dimensional foliation $\mathcal{H}$.

Now we would like to make a simple observation. It if we complexify Σ factor in $\Pi H \times \Sigma$ we will get back to construction (16).

With such identification in light of proposition (15) it becomes obvious that any real analytic solution of $Da + \frac{1}{2}\{a, a\} = 0$ on $\Pi F \times \Sigma$ is **equivalent** to solution of the same equation in a small neighborhood $U(\Pi F \times \Sigma) \subset ST$.

We can use the restriction map $\Omega^\bullet(ST) \to \Omega^\bullet_{\overline{\mathcal{F}}}(Z)$ to pullback the trace tr_Z on algebra $\Omega^\bullet(ST)$. Thus we have a triple $(\Omega^\bullet(ST), \bar{\partial}, tr_Z)$, which could be used to construct CS theory

Conjecture 26 *It is reasonable to conjecture that CS theory constructed by* $(\Omega^\bullet(ST) \otimes Mat_n, \bar{\partial}, tr_Z \otimes tr_{Mat_n})$ *is equivalent to N=3 D=4 YM-theory.*

4.5 Proofs of some statements.

In this section we do some proofs.

Proof of Proposition 2 Let us understand the structure of the representation $\mathfrak{gl}(4|3)/\mathfrak{p}$. It is easy to see that for any $l \in \mathfrak{rad}$ and element $v \in \mathfrak{m}$ we have $[v, l] \in \mathfrak{p}$. It implies that the action of $\mathfrak{rad}$ on $\mathfrak{gl}(4|3)/\mathfrak{p}$ is trivial(it is not always that case for homogeneous spaces).

What is left is to analyze the action of $\mathbb{C}^* \times \mathbb{C}^* \times GL(2|3)$ on the space $\mathfrak{gl}(4|3)/\mathfrak{p}$. It is easy to see that the map $\mathfrak{m} \to \mathfrak{gl}(4|3)/\mathfrak{p}$ is an (isomorphic) map of $\mathbb{C}^* \times \mathbb{C}^* \times GL(2|3)$ representations. It mean that we need to examine the action of $\mathbb{C}^* \times \mathbb{C}^* \times GL(2|3)$ on $\mathfrak{m}$.

There is a number of identifications between representations(bundles ...) which involve Ber. These are $Ber(A + B) = Ber(A) \otimes Ber(B)$, $Ber(A^*) = Ber(A)^*$ and $Ber(\Pi A) = Ber(A)^*$, where Π is the parity inversion, $*$ notation for contragradient representation. Using this observations we infer:

$$Ber(\mathbb{C}^{2|3} + \mathbb{C}^{*2|3} + \mathbb{C}^{1|0}) = Ber(\mathbb{C}^{2|3}) \otimes Ber(\mathbb{C}^{2|3})^* \otimes Ber(\mathbb{C}^{1|0}) = Ber(\mathbb{C}^{1|0}) \tag{22}$$

is an isomorphism of $GL(2|3)$ representations. The action of $GL(2|3)$ is trivial on $\mathbb{C}^{1|0}$. The formula (22) implies that element (8) is invariant with respect to $GL(2|3)$.

We need to analyze the action $\mathbb{C}^* \times \mathbb{C}^*$ on $Ber(\mathfrak{gl}(4|3)/\mathfrak{p})^*$.

For example an element $z \in \mathbb{C}^* \times 1 \subset \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}^{2|3} + \mathbb{C}^{1|0}$ by multiplication of all coordinates on z and acts trivially on $\mathbb{C}^{*2|3}$. Supertrace of the derivative of such transformation is equal to zero. The second case is proved similarly.

■ **Proof of Proposition 11** The action of $SU(2) \times SU(2) \subset SP$ is unmodified by α. The vector field θ_γ- the odd generator of SP gets modified to $\theta_\gamma + \phi_\gamma$, where $\phi_\gamma \in \Omega^{0,1}\Lambda^1(H^*) \otimes \Theta^*$ is determined from the equations $\{\theta_\gamma + \phi_\gamma, \bar{\partial} + \alpha\} = 0$. The element

$$x_{\gamma\delta} = \{\theta_\gamma, \phi_\delta\} + \{\phi_\gamma, \theta_\delta\} \in \Omega^{0,1}(\Theta^*) \tag{23}$$

commutes with arbitrary odd generator $\theta_\varepsilon + \phi_\varepsilon$. The element $x_{\gamma\delta}$ corresponds to the action of some generator from V. It is $\bar{\partial}$ exact and $x_{\gamma\delta} = \bar{\partial}\kappa_{\gamma\delta}$. The element $\kappa_{\gamma\delta} \in \Omega^{0,0}(\Theta^*)$ corresponds to the action of some odd generator from ΠV. We skip the part of the proof which shows that the action of SP_G is unambiguously defined.

The action of SP_G preserves the ghost number grading of $\Omega^{0,\bullet}(\Lambda(H^*)\otimes\Lambda(\Theta))$ and acts on cohomology of $\bar{\partial} + \alpha$. All cohomology groups are listed in the table (14). The third cohomology group ($gh = 3$) is one-dimensional. Simple representation theory arguments show that action of SP_G must be trivial on this group. It is easy to see that $a \to \int aVol = Tr(a)$ is $\bar{\partial} + \alpha$-closed functional. Then we must have $\xi Tr = \bar{\partial} + \alpha l_\xi$. ■

Proof of Proposition 18 The cohomology $H^\bullet(V(-1,-1))$ is equal to zero in all dimensions,with this fact and with the aid of $\mathcal{O}(-1,-1)$-twisted sequence

$$0 \to \mathcal{O}(-1,-1) \xrightarrow{\iota} V \to M^* \to 0 \tag{24}$$

dual to (15) we conclude that

$$H^1(M^*(-1,-1)) \cong H^2(\mathcal{O}(-2,-2)) \tag{25}$$

The sheaf $\mathcal{O}(-2,-2)$ is the canonical class of X, hence $H^2(\mathcal{O}(-2,-2)) = \mathbb{C}$. Denote a $SO(4,\mathbb{R})$-invariant representative, whose value on fundamental class of X equal to one by ν The isomorphism (25) is defines in the following manner.

Since $V(-1,-1)$ is acyclic, then there is $\beta \in \Omega^{0,2}V(-1,-1)$ such that $\bar{\partial}\beta = \iota(\nu)$. Consider the image of β in $\Omega^{0,1}(M^*(-1,-1))$. This is our cocycle; all choices in transition from ν to β can be fixed by requiring $SO(4,\mathbb{R})$-invariance. ■

Proof of Proposition 19 The cocycle β extends to a derivation $\Omega^{0,\bullet}(GrST)$. It coincides with the μ_{ij} from section 1.4.

We have a chain of inclusions

$$\Omega^{0,1}(M^*(-1,-1) \overset{id}{\subset} \Omega^{0,1}(T \otimes T^*(-1,-1) \otimes M^*) \subset \Omega^{0,1}(\Lambda^2(M^*) \otimes M^*) \subset$$
$$\subset \Omega^{0,1}(\Lambda(M^*) \otimes M^*) \subset \Omega^{0,1}(GrST, \mathcal{T}_{\mathbb{C}}) \tag{26}$$

$\mathcal{T}_{\mathbb{C}}$ is a holomorphic tangent bundle. It is obvious that the inclusions induce inclusions in homology. The group $H^{0,1}(GrST, \mathcal{T})$ classifies infinitesimal deformations. ■

Proof of Proposition 20 The only nontrivial relation of β with the symmetry group $GrSP$ is through the $\Pi(W_l \otimes T + W_r \otimes T^*)$-factor. If we choose an element $\theta_\alpha \in \mathfrak{susy}_0(3)$, then the element $\{\theta_\alpha, \beta\} \in \Omega^1(X, \Lambda^1(H^*) \otimes M^*)$.

Lemma 27 *The group $H^1(\Lambda^1(H^*) \otimes M^*)$ is equal to zero.*

Proof. From the structure of $\Lambda^1(H^*)$ (see formula (13)) we may assume that we need to compute groups $H^1(M^*(-1,0))$ or $H^1(M^*(0,1))$. For example from the Serre duality the first group is dual to $H^1(M(-1,-2))$. Tensoring (15) on $\mathcal{O}(-1,-2)$ and knowing that $H^0(\mathcal{O}(0,-1)) = H^1(\mathcal{O}(-1,-2)) = 0$ we prove the claim. ■

From this result we conclude that the equation $\{\bar{\partial} + \beta, \theta_\gamma + \phi_\gamma\} = 0$, where $\phi_\gamma \in \Omega^0(\Lambda^1(H^*) \otimes M^*)$ is resolvable with respect to ϕ_γ. It means that $\theta_\gamma + \phi_\gamma$ is holomorphic with respect to the new complex structure with $\bar{\partial}_{new}$ operator equal to $\bar{\partial} + \beta$. Denote

$$x_{\gamma\delta} = \{\theta_\gamma + \phi_\gamma, \theta_\delta + \phi_\delta\} \tag{27}$$

The following identity holds:

$$\bar{\partial}\{\theta_\gamma + \phi_\gamma, \theta_\delta + \phi_\delta\} = \bar{\partial}(\{\phi_\gamma, \theta_\delta\} + \{\theta_\gamma, \phi_\delta\}) = \\ = \{\{\theta_\gamma, \beta\}, \theta_\delta\} - \{\theta_\gamma, \{\theta_\delta, \beta\}\} = -\beta\{\theta_\gamma, \theta_\delta\} = 0 \tag{28}$$

This computations shows that $x_{\gamma\beta}$ is a holomorphic section of M^*.

Lemma 28 $H^0(M^*) = V$

Proof. The group of sections can be computed using (24), knowing that $H^1(\mathcal{O}(-1,-1)) = 0$. This computation shows that $H^0(M^*) = V$. ■

The group $H^0(M^*)$ can be interpreted as a subalgebra of global symmetries of $GrST$ and is spanned by translations. The map $\mathrm{Sym}^2\mathfrak{susy}_1 \to V$, which we just described, is $SO(4,\mathbb{R})$ invariant. It is uniquely determined (up to a constant c) from representation theory and can be expressed via Γ-matrices. The only question remains: what if the constant c is equal to zero?

Suppose that the map $\mathrm{Sym}^2\mathfrak{susy}_1 \to V$ is equal to zero. Then notice that the constructed deformed manifold is a homogeneous space of $GrSP$. Our space is completely determined by the isotropy group $\tilde{H}$. The latter is a rigid subgroup inside of $GrSP$.

The R-group GL_3 preserves the cocycle β. From this we can uniquely identify subgroup $\tilde{H}$ of $GrSP$. It is equal to A, except it has abelian odd part. It means that our manifold is isomorphic to $GrSP/\tilde{H} = GrST$ and hence is undeformed. It means the cohomology class of β is equal to zero Our manifold is a homogeneous space of super Poincare group SP. The manifold has isotropy group A. Hence it is isomorphic to ST.

■

5 Appendix

5.1 On integral forms.

On a complex super manifold a real line bundle Ber is a tensor product $Ber_{\mathbb{C}} \otimes \overline{Ber}_{\mathbb{C}}$, where $Ber_{\mathbb{C}}$ is a holomorphic line bundle, called a holomorphic Berezinian, the bar symbol means a complex conjugation. The symbol $\otimes$ in context of vector bundles will always mean their fiberwise tensor product.

A complexification of the tangent vector bundle $\mathcal{T}$ of a complex manifold splits into a sum $\mathcal{T}_{\mathbb{C}} + \overline{\mathcal{T}}_{\mathbb{C}}$, where $\mathcal{T}_{\mathbb{C}}$ is a holomorphic and $\overline{\mathcal{T}}_{\mathbb{C}}$ is an antiholomorphic component. This implies that

$$\Omega^{-k}_{int} = Ber \otimes \Lambda^k(\mathcal{T}) = \bigoplus_{i+j=k} Ber_{\mathbb{C}} \otimes \Lambda^i(\mathcal{T}_{\mathbb{C}}) \otimes \overline{Ber}_{\mathbb{C}} \otimes \Lambda^j(\overline{\mathcal{T}}_{\mathbb{C}})$$

In other words, on complex manifold we can consider not only differential (i, j)-forms, but also integral (i, j)-forms.

Suppose M is a n-dimensional manifold, V is an k-dimensional vector bundle. On the total space ΠV there is a canonical section of $c_{\Pi V} \in Ber \otimes \Lambda^n(\mathcal{T})$. In local odd fiberwise coordinates θ_i it is equal to

$$c_{\Pi V} = \theta_1 \dots \theta_k d\theta_1 \wedge \dots \wedge d\theta_k. \tag{29}$$

If M is complex then $c \in Ber_{\mathbb{C}} \otimes \Lambda^n(\mathcal{T}_{\mathbb{C}})$.

Proposition 29 *The forms $c_{\Pi V}$ and $\bar{c}_{\Pi V}$ are $\bar{\partial}$-closed.*

Proof. Since the formula (29) does not depend on the choice of coordinates on M one can do a local computation, which is trivial. ■

5.2 Algebra related to a foliation

Definition 30 *In this subsection we adopt an algebraic point of view on foliations. We think about a foliation on a (super)manifold Y as of a direct summand $\mathcal{F}$ in the complexified tangent bundle $\mathcal{T} \otimes \mathbb{C}$, such that for any sections $a, b \in \mathcal{F}$, $[a, b] \in \mathcal{F}$.*

Example 31 *An ordinary foliation (local fibering of Y by leaves) gives an example of such structure. $\mathcal{F}$ is a complexification of a bundle formed by vector fields tangent to the leaves. In this case we say that $\mathcal{F}$ is purely real.*

Example 32 *Suppose Y is a complex manifold. Then $\mathcal{T} \otimes \mathbb{C} = \mathcal{T}_{\mathbb{C}} + \overline{\mathcal{T}_{\mathbb{C}}}$. Then $\mathcal{F} = \mathcal{T}_{\mathbb{C}}$.*

Define an algebra of differential leafwise forms $\Omega^*_{\mathcal{F}}$ as a quotient $\Omega/I_{\mathcal{F}}$, where Ω is the algebra of differential forms on Y, $I_{\mathcal{F}}$ is an ideal in Ω generated by sections of conormal subbundle $(\Omega^1)^{\perp\mathcal{F}}$. In this case $I_{\mathcal{F}}$ is closed under the differential and $\Omega^*_{\mathcal{F}}$ is a differential algebra.

We say that a bundle G is equipped with connection along foliation $\mathcal{F}$ if there is an operator $\nabla : \Omega^i_{\mathcal{F}} G \to \Omega^{i+1}_{\mathcal{F}} G$, which satisfies

$$\nabla(ab) = (da)b + (-1)^{dega} a\nabla b, \tag{30}$$

where $a \in \Omega_{\mathcal{F}}$ and $b \in \Omega_{\mathcal{F}} G$. A connection ∇ is flat if $\nabla^2 = 0$.

The vector bundle $(\Omega^1)^{\perp\mathcal{F}}$ has a nontrivial $\mathcal{F}$-flat connection - the holonomy connection. The dual bundle $N_{\mathcal{F}}$ also carries a flat connection.

Suppose $\mathcal{F}, \mathcal{G}$ are transverse foliations. There are inclusions $\mathcal{G} \to N_{\mathcal{F}}$, $\mathcal{F} \to N_{\mathcal{G}}$. If holonomy of $\mathcal{F}$ acts trivially on $\mathcal{G}$ and holonomy of $\mathcal{G}$ acts trivially on $\mathcal{F}$ we say that $\mathcal{F}, \mathcal{G}$ is a commuting pair of foliations.

Example 33 *The subbundles $\mathcal{T}_{\mathbb{C}}$ and $\overline{\mathcal{T}_{\mathbb{C}}}$ of $\mathcal{T} \otimes \mathbb{C}$ form a commuting pair of foliations.*

Definition 34 *The subbundle $\mathcal{F} + \mathcal{G}$ defines a foliation, $\mathcal{F}$ and $\mathcal{G}$ are subfoliations. The complex $\Omega_{\mathcal{G}+\mathcal{F}}$ has a bigrading because it can be identified with $\Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{G}}$ and the differential $d_{\mathcal{F}+\mathcal{G}}$ splits into a sum of two commuting differentials $d_{\mathcal{F}}$, $d_{\mathcal{G}}$. It define a complex of forms along a pair of commuting foliations.*

Definition 35 *Denote Ber the Berezinian line bundle of Y. In case of purely even manifold $Ber = \Omega^{top}$-the space of differential forms of top degree. The spaces $\Lambda(\mathcal{F}) \otimes \Lambda(\mathcal{G}) Ber$ is a subcomplex of $\Omega_{int}(Y)$. This subcomplex is in fact a bicomplex and the differential splits into a sum of commuting differentials $d = d_{\mathcal{F}} + d_{\mathcal{G}}$.*

6 Acknowledgments

The author would like to thank M.Kontsevich, A.S. Schwarz for useful comments. The note was written while the author was staying at Mittag-Leffler

Institut and Max Plank Institute for Mathematics.

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