Picard-Lefschetz theory for the universal coverings of complements to affine hypersurfaces

Ichiro Shimada

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060

JAPAN

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

GERMANY

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§0. Introduction

Let Γ denote the vector space $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$, and $\Gamma^\times$ the space $\Gamma \setminus \{0\}$. We assume that $n \geq 2$ and $d \geq 3$. Let $\mathbb{P}_*(\Gamma)$ stand for the projective space $\Gamma^\times/\mathbb{C}^\times$, and $pr : \Gamma^\times \to \mathbb{P}_*(\Gamma)$ the natural projection. This space $\mathbb{P}_*(\Gamma)$ parameterizes all projective hypersurfaces of degree d in $\mathbb{P}^n$. We fix a hyperplane at infinity H_∞ in $\mathbb{P}^n$, and consider the affine space $\mathbb{A}^n := \mathbb{P}^n \setminus H_\infty$. We define $U \subset \mathbb{P}_*(\Gamma)$ to be the locus of all projective hypersurfaces of degree d which are non-singular and intersect H_∞ transversely, and define $\mathcal{U}$ to be the pull-back of U by the projection:

$$\mathcal{U} \quad := \quad pr^{-1}(U) \quad \subset \quad \Gamma^\times.$$

For $u \in \Gamma^\times$, let f_u denote the corresponding homogeneous polynomial of degree d. We put

$$\overline{X}_u := \{f_u = 0\}, \quad Y_u := \overline{X}_u \cap H_\infty, \quad X_u := \overline{X}_u \setminus Y_u, \quad \text{and} \quad E_u := \mathbb{A}^n \setminus X_u.$$

Then we have the monodromy representation

$$\rho \quad : \quad \pi_1(U, \bar{b}) \quad \longrightarrow \quad \mathrm{Aut}_{\mathbf{Z}}(H_n(X_b; \mathbb{Z})), \tag{0.1}$$

where $b \in \mathcal{U}$ is a base point and $\bar{b} \in U$ is the point $pr(b)$. This representation has been well investigated by the classical Picard-Lefschetz theory.

The purpose of this paper is to construct a certain kind of deformation of this classical monodromy representation.

The idea is to consider the middle homology group $H_n(F_b; \mathbb{Z})$ of the universal covering

$$F_b \quad \longrightarrow \quad E_b$$

of the complement E_b. We cannot, however, define the action of $\pi_1(U, \bar{b})$ on $H_n(F_b; \mathbb{Z})$ in a naive way, because the universal coverings $F_u \to E_u$ cannot be constructed universally over U. In order to construct the universal family of F_u, it is necessary to enlarge the base space U to $\mathcal{U} = pr^{-1}(U)$.

Since $\mathrm{Gal}\,(F_u/E_u) \cong \pi_1(E_u)$ is an infinite cyclic group, we can consider $H_n(F_u; \mathbb{Z})$ as a module over the ring of Laurant polynomials $\mathbb{Z}[q, q^{-1}]$, where the multiplication by q is identified with the action of a generator of $\mathrm{Gal}\,(F_u/E_u) \cong \mathbb{Z}$ on $H_n(F_u; \mathbb{Z})$. This action is also defined globally over $\mathcal{U}$.

Therefore, we get a monodromy representation

$$\tilde{\rho} \;:\; \pi_1(\mathcal{U}, b) \;\longrightarrow\; \mathrm{Aut}_{\mathbf{Z}[q,q^{-1}]}\,(H_n(F_b;\mathbb{Z})) \tag{0.2}$$

of $\pi_1(\mathcal{U}, b)$ on the $\mathbb{Z}[q, q^{-1}]$-module $H_n(F_b;\mathbb{Z})$.

This representation $\tilde{\rho}$ can be regarded as a deformation of the classical monodromy ρ in (0.1), because there is an isomorphism

$$H_n(F_b;\mathbb{Z}) \;\cong\; H_{n-1}(X_b;\mathbb{Z}) \otimes_{\mathbf{Z}} \mathbb{Z}[q, q^{-1}] \tag{0.3}$$

of $\mathbb{Z}[q, q^{-1}]$-modules such that the homomorphism $H_n(F_b;\mathbb{Z}) \to H_{n-1}(X_b;\mathbb{Z})$ obtained from (0.3) combined with the homomorphism $H_{n-1}(X_b;\mathbb{Z}) \otimes \mathbb{Z}[q, q^{-1}] \to H_{n-1}(X_b;\mathbb{Z})$ given by $q \mapsto 1$ is $\pi_1(\mathcal{U}, b)$-equivariant (see Theorems 6.1 and 7.1). Here $\pi_1(\mathcal{U}, b)$ acts on $H_n(F_b;\mathbb{Z})$ by $\tilde{\rho}$, and on $H_{n-1}(X_b;\mathbb{Z})$ by ρ composed with the natural surjective homomorphism $\pi_1(\mathcal{U}, b) \to \pi_1(U, \bar{b})$ induced by the projection $pr : \mathcal{U} \to U$.

Suppose that we are given a non-zero complex number α. We can consider $\mathbb{C}$ as a $\mathbb{Z}[q, q^{-1}]$-module by identifying q with α. Then the isomorphism (0.3) implies the isomorphism between complex vector spaces

$$H_n(F_b;\mathbb{Z}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \;\cong\; H_{n-1}(X_b;\mathbb{Z}) \otimes_{\mathbf{Z}} \mathbb{C} \;\cong\; H_{n-1}(X_b;\mathbb{C}).$$

Evaluationg $\tilde{\rho}$ at $q = \alpha$ and using this isomorphism, we obtain a representation

$$\rho(\alpha) \;:\; \pi_1(\mathcal{U}, b) \;\longrightarrow\; \mathrm{Aut}_{\mathbf{C}}\,(H_{n-1}(X_b;\mathbb{C})),$$

and thus we get a family of representations $\{\rho(\alpha)\}$ parameterized by all non-zero complex numbers. The property of the isomorphism (0.3) implies that $\rho(1)$ is nothing but the complexified classical representation $\rho \otimes_{\mathbf{Z}} \mathbb{C}$ composed with the homomorphism $\pi_1(\mathcal{U}) \to \pi_1(U)$.

The main theorem of this paper is as follows. Let $\mathbb{Q}(q)$ denote the quotient field of $\mathbb{Z}[q, q^{-1}]$.

Irreducibility Theorem. *The monodromy representation of $\pi_1(\mathcal{U}, b)$ on the vector space $H_n(F_b;\mathbb{Z}) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbb{Q}(q)$ induced from $\tilde{\rho}$ is absolutely irreducible.*

Corollary. *If α is general enough, then $\rho(\alpha)$ is irreducible.*

This shows that our deformation is non-trivial, because the classical representation $\rho \otimes_{\mathbf{Z}} \mathbb{C}$ is *not* irreducible. In fact, $\rho \otimes_{\mathbf{Z}} \mathbb{Q}$ is composed of the following two representations on the primitive parts of middle cohomology groups:

$$\begin{aligned} \rho_0 \;&:\; \pi_1(U, \bar{b}) \;\longrightarrow\; \mathrm{Aut}_{\mathbf{Q}}\,(H^{n-1}_{\mathrm{prim}}(\overline{X}_b;\mathbb{Q})), \quad \text{and} \\ \rho_\infty \;&:\; \pi_1(U, \bar{b}) \;\longrightarrow\; \mathrm{Aut}_{\mathbf{Q}}\,(H^{n-2}_{\mathrm{prim}}(Y_b;\mathbb{Q})); \end{aligned}$$

that is, there exists an exact sequence

$$0 \longrightarrow H^{n-2}_{\mathrm{prim}}(Y_b;\mathbb{Q}) \longrightarrow H_{n-1}(X_b;\mathbb{Q}) \longrightarrow H^{n-1}_{\mathrm{prim}}(\overline{X}_b;\mathbb{Q}) \longrightarrow 0 \tag{0.4}$$

which is preserved by the monodromy action of $\pi_1(U, \bar{b})$. This exact sequence follows from the isomorphism $H_{n-1}(X_b; \mathbb{Q}) \cong H^{n-1}(\overline{X}_b, Y_b; \mathbb{Q})$. It corresponds to the weight filtration of the mixed Hodge structure on the middle term, and hence is preserved by the monodromy action. The old Picard-Lefschetz theory tells us the following:

Theorem. *Both of ρ_0 and ρ_∞ are absolutely irreducible.*

Therefore, our deformation *fuses* these two irreducible representations into one big irreducible representation.

The complement $\Gamma^\times \setminus \mathcal{U}$ consists of the following two irreducible divisors:

$$\begin{aligned} \mathcal{D}_0 &:= \{\ u \in \Gamma^\times\ ;\quad \overline{X}_u \text{ is singular}\}, \quad \text{and} \\ \mathcal{D}_\infty &:= \{\ u \in \Gamma^\times\ ;\quad \overline{X}_u \text{ does } \textit{not} \text{ intersect } H_\infty \text{ transversely}\}. \end{aligned}$$

The main tool of the proof of Irreducibility Theorem is the *Picard-Lefschetz formula*, which describes the local monodromy action on $H_n(F_b; \mathbb{Z})$ along simple loops around these divisors. Roughly speaking, we proceed as follows. First, we define a boundary ∂F_u of F_u, and a "hermitian" intersection pairing

$$(\ ,\ )\ :\ H_n(F_b; \mathbb{Z}) \times H_n(F_b, \partial F_b; \mathbb{Z}) \longrightarrow \mathbb{Z}[q, q^{-1}]$$

in appropriate ways. Let $[\gamma] \in \pi_1(\mathcal{U}, b)$ be the homotopy class of a simple loop around $\mathcal{D}_0$ or $\mathcal{D}_\infty$. Then there exists a pair of $v[\gamma] \in H_n(F_b; \mathbb{Z})$ and $v^\sim[\gamma] \in H_n(F_b, \partial F_b; \mathbb{Z})$ such that the action of $[\gamma]_*$ on $H_n(F_b; \mathbb{Z})$ is given by

$$x \quad \mapsto \quad x + (x, v^\sim[\gamma]) \cdot v[\gamma].$$

This is a natural generalization of the classical Picard-Lefschetz formula with $\mathbb{Z}$ replaced by $\mathbb{Z}[q, q^{-1}]$. The homology class $v[\gamma]$ is the *vanishing cycle* associated with $[\gamma]$.

Moreover, we have the following two facts:

(1) As a module over the group ring $\mathbb{Z}[q, q^{-1}][\pi_1(\mathcal{U}, b)]$, $H_n(F_b; \mathbb{Z})$ is generated by one element $v[\gamma_0]$, where γ_0 is an arbitrary simple loop around $\mathcal{D}_0$.

(2) Let γ_∞ be a simple loop around $\mathcal{D}_\infty$. Then there exists a simple loop γ_0 around $\mathcal{D}_0$ such that

$$[\gamma_0]_*(v[\gamma_\infty]) \quad \neq \quad v[\gamma_\infty]. \tag{0.5}$$

The first fact just corresponds to the classically known fact that the space of vanishing cycles in the sense of [6; §3] is generated, as a module over the group ring of the monodromy group, by one vanishing cycle for a simple loop, if the coefficients of the homology groups are in $\mathbb{Q}$ (see [6; §7]).

On the other hand, the second fact causes the crucial difference between the classical representation ρ and our representation $\tilde{\rho}$. Indeed, for the classical monodromy $\rho(1)$, the inequality (0.5) does not hold; that is, we alway have

$$[\gamma_0]_*(v[\gamma_\infty]) \quad \equiv \quad v[\gamma_\infty] \qquad \bmod q-1$$

for arbitrary simple loops γ_0 and γ_∞ around $\mathcal{D}_0$ and $\mathcal{D}_\infty$, respectively. This equality modulo $q-1$ gurantees the stability of the subspace $H^{n-2}_{\rm prim}(Y_b;\mathbb{Q})$ of $H_{n-1}(X_b;\mathbb{Q})$ under the monodromy action, because this subspace is generated by vanishing cycles $v[\gamma_\infty]$*module* $q-1$ associated with simple loops γ_∞ around $\mathcal{D}_\infty$.

The idea to look at the universal covering of the complement comes from [5]. In this paper, Givental' considered the versal deformation family of a hypersurface singularity, and studied the monodromy action on the middle homology group of the universal covering of the complement to the Milnor fiber. In the case of simple singularity, the fundamental group of the complement to the discriminant locus in the base space of the versal deformation family is known to be isomorphisc to the generalized braid group corresponding to the Dynkin diagram of the simple singularity. What he obtained is a representation of the Iwahori-Hecke algebra, which connects the classical representations on the module of vanishing cycles in odd dimensions and in even dimensions.

A similar investigation had been done in [10] in a more general setting than ours.

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve, and let $L \subset \mathbb{P}^2$ be a general line. We put $X := \mathbb{P}^2 \setminus (C \cup L)$. Let $\tilde{X} \to X$ be the infinite cyclic covering corresponding to the Hurwicz map $\pi_1(X) \to H_1(X;\mathbb{Z}) \cong \mathbb{Z}$. Then the first homology group $H_1(\tilde{X};\mathbb{Z})$ of $\tilde{X}$, as a $\mathbb{Z}[q,q^{-1}]$-module, plays an important role in the study of singular plane curves [8]. Here in this paper, we treated $H_2(\tilde{X};\mathbb{Z})$ when C is non-singular. Thus we hope that it would be interesting to study the structure of $H_n(F_b;\mathbb{Z})$ as a $\mathbb{Z}[q,q^{-1}]$-module when the hypersurface X_b is singular.

This paper is organized as follows.

In §1, we construct the universal family of the universal coverings $F_u \to E_u$ of the complement $E_u = \mathbb{A}^n \setminus X_u$ over the extended base space $\mathcal{U} \subset \Gamma^\times$. We shall show that the deck transformation $T_u : F_u \to F_u$ over E_u corresponding to a generator of $\mathrm{Gal}\,(F_u/E_u) \cong \mathbb{Z}$ is also constructed universally over $\mathcal{U}$. Thus we obtain the representation $\tilde{\rho}$.

In §2, we investigate the polynomial map $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ which defines the affine hypersurface X_u; that is, $X_u = \hat{\phi}_u^{-1}(0)$ and $E_u = \hat{\phi}_u^{-1}(\mathbb{C}^\times)$. We shall study the critical points of $\hat{\phi}_u$ and the behavior of the fibers $\hat{\phi}_u^{-1}(t)$ "at infinity". We introduce a Zariski open dense subset $\mathcal{U}_N \subset \mathcal{U}$, over which the topology of the polynomial maps $\hat{\phi}_u$ does not vary locally.

In §3, we introduce a continuous function $\varepsilon : \mathcal{U} \to \mathbb{R}_{>0}$ which is "small enough", and define two boundaries $\partial_0 E_u$ and $\partial_\infty E_u$ of E_u as $\phi_u^{-1}(\Delta^\times(0))$ and $\phi_u^{-1}(\Delta^\times(\infty))$, where $\Delta^\times(0) := \{\, z \in \mathbb{C}\,;\; 0 < |z| \le \varepsilon(u)\,\}$ and $\Delta^\times(\infty) := \{\, z \in \mathbb{C}\,;\; |z|^{-1} \le \varepsilon(u)\,\}$. We then define two boundaries $\partial_0 F_u$ and $\partial_\infty F_u$ of F_u as the pull-backs of the boundaries of E_u by the covering map $F_u \to E_u$. It turns out that the relative homology groups $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$, both of which are also $\mathbb{Z}[q,q^{-1}]$-modules, are easier to describe than $H_n(F_u)$. The pleasant feature of this theory is that there is a certain kind of duality between $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$.

In §4, we review the classical theory of Lefschetz [7], and fix some notion and notation about vanishing cycles and thimbles. In this paper, a vanishing cycle in X_u, for example, is defined as a homotopy class of continuous maps from S^{n-1} to X_u which satisfies certain conditions, and a thimble in $(E_u, \partial_0 E_u)$, for example, is defined as a homotopy class of continuous maps from the pair (CS^{n-1}, S^{n-1}), where CS^{n-1} is the cone over S^{n-1}, to $(E_u, \partial_0 E_u)$ which possesses certain properties.

In §5, we investigate the homology groups $H_{n-1}(X_u)$, $H_n(E_u)$ and $H_n(E_u, \partial_0 E_u)$. The main results are that, if $u \in \mathcal{U}_N$, then the homology classes of the vanishing cycles corresponding to the critical points of $\hat{\phi}_u$ form a basis of $H_{n-1}(X_u)$, and the homology classes of the associated thimbles form a basis of $H_n(E_u, \partial_0 E_u)$. In particular, $H_{n-1}(X_u)$ and $H_n(E_u, \partial_0 E_u)$ are canonically isomorphic, and the rank of them is equal with the number of the critical points of $\hat{\phi}_u$. These facts seem to be well-known. However, we present them with complete proofs in order for the paper to be self-contained.

In §6 and §7, we study the structure of $H_n(F_u)$ and $H_n(F_u, \partial_0 F_u)$, $H_n(F_u, \partial_\infty F_u)$. We show that $H_n(F_u)$ is embedded in $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$ by the natural homomorphisms. We also show that that the homology classes of the thimbles lifted from $(E_u, \partial_0 E_u)$ (resp. $(E_u, \partial_\infty E_u)$) form a set of basis of $H_n(F_u, \partial_0 F_u)$ (resp. $H_n(F_u, \partial_\infty F_u)$) over $\mathbb{Z}[q, q^{-1}]$. In particular, we obtain isomorphisms

$$\begin{aligned} H_n(F_u, \partial_0 F_u) &\cong H_n(E_u, \partial_0 E_u) \otimes_{\mathbf{Z}} \mathbb{Z}[q, q^{-1}] \cong H_{n-1}(X_u) \otimes_{\mathbf{Z}} \mathbb{Z}[q, q^{-1}], \quad \text{and} \\ H_n(F_u, \partial_\infty F_u) &\cong H_n(E_u, \partial_\infty E_u) \otimes_{\mathbf{Z}} \mathbb{Z}[q, q^{-1}]. \end{aligned} \tag{0.6}$$

These isomorphisms are, however, *not* canonical by any means, because there is ambiguity of the way of lifting of a given thimble in $(E_u, \partial_0 E_u)$ (resp. $(E_u, \partial_\infty E_u)$) up to $(F_u, \partial_0 F_u)$ (resp. $(F_u, \partial_\infty F_u)$). In order to state the isomorphisms (0.6) precisely, we have to restrict ourselves to a smaller locus $\mathcal{U}_N^\sim \subset \mathcal{U}_N$, over which a canonical lifting can be assigned to each thimble in $(E_u, \partial_0 E_u)$ or in $(E_u, \partial_\infty E_u)$. However, $\mathcal{U} \setminus \mathcal{U}_N^\sim$ is a real semi-algebraic subset of real codimension 1, and $\mathcal{U}_N^\sim$ is not path-connected. Hence these isomorphisms cannot be $\pi_1(\mathcal{U})$-equivariant. (Otherwise, we would get a contradiction to Irreducible Theorem above.)

In §8, we introduce two hermitian intersection pairings between the two relative homology groups $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$, which take values in $\mathbb{Z}[q, q^{-1}]$, and prove that they are non-degenerate. The idea of these pairings is also due to [5].

In §9, we formulate and state the Picard-Lefschetz formula. Let γ_0 be a simple loop around $\mathcal{D}_0$, and γ_∞ a simple loop around $\mathcal{D}_\infty$. The precise definition of simple loops is given in §9.1. We describe the action of $[\gamma_0] \in \pi_1(\mathcal{U}, b)$ on $H_n(F_b, \partial_\infty F_b)$ in Theorem 9.2.1, and the action of $[\gamma_\infty] \in \pi_1(\mathcal{U}, b)$ on $H_n(F_b, \partial_0 F_b)$ in Theorem 9.2.2, with the help of the hermitian intersection pairings defined in §8. As is seen from the proofs, which are given in §9.4 and §9.7 respectively, this is a more appropriate way to state Picard-Lefschetz formula than to describe the action on $H_n(F_b)$. The action on $H_n(F_b)$, however, can be derived from these two theorems, because $H_n(F_b)$ is embedded in $H_n(F_b, \partial_\infty F_b)$ and $H_n(F_b, \partial_0 F_b)$ by the natural homomorphisms.

As can be guessed from the fact that the basis of $H_n(F_b, \partial_\infty F_b)$ or $H_n(F_b, \partial_0 F_b)$ over $\mathbb{Z}[q, q^{-1}]$ consists of the homology classes of lifted thimbles, each of which corresponds to a critical value of $\hat{\phi}_b$ in a bijective way, the main ingredient of the proof is to study the movements of the critical values of $\hat{\phi}_u$ when u makes a round trip along γ_0 or γ_∞. In the case of γ_0, it is quite easy to see how the critical values moves on the complex plane. On the contrary, it takes the whole subsection §9.6 in the case of γ_∞.

There is one more important result in §9. In §9.5, we give a proof to Theorem 9.5.1, which states that $H_n(F_b)$ is generated, as a module over the group ring $\mathbb{Z}[q, q^{-1}][\pi_1(\mathcal{U}, b)]$, by one "vanishing cycle" $v[\gamma_0]$ associated with an arbitrary simple loop γ_0 around $\mathcal{D}_0$.

By Zariski's hyperplane section theorem, $\pi_1(\mathcal{U})$ is generated by the homotopy classes of simple loops around $\mathcal{D}_0$ and $\mathcal{D}_\infty$. Hence, using the results in §9, we can prove Irreducibility Theorem in §10.

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Conventions.

(1) The symbol I always denotes the closed interval $[0, 1] \subset \mathbb{R}$.

(2) A path $I \to V$ on a $\mathcal{C}^\infty$-manifold V is always assumed to be *piece-wise smooth*.

(3) Let $\alpha : I \to V$ and $\beta : I \to V$ be two paths on a topological space V. We define the order of the product of paths in such a way that $\alpha \cdot \beta$ is well-defined if and only if $\beta(1) = \alpha(0)$.

(4) Let V_1 and V_2 be topological spaces, or pairs of topological spaces. Then $[V_1, V_2]$ denotes the set of homotopy classes of continuous maps from V_1 to V_2.

(5) Let V, W and W' be topological spaces, and $f : V \to W$, $g : W' \to W$ continuous maps. We say that f is locally trivial over $g : W' \to W$ (or simply over W') if the pull-back $W' \times_W V \to W'$ of f by g is locally trivial.

(6) Let X_1 and X_2 be complex manifolds, and let $h : X_1 \to X_2$ be a holomorphic map. We say that h is locally trivial if it is locally trivial *in the category of topological spaces and continuous maps.*

(7) In this paper, we work with homology groups in $\mathbb{Z}$-coefficients unless otherwise stated, and we omit $\mathbb{Z}$ in the notation.

§1. Construction of the universal family

Our first task is to construct the universal family of the universal coverings $F_u \to E_u$ over the extended base space $\mathcal{U}$.

The complement $\mathbb{P}_*(\Gamma) \setminus U$ consists of two irreducible divisors D_0 and D_∞, where D_0 consists of all singular hypersurfaces, while D_∞ consists of all hypersurfaces whose intersections with H_∞ are not transverse. Then a general point of D_0 corresponds to a hypersurface possessing one ordinary double point as its only singularity, while a general point of D_∞ corresponds to a non-singular hypersurface $\overline{X}$ such that $H_\infty \cap \overline{X}$ is a hypersurface in H_∞ possessing only one ordinary double point as its singularity.

Then the divisors $\mathcal{D}_0$ and $\mathcal{D}_\infty$ of $\Gamma^\times$ defined in Introduction are the pull-backs of D_0 and D_∞, respectively, by the natural projection $\Gamma^\times \to \mathbb{P}_*(\Gamma)$.

We choose $h \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ which defines the hyperplane $H_\infty = \{h = 0\}$, and fix it throughout this paper. Then $h^d \in \Gamma^\times$. Recall that f_u denote the homogeneous polynomial of degree d corresponding to $u \in \Gamma^\times$. Using the fixed homogeneous polynomial h^d defining the multiple hyperplane $d \cdot H_\infty$, we get a morphism

$$\phi_u := f_u/h^d \ : \ E_u \longrightarrow \mathbb{C}^\times,$$

which is the restriction of the polynomial map

$$\hat{\phi}_u := f_u/h^d \ : \ \mathbb{A}^n \longrightarrow \mathbb{C}$$

to $E_u = \hat{\phi}_u^{-1}(\mathbb{C}^\times)$. The following lemma is easy to prove by using Zariski's hyperplane section theorem [11], and the theorem of Deligne-Fulton on Zariski's conjecture ([2], [3], [4]):

Lemma 1.1. *Suppose that $u \in \mathcal{U}$. Then ϕ_u induces an isomorphism $\pi_1(E_u) \cong \pi_1(\mathbb{C}^\times)$ on the fundamental groups.* □

Let $e : \mathbf{C} \to \mathbb{C}^\times$ be the universal covering given by $z \mapsto \exp z$. (We distinguish two complex planes $\mathbb{C}$ and $\mathbf{C}$. This distinction will help to avoid confusions.) For every $u \in \Gamma^\times$, we define a complex space F_u by the fiber product

$$\begin{tikzcd} F_u \arrow[r, "\tilde{e}"] \arrow[d, "\psi_u"'] \arrow[dr, phantom, "\square"] & E_u \arrow[d, "\phi_u"] \\ \mathbf{C} \arrow[r, "e"'] & \mathbb{C}^\times. \end{tikzcd} \tag{1.1}$$

If $u \in \mathcal{U}$, then Lemma 1.1 implies that the covering $\tilde{e} : F_u \to E_u$ is the universal covering of E_u whose Galois group is canonically isomorphic to $\pi_1(\mathbb{C}^\times)$. Let $T_u : F_u \to F_u$ denote the deck transformation over E_u corresponding to the counter-clockwise generator of $\pi_1(\mathbb{C}^\times)$.

The construction of the universal covering $F_u \to E_u$ can be carried out universally over the base space $\mathcal{U}$. Let $\mathcal{X} \subset \mathbb{A}^n \times \mathcal{U}$ denote the universal family of the affine hypersurfaces $\{ X_u \,;\, u \in \mathcal{U} \}$, with the natural projection $\mathcal{X} \to \mathcal{U}$, and let $\mathcal{E}$ stand for the complement $(\mathbb{A}^n \times \mathcal{U}) \setminus \mathcal{X}$, which is the universal family of $\{ E_u \,;\, u \in \mathcal{U} \}$ with the natural projection $\mathcal{E} \to \mathcal{U}$. By putting $\phi_u : E_u \to \mathbb{C}^\times$ together, we get a morphism

$$\Phi \ : \ \mathcal{E} \longrightarrow \mathbb{C}^\times,$$

which maps $(P, u) \in \mathcal{E}$ to $\phi_u(P) \in \mathbb{C}^\times$. Let $\mathcal{F}$ be the fiber product $\mathcal{E} \times_{\mathbb{C}^\times} \mathbf{C}$, where $\mathbf{C} \to \mathbb{C}^\times$ is given by the exponential map e. Then this $\mathcal{F}$ with the natural projection onto $\mathcal{U}$ is the universal family of $\{ F_u \,;\, u \in \mathcal{U} \}$. Again, the natural map $\mathcal{F} \to \mathcal{E}$ is the Galois covering with the Galois group $\pi_1(\mathbb{C}^\times)$. Let $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ be the deck transformation over $\mathcal{E}$ corresponding to the counter-clockwise generator of $\pi_1(\mathbb{C}^\times)$. Then the restriction of $\mathcal{T}$ to a fiber $F_u \subset \mathcal{F}$ over $u \in \mathcal{U}$ gives the deck transformation $T_u : F_u \to F_u$.

Now it is easy to see that the families $\mathcal{X} \to \mathcal{U}$, $\mathcal{E} \to \mathcal{U}$ and hence $\mathcal{F} \to \mathcal{U}$ are all locally trivial. Therefore we obtain a natural monodromy representation of $\pi_1(\mathcal{U}, b)$ on $H_n(F_b)$, where $b \in \mathcal{U}$ is a base point. Since the deck transformations T_u are defined globally over $\mathcal{U}$, we get the following:

Lemma 1.2. *The monodromy action of $\pi_1(\mathcal{U}, b)$ on $H_n(F_b)$ commutes with the automorphism $T_{b*} : H_n(F_b) \to H_n(F_b)$ induced by the deck transformation.* $\square$

We fix an isomorphism between the group ring $\mathbb{Z}[\pi_1(\mathbb{C}^\times)]$ and the ring of Laurant polynomials $\mathbb{Z}[q, q^{-1}]$ by identifying the counter-clockwise generator of $\pi_1(\mathbb{C}^\times)$ with q. Then $H_n(F_u)$ becomes a $\mathbb{Z}[q, q^{-1}]$-modules for each $u \in \mathcal{U}$, in which the multiplication by q is nothing but the automorphism $T_{u*} : H_n(F_u) \to H_n(F_u)$. Lemma 1.2 implies that the monodromy representation of $\pi_1(\mathcal{U}, b)$ on $H_n(F_b)$ is a representation on the $\mathbb{Z}[q, q^{-1}]$-module, and thus we get

$$\pi_1(\mathcal{U}, b) \longrightarrow \operatorname{Aut}_{\mathbb{Z}[q,q^{-1}]} H_n(F_b). \tag{1.2}$$

This monodromy representation is the central theme of this article.

The natural projection $\mathcal{U} \to U$ is a $\mathbb{C}^\times$-bundle. Hence the kernel of $\pi_1(\mathcal{U}) \to \pi_1(U)$ is generated by an element $\mathfrak{c} \in \pi_1(\mathcal{U})$, which is the counter-clockwise generator of the fundamental group of the fiber $\cong \mathbb{C}^\times$. It is obvious that $\mathfrak{c}$ is contained in the center of $\pi_1(\mathcal{U})$.

Proposition 1.1. *The action of $\mathfrak{c}$ on $H_n(F_b)$ is equal with the multiplication by q.*

Proof. The element $\mathfrak{c} \in \pi_1(\mathcal{U}, b)$ is represented by the loop $\{ f[\theta] \,;\, \theta \in I \}$ in $\mathcal{U}$, where $f[\theta] := e^{2\pi\sqrt{-1}\theta} f_b$. Let $E[\theta] \subset \mathbb{A}^n$ be the complement to the affine hypersurface defined by $f[\theta] = 0$. Then $E[\theta]$ does not change even when θ varies. The function $\phi[\theta] : E[\theta] \to \mathbb{C}^\times$ on it, however, varies as $\phi[\theta] = e^{2\pi\sqrt{-1}\theta}\phi[0]$. This is equivalent to rotate E_b over $\mathbb{C}^\times$ once in the counter-clockwise direction. Therefore it induces the deck transformation T_b on F_b, and hence the multiplication by q on $H_n(F_b)$. $\square$

This proposition justifies us in working, not with $\pi_1(U)$, but with $\pi_1(\mathcal{U})$. Later on, we shall prove that $H_n(F_b)$ is torsion free as a $\mathbb{Z}[q, q^{-1}]$-module (Corollary 6.1). Hence $\mathfrak{c}$ has an infinite order in $\pi_1(\mathcal{U}, b)$; that is, the kernel of $\pi_1(\mathcal{U}) \to \pi_1(U)$ is isomorphic to $\mathbb{Z}$.

§2. Structure of the polynomial map $\hat{\phi}_u$

Let $\mathcal{V} \subset \Gamma^\times$ denote the locus of all points $u \in \Gamma^\times$ such that $\overline{X}_u$ does not contain H_∞ as an irreducible component. It is obvious that $\mathcal{U} \subset \mathcal{V}$. If $u \in \mathcal{V}$, then the morphism $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ is expressed as a polynomial of degree d in terms of affine coordinates of $\mathbb{A}^n$. In this section, we always assume $u \in \mathcal{V}$.

Convention. We do not distinguish a point on Γ and the corresponding homogeneous polynomial meticulously. For example, we use both of the notations $u \in \Gamma$ and $f_u \in \Gamma$ in the same meaning.

Let $\operatorname{Cr}(u) \subset \mathbb{C}$ denote the set of critical values of $\hat{\phi}_u$. For $t \in \mathbb{C}$, we have

$$t \notin \operatorname{Cr}(u) \iff \hat{\phi}_u^{-1}(t) \text{ is non-singular.} \tag{2.1}$$

We write by $\mathfrak{h}_\infty \in \mathbb{P}_*(\Gamma)$ the point corresponding to $d \cdot H_\infty$. Let $\mathcal{L}_u \subset \Gamma^\times$ denote the affine line $\{ f_u - t \cdot h^d \,;\, t \in \mathbb{C} \}$, and let $L_u \subset \mathbb{P}_*(\Gamma)$ denote the projective line spanned by $\mathfrak{h}_\infty$ and the point $pr(u) \in \mathbb{P}_*(\Gamma)$ corresponding $\overline{X}_u$. We put

$$L_u^o \;:=\; L_u \setminus \{\mathfrak{h}_\infty\}.$$

Then the projection $pr : \Gamma^\times \to \mathbb{P}_*(\Gamma)$ induces an isomorphism between $\mathcal{L}_u$ and L_u^o. There are natural parameterizations

$$\iota_u \;:\; \mathbb{C} \xrightarrow{\sim} \mathcal{L}_u, \quad \text{and} \quad \bar{\iota}_u := pr \circ \iota_u \;:\; \mathbb{C} \xrightarrow{\sim} L_u^o$$

given by $\iota_u(t) := f_u - t \cdot h^d$. The following remark will be used frequently throughout this paper.

Remark 2.1. By definition, the morphism $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ is nothing but the pull-back of the universal family $\mathcal{X}_\Gamma \to \Gamma^\times$ by

$$\mathbb{C} \underset{\iota_u}{\xrightarrow{\sim}} \mathcal{L}_u \hookrightarrow \Gamma^\times,$$

where $\mathcal{X}_\Gamma := \{ (P, u) \in \mathbb{A}^n \times \Gamma^\times \,;\, P \in X_u \}$, and $\mathcal{X}_\Gamma \to \Gamma^\times$ is the second projection.

Proposition 2.1. *If $u \in \mathcal{U}$, then $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ is locally trivial over $\mathbb{C} \setminus \mathrm{Cr}(u)$.*

Proof. By (2.1), it is enough to show that $\hat{\phi}_u$ is locally trivial "at infinity" over the complex plane $\mathbb{C}$; that is, if $u \in \mathcal{U}$, then, for all $t \in \mathbb{C}$, the projective compactification of the affine hypersurface $\hat{\phi}_u^{-1}(t)$ is non-singular at every point of the intersection with H_∞, and moreover, the intersection is transverse. This follows directly from two Lemmas below and Remark 2.1. $\square$

Note that $\mathcal{L}_u \subset \mathcal{V}$ because of the assumption $u \in \mathcal{V}$. Note also that

$$\overline{X}_w \cap H_\infty = \overline{X}_u \cap H_\infty \quad \text{for all} \quad w \in \mathcal{L}_u, \tag{2.2}$$

by the definition of $\mathcal{L}_u$.

Lemma 2.1. *Suppose that $\overline{X}_w$ is non-singular at a point $P \in \overline{X}_w \cap H_\infty$ for one $w \in \mathcal{L}_u$. Then $\overline{X}_{w'}$ is non-singular at P for all $w' \in \mathcal{L}_u$.*

Lemma 2.2. *Suppose that $\overline{X}_w$ intersects H_∞ transversely at a point $P \in \overline{X}_w \cap H_\infty$ for one $w \in \mathcal{L}_u$. Then $\overline{X}_{w'}$ intersects H_∞ transversely at P for all $w' \in \mathcal{L}_u$.*

Proof of Lemmas 2.1 and 2.2. Let $(z_1, \ldots, z_n)$ be an affine coordinate system on an affine open subset $\mathbb{A}^{n\prime}$ of $\mathbb{P}^n$ with the origin P such that $H_\infty = \{z_n = 0\}$. Suppose that $\overline{X}_u$ is defined by

$$f_u(z_1, \ldots, z_n) = 0$$

in $\mathbb{A}^{n\prime}$, where $f_u(z_1, \ldots, z_n)$ is an inhomogeneous polynomial of degree d with zero constant term. If $w = \iota_u(t)$, then, after replacing z_n with αz_n where α is an appropriate non-zero constant, an inhomogeneous polynomial defining $\overline{X}_w$ is given by

$$f_w(z_1, \ldots, z_n) \;:=\; f_u(z_1, \ldots, z_n) - t \cdot z_n^d.$$

The projective hypersurface $\overline{X}_w$ is non-singular at P if and only if the homogeneous part $f_w^{[1]}(z_1, \ldots, z_n)$ of degree 1 in $f_w(z_1, \ldots, z_n)$ is non-zero. Since $d \geq 2$, if it holds for one $w \in \mathcal{L}_u$, then it holds for all $w \in \mathcal{L}_u$. The condition that the intersection of $\overline{X}_w$ and H_∞ is transverse at P is equivalent to the condition that $f_w^{[1]}(z_1, \ldots, z_{n-1}, 0)$ is non-zero. Again, since $d \geq 2$, if it holds for one $w \in \mathcal{L}_u$, then it holds for all $w \in \mathcal{L}_u$. □

These two lemmas imply the following:

Proposition 2.2. *If $u \notin \mathcal{D}_\infty$, then $\mathcal{L}_u \cap \mathcal{D}_\infty = \emptyset$. If $u \in \mathcal{D}_\infty$, then $\mathcal{L}_u \subset \mathcal{D}_\infty$.* □

Proposition 2.3. *If $\overline{X}_u$ is non-singular at every point of $\overline{X}_u \cap H_\infty$, then $\mathrm{Cr}(u) \subset \mathbb{C}$ is equal with $\iota_u^{-1}(\mathcal{L}_u \cap \mathcal{D}_0)$ and with $\bar{\iota}_u^{-1}(L_u^o \cap D_0)$.* □

Corollary 2.1. *If $u \in \mathcal{U}$, then $\mathcal{L}_u \cap \mathcal{D}_\infty = \emptyset$, and $\mathrm{Cr}(u) = \iota_u^{-1}(\mathcal{L}_u \cap \mathcal{D}_0)$.* □

Let $\hat{\phi}_u(x_1, \ldots, x_n)$ be the polynomial of degree d expressing $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ in terms of affine coordinates $(x_1, \ldots, x_n)$ of $\mathbb{A}^n$. The critical points of $\hat{\phi}_u$ are then given by the solutions of

$$\frac{\partial \hat{\phi}_u}{\partial x_1} = \cdots = \frac{\partial \hat{\phi}_u}{\partial x_n} = 0.$$

Hence, if $u \in \mathcal{U}$ is chosen generally, the number of the distinct critical points of $\hat{\phi}_u$ is

$$N \; := \; (d-1)^n.$$

Definition 2.1. Let $\mathcal{U}_N \subset \mathcal{U}$ denote the locus of all $u \in \mathcal{U}$ which satisfies the following; (i) $\mathrm{Cr}(u)$ consists of distinct N values, and (ii) over each $p \in \mathrm{Cr}(u)$, $\hat{\phi}_u$ has only one critical point and that critical point is non-degenerate.

Since both of (i) and (ii) are algebraically open conditions, the locus $\mathcal{U}_N$ is a Zariski open subset of $\mathcal{U}$. It is easy to see that $\mathcal{U}_N \neq \emptyset$. Hence $\mathcal{U}_N \subset \mathcal{U}$ is dense.

Note that N is the maximal number which can be attained by the number of elements of $\mathrm{Cr}(u)$. Hence Corollary 2.1 implies the following:

Proposition 2.4. *If $u \in \mathcal{U}_N$, then $\mathcal{L}_u$ intersects $\mathcal{D}_0$ transversely at distinct N points of the non-singular locus of $\mathcal{D}_0$.* □

Lemma 2.3. *Let u be a point on $\mathcal{U}_N$. Then we have $\mathcal{L}_u \setminus \mathcal{D}_0 = \mathcal{L}_u \cap \mathcal{U}_N$.*

Proof. Let w be an arbitrary point on $\mathcal{L}_u$. By definition, the affine line $\mathcal{L}_w$ is equal with $\mathcal{L}_u$, and we write this affine line simply by $\mathcal{L}$. By Remark 2.1 , we have

$$\iota_u \circ \hat{\phi}_u \; = \; \iota_w \circ \hat{\phi}_w \tag{2.3}$$

as a morphism from $\mathbb{A}^n$ to $\mathcal{L}$. In particular, the morphism $\hat{\phi}_w$ also satisfies the conditions (i) and (ii) in Definition 2.1. This implies that, if $w \in \mathcal{U}$, then $w \in \mathcal{U}_N$. On the other hand, because of Corollary 2.1, we have $\mathcal{L} \cap \mathcal{D}_\infty = \emptyset$ and hence $\mathcal{L} \setminus \mathcal{D}_0 = \mathcal{L} \cap \mathcal{U} = \mathcal{L} \cap \mathcal{U}_N$.
□

Suppose that $u \in \mathcal{U}_N$. Let $p \in \mathbb{C}$ be one of the critical values of $\hat{\phi}_u$. and let $q \in \mathbb{A}^n$ be the critical point of $\hat{\phi}_u$ on $\hat{\phi}_u^{-1}(p)$. Then there exists an analytically local coordinate system $(w_1, \dots, w_n)$ on a small neighborhood of q in $\mathbb{A}^n$ with the center q such that $\hat{\phi}_u$ is given by

$$(w_1, \dots, w_n) \mapsto p + w_1^2 + \cdots + w_n^2 \tag{2.4}$$

locally around q. Let ϵ be a small positive real number. We put

$$B := \{ (w_1, \dots, w_n) \ ; \ |w_1|^2 + \cdots + |w_n|^2 \leq \epsilon \} \subset \mathbb{A}^n.$$

Lemmas 2.1 and 2.2 imply the following (cf. [6; §3. Ehresmann's Fibration Theorem]):

Proposition 2.5. *Let η be a positive real number small enough compared with ϵ, and let $\Delta \subset \mathbb{C}$ be the closed disk with the center p and of radius η. (1) By the restriction of $\hat{\phi}_u$, the pair $(\ \phi_u^{-1}(\Delta) \setminus B, \quad \phi_u^{-1}(\Delta) \cap \partial B\ )$ is a trivial fiber space with boundary over Δ. (2) Moreover, $\phi_u^{-1}(p)$ is a strong deformation retract of $\phi_u^{-1}(\Delta)$.*

Proof. The situation near H_∞ can be checked by Lemmas 2.1 and 2.2. The situation near the point q can be studied by the explicit formula (2.4) of $\hat{\phi}_u$. $\square$

§3. Boundaries of F_u

In this section, we always assume $u \in \mathcal{U}$.

Note that $0 \notin \mathrm{Cr}(u)$ by (2.1), and $\mathrm{Cr}(u) = \iota_u^{-1}(\mathcal{L}_u \cap \mathcal{D}_0)$ by Corollary 2.1. These imply that

$$\tilde{\varepsilon}(u) \quad := \quad \min \{ |p|, |p|^{-1} \ ; \ p \in \mathrm{Cr}(u) \} \tag{3.1}$$

defines a continuous function $\tilde{\varepsilon} : \mathcal{U} \to \mathbb{R}_{>0}$. Suppose that $\varepsilon : \mathcal{U} \to \mathbb{R}_{>0}$ is a continuous function which satisfies

$$\varepsilon(u) < \tilde{\varepsilon}(u) \quad \text{for all} \quad u \in \mathcal{U}. \tag{3.2}$$

We put

$$B_u^0 := \{ z \in \mathbb{C}^\times \ ; \ 0 < |z| \leq \varepsilon(u) \}, \quad \text{and} \quad B_u^\infty := \{ z \in \mathbb{C}^\times \ ; \ |z|^{-1} \leq \varepsilon(u) \},$$

each of which is a punctured closed disk on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We also put

$$\partial_0 E_u := \phi_u^{-1}(B_u^0) \subset E_u, \qquad \partial_\infty E_u := \phi_u^{-1}(B_u^\infty) \subset E_u,$$

and

$$\begin{aligned} \partial_0 F_u &:= \tilde{e}^{-1}(\partial_0 E_u) = \psi_u^{-1}(e^{-1}(B_u^0)) \subset F_u, \\ \partial_\infty F_u &:= \tilde{e}^{-1}(\partial_\infty E_u) = \psi_u^{-1}(e^{-1}(B_u^\infty)) \subset F_u. \end{aligned}$$

(See (1.1) for the definition of $\tilde{e}$ and ψ_u.) Note that the set of critical values of $\psi_u : F_u \to \mathbf{C}$ is

$$\mathcal{C}r(u) := e^{-1}(\mathrm{Cr}(u)),$$

and that $\psi_u : F_u \to \mathbf{C}$ is locally trivial over $\mathbf{C} \setminus Cr(u)$ by Proposition 2.1. By the definition (3.1) of $\tilde{\varepsilon}$ and the condition (3.2), there are no critical points of $\psi_u : F_u \to \mathbf{C}$ in $\partial_0 F_u$ and in $\partial_\infty F_u$. Moreover, each of the subspaces

$$\mathbf{C} \setminus e^{-1}(B_u^0) \subset \mathbf{C}, \quad \mathbf{C} \setminus e^{-1}(B_u^\infty) \subset \mathbf{C}, \quad \text{and} \quad \mathbf{C} \setminus (e^{-1}(B_u^0) \cup e^{-1}(B_u^\infty)) \subset \mathbf{C}$$

is a strong deformation retract of $\mathbf{C}$. Hence each of the subspaces

$$F_u \setminus \partial_0 F_u \subset F_u, \quad F_u \setminus \partial_\infty F_u \subset F_u, \quad \text{and} \quad F_u \setminus (\partial_0 F_u \cup \partial_\infty F_u) \subset F_u$$

is also a strong deformation retract of F_u. Therefore, we can call $\partial_0 F_u$ and $\partial_\infty F_u$ *the boundaries of* F_u. In particular, since $\partial_0 F_u \cap \partial_\infty F_u = \emptyset$, the intersection pairing

$$\langle \ , \ \rangle \ : \ H_n(F_u, \partial_0 F_u) \times H_n(F_u, \partial_\infty F_u) \longrightarrow \mathbb{Z}$$

between the relative homology groups is well defined.

It is obvious that each of the pairs $(F_u, \partial_0 F_u)$ and $(F_u, \partial_\infty F_u)$ forms a locally trivial family over $\mathcal{U}$ when u varies. Moreover, the deck transformation $T_u : F_u \to F_u$ induces automorphisms of $\partial_0 F_u$ and $\partial_\infty F_u$. Hence $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$ can be regarded as $\mathbb{Z}[q, q^{-1}]$-modules in the same way as $H_n(F_u)$. Therefore each of $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$ forms a locally constant system of $\mathbb{Z}[q, q^{-1}]$-modules over $\mathcal{U}$. We thus obtain natural monodromy representations

$$\pi_1(\mathcal{U}, b) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[q,q^{-1}]}(H_n(F_b, \partial_0 F_b)) \quad \text{and} \quad \pi_1(\mathcal{U}, b) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[q,q^{-1}]}(H_n(F_b, \partial_\infty F_b)),$$

which are compatible with (1.2) via the natural homomorphisms $H_n(F_b) \to H_n(F_b, \partial_0 F_b)$ and $H_n(F_b) \to H_n(F_b, \partial_\infty F_b)$ of $\mathbb{Z}[q, q^{-1}]$-modules.

Remark 3.1. The homeomorphism types of all spaces $(E_u, \partial_0 E_u)$, $(E_u, \partial_\infty E_u)$, $(F_u, \partial_0 F_u)$, $(F_u, \partial_\infty F_u)$, and so on, or of the maps between them are independent of the choice of the function ε, provided that (3.2) is fulfilled. In order to make the argument concrete, we put

$$\varepsilon := \tilde{\varepsilon}/2, \tag{3.3}$$

and use this ε unless otherwise stated. Sometimes, however, we pick up a sufficiently small positive real number r, and use

$$\varepsilon' := \min\{\tilde{\varepsilon}/2, r\} = \min\{\varepsilon, r\}$$

instead of ε, so that ε' is a constant function in a neighborhood of a given point of $\mathcal{U}$.

§4. Vanishing cycles and thimbles

In this section, we fix notion and notation concerned with vanishing cycles for ordinary double points and associated thimbles. For the proofs of the facts stated in this section, we refer the reader to [6].

Let S^{n-1} be an oriented $(n-1)$-sphere, and let $r \in [S^{n-1}, S^{n-1}]$ be the homotopy class of orientation reversing self-homeomorphisms. Note that $r^2 \in [S^{n-1}, S^{n-1}]$ is the homotopy class of the identity. For a topological space T and a homotopy class $f \in [S^{n-1}, T]$, we write by $-f \in [S^{n-1}, T]$ the homotopy class $f \circ r$. Note that, since S^{n-1} is oriented, we have a natural map $[S^{n-1}, T] \to H_{n-1}(T)$.

We denote by CS^{n-1} the cone over S^{n-1}; that is, the space obtained from $I \times S^{n-1}$ by contracting $\{1\} \times S^{n-1}$ to a point, which is the vertex of the cone. We equip CS^{n-1} with the orientation induced from that of the product space $I \times S^{n-1}$. Hence we have

$$\partial\, CS^{n-1} = -S^{n-1}.$$

Therefore, for a pair (T, S) of a topological space T and its subspace S, there is a natural map $[(CS^{n-1}, S^{n-1}), (T, S)] \to H_n(T, S)$, which makes the following diagram *anti-commutative*;

$$\begin{array}{ccc} [(CS^{n-1}, S^{n-1}), (T, S)] & \longrightarrow & H_n(T, S) \\ \big\downarrow & & \big\downarrow \partial \\ [S^{n-1}, S] & \longrightarrow & H_{n-1}(S). \end{array} \tag{4.1}$$

There is a unique class $\tilde{r} \in [(CS^{n-1}, S^{n-1}), (CS^{n-1}, S^{n-1})]$ which is represented by an orientation reversing self-homeomorphism. For $f \in [(CS^{n-1}, S^{n-1}), (T, S)]$, we write by $-f$ the homotopy class $f \circ \tilde{r}$.

Now we consider the following situation. Let W be a non-singular connected complex manifold of dimension n, Z a Riemann surface, and $g : W \to Z$ a surjective holomorphic map. For a point $z \in Z$, let W_z denote the fiber $g^{-1}(z)$. Suppose that the following conditions (wz-1)-(wz-3) are satisfied.

(wz-1) The map g has only one critical point $q \in W$, which is non-degenerate.
(wz-2) Moreover, g is locally trivial over $Z \setminus \{p\}$, where $p = g(q)$.

Because of (wz-1), there exist local analytic coordinates $(w_1, \ldots, w_n)$ on W with the center q and an analytic coordinate t on Z with the center p such that g is given by

$$(w_1, \ldots, w_n) \;\mapsto\; t = w_1^2 + \cdots + w_n^2. \tag{4.2}$$

We choose a small positive real number ϵ and a positive real number η which is small enough even compared with ϵ, and put

$$B_\epsilon := \{\ (w_1, \ldots, w_n) \in W\ ;\ |w_1|^2 + \cdots + |w_n|^2 \le \epsilon\} \quad \text{and} \quad \Delta_\eta := \{\ t \in Z\ ;\ |t| \le \eta\}.$$

The third condition we impose is the following;

(wz-3) the restriction of g to $(\ g^{-1}(\Delta_\eta) \setminus B_\epsilon,\ g^{-1}(\Delta_\eta) \cap \partial B_\epsilon\ )$ is trivial over Δ_η.

The meaning of (wz-3) is that, "at infinity", g is locally trivial even over the critical value p.

The cases we are going to apply the facts explained in this section are, for example, as follows. Let u be a point on $\mathcal{U}_N$, and $p \in \mathbb{C}$ a value in $\mathrm{Cr}(u)$. Then the situation

$$Z = \mathbb{C}^\times \setminus (\mathrm{Cr}(u) \setminus \{p\}), \quad W = E_u \setminus \bigcup_{p' \in \mathrm{Cr}(u)\setminus\{p\}} \phi_u^{-1}(p'), \quad \text{and} \quad g = \phi_u|_W$$

satisfies the conditions (wz-1)-(wz-3) because of Propositions 2.1, 2.5 and the definition of $\mathcal{U}_N$. We will also consider the following situation. Let u be as above, and let $p \in \mathbf{C}$ be a value in $Cr(u)$. Then the data

$$Z = \mathbf{C} \setminus (Cr(u) \setminus \{p\}), \quad W = F_u \setminus \bigcup_{p' \in Cr(u)\setminus\{p\}} \psi_u^{-1}(p'), \quad \text{and} \quad g = \psi_u|_W$$

satisfy the conditions (wz-1)-(wz-3) because ψ_u is the pull-back of ϕ_u by the étale covering $e : \mathbf{C} \to \mathbb{C}^\times$.

Now we go back to the general situation.

Definition 4.1. Let a be a point on $Z\setminus\{p\}$, and let $\mathcal{P}_a$ be the space of all paths $\omega : I \to Z$ from a to p such that $p \notin \omega([0,1))$. We equip $\mathcal{P}_a$ with the compact-open topology, and let $[\,\mathcal{P}_a\,]$ denote the set of path-connected components of $\mathcal{P}_a$. For $\omega \in \mathcal{P}_a$, let $[\omega] \in [\,\mathcal{P}_a\,]$ denote the path-connected component containing ω; that is, the homotopy class of paths in $\mathcal{P}_a$ represented by ω.

Proposition 4.1. *For a point $a \in Z \setminus \{p\}$ and a homotopy class $[\omega] \in [\,\mathcal{P}_a\,]$, there exists a homotopy class $\sigma[\omega] \in [S^{n-1}, W_a]$, unique up to sign, which satisfies the following properties. (i) Let a' be another point on $Z \setminus \{p\}$, and $\tau : I \to Z \setminus \{p\}$ a path from a' to a. Then we have*

$$\sigma[\omega \cdot \tau] = \pm[\tau]_*^{-1}(\sigma[\omega]),$$

where $[\tau]_* : [S^{n-1}, W_{a'}] \to [S^{n-1}, W_a]$ is the bijective map induced from the triviality of $g : W \to Z$ over $\tau : I \to Z \setminus \{p\}$. (ii) Suppose that $a \in \Delta_\eta \setminus \{p\}$ and $\omega(I) \subset \Delta_\eta$. Then $\sigma[\omega] \in [S^{n-1}, W_a]$ is represented by a continuous map

$$S^{n-1} \longrightarrow B_\epsilon \cap W_a \hookrightarrow W_a$$

such that the map $S^{n-1} \to B_\epsilon \cap W_a$ induces a homotopy equivalence.

Sketch of Proof. Let a be a point on $\Delta_\eta \setminus \{p\}$. The fact that $B_\epsilon \cap W_a$ is homotopically equivalent to S^{n-1} follows from (4.2). Hence $\sigma[\omega] \in [S^{n-1}, W_a]$ is uniquely determined, up to sign, by the property (ii), when ω is a path in Δ_η. For an arbitrary $a \in Z \setminus \{p\}$ and an arbitrary $\omega \in \mathcal{P}_a$, there exists $t \in (0,1)$ such that $\omega([t,1]) \subset \Delta_\eta$. We decompose ω into $\omega_2 \cdot \omega_1$ at t; that is, $\omega_1(s) = \omega(ts)$ and $\omega_2(s) = \omega(t + s(1-t))$. By the above argument, we have $\sigma[\omega_2] \in [S^{n-1}, W_{\omega(t)}]$. The class $\sigma[\omega] \in [S^{n-1}, W_a]$ is derived from $\sigma[\omega_2]$ via the bijective map between $[S^{n-1}, W_a]$ and $[S^{n-1}, W_{\omega(t)}]$ induced by the triviality of g over ω_1, using property (i). $\square$

Definition 4.2. We call the class $\sigma[\omega] \in [S^{n-1}, W_a]$ the *vanishing cycle for* $[\omega]$. Let $\bar{\sigma}[\omega] \in H_{n-1}(W_a)$ denote the corresponding homology class.

Remark 4.1. Traditionally, the homology class $\bar{\sigma}[\omega]$ has been called the vanishing cycle for $[\omega]$.

Remark 4.2. There are usually two vanishing cycles $\sigma[\omega]$ and $-\sigma[\omega] = \sigma[\omega] \circ r$ for a given $[\omega]$.

Let $W \times_Z I_\omega$ be the pull-back of $g : W \to Z$ by $\omega : I \to Z$, where $\omega \in \mathcal{P}_a$. Then the embedding $W_p \hookrightarrow W \times_Z I_\omega$ induces a homotopy equivalence because of (wz-2) and (wz-3). Combining the embedding $W_a \hookrightarrow W \times_Z I_\omega$ with the homotopy inverse $W \times_Z I_\omega \to W_p$, we get a *contraction map*

$$C_\omega \;:\; W_a \longrightarrow W_p$$

along ω. Let $\zeta : I \to Z \setminus \{p\}$ be a loop from a to a as follows; ζ goes along ω from a to a point $p' := \omega(1-\lambda) \in \Delta_\eta$, where λ is a positive real number small enough, draws a circle in the punctured disk $\Delta_\eta \setminus \{p\}$ from p' to p' in a counter-clockwise direction, and goes back to a along ω^{-1}.

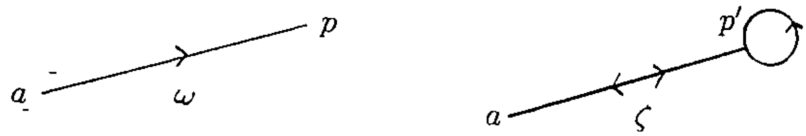

Figure 1

Then we have the monodromy action

$$[\zeta]_* \;:\; H_{n-1}(W_a) \longrightarrow H_{n-1}(W_a)$$

induced by $[\zeta] \in \pi_1(Z \setminus \{p\}, a)$. The classical theory of Lefschetz states the following theorem.

Theorem L1. (1) *The kernel of* $C_{\omega *} : H_{n-1}(W_a) \to H_{n-1}(W_p)$ *is generated by the homology class* $\bar{\sigma}[\omega]$ *of a vanishing cycle for* $[\omega]$. (2) *The image of the endomorphism* $\mathrm{Id} - [\zeta]_*$ *of* $H_{n-1}(W_a)$ *coincides with the kernel of* $C_{\omega *}$. $\square$

Now we describe the notion of *thimbles*. Let

$$\rho \;:\; CS^{n-1} \longrightarrow I$$

be the natural projection induced from the first projection $I \times S^{n-1} \to I$.

Proposition 4.2. *Suppose that* $a \in Z \setminus \{p\}$ *and* $\omega \in \mathcal{P}_a$ *are given. Suppose also that the sign of the vanishing cycle* $\sigma[\omega]$ *is specified. Then there exists a unique homotopy class*

$$\theta([\omega], \sigma[\omega]) \;\in\; [(CS^{n-1}, S^{n-1}), (W, W_a)]$$

with the following properties. (i) The image of $\theta([\omega],\sigma[\omega])$ *by the natural map*

$$[(CS^{n-1},S^{n-1}),(W,W_a)] \longrightarrow [S^{n-1},W_a]$$

is $\sigma[\omega]$. *(ii) The homotopy class* $\theta([\omega],\sigma[\omega])$ *is represented by a continuous map* $T : CS^{n-1} \to W$ *which makes the following diagram commutative*

$$\begin{array}{ccc} CS^{n-1} & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle g} \\ I & \xrightarrow{\omega} & Z, \end{array}$$

and which maps the vertex of the cone CS^{n-1} *to the critical point* q.

Sketch of Proof. Suppose that $\sigma[\omega] \in [S^{n-1},W_a]$ is represented by $s_0 : S^{n-1} \to W_a$. Then s_0 deforms continuously to $s_t : S^{n-1} \to W_{\omega(t)}$ for $t \in [0,1]$. We see that s_1 is homotopically equivalent to the constant map $S^{n-1} \to \{q\} \hookrightarrow W_p$, because $B_\epsilon \cap W_p$ is contractible by (4.2). Therefore, by changing the deformation s_t homotopically, we may assume that s_1 is the constant map through $\{q\}$. The continuous map T is constructed by putting these s_t together. $\square$

Definition 4.3. We call the homotopy class $\theta([\omega],\sigma[\omega])$ the *thimble for* $[\omega]$ *starting from* $\sigma[\omega]$. When the orientation does not need to be specified, we write this thimble simply $\theta([\omega])$. (Note that $\theta([\omega],-\sigma[\omega)) = -\theta([\omega],\sigma[\omega])$.) We denote its homology class by $\bar{\theta}([\omega],\sigma[\omega]) \in H_n(W,W_a)$.

Definition 4.4. Suppose that $\omega' \in \mathcal{P}_a$ is a path representing a homotopy class $[\omega] \in [\,\mathcal{P}_a\,]$. We say that a continuous map $T : CS^{n-1} \to W$ *represents the thimble* $\theta([\omega],\sigma[\omega])$ *over the path* ω', if the diagram

$$\begin{array}{ccc} CS^{n-1} & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle g} \\ I & \xrightarrow{\omega'} & Z \end{array}$$

is commutative (in particular, $T(\{0\} \times S^{n-1})$ is contained in W_a), and if T represents $\theta([\omega],\sigma[\omega])$ in $[(CS^{n-1},S^{n-1}),(W,W_a)]$.

It is obvious that, for any $\omega' \in [\omega]$, there exists a continuous map $T : CS^{n-1} \to W$ which represents the thimble $\theta([\omega],\sigma[\omega])$ over ω'.

Definition 4.5. Let ξ be a sub-path of ω; that is, there is a continuous increasing map $i : I \to I$ such that $\xi = \omega \circ i$. Let $T : CS^{n-1} \to W$ be a continuous map representing the thimble $\theta([\omega],\sigma[\omega])$ over ω. The *restriction* $T|_\xi$ *of* T *to* ξ is the composition of $\tilde{i} : (CS^{n-1}) \times_I I \to CS^{n-1}$ and T, where $\tilde{i}$ is the pull-back of i by $\rho : CS^{n-1} \to I$. If $i(1) = 1$, then $T|_\xi$ is a continuous map from CS^{n-1} to W, which represents the thimble $\theta([\xi])$ over the path ξ. If $i(1) < 1$, then $T|_\xi$ is a map from $I \times S^{n-1}$ to W.

Now we choose two points a and a' in $\Delta_\eta \setminus \{p\}$ such that the two radii of the disk Δ_η passing through a and a', respectively, are distinct. Let ω and ω' be the paths from

a and a', respectively, to the center p along the radius of the disk Δ_η. Let ι_+ and ι_- be the paths in $\Delta_\eta \setminus \{p\}$ from a to a' described as follows; the path ι_+ (resp. ι_-) start from a, goes to a point on the boundary $\partial\Delta_\eta$ along the radius, draws an arc on $\partial\Delta_\eta$ in the counter-clockwise direction (resp. in the clockwise direction) to the end point of the radius passing through a', and then goes to a' along this radius.

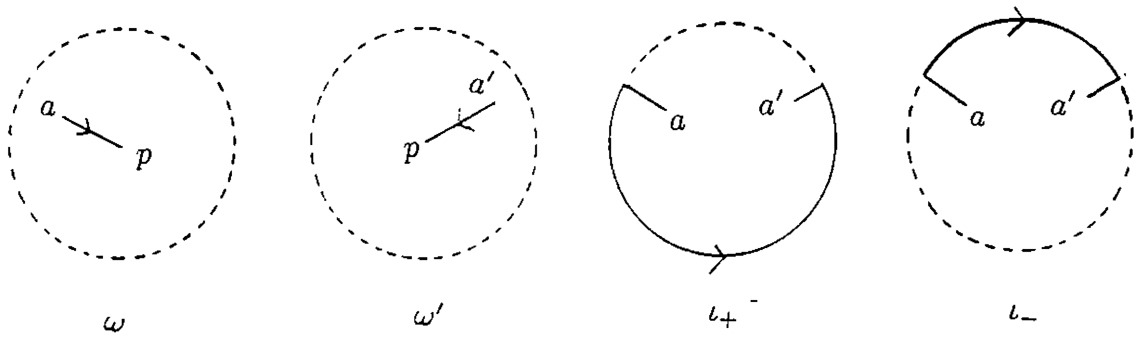

Figure 2

Suppose that a vanishing cycle $\sigma[\omega] \in [S^{n-1}, W_a]$ for $[\omega]$ is chosen from among the two possibilities. We put

$$\sigma_+[\omega'] := [\iota_+]_*(\sigma[\omega]), \quad \text{and} \quad \sigma_-[\omega'] := [\iota_-]_*(\sigma[\omega]),$$

both of which are vanishing cycles for $[\omega']$, because $[\omega \cdot \iota_+^{-1}] = [\omega \cdot \iota_-^{-1}] = [\omega']$. Then we have

$$\sigma_+[\omega'] = (-1)^n \sigma_-[\omega'] \quad \text{in} \quad [S^{n-1}, W_{a'}].$$

Let T, T_+ and T_- be continuous maps from CS^{n-1} to W which represent the thimbles $\theta([\omega], \sigma[\omega])$, $\theta([\omega'], \sigma_+[\omega'])$ and $\theta([\omega'], \sigma_-[\omega'])$, respectively, over ω, ω' and ω', respectively. With the orientation of CS^{n-1}, we can consider these maps as n-chains in W.

Lemma 4.1. *We can choose the maps T, T_+ and T_- in such a way that the n-chains T and T_+ (resp. T and T_-) intersect at only one point q transversely with the intersection number $(-1)^{n(n-1)/2}$ (resp. $(-1)^{n(n+1)/2}$).*

Proof. This lemma can be checked by direct calculation using the explicit form (4.2) of g. $\square$

§5. Structures of $H_{n-1}(X_u)$, $H_n(E_u)$ and $H_n(E_u, \partial_0 E_u)$

In this section, we always assume that $u \in \mathcal{U}$. We define two points

$$a_u^0 := \varepsilon(u), \qquad \text{and} \qquad a_u^\infty := 1/\varepsilon(u)$$

on $\mathbb{C}^\times$, and consider the fibers

$$X_u^0 := \phi_u^{-1}(a_u^0), \qquad \text{and} \qquad X_u^\infty := \phi_u^{-1}(a_u^\infty).$$

By definition (3.3) of $\varepsilon : \mathcal{U} \to \mathbb{R}_{>0}$, there are no critical values of $\hat{\phi}_u$ on the interval $[0, \varepsilon(u)] \subset \mathbb{R}$. Hence, by Proposition 2.1, there is a diffeomorphism, unique up to homotopy,

$$X_u = \hat{\phi}_u^{-1}(0) \ \cong \ X_u^0 , \tag{5.1}$$

which is induced by the path from 0 to $\varepsilon(u)$ along $\mathbb{R}$. It is obvious that $\pi_1(\mathcal{U}, b)$ acts also on $H_{n-1}(X_b^0)$ and $H_{n-1}(X_b^\infty)$. The Lemma below follows immediately from the definition of ε.

Lemma 5.1. *The isomorphism $H_{n-1}(X_b) \cong H_{n-1}(X_b^0)$ induced by (5.1) is $\pi_1(\mathcal{U}, b)$-equivariant.* $\square$

Since $a_u^\infty \notin \mathrm{Cr}(u)$, Proposition 2.1 implies that X_u and X_u^∞ are also diffeomorphic. However the homotopy class of the diffeomorphism is not uniquely determined, and we cannot expect that $H_{n-1}(X_b) \cong H_{n-1}(X_b^\infty)$ is $\pi_1(\mathcal{U}, b)$-equivariant by any means.

Note the following:

Theorem L2 (Lefschetz Hyperplane Section Theorem). *The homology groups $H_i(X_u) \cong H_i(X_u^0) \cong H_i(X_u^\infty)$ are zero for $i > n-1$.*

Proof. See, for example, [9]. $\square$

Definition 5.1. For a point $a \in \mathbb{C}^\times \setminus \mathrm{Cr}(u)$ and $p \in \mathrm{Cr}(u)$, let $\mathcal{P}_u(a, p)$ denote the space of all paths $\omega : I \to \mathbb{C}^\times$ which satisfy the following; (i) $\omega(0) = a$, $\omega(1) = p$, and (ii) $\omega([0,1)) \cap \mathrm{Cr}(u) = \emptyset$. We equip $\mathcal{P}_u(a,p)$ with the compact-open topology. Let $[\,\mathcal{P}_u(a,p)\,]$ denote the set of path connected components of $\mathcal{P}_u(a,p)$. For $\omega \in \mathcal{P}_u(a,p)$, let $[\omega] \in [\,\mathcal{P}_u(a,p)\,]$ denote the path connected component containing ω; that is, $[\omega]$ denotes the homotopy class of paths in $\mathcal{P}_u(a,p)$ represented by ω.

Suppose that $u \in \mathcal{U}_N$. Then $\mathrm{Cr}(u)$ consists of distinct N values $\{p_1, \dots, p_N\}$.

Definition 5.2. Suppose that $a \in \mathbb{C}^\times \setminus \mathrm{Cr}(u)$ is given. A set of paths$\{\xi_1, \dots, \xi_N\}$, where $\xi_i \in \mathcal{P}_u(a, p_i)$, is called a *regular system of paths from a* if the following are satisfied; (i) each $\xi_i : I \to \mathbb{C}^\times$ is injective, and (ii) $\xi_i(I) \cap \xi_j(I) = \{a\}$ if $i \neq j$.

Since $u \in \mathcal{U}_N$, the morphism ϕ_u has only one critical point q_i over each p_i. Moreover, these critical points are all non-degenerate. Therefore, if we are given a regular system $\{\xi_1, \dots, \xi_N\}$ of paths from a, we obtain vanishing cycles $\pm\sigma[\xi_i] \in [S^{n-1}, \phi_u^{-1}(a)]$ for each $[\xi_i]$, and the associated thimbles

$$\pm\theta([\xi_i], \sigma[\xi_i]) \ \in \ [(CS^{n-1}, S^{n-1}), (E_u, \phi_u^{-1}(a))]$$

for each $[\xi_i]$.

We are going to use regular systems of paths from a_u^0 and from a_u^∞ exclusively. It is obvious that there always exist regular systems of paths from a_u^0 and from a_u^∞ for every $u \in \mathcal{U}_N$.

Proposition 5.1. *Suppose that $u \in \mathcal{U}_N$.*

(0) *Suppose that* $\{\xi_1^0, \ldots, \xi_N^0\}$ *is a regular system of paths from* a_u^0. *We choose a vanishing cycle* $\sigma[\xi_i^0] \in [S^{n-1}, X_u^0]$ *for each* $[\xi_i^0]$ *from among the two possibilities. Then the homology classes* $\bar{\sigma}[\xi_1^0], \ldots, \bar{\sigma}[\xi_N^0]$ *form a set of basis for the free* $\mathbb{Z}$*-module* $H_{n-1}(X_u^0)$.

(∞) *Suppose that* $\{\xi_1^\infty, \ldots, \xi_N^\infty\}$ *is a regular system of paths from* a_u^∞. *We choose a vanishing cycle* $\sigma[\xi_i^\infty] \in [S^{n-1}, X_u^\infty]$ *for each* $[\xi_i^\infty]$ *from among the two possibilities. Then the homology classes* $\bar{\sigma}[\xi_1^\infty], \ldots, \bar{\sigma}[\xi_N^\infty]$ *form a set of basis for the free* $\mathbb{Z}$*-module* $H_{n-1}(X_u^\infty)$.

Proof. Since these two assertions can be proved in completely parallel ways, we prove only the assertion (0).

Let $\Delta_i \subset \mathbb{C}^\times$ be a small closed disk with the center p_i. Since $\{\xi_1^0, \ldots, \xi_N^0\}$ is a regular system of paths, the union $\cup_{i=1}^N(\xi_i^0(I) \cup \Delta_i)$ is a strong deformation retract of $\mathbb{C}$, and it contains $\mathrm{Cr}(u)$ in its interior. By Proposition 2.1, the space

$$A \quad := \quad \hat{\phi}_u^{-1}\Big(\bigcup_{i=1}^{N}(\xi_i^0(I) \cup \Delta_i)\Big) \tag{5.2}$$

is also a strong deformation retract of $\mathbb{A}^n$. Hence A is contractible. We decompose A into the union of the two parts

$$A_1 \quad := \quad \hat{\phi}_u^{-1}\Big(\ \bigcup_{i=1}^{N} \xi_i^0([0,1/2])\ \Big), \quad \text{and} \quad A_2 \quad := \quad \hat{\phi}_u^{-1}\Big(\ \bigcup_{i=1}^{N}\ (\xi_i^0([1/2,1]) \cup \Delta_i)\ \Big).$$

By applying the Mayer-Vietoris sequence to this decomposition of the contractible space A, we obtain an isomorphism

$$H_{n-1}(A_1 \cap A_2) \xrightarrow{\sim} H_{n-1}(A_1) \oplus H_{n-1}(A_2) \tag{5.3}$$

induced by the inclusions. Using Propositions 2.1 and 2.5(2), we have canonical homotopy equivalences

$$\begin{array}{rcllr} A_1 & \sim & X_u^0, & & \\ A_2 & \sim & \coprod_{i=1}^N \phi_u^{-1}(\Delta_i) & \sim \quad \coprod_{i=1}^N \phi_u^{-1}(p_i), & \text{and} \\ A_1 \cap A_2 & \sim & \coprod^N X_u^0 & \quad \text{(the disjoint union of } N \text{ copies of } X_u^0\text{)}, & \end{array}$$

through which the isomorphism (5.3) is written as follows;

$$s \oplus (c_1 \oplus \cdots \oplus c_N) \ : \ \bigoplus_{i=1}^{N} H_{n-1}(X_u^0) \xrightarrow{\sim} H_{n-1}(X_u^0) \oplus \bigoplus_{i=1}^{N} H_{n-1}(\phi_u^{-1}(p_i)),$$

where $s : \bigoplus_{i=1}^N H_{n-1}(X_u^0) \to H_{n-1}(X_u^0)$ is the summation $(x_1, \ldots, x_N) \mapsto x_1 + \cdots + x_N$, and $c_i : H_{n-1}(X_u^0) \to H_{n-1}(\phi_u^{-1}(p_i))$ is the homomorphism induced by the contraction map $X_u^0 \to \phi_u^{-1}(p_i)$ along ξ_i^0. Thus we get an isomorphism

$$H_{n-1}(X_u^0) \ \cong \ \bigoplus_{i=1}^{N} \mathrm{Ker}\ c_i.$$

By Theorem L1, the kernel of c_i is generated by the homology class $\bar{\sigma}[\xi_i^0]$ of a vanishing cycle for $[\xi_i^0]$. Hence all we have to do now is to show that the $\mathbb{Z}$-module $H_{n-1}(X_u^0)$ is torsion free of rank N; that is,

$$b_{n-1}(X_u^0) = b_{n-1}(X_u) = N = (d-1)^n. \tag{5.4}$$

This is a well-known formula. $\square$

Next we shall investigate $H_n(E_u)$ and $H_n(E_u, \partial_0 E_u)$.

Proposition 5.2. *Suppose that $u \in \mathcal{U}$.*
(1) There is an isomorphism between $H_{n-1}(X_u^0)$ and $H_n(\partial_0 E_u)$.
(2) The inclusion $\partial_0 E_u \hookrightarrow E_u$ induces an isomorphism $H_n(\partial_0 E_u) \xrightarrow{\sim} H_n(E_u)$.
(3) The natural homomorphism $H_n(E_u) \to H_n(E_u, \partial_0 E_u)$ is a zero map.
(4) The boundary homomorphism $H_n(E_u, \partial_0 E_u) \to H_{n-1}(\partial_0 E_u)$ is an isomorphism.
(5) The inclusion $X_u^0 \hookrightarrow \partial_0 E_u$ induces an isomorphism $H_{n-1}(X_u^0) \xrightarrow{\sim} H_{n-1}(\partial_0 E_u)$.
(6) Moreover, when $u = b$, all the isomorphisms above between the homology groups are $\pi_1(\mathcal{U}, b)$-equivariant.

The assertions can be summarized in the following diagram of $\pi_1(\mathcal{U}, b)$-equivariant homomorphisms:

$$\begin{array}{ccccc} H_n(\partial_0 E_b) & \underset{\text{incl.}}{\xrightarrow{\sim}} & H_n(E_b) & \underset{\text{zero}}{\longrightarrow} & H_n(E_b, \partial_0 E_b) \\ \uparrow \wr & & & & \partial \downarrow \wr \\ H_{n-1}(X_b^0) & & \underset{\text{incl.}}{\xrightarrow{\sim}} & & H_{n-1}(\partial_0 E_b), \end{array}$$

where incl$_*$ means the homomorphisms induced by the inclusions.

Proof. Since the isomorphisms in (2), (4) and (5) are defined by natural topological operations, they are obviously $\pi_1(\mathcal{U})$-equivariant. The remaining isomorphism in (1) being $\pi_1(\mathcal{U})$-equivariant can be seen from the construction below.

Let $\Delta_{\varepsilon(u)}(0) \subset \mathbb{C}$ be the closed disk of radius $\varepsilon(u)$ with the center 0. We have

$$B_u^0 = \Delta_{\varepsilon(u)}(0) \setminus \{0\}.$$

Since there are no critical values of $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ on $\Delta_{\varepsilon(u)}(0)$, Proposition 2.1 implies that there is a diffeomorphism

$$\hat{\phi}_u^{-1}(\Delta_{\varepsilon(u)}(0)) \;\cong\; \Delta_{\varepsilon(u)}(0) \times X_u^0 \tag{5.5}$$

over $\Delta_{\varepsilon(u)}(0)$ which induces the identity on X_u^0. By restricting it, we obtain a diffeomorphism

$$\partial_0 E_u = \phi_u^{-1}(B_u^0) \;\cong\; B_u^0 \times X_u^0 \tag{5.6}$$

over B_u^0. Each of these diffeomorphisms is unique up to homotopy. Using Theorem L2 and Künneth formula, we obtain canonical isomorphisms

$$H_n(\partial_0 E_u) \;\cong\; H_{n-1}(X_u^0), \quad \text{and} \quad H_{n-1}(\partial_0 E_u) \;\cong\; H_{n-1}(X_u^0), \tag{5.7}$$

from (5.6). The second isomorphism of (5.7) is induced from the inclusion $X_u^0 \hookrightarrow \partial_0 E_u$. Thus (1) and (5) are proved. Using the excision property of homology groups and the diffeomorphism (5.5), we get

$$H_n(E_u, \partial_0 E_u) \;\cong\; H_n(\mathbb{A}^n, \hat{\phi}_u^{-1}(\Delta_{\varepsilon(u)}(0))) \;\cong\; H_{n-1}(\hat{\phi}_u^{-1}(\Delta_{\varepsilon(u)}(0))) \;\cong\; H_{n-1}(X_u^0).$$

We can easily see that this isomorphism coincides with the composition of the boundary map from $H_n(E_u, \partial_0 E_u)$ to $H_{n-1}(\partial_0 E_u)$ and the second isomorphism of (5.7). Hence (4) is proved. The assertion (3) is a consequence of (2) and (4). Therefore only (2) remains to be proved.

It is enough to prove (2) when u is a point of $\mathcal{U}_N$, because each of $H_n(\partial_0 E_u)$ and $H_n(E_u)$ forms a locally constant system over $\mathcal{U}$ when u varies. Let $\Delta_i \subset \mathbb{C}^\times$ be a small closed disk with the center p_i. We can take a regular system $\{\xi_1^0, \dots, \xi_N^0\}$ of paths from a_u^0 in such a way that

$$\xi_i^0(I) \cap \Delta_{\varepsilon(u)}(0) = \{a_u^0\}. \tag{5.8}$$

Then the space

$$B_u^0 \;\cup\; \bigcup_{i=1}^{N} (\xi_i^0(I) \cup \Delta_i) \quad \subset \quad \mathbb{C}^\times$$

is a strong deformation retract of $\mathbb{C}^\times$, and it contains $\mathrm{Cr}(u)$ in its interior. Hence the space

$$A^\times \quad := \quad \phi_u^{-1}(\; B_u^0 \cup \bigcup_{i=1}^{N} (\xi_i^0(I) \cup \Delta_i) \;)$$

is also a strong deformation retract of E_u by Proposition 2.1. Thus $H_n(E_u)$ is canonically isomorphic to $H_n(A^\times)$. We decompose $A^\times$ into the union of A in (5.2) and $\partial_0 E_u = \hat{\phi}_u^{-1}(B_u^0)$. Because of (5.8), we have

$$A \cap \partial_0 E_u = X_u^0.$$

Recall that A is contractible. Hence the Mayer-Vietoris sequence for this decomposition is written as follows;

$$\begin{array}{cccccc} \longrightarrow & H_n(X_u^0) & \longrightarrow & H_n(\partial_0 E_u) & \longrightarrow & H_n(A^\times) \\ \longrightarrow & H_{n-1}(X_u^0) & \longrightarrow & H_{n-1}(\partial_0 E_u) & \longrightarrow & \cdots. \end{array}$$

Because of the second isomorphism in (5.7) and because of $H_n(X_u^0) = 0$ by Theorem L2, we see that the inclusion $\partial_0 E_u \hookrightarrow A^\times$ induces an isomorphism between $H_n(\partial_0 E_u)$ and $H_n(A^\times) \cong H_n(E_u)$. □

As in Proposition 5.1, we will describe geometrically a set of basis for the free $\mathbb{Z}$-module $H_n(E_u, \partial_0 E_u)$ when $u \in \mathcal{U}_N$.

Proposition 5.3. *Suppose that $u \in \mathcal{U}_N$. Let $\{\xi_1^0, \ldots, \xi_N^0\}$ be a regular system of paths from a_u^0. Let $\sigma[\xi_i^0] \in [S^{n-1}, X_u^0]$ be a vanishing cycle for $[\xi_i^0]$, and let*

$$\theta([\xi_i^0], \sigma[\xi_i^0]) \in [(CS^{n-1}, S^{n-1}), (E_u, X_u^0)]$$

be the thimble for $[\xi_i^0]$ starting from $\sigma[\xi_i^0]$. Then the homology classes $\bar{\theta}([\xi_1^0], \sigma[\xi_1^0])$, ..., $\bar{\theta}([\xi_N^0], \sigma[\xi_N^0])$ form a set of basis for $H_n(E_u, \partial_0 E_u)$.

Proof. Note that, by the isomorphism from $H_n(E_u, \partial_0 E_u)$ to $H_{n-1}(X_u^0)$ given by the composition of the isomorphisms of (4) and (5) in Proposition 5.2, the homology class $\bar{\theta}([\xi_i^0], \sigma[\xi_i^0])$ is mapped to $-\bar{\sigma}[\xi_i^0]$ because of the anti-commutativity of (4.1). Hence the assertion follows from Proposition 5.1. $\square$

Now we fix a base point $b \in \mathcal{U}$. We shall review the classical theory of Lefschetz about monodromy representations, and study the structure of $H_{n-1}(X_b^0)$ as a $\pi_1(\mathcal{U}, b)$-module. Again, we refer the reader to [6] for the proof.

Let $\overline{X}_b^0 \subset \mathbb{P}^n$ be the projective compactification of the affine hypersurface $X_b^0 \subset \mathbb{A}^n$. Taking Remark 2.1 into account, we see that $\overline{X}_b^0$ is non-singular from Lemma 2.1 and the definition of a_b^0. Moreover, the intersection $H_\infty \cap \overline{X}_b^0$ coincides with $Y_b := H_\infty \cap \overline{X}_b$ from (2.2). There is a canonical isomorphism

$$H_{n-1}(X_b^0) \cong H^{n-1}(\overline{X}_b^0, Y_b). \tag{5.9}$$

We put

$$\begin{aligned} H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0) &:= \mathrm{Ker}\,(H^{n-1}(\overline{X}_b^0) \xrightarrow{r} H^{n-1}(Y_b)), \quad \text{and} \\ H^{n-2}_{\mathrm{prim}}(Y_b) &:= \mathrm{Coker}\,(H^{n-2}(\overline{X}_b^0) \xrightarrow{r} H^{n-2}(Y_b)), \end{aligned}$$

where r is the restriction homomorphism. Then, from (5.9), we obtain an exact sequence

$$0 \longrightarrow H^{n-2}_{\mathrm{prim}}(Y_b) \longrightarrow H_{n-1}(X_b^0) \longrightarrow H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0) \longrightarrow 0. \tag{5.10}$$

The fundamental group $\pi_1(\mathcal{U}, b)$ acts on this exact sequence. The action on $H^{n-2}_{\mathrm{prim}}(Y_b)$ factors through the natural homomorphism

$$\pi_1(\mathcal{U}) \longrightarrow \pi_1(\Gamma^\times \setminus \mathcal{D}_\infty) \longrightarrow \pi_1(\mathbb{P}_*(\Gamma) \setminus D_\infty),$$

while the action on $H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0)$ factors through

$$\pi_1(\mathcal{U}) \longrightarrow \pi_1(\Gamma^\times \setminus \mathcal{D}_0) \longrightarrow \pi_1(\mathbb{P}_*(\Gamma) \setminus D_0).$$

Note that $H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0) \otimes_{\mathbf{Z}} \mathbb{Q}$ coincides with the Poincaré dual of "the module of vanishing cycles" in $H_{n-1}(\overline{X}_b^0) \otimes_{\mathbf{Z}} \mathbb{Q}$ in the sense of [6; \$3]. Hence the classical theory of Lefschetz tells us the following:

Theorem L3. *Let p be a value in* $\mathrm{Cr}(b)$, *and let ω be a path in $\mathcal{P}_b(a_b^0, p)$. Let $\bar{\sigma}[\omega]' \in H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0)$ denote the image of the homology class $\bar{\sigma}[\omega] \in H_{n-1}(X_b^0)$ of a vanishing cycle $\sigma[\omega]$ for $[\omega]$ by the homomorphism in (5.10). Then $H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0) \otimes_{\mathbf{Z}} \mathbb{Q}$ is generated by $\bar{\sigma}[\omega]'$ as a module over the group ring $\mathbb{Q}[\pi_1(\mathbb{P}_*(\Gamma) \setminus D_0, b)]$.* $\square$

§6. Structures of $H_n(F_u)$, $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$

In order to state the main theorem of this section, we need two definitions. First, we put

$$\mathcal{U}_N^{\sim} := \{ \ u \in \mathcal{U}_N \ ; \ \mathrm{Cr}(u) \cap \mathbb{R}_{\leq 0} = \emptyset \}.$$

The complement $\Gamma \setminus \mathcal{U}_N^{\sim}$ is a real semi-algebraic subset of real codimension 1 in the affine space Γ. Second, we define the automorphism

$$j \ : \ H_{n-1}(X_u^\infty) \longrightarrow H_{n-1}(X_u^\infty)$$

for $u \in \mathcal{U}$ as follows. We set

$$C_u^0 \ := \ \{ \ z \in \mathbb{C} \ ; \ |z| = \varepsilon(u) \}, \quad \text{and} \quad C_u^\infty \ := \ \{ \ z \in \mathbb{C} \ ; \ |z| = 1/\varepsilon(u) \}.$$

Note that $\hat{\phi}_u$ has no critical values on the circle C_u^∞. Then j is defined as the monodromy on $H_{n-1}(X_u^\infty)$ along the loop from a_u^∞ to a_u^∞ which draws the circle C_u^∞ in the counter-clockwise direction.

Theorem 6.1. (1) *If $u \in \mathcal{U}$, then the natural homomorphisms $H_n(F_u) \to H_n(F_u, \partial_0 F_u)$ and $H_n(F_u) \to H_n(F_u, \partial_\infty F_u)$ are injective.* (2) *Suppose that $u \in \mathcal{U}_N^{\sim}$. Then there is a canonical isomorphism*

$$\Psi_u^0 \ : \ H_{n-1}(X_u^0) \otimes \mathbb{Z}[q, q^{-1}] \xrightarrow{\sim} H_n(F_u, \partial_0 F_u)$$

of $\mathbb{Z}[q, q^{-1}]$-modules through which the image of $H_n(F_u) \hookrightarrow H_n(F_u, \partial_0 F_u)$ is identified with $H_{n-1}(X_u^0) \otimes (1 - q)$, where $(1 - q) \subset \mathbb{Z}[q, q^{-1}]$ is the principal ideal generated by $1 - q$. There also exists a canonical isomorphism

$$\Psi_u^\infty \ : \ H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q, q^{-1}] \xrightarrow{\sim} H_n(F_u, \partial_\infty F_u)$$

of $\mathbb{Z}[q, q^{-1}]$-modules through which the image of $H_n(F_u) \hookrightarrow H_n(F_u, \partial_\infty F_u)$ is identified with the image of the endomorphism $\mathrm{Id} - j \otimes q$ of $H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q, q^{-1}]$.

Since each of the $\mathbb{Z}[q, q^{-1}]$-modules $H_{n-1}(X_u^0) \otimes \mathbb{Z}[q, q^{-1}]$, $H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q, q^{-1}]$, $H_n(F_u, \partial_0 F_u)$, $H_n(F_u, \partial_\infty F_u)$ and $H_n(F_u)$ forms a locally constant system of $\mathbb{Z}[q, q^{-1}]$-modules over $\mathcal{U}$, Theorem 6.1 and Proposition 5.1 imply the following:

Corollary 6.1. *For an arbitrary $u \in \mathcal{U}$, each of $H_n(F_u, \partial_0 F_u)$, $H_n(F_u, \partial_\infty F_u)$ and $H_n(F_u)$ is a free $\mathbb{Z}[q, q^{-1}]$-module of rank N.*

Remark 6.1. The assertion that the isomorphisms Ψ_u^0 and Ψ_u^∞ are canonical for $u \in \mathcal{U}_N^\sim$ means that, when u moves on $\mathcal{U}_N^\sim$, they form isomorphisms between the corresponding locally constant systems *restricted over* $\mathcal{U}_N^\sim$. Even though $\mathcal{U}_N^\sim$ is dense in $\mathcal{U}$, these isomorphisms of locally constant systems cannot be extended to the whole space $\mathcal{U}$. Otherwise, the isomorphisms Ψ_u^0 and Ψ_u^∞ would be isomorphisms of $\pi_1(\mathcal{U})$-modules, but this would contradict to Irreducibility Theorem in Introduction, which will be proved in §10. In particular, this argument shows that $\mathcal{U}_N^\sim$ is *not* path-connected.

Before starting the proof, we prepare some notation. Suppose that $u \in \mathcal{U}$. Note that the circles C_u^0 and C_u^∞ are disjoint from $\mathrm{Cr}(u)$. We define the loops

$$\delta_u^0 \;:\; I \longrightarrow \mathbb{C}^\times \setminus \mathrm{Cr}(u), \quad \text{and} \quad \delta_u^\infty \;:\; I \longrightarrow \mathbb{C}^\times \setminus \mathrm{Cr}(u)$$

with the base point a_u^0 and a_u^∞, respectively, by

$$\delta_u^0(t) \;:=\; e^{2\pi\sqrt{-1}t}\varepsilon(u), \quad \text{and} \quad \delta_u^\infty(t) \;:=\; e^{2\pi\sqrt{-1}t}\varepsilon(u)^{-1}.$$

Remark 6.2. Then the automorphism $j : H_{n-1}(X_u^\infty) \to H_{n-1}(X_u^\infty)$ is nothing but the monodromy operator $[\delta_u^\infty]_*$. On the other hand, since $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ is locally trivial on

$$\overline{B}_u^0 \;:=\; \{\; z \in \mathbb{C} \;;\; 0 \leq |z| \leq \varepsilon(u)\} \;=\; B_u^0 \cup \{0\},$$

the monodromy action $[\delta_u^0]_* : H_{n-1}(X_u^0) \to H_{n-1}(X_u^0)$ is trivial by Proposition 2.1.

We put

$$R_u^0 \;:=\; e^{-1}(C_u^0) = \log\varepsilon(u) + \sqrt{-1}\mathbb{R}, \qquad R_u^\infty \;:=\; e^{-1}(C_u^\infty) = \log\varepsilon(u)^{-1} + \sqrt{-1}\mathbb{R},$$

and

$$Z_u^0 \;:=\; e^{-1}(a_u^0) = \log\varepsilon(u) + \sqrt{-1}\mathbb{Z}, \qquad Z_u^\infty \;:=\; e^{-1}(a_u^\infty) = \log\varepsilon(u)^{-1} + \sqrt{-1}\mathbb{Z},$$

where $e : \mathbf{C} \to \mathbb{C}^\times$ is the exponential map. For each $\nu \in \mathbb{Z}$, we put

$$a_u^0\langle\nu\rangle \;:=\; \log\varepsilon(u) + \sqrt{-1}\nu \;\in\; Z_u^0, \quad \text{and} \quad a_u^\infty\langle\nu\rangle \;:=\; \log\varepsilon(u)^{-1} + \sqrt{-1}\nu \;\in\; Z_u^\infty.$$

We also put

$$X_u^0\langle\nu\rangle \;:=\; \psi_u^{-1}(a_u^0\langle\nu\rangle), \quad \text{and} \quad X_u^\infty\langle\nu\rangle \;:=\; \psi_u^{-1}(a_u^\infty\langle\nu\rangle).$$

Then we have the natural isomorphisms

$$X_u^0 \cong X_u^0\langle\nu\rangle, \quad \text{and} \quad X_u^\infty \cong X_u^\infty\langle\nu\rangle \tag{6.1}$$

induced from the covering map $\tilde{e} : F_u \to E_u$.

Now suppose that $u \in \mathcal{U}_N^{\sim}$. For each $\nu \in \mathbb{Z}$, there exists a unique connected component

$$(\mathbb{C} \setminus \mathbb{R}_{\leq 0})\langle \nu \rangle \ \subset \ e^{-1}(\mathbb{C} \setminus \mathbb{R}_{\leq 0})$$

which contains $a_u^0\langle \nu \rangle$ and $a_u^\infty\langle \nu \rangle$. Let $\{p_1, \ldots, p_N\}$ be the set $\mathrm{Cr}(u)$, which is contained in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. For each $\nu \in \mathbb{Z}$, let $p_i\langle \nu \rangle$ denote the unique point on $(\mathbb{C} \setminus \mathbb{R}_{\leq 0})\langle \nu \rangle$ which is mapped to p_i by e. Therefore, we have

$$Cr(u) = \coprod_{\nu \in \mathbf{Z}} \mathrm{Cr}(u)\langle \nu \rangle,$$

where

$$\mathrm{Cr}(u)\langle \nu \rangle \ := \ \{ \ p_i\langle \nu \rangle \ ; \ i = 1 \ldots, N\}.$$

Note that $-\pi < \arg p_i < \pi$ for $i = 1, \ldots, N$. We put

$$\eta(u) \quad := \quad \frac{1}{2} \ \min \ \{ \ \pi - \arg p_i, \ \pi + \arg p_i \ ; \ i = 1, \ldots, N\}. \tag{6.2}$$

Then $\eta : \mathcal{U}_N^{\sim} \to \mathbb{R}_{>0}$ is a continuous function on $\mathcal{U}_N^{\sim}$. We put

$$K_u \ := \ \{ \ z \in \mathbb{C}^\times \ ; \ \varepsilon(u) \leq |z| \leq \varepsilon(u)^{-1}, \ \text{ and } \ -\pi + \eta(u) \leq \arg z \leq \pi - \eta(u)\},$$

and

$$K_u\langle \nu \rangle \ := \ \text{the unique connected component of } e^{-1}(K_u) \text{ containing } a_u^0\langle \nu \rangle \text{ and } a_u^\infty\langle \nu \rangle.$$

Then, for each ν, the exponential map $e : \mathbf{C} \to \mathbb{C}^\times$ induces an isomorphism between $K_u\langle \nu \rangle$ and K_u, and $e^{-1}(K_u)$ is the disjoint union of all $K_u\langle \nu \rangle$. Moreover, each $\mathrm{Cr}(u)\langle \nu \rangle$ is contained in the interior of $K_u\langle \nu \rangle$. We put

$$M_u^0 \ := \ e^{-1}(K_u \cup C_u^0) \ = (\coprod_{\nu \in \mathbf{Z}} K_u\langle \nu \rangle) \cup R_u^0 \ \ \subset \ \mathbf{C} \quad \text{and}$$
$$M_u^\infty \ := \ e^{-1}(K_u \cup C_u^\infty) = (\coprod_{\nu \in \mathbf{Z}} K_u\langle \nu \rangle) \cup R_u^\infty \ \ \subset \ \mathbf{C}.$$

We also put

$$N_u^0 \ := \ K_u \cap C_u^0, \quad \text{and} \quad N_u^\infty \ := \ K_u \cap C_u^\infty,$$

both of which are arcs in $\mathbb{C}^\times$. Each $K_u\langle \nu \rangle$ is a rectangle in $\mathbf{C}$, whose vertical sides are given by

$$N_u^0\langle \nu \rangle \ := \ K_u\langle \nu \rangle \cap R_u^0, \quad \text{and} \quad N_u^\infty\langle \nu \rangle \ := \ K_u\langle \nu \rangle \cap R_u^\infty.$$

Then we have

$$e^{-1}(N_u^0) \ = \ \coprod_{\nu \in \mathbf{Z}} N_u^0\langle \nu \rangle, \quad \text{and} \quad e^{-1}(N_u^\infty) \ = \ \coprod_{\nu \in \mathbf{Z}} N_u^\infty\langle \nu \rangle.$$

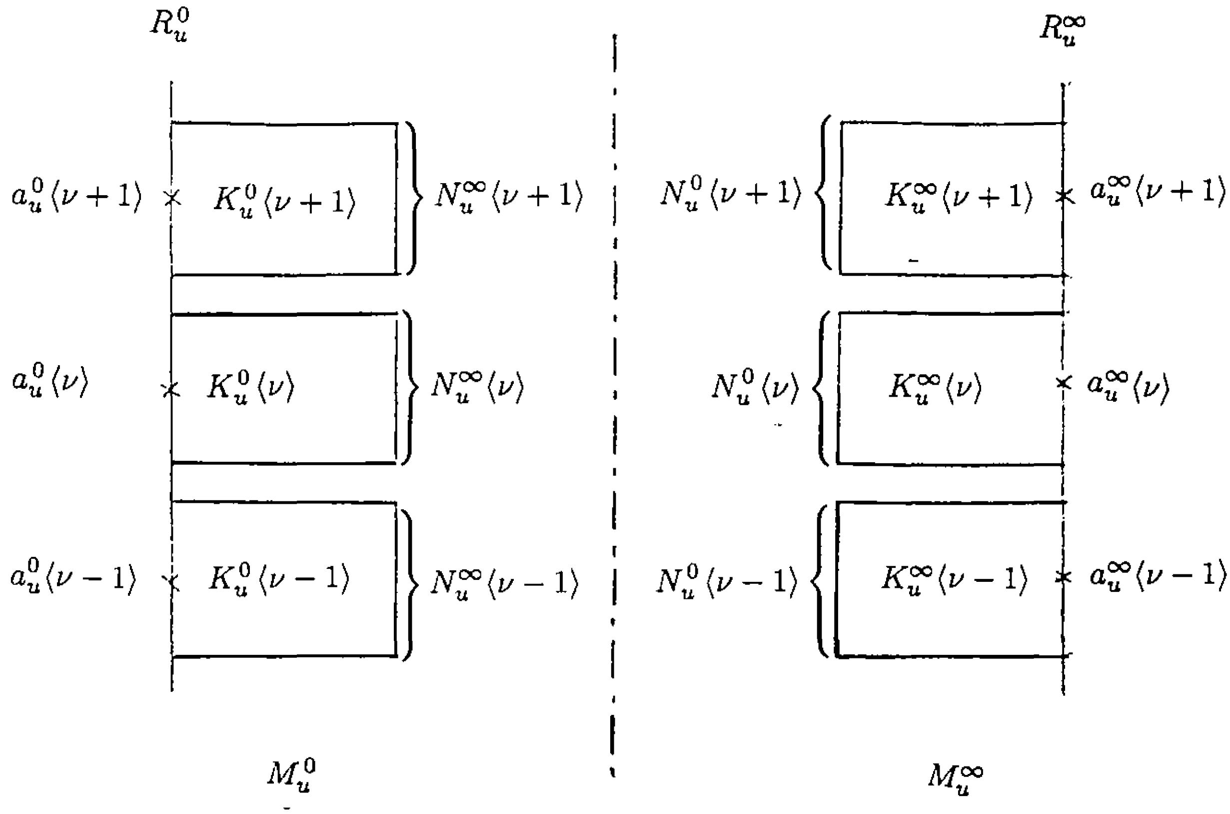

Figure 3

Proof of Theorem 6.1. We will give a proof only to the assertions concerned with $H_n(F_u, \partial_\infty F_u)$ and Ψ_u^∞. The assertions concerned with $H_n(F_u, \partial_0 F_u)$ and Ψ_u^0 can be proved completely in the same way. All we have to do is just to replace every ∞ appearing in the argument with 0, and to notice that the monodromy action on $H_{n-1}(X_u^0)$ associated to the loop δ_u^0 is the identity. (See Remark 6.2.)

Since $H_n(F_u)$ and $H_n(F_u, \partial_\infty F_u)$ form locally constant systems over $\mathcal{U}$, and $H_n(F_u) \to H_n(F_u, \partial_\infty F_u)$ is natural, it is enough to prove the injectivity of $H_n(F_u) \to H_n(F_u, \partial_\infty F_u)$ for one u. Thus we assume $u \in \mathcal{U}_N^\sim$ throughout the proof.

By Proposition 2.1, the holomorphic map $\psi_u : F_u \to \mathbf{C}$ is locally trivial over $\mathbf{C} \setminus Cr(u)$. On the other hand, the inclusion $(M_u^\infty, R_u^\infty) \hookrightarrow (\mathbf{C}, e^{-1}(B_u^\infty))$ induce a homotopy equivalence. Since $Cr(u)$ is contained in the interior of M_u^∞, the inclusion

$$(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) \quad \hookrightarrow \quad (F_u, \partial_\infty F_u)$$

also induces a homotopy equivalence. Hence there exists a strong deformation retraction

$$(F_u, \partial_\infty F_u) \quad \longrightarrow \quad (\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)); \tag{6.3}$$

which is the homotopy inverse of the inclusion. Note that the deck transformation T_u on $(F_u, \partial_\infty F_u)$ induces an automorphism of the pair of subspaces $(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty))$.

Thus both of $H_n(\psi_u^{-1}(M_u^\infty))$ and $H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty))$ can be considered as $\mathbb{Z}[q, q^{-1}]$-modules, and we obtain a commutative diagram of $\mathbb{Z}[q, q^{-1}]$-modules;

$$\begin{array}{ccc} H_n(\psi_u^{-1}(M_u^\infty)) & \longrightarrow & H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) \\ \downarrow \wr & & \downarrow \wr \\ H_n(F_u) & \longrightarrow & H_n(F_u, \partial_\infty F_u). \end{array} \tag{6.4}$$

By the excision property of homology groups, we have

$$\begin{aligned} H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) &\cong H_n(\psi_u^{-1}(\coprod_{\nu \in \mathbf{Z}} K_u\langle\nu\rangle), \psi_u^{-1}(\coprod_{\nu \in \mathbf{Z}} N_u^\infty\langle\nu\rangle)) \\ &\cong \bigoplus_{\nu \in \mathbf{Z}} H_n(\psi_u^{-1}(K_u\langle\nu\rangle), \psi_u^{-1}(N_u^\infty\langle\nu\rangle)). \end{aligned} \tag{6.5}$$

On the other hand, the deck transformation T_u on $(F_u, \partial_\infty F_u)$ induces isomorphisms

$$(\psi_u^{-1}(K_u\langle\nu\rangle), \psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \xrightarrow{\sim} (\psi_u^{-1}(K_u\langle\nu+1\rangle), \psi_u^{-1}(N_u^\infty\langle\nu+1\rangle))$$

for all $\nu \in \mathbb{Z}$, and these isomorphisms are compatible with the isomorphisms

$$(\psi_u^{-1}(K_u\langle\nu\rangle), \psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \cong (\phi_u^{-1}(K_u), \phi_u^{-1}(N_u^\infty)) \tag{6.6}$$

given by the restriction of the covering map $\tilde{e} : F_u \to E_u$. Hence the multiplication by q in the decomposition (6.5) into the direct sum is given by the shift of the numbering $\langle\nu\rangle$;

$$H_n(\psi_u^{-1}(K_u\langle\nu\rangle), \psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \xrightarrow{\sim} H_n(\psi_u^{-1}(K_u\langle\nu+1\rangle), \psi_u^{-1}(N_u^\infty\langle\nu+1\rangle)),$$

which commutes with the isomorphisms

$$H_n(\psi_u^{-1}(K_u\langle\mu\rangle), \psi_u^{-1}(N_u^\infty\langle\mu\rangle)) \cong H_n(\phi_u^{-1}(K_u), \phi_u^{-1}(N_u^\infty))$$

for $\mu = \nu$ and $\mu = \nu + 1$ induced by (6.6). Therefore, we get an isomorphism

$$H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) \cong H_n(\phi_u(K_u), \phi_u(N_u^\infty)) \otimes \mathbb{Z}[q, q^{-1}] \tag{6.7}$$

of $\mathbb{Z}[q, q^{-1}]$-modules.

On the other hand, since $K_u \subset \mathbb{C}$ is a strong deformation retract of $\mathbb{C}$, which contains all of the critical values $\mathrm{Cr}(u)$ of $\hat{\phi}_u : \mathbb{A}^n \to \mathbb{C}$ in its interior, the pull-back $\phi_u^{-1}(K_u) = \hat{\phi}_u^{-1}(K_u)$ is also a strong deformation retract of $\mathbb{A}^n$ by Proposition 2.1. Combining this with the isomorphisms (6.6), we see that

$$\phi_u^{-1}(K_u) \text{ and } \psi_u^{-1}(K_u\langle\nu\rangle) \text{ are all contractible spaces.} \tag{6.8}$$

This implies that we get isomorphisms

$$\begin{aligned} H_n(\phi_u^{-1}(K_u), \phi_u^{-1}(N_u^\infty)) &\xrightarrow{\sim} H_{n-1}(\phi_u^{-1}(N_u^\infty)) \quad \text{and} \\ H_n(\psi_u^{-1}(K_u\langle\nu\rangle), \psi_u^{-1}(N_u^\infty\langle\nu\rangle)) &\xrightarrow{\sim} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \end{aligned} \tag{6.9}$$

induced by the boundary homomorphisms. Combining these with (6.5) and (6.7), we obtain the isomorphisms

$$\begin{aligned} H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) &\cong \bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \\ &\cong H_{n-1}(\phi_u^{-1}(N_u^\infty)) \otimes \mathbb{Z}[q,q^{-1}] \end{aligned} \tag{6.10}$$

of $\mathbb{Z}[q,q^{-1}]$-modules. Lastly, since ϕ_u and ψ_u are locally trivial over the arc $N_u^\infty \subset \mathbb{C}^\times$ and the line segment $N_u^\infty\langle\nu\rangle \subset \mathbf{C}$, respectively, the inclusions

$$X_u^\infty \hookrightarrow \phi_u^{-1}(N_u^\infty) \quad \text{and} \quad X_u^\infty\langle\nu\rangle \hookrightarrow \psi_u^{-1}(N_u^\infty\langle\nu\rangle) \tag{6.11}$$

induce homotopy equivalences. Therefore (6.10) can be written as

$$\begin{aligned} H_n(\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty)) &\cong \bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(X_u^\infty\langle\nu\rangle) \\ &\cong H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q,q^{-1}]. \end{aligned} \tag{6.12}$$

Combining this with (6.4), we get the hoped-for isomorphism Ψ_u^∞ of $\mathbb{Z}[q,q^{-1}]$-modules. Note that the homeomorphism types of all spaces and continuous maps which have appeared in the course of the construction of Ψ_u^∞ *do not* change when u varies continuously in a path-connected component of $\mathcal{U}_N^\sim$. Hence the isomorphisms Ψ_u^∞ with $u \in \mathcal{U}_N^\sim$ yield an isomorphism between the corresponding locally constant systems over $\mathcal{U}_N^\sim$.

Now we shall calculate $H_n(F_u) \cong H_n(\psi_u^{-1}(M_u^\infty))$ by applying Mayer-Vietoris sequence to the decomposition

$$\psi_u^{-1}(M_u^\infty) = \psi_u^{-1}(\coprod_{\nu\in\mathbf{Z}} K_u\langle\nu\rangle) \ \cup\ \psi_u^{-1}(R_u^\infty).$$

Note that

$$\psi_u^{-1}(\coprod_{\nu\in\mathbf{Z}} K_u\langle\nu\rangle) \cap \psi_u^{-1}(R_u^\infty) \ = \ \coprod_{\nu\in\mathbf{Z}} \psi_u^{-1}(N_u^\infty\langle\nu\rangle).$$

Since $\psi_u^{-1}(K_u\langle\nu\rangle)$ is contractible for each $\nu \in \mathbb{Z}$ by (6.8), the Mayer-Vietoris sequence is of the form

$$\begin{aligned} &\longrightarrow \bigoplus_{\nu\in\mathbf{Z}} H_n(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \longrightarrow H_n(\psi_u^{-1}(R_u^\infty)) \longrightarrow H_n(\psi_u^{-1}(M_u^\infty)) \\ &\xrightarrow[\partial]{} \bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \xrightarrow[\iota]{} H_{n-1}(\psi_u^{-1}(R_u^\infty)) \longrightarrow \quad \cdots . \end{aligned} \tag{6.13}$$

Since $\psi_u : F_u \to \mathbf{C}$ is locally trivial over $R_u^\infty \subset \mathbf{C}$ and R_u^∞ is contractible, the inclusion of $X_u^\infty\langle\nu\rangle$ into $\psi_u^{-1}(R_u^\infty)$ induces a homotopy equivalence for each ν. By Theorem L2 and the isomorphism (6.1), we have

$$H_n(\psi_u^{-1}(R_u^\infty)) = 0.$$

Therefore the boundary map ∂ in (6.13) is injective. Recall the construction of the isomorphism (6.10). It follows from (6.5) through the boundary map (6.9). Then it can be easily checked that the following diagram is commutative;

$$\begin{array}{ccc} H_n(\psi_u^{-1}(M_u^\infty)) & \overset{\partial \ \text{in (6.13)}}{\longrightarrow} & \bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \\ {\scriptstyle(6.4)}\big\downarrow\wr & & {\scriptstyle(6.10)}\big\uparrow\wr \\ H_n(F_u) & \overset{\text{the natural map}}{\longrightarrow} & H_n(F_u, \partial_\infty F_u). \end{array}$$

Hence the natural homomorphism $H_n(F_u) \to H_n(F_u, \partial_\infty F_u)$ is also injective. Moreover the image of this injection is identified, via (6.10), with the kernel of the homomorphism

$$\iota \ : \ \bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \ \longrightarrow \ H_{n-1}(\psi_u^{-1}(R_u^\infty))$$

in (6.13) induced by the inclusions. Recall that the inclusion $X_u^\infty\langle 0\rangle \hookrightarrow \psi_u^{-1}(R_u^\infty)$ induces a homotopy equivalence. Let

$$\psi_u^{-1}(R_u^\infty) \ \longrightarrow \ X_u^\infty\langle 0\rangle \tag{6.14}$$

is a continuous map which represents the homotopy inverse of the inclusion. Consider the composition

$$X_u^\infty \ \underset{(6.1)}{\overset{\sim}{\longrightarrow}} \ X_u^\infty\langle\nu\rangle \ \hookrightarrow \ \psi_u^{-1}(R_u^\infty) \ \underset{(6.14)}{\longrightarrow} \ X_u^\infty\langle 0\rangle \ \underset{(6.1)}{\overset{\sim}{\longrightarrow}} \ X_u^\infty,$$

of continuous maps, each of which induces a homotopy equivalence. The induced automorphism $H_{n-1}(X_u^\infty) \to H_{n-1}(X_u^\infty)$ is nothing but the monodromy operator $j^{-\nu}$, because the path on $\mathbf{C}$ from $a_u^\infty\langle\nu\rangle$ to $a_u^\infty\langle 0\rangle$ along R_u^∞ is mapped to the loop $(\delta_u^\infty)^{-\nu}$ on $\mathbb{C}^\times$ by e. Therefore, through the isomorphism

$$\bigoplus_{\nu\in\mathbf{Z}} H_{n-1}(\psi_u^{-1}(N_u^\infty\langle\nu\rangle)) \ \cong \ H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q, q^{-1}] \tag{6.15}$$

by the homotopy equivalence induced from (6.11) and the isomorphism (6.1), and the isomorphism

$$H_{n-1}(\psi_u^{-1}(R_u^\infty)) \ \cong \ H_{n-1}(X_u^\infty)$$

by the homotopy equivalence induced from (6.14) and the isomorphism (6.1), we can identify ι in (6.13) with $\tilde{\iota} : H_{n-1}(X_u^\infty) \otimes \mathbb{Z}[q, q^{-1}] \to H_{n-1}(X_u^\infty)$ given by

$$\tilde{\iota}\ \Big(\sum_{\nu\in\mathbf{Z}} (x_\nu \otimes q^\nu) \ \Big) = \sum_{\nu\in\mathbf{Z}} j^{-\nu}(x_\nu), \quad \text{where} \quad x_\nu \in H_{n-1}(X_u^\infty).$$

Then it can be easily checked that the kernel of this $\tilde{\iota}$ coincides with the image of the endomorphism $\mathrm{Id} - j \otimes q$. Since Ψ_u^∞ is given by (6.15) combined with (6.10) and (6.4), we complete the proof. □

By looking back at the constructions, we can describe the isomorphisms Ψ_u^0 and Ψ_u^∞ in a geometric way.

Corollary 6.2. *Let Λ be an $(n-1)$-cycle in X_u^0 (resp. X_u^∞), and let Γ be an n-chain in $\phi_u^{-1}(K_u)$ such that $\partial\Gamma = \Lambda$. Let $\Gamma\langle\nu\rangle$ be the n-chain in $\psi_u^{-1}(K_u\langle\nu\rangle)$ corresponding to Γ via the isomorphism $\psi_u^{-1}(K_u\langle\nu\rangle) \cong \phi_u^{-1}(K_u)$. Then*

$$[\Gamma\langle\nu\rangle] = \Psi_u^0([\Lambda]\otimes q^\nu) \qquad (\textit{resp.} \qquad [\Gamma\langle\nu\rangle] = \Psi_u^\infty([\Lambda]\otimes q^\nu)\ )$$

holds in $H_n(F_u, \partial_0 F_u)$ (resp. in $H_n(F_u, \partial_\infty F_u)$). □

Remark 6.3. Since $\phi_u^{-1}(K_u)$ is contractible, there always exists an n-chain $\Gamma \subset \phi_u^{-1}(K_u)$ such that $\partial\Gamma = \Lambda$ for any $(n-1)$-cycle $\Lambda \subset X_u^0$ (resp. $\Lambda \subset X_u^\infty$).

Corollary 6.3. *Suppose, the other way around, that we are given an n-cycle Γ in $(F_u, \partial_0 F_u)$ (resp. in $(F_u, \partial_\infty F_u)$). Let Γ' be the image of Γ by the retraction*

$$(F_u, \partial_0 F_u) \to (\psi_u^{-1}(M_u^0), \psi_u^{-1}(R_u^0)) \qquad (\text{resp.} \quad (F_u, \partial_\infty F_u) \to (\psi_u^{-1}(M_u^\infty), \psi_u^{-1}(R_u^\infty))),$$

which is the homotopy inverse of the inclusion. We put $\Gamma'_\nu := \Gamma' \cap \psi_u^{-1}(K_u\langle\nu\rangle)$. Then, since $\partial\Gamma' \subset \psi_u^{-1}(R_u^0)$ (resp. $\partial\Gamma' \subset \psi_u^{-1}(R_u^\infty)$), we have $\partial\Gamma'_\nu \subset \psi_u^{-1}(N_u^0\langle\nu\rangle)$ (resp. $\partial\Gamma'_\nu \subset \psi_u^{-1}(N_u^\infty\langle\nu\rangle)$). Let $\Lambda_\nu \subset X_u^0$ (resp. $\Lambda_\nu \subset X_u^\infty$) be the image of $\partial\Gamma'_\nu$ by the continuous map

$$\psi_u^{-1}(N_u^0\langle\nu\rangle) \cong \phi_u^{-1}(N_u^0) \underset{rt}{\longrightarrow} X_u^0, \qquad (\text{resp.} \quad \psi_u^{-1}(N_u^\infty\langle\nu\rangle) \cong \phi_u^{-1}(N_u^\infty) \underset{rt}{\longrightarrow} X_u^\infty,)$$

where rt is the homotopy inverse of the inclusion. Then

$$[\Gamma] = \Psi_u^0\Big(\sum_{\nu\in\mathbf{Z}} ([\Lambda_\nu]\otimes q^\nu)\Big) \qquad (\text{resp.} \quad [\Gamma] = \Psi_u^\infty\Big(\sum_{\nu\in\mathbf{Z}} ([\Lambda_\nu]\otimes q^\nu)\Big)\ )$$

holds in $H_n(F_u, \partial_0 F_u)$ (resp. in $H_n(F_u, \partial_\infty F_u)$). □

From now on, we consider $H_n(F_u)$ as $\mathbb{Z}[q, q^{-1}]$-submodules of $H_n(F_u, \partial_0 F_u)$ and of $H_n(F_u, \partial_\infty F_u)$. For $u \in \mathcal{U}_N^\sim$, we define

$$\tilde{q} := \Psi_u^\infty \circ (j \otimes q) \circ (\Psi_u^\infty)^{-1} \quad : \quad H_n(F_u, \partial_\infty F_u) \longrightarrow H_n(F_u, \partial_\infty F_u).$$

Then we have

$$H_n(F_u) = (1-q)H_n(F_u, \partial_0 F_u), \quad \text{and} \quad H_n(F_u) = (1-\tilde{q})H_n(F_u, \partial_\infty F_u). \tag{6.16}$$

The following Lemma 6.1 will be used in §10.

Lemma 6.1. *Suppose $u \in \mathcal{U}_N^\sim$. Suppose that an element $\lambda \in H_{n-1}(X_u^0)$ is given. Then there exist elements $\lambda_1, \lambda_2 \in H_{n-1}(X_u^\infty)$ such that*

$$(1-q)\Psi_u^0(\lambda\otimes 1) = \Psi_u^\infty(\lambda_1\otimes 1) + \Psi_u^\infty(\lambda_2\otimes q)$$

holds in $H_n(F_u)$.

Proof. First we shall describe an n-cycle in F_u which represents the homology class

$$(1-q)\Psi_u^0(\lambda \otimes q^\nu) \in H_n(F_u).$$

Let $\Lambda \subset X_u^0$ be an $(n-1)$-cycle which represents λ, and let $\Lambda\langle\nu\rangle \subset X_u^0\langle\nu\rangle$ be the lifting of Λ by (6.1). By Remark 6.3, we have an n-chain Γ in $\phi_u^{-1}(K_u)$ such that its lifting $\Gamma\langle\nu\rangle \subset \psi_u^{-1}(K_u\langle\nu\rangle)$ satisfies $\partial\Gamma\langle\nu\rangle = \Lambda\langle\nu\rangle$ for all $\nu \in \mathbb{Z}$. Recall that there exists a diffeomorphism

$$\phi_u^{-1}(C_u^0) \cong C_u^0 \times X_u^0 \tag{6.17}$$

over the circle C_u^0 which induces the identity on X_u^0. (See (5.6).) Such a diffeomorphism is unique up to homotopy. By taking the covering of (6.17), we get a diffeomorphism

$$\psi_u^{-1}(R_u^0) \cong R_u^0 \times X_u^0, \tag{6.18}$$

which induces the isomorphism (6.1) over each point $a_u^0\langle\nu\rangle \in R_u^0$. Let

$$J\langle\nu\rangle \;:\; I \times \Lambda \longrightarrow \psi_u^{-1}(R_u^0)$$

be the composition of the diffeomorphism (6.18) with

$$\delta_u^0\langle\nu+1\rangle \times \text{ inclusion } \;:\; I \times \Lambda \longrightarrow R_u^0 \times X_u^0$$

where $\delta_u^0\langle\nu+1\rangle : I \to R_u^0$ is the lifting of the path δ_u^0 such that $\delta_u^0\langle\nu+1\rangle(1) = a_u^0\langle\nu+1\rangle$. Then we have

$$\partial J\langle\nu\rangle = \Lambda\langle\nu+1\rangle - \Lambda\langle\nu\rangle = \partial(\Gamma\langle\nu+1\rangle - \Gamma\langle\nu\rangle).$$

Hence

$$T_\nu := J\langle\nu\rangle - \Gamma\langle\nu+1\rangle + \Gamma\langle\nu\rangle$$

is an n-cycle in F_u. Since $J\langle\nu\rangle$ is contained in $\partial_0 F_u$, we see from Corollary 6.2that

$$[T_\nu] = -[\Gamma\langle\nu+1\rangle] + [\Gamma\langle\nu\rangle] = -\Psi_u^0(\lambda \otimes q^{\nu+1}) + \Psi_u^0(\lambda \otimes q^\nu) \quad \text{in} \quad H_n(F_u, \partial_0 F_u),$$

and hence

$$[T_\nu] = (1-q)\Psi_u^0(\lambda \otimes q^\nu) \quad \text{in} \quad H_n(F_u).$$

Note that the n-cycle T_0 in F_u is contained in the subspace $\psi_u^{-1}(K_{01}^0)$ of $\psi_u^{-1}(M_u^0)$, where

$$K_{01}^0 := K_u\langle 0\rangle \cup \delta_u^0\langle 1\rangle(I) \cup K_u\langle 1\rangle.$$

Consider the composition

$$\psi_u^{-1}(M_u^0) \hookrightarrow F_u \to \psi_u^{-1}(M_u^\infty) \tag{6.19}$$

of the inclusion and the retraction (6.3), both of which induce homotopy equivalence. The maps in (6.19) are liftings of the continuous maps on the base space

$$M_u^0 \hookrightarrow \mathbf{C} \to M_u^\infty \tag{6.20}$$

which are the inclusion and a retraction. By choosing an appropriate retraction, we can assume that $K^0_{01} \subset M^0_u$ is mapped to

$$K^{\infty}_{01} \quad := \quad K_u\langle 0\rangle \cup \delta^{\infty}_u\langle 1\rangle(I) \cup K_u\langle 1\rangle \quad \subset \quad M^{\infty}_u$$

by (6.20), where $\delta^{\infty}_u\langle 1\rangle(I)$ is the segment of R^{∞}_u between $a^{\infty}_u\langle 0\rangle$ and $a^{\infty}_u\langle 1\rangle$. Hence the n-cycle $T_0 \subset \psi_u^{-1}(K^0_{01})$ is mapped by (6.19) to an n-cycle T'_0 contained in $\psi_u^{-1}(K^{\infty}_{01})$. In particular, we have

$$T'_0 \cap \psi_u^{-1}(K_u\langle\nu\rangle) = \emptyset \quad \text{if} \quad \nu \neq 0, 1.$$

Hence Corollary 6.3 implies that the homology class $[T_0] = [T'_0] \in H_n(F_u)$ is written in the form $\Psi^{\infty}_u(\lambda_1 \otimes 1 + \lambda_2 \otimes q)$ by some $\lambda_1, \lambda_2 \in H_{n-1}(X^{\infty}_u)$. □

§7. Description of the basis of $H_n(F_u)$, $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$

In this section, we will describe explicitly n-cycles representing the basis of the free $\mathbb{Z}[q, q^{-1}]$-modules $H_n(F_u)$, $H_n(F_u, \partial_0 F_u)$ and $H_n(F_u, \partial_\infty F_u)$. Throughout this section, we assume $u \in \mathcal{U}^{\sim}_N$.

First we define the notion of a *K-regular system of paths*. Recall that we have defined the closed subset K_u of $\mathbb{C}^\times$ for $u \in \mathcal{U}^{\sim}_N$.

Definition 7.1. Suppose that a point $a \in K_u \setminus \mathrm{Cr}(u)$ is given. A regular system $\{\xi_1, \dots, \xi_N\}$ of paths from a (see Definition 5.2) is said to be *K-regular* if and only if $\xi_i(I)$ is contained in K_u for $i = 1, \dots, N$.

It is obvious that a K-regular system of paths from a always exists for every $u \in \mathcal{U}^{\sim}_N$ and every $a \in K_u \setminus \mathrm{Cr}(u)$.

Next, we fix some notation concerned with the lifting of objects on $\mathbb{C}^\times$ and E_u by the étale coverings $e : \mathbf{C} \to \mathbb{C}^\times$ and $\tilde{e} : F_u \to E_u$.

Recall that $(\mathbb{C}\setminus\mathbb{R}_{\le 0})\langle\nu\rangle$ is the unique connected component of $e^{-1}(\mathbb{C}\setminus\mathbb{R}_{\le 0})$ containing $K_u\langle\nu\rangle$. Recall also that, for a point $c \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$, $c\langle\nu\rangle$ denotes the intersection point of $e^{-1}(c)$ and $(\mathbb{C} \setminus \mathbb{R}_{\le 0})\langle\nu\rangle$.

Definition 7.2. Suppose that a point $a \in \mathbf{C} \setminus \mathcal{C}r(u)$ is given. For $p_i\langle\nu\rangle \in \mathcal{C}r(u)$, let $\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)$ denote the space of all paths $\omega : I \to \mathbf{C}$ which satisfy the following; (i) $\omega(0) = a$, $\omega(1) = p_i\langle\nu\rangle$, and (ii) $\omega([0,1)) \cap \mathcal{C}r(u) = \emptyset$. We equip this space with the compact-open topology, and denote by $[\,\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)\,]$ the set of path-connected components of $\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)$. For a path $\omega \in \mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)$, let $[\omega] \in [\,\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)\,]$ denote the path-connected component of $\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)$ containing ω; or equivalently, the homotopy class of paths in $\mathcal{P}^{\sim}_u(a, p_i\langle\nu\rangle)$ represented by ω.

Definition 7.3. Suppose that a path $\omega : I \to \mathbb{C}^\times$ with $\omega(1) \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$ is given. Then $\omega\langle\nu\rangle : I \to \mathbf{C}$ is the unique lifting of ω to $\mathbf{C}$ by $e : \mathbf{C} \to \mathbb{C}^\times$ such that

$$\omega\langle\nu\rangle(1) \;=\; \omega(1)\langle\nu\rangle \quad\in\quad (\mathbb{C} \setminus \mathbb{R}_{\le 0})\langle\nu\rangle \;\subset\; e^{-1}(\mathbb{C} \setminus \mathbb{R}_{\le 0}).$$

Note that, if $\omega \in \mathcal{P}_u(a, p_i)$ with $a \notin \mathbb{R}_{\leq 0}$ is given, then we have

$$\omega\langle\nu\rangle \in \mathcal{P}_u^{\sim}(a\langle\nu+\mu\rangle, p_i\langle\nu\rangle)$$

for all $\nu \in \mathbb{Z}$ with a fixed integer $\mu \in \mathbb{Z}$. Moreover, if $[\omega_1] = [\omega_2]$ in $[\mathcal{P}_u(a, p_i)]$, then $[\omega_1\langle\nu\rangle] = [\omega_2\langle\nu\rangle]$ in $[\mathcal{P}_u^{\sim}(a\langle\nu+\mu\rangle, p_i\langle\nu\rangle)]$.

For the path ω as above, we have a vanishing cycle

$$\sigma[\omega] \in [S^{n-1}, \phi_u^{-1}(a)]$$

for $[\omega]$, unique up to sign, and the thimble

$$\theta([\omega], \sigma[\omega]) \in [(CS^{n-1}, S^{n-1}), (E_u, \phi_u^{-1}(a))]$$

for $[\omega]$ starting from $\sigma[\omega]$.

Definition 7.4. For each $\nu \in \mathbb{Z}$, the vanishing cycle $\sigma[\omega]$ lifts uniquely to a vanishing cycle

$$\sigma[\omega]\langle\nu\rangle \in [S^{n-1}, \psi_u^{-1}(a\langle\nu+\mu\rangle)],$$

which is one of the two vanishing cycles for $[\omega\langle\nu\rangle] \in [\mathcal{P}_u^{\sim}(a\langle\nu+\mu\rangle, p_i\langle\nu\rangle)]$. Also the thimble $\theta([\omega], \sigma[\omega])$ lifts uniquely to the thimble

$$\theta([\omega], \sigma[\omega])\langle\nu\rangle := \theta([\omega\langle\nu\rangle], \sigma[\omega]\langle\nu\rangle) \in [(CS^{n-1}, S^{n-1}), (F_u, \psi_u^{-1}(a\langle\nu+\mu\rangle))]$$

for $[\omega\langle\nu\rangle]$ starting from $\sigma[\omega]\langle\nu\rangle$. Its homology class is denoted by

$$\bar{\theta}([\omega], \sigma[\omega])\langle\nu\rangle \in H_n(F_u, \psi_u^{-1}(a\langle\nu+\mu\rangle)).$$

As before, when the orientation is irrelevant, we write simply $\theta([\omega])\langle\nu\rangle$ and $\bar{\theta}([\omega])\langle\nu\rangle$.

When $a = a_u^0$ (resp. $a = a_u^\infty$), then these homology classes can be considered as elements in $H_n(F_u, \partial_0 F_u)$ (resp. in $H_n(F_u, \partial_\infty F_u)$), which will be denoted by the same symbol $\bar{\theta}([\omega], \sigma[\omega])\langle\nu\rangle$. By definition, we have

$$q\bar{\theta}([\omega], \sigma[\omega])\langle\nu\rangle = \bar{\theta}([\omega], \sigma[\omega])\langle\nu+1\rangle \tag{7.1}$$

in the $\mathbb{Z}[q, q^{-1}]$-module $H_n(F_u, \partial_0 F_u)$ (resp. $H_n(F_u, \partial_\infty F_u)$). Moreover, suppose that $\omega(I) \subset K_u$. Then we have $\omega\langle\nu\rangle(I) \subset K_u\langle\nu\rangle$, and therefore, the starting point of $\omega\langle\nu\rangle$ is $\omega(0)\langle\nu\rangle$. Hence the thimble $\theta([\omega])\langle\nu\rangle$ is an element of $[(CS^{n-1}, S^{n-1}), (F_u, X_u^0\langle\nu\rangle)]$ (resp. of $[(CS^{n-1}, S^{n-1}), (F_u, X_u^\infty\langle\nu\rangle)]$). We apply these considerations to the case when ω is a member of a K-regular system of paths from a_u^0 (resp. from a_u^∞).

Proposition 7.1. *Suppose that $u \in \mathcal{U}_N^{\sim}$.*

(0) *Let $\{\xi_1^0, \ldots, \xi_N^0\}$ be a K-regular system of paths from a_u^0, and let $\sigma[\xi_i^0] \in [S^{n-1}, X_u^0]$ be a vanishing cycle for $[\xi_i^0]$. Then the homology classes*

$$\bar{\theta}([\xi_1^0], \sigma[\xi_1^0])\langle 0\rangle, \quad \cdots\cdots\cdots \quad , \bar{\theta}([\xi_N^0], \sigma[\xi_N^0])\langle 0\rangle$$

of the lifted thimbles form a set of basis for the free $\mathbb{Z}[q,q^{-1}]$*-module* $H_n(F_u, \partial_0 F_u)$.

(∞) *Let* $\{\xi_1^\infty, \dots, \xi_N^\infty\}$ *be a* K*-regular system of paths from* a_u^∞*, and let* $\sigma[\xi_i^\infty] \in [S^{n-1}, X_u^\infty]$ *be a vanishing cycle for* $[\xi_i^\infty]$*. Then the homology classes*

$$\bar{\theta}([\xi_1^\infty], \sigma[\xi_1^\infty])\langle 0\rangle, \quad \dots\dots\dots \quad , \bar{\theta}([\xi_N^\infty], \sigma[\xi_N^\infty])\langle 0\rangle$$

of the lifted thimbles form a set of basis for the free $\mathbb{Z}[q,q^{-1}]$*-module* $H_n(F_u, \partial_\infty F_u)$.

Proof. Let $\Gamma_i^0 : CS^{n-1} \to E_u$ be a continuous map representing the thimble $\theta([\xi_i^0], \sigma[\xi_i^0])$ over the path ξ_i^0. Since $\xi_i^0(I) \subset K_u$ by the assumption of the K-regularity, Γ_i^0 can be considered as an n-chain in $\phi_u^{-1}(K_u)$. Let $\Gamma_i^0\langle\nu\rangle \subset \psi_u^{-1}(K_u\langle\nu\rangle)$ be the lifting of Γ_i^0 by the isomorphism $\psi_u^{-1}(K_u\langle\nu\rangle) \cong \phi_u^{-1}(K_u)$. Then we have

$$[\Gamma_i^0\langle\nu\rangle] \;=\; \bar{\theta}([\xi_i^0], \sigma[\xi_i^0])\langle\nu\rangle$$

in $H_n(F_u, \partial_0 F_u)$ by the definition of lifting. On the other hand, the $(n-1)$-cycle $\partial\Gamma_i^0$ in X_u^0 represents $-\bar{\sigma}[\xi_i^0] \in H_{n-1}(X_u^0)$ by the anti-commutativity of (4.1). Hence, by Corollary 6.2, we get

$$\bar{\theta}([\xi_i^0], \sigma[\xi_i^0])\langle\nu\rangle \;=\; -\,\Psi_u^0(\bar{\sigma}[\xi_i^0] \otimes q^\nu). \tag{7.2}$$

By the same argument, we have

$$\bar{\theta}([\xi_i^\infty], \sigma[\xi_i^\infty])\langle\nu\rangle \;=\; -\,\Psi_u^\infty(\bar{\sigma}[\xi_i^\infty] \otimes q^\nu). \tag{7.2$'$}$$

Since $\{\xi_1^0, \dots, \xi_N^0\}$ is regular, Proposition 5.1 implies that $\bar{\sigma}[\xi_1^0]$, $\dots$, $\bar{\sigma}[\xi_N^0]$ form a set of basis of the free $\mathbb{Z}$-module $H_{n-1}(X_u^0)$. Hence the assertion (0) follows from (7.2) and Theorem 6.1 . The assertion (∞) can be proved by the same argument. □

Theorem 7.1. *Let* $b \in \mathcal{U}$ *be a base point which is contained in* $\mathcal{U}_N^\sim$*. The homomorphism*

$$H_n(F_b, \partial_0 F_b) \;\longrightarrow\; H_{n-1}(X_b^0) \tag{7.3}$$

which is the composition of the inverse of Ψ_b^0 *and the homomorphism* $H_{n-1}(X_b^0) \otimes \mathbb{Z}[q,q^{-1}] \to H_{n-1}(X_b^0)$ *given by* $q \mapsto 1$ *is* $\pi_1(\mathcal{U}, b)$*-equivariant.*

Proof. We will prove this theorem by showing that (7.3) is equal with the composition map

$$H_n(F_b, \partial_0 F_b) \;\underset{\tilde{e}_*}{\longrightarrow}\; H_n(E_b, \partial_0 E_b) \;\underset{(A)}{\xrightarrow{\sim}}\; H_{n-1}(X_b^0), \tag{7.4}$$

where $\tilde{e}_*$ is the homomorphism induced from the covering map $\tilde{e} : F_b \to E_b$ and (A) is obtained from the isomorphisms in Proposition 5.2 (4) and (5). It is obvious that $\tilde{e}_*$ is $\pi_1(\mathcal{U}, b)$-equivariant. By Proposition 5.2 (6), (A) is also $\pi_1(\mathcal{U}, b)$-equivariant.

We fix a K-regular system $\{\xi_1^0, \dots, \xi_N^0\}$ of paths from a_b^0 and, for each $i = 1, \dots, N$, we choose a vanishing cycle $\sigma[\xi_i^0] \in [S^{n-1}, X_b^0]$. We put

$$\bar{\theta}_i^0\langle\nu\rangle \;:=\; \bar{\theta}([\xi_i^0], \sigma[\xi_i^0])\langle\nu\rangle \;\in\; H_n(F_b, \partial_0 F_b).$$

We have $\bar{\theta}_i^0\langle\nu\rangle = q^\nu\bar{\theta}_i^0\langle 0\rangle$ for all $\nu \in \mathbb{Z}$. By Proposition 7.1, the set

$$\{\ \bar{\theta}_i^0\langle\nu\rangle\ ;\ \nu \in \mathbb{Z} \text{ and } i = 1, \dots, N\}$$

form a set of basis for the free $\mathbb{Z}$-modules of $H_n(F_b, \partial_0 F_b)$. Therefore, it is enough to show that the two homomorphisms (7.3) and (7.4) map each $\bar{\theta}_i^0\langle\nu\rangle$ to a same element in $H_{n-1}(X_b^0)$.

By (7.2), the homomorphism (7.3) maps $\bar{\theta}_i^0\langle\nu\rangle$ to $-\bar{\sigma}[\xi_i^0]$. On the other hand, $\tilde{e}_*$ maps to $\bar{\theta}_i^0\langle\nu\rangle$ to $\bar{\theta}([\xi_i^0], \sigma[\xi_i^0]) \in H_n(E_b, \partial_0 E_b)$ because of the definition of the lifting. By the boundary map $H_n(F_b, \partial_0 F_b) \xrightarrow{\sim} H_{n-1}(\partial_0 E_u)$, this element is mapped to

$$\partial\bar{\theta}([\xi_i^0], \sigma[\xi_i^0]) \;=\; -\bar{\sigma}[\xi_i^0]$$

because of the anti-commutativity of (4.1). and this homology class can be regarded as already contained in $H_{n-1}(X_b^0)$. Hence (7.4) also maps $\theta_i^0\langle\nu\rangle$ to $-\bar{\sigma}[\xi_i^0] \in H_{n-1}(X_b^0)$. $\square$

Remark 7.1. The isomorphism (0.3) in Introduction is obtained as follows;

$$H_n(F_b) \underset{(B)}{\xleftarrow{\sim}} H_n(F_b, \partial_0 F_b) \underset{(7.3)=(7.4)}{\xrightarrow{\sim}} H_{n-1}(X_b^0) \underset{(C)}{\xrightarrow{\sim}} H_{n-1}(X_b),$$

where (B) is the multiplication by $(1-q)$ (see (6.16)), and (C) is induced from (5.1). Through Theorem 7.1 and Lemma 5.1, we see that (0.3) has the required property.

§8. Intersection form on $H_n(F_u, \partial_0 F_u) \times H_n(F_u, \partial_\infty F_u)$

As in [5], we introduce hermitian intersection forms

$$\begin{aligned} (\ ,\ )_0 \quad &: \quad H_n(F_u, \partial_\infty F_u) \quad \times \quad H_n(F_u, \partial_0 F_u) \quad \longrightarrow \quad \mathbb{Z}[q, q^{-1}], \qquad \text{and} \\ (\ ,\ )_\infty \quad &: \quad H_n(F_u, \partial_0 F_u) \quad \times \quad H_n(F_u, \partial_\infty F_u) \quad \longrightarrow \quad \mathbb{Z}[q, q^{-1}], \end{aligned}$$

for $u \in \mathcal{U}$. Note that the usual intersection form

$$\langle\ ,\ \rangle \;:\; H_n(F_u, \partial_\infty F_u) \times H_n(F_u, \partial_0 F_u) \;\longrightarrow\; \mathbb{Z}$$

is well-defined. (See §3.) For $x \in H_n(F_u, \partial_\infty F_u)$ and $y \in H_n(F_u, \partial_0 F_u)$, we put

$$(x, y)_0 \;:=\; \sum_{\nu\in\mathbb{Z}} \langle x, q^\nu y\rangle q^\nu \;\in\; \mathbb{Z}[q, q^{-1}].$$

Let $* : \mathbb{Z}[q, q^{-1}] \to \mathbb{Z}[q, q^{-1}]$ be the ring automorphism given by $*q = q^{-1}$. It is obvious that $\langle q^\nu x, q^\nu y\rangle = \langle x, y\rangle$ for all $\nu \in \mathbb{Z}$. Therefore, for arbitrary a, a', b, $b' \in \mathbb{Z}[q, q^{-1}]$, we have,

$$\begin{aligned} (ax + a'x', y)_0 &= a(x, y)_0 + a'(x', y)_0, \quad \text{and} \\ (x, by + b'y')_0 &= *b(x, y)_0 + *b'(x, y')_0. \end{aligned} \tag{8.1}$$

We define the hermitian form $(\ ,\ )_\infty$ by

$$(x,y)_\infty := *(y,x)_0 .$$

Remark 8.1. For any $[\gamma] \in \pi_1(\mathcal{U}, b)$, we have $\langle [\gamma]_* x, [\gamma]_* y\rangle = \langle x, y\rangle$. Combining this with Lemma 1.2, we get

$$([\gamma]_* x, [\gamma]_* y)_0 = (x,y)_0 .$$

This implies that $(\ ,\ )_0$ and $(\ ,\ )_\infty$ are hermitian intersection forms between the locally constant systems on $\mathcal{U}$ corresponding to $H_n(F_b, \partial_0 F_b)$ and $H_n(F_b, \partial_\infty F_b)$.

Lemma 8.1. *Suppose that $u \in \mathcal{U}_N^{\sim}$. Let α and β be elements of $H_{n-1}(X_u^\infty)$ and $H_{n-1}(X_u^0)$, respectively. Then the integer $\langle \Psi_u^\infty(\alpha\otimes q^\nu), \Psi_u^0(\beta\otimes q^\mu)\rangle$ is zero unless $\nu = \mu$.*

Proof. By Corollary 6.2, $\Psi_u^\infty(\alpha \otimes q^\nu)$ is represented by an n-chain $\Gamma_\alpha\langle\nu\rangle$ contained in $\psi_u^{-1}(K_u\langle\nu\rangle)$, while $\Psi_u^0(\beta\otimes q^\mu)$ is represented by an n-chain $\Gamma_\beta\langle\mu\rangle$ contained in $\psi_u^{-1}(K_u\langle\mu\rangle)$. If $\nu \neq \mu$, then $K_u\langle\nu\rangle \cap K_u\langle\mu\rangle = \emptyset$, and hence $\langle[\Gamma_\alpha\langle\nu\rangle], [\Gamma_\beta\langle\mu\rangle]\rangle = 0$. $\square$

Combining Lemma 8.1 with (8.1), we get the following formula. Let α_ν ($\nu \in \mathbb{Z}$) and β_μ ($\mu \in \mathbb{Z}$) be elements of $H_{n-1}(X_u^\infty)$ and $H_{n-1}(X_u^0)$, respectively, such that almost all of them are zero. Then

$$\begin{aligned} &\Big(\ \Psi_u^\infty\big(\sum_{\nu\in\mathbf{Z}} \alpha_\nu \otimes q^\nu\big)\ ,\ \Psi_u^0\big(\sum_{\mu\in\mathbf{Z}} \beta_\mu \otimes q^\mu\big)\ \Big)_0 \\ &= \sum_{k\in\mathbf{Z}} \Big(\ \sum_{\nu-\mu=k} \langle\ \Psi_u^\infty(\alpha_\nu\otimes 1)\ ,\ \Psi_u^0(\beta_\mu \otimes 1)\ \rangle\ \Big)\cdot q^k . \end{aligned} \tag{8.2}$$

Lemma 8.2. *Suppose that $u \in \mathcal{U}_N^{\sim}$. Let p and p' be values in $\mathrm{Cr}(u)$, and let ξ^0 and ξ^∞ be paths in $\mathcal{P}_u(a_u^0, p)$ and $\mathcal{P}_u(a_u^\infty, p')$, respectively. Suppose that $\xi^0(I) \subset K_u$ and $\xi^\infty(I) \subset K_u$. (1) Suppose that $p = p'$ and $\xi^0(I)\cap\xi^\infty(I) = \{p\}$. Then*

$$(\ \bar\theta([\xi^\infty])\langle\nu\rangle\ ,\ \bar\theta([\xi^0]\langle\mu\rangle\ )_0 = \pm q^{\nu-\mu}.$$

(2) Suppose that $p \neq p'$ and $\xi^0(I) \cap \xi^\infty(I) = \emptyset$. Then

$$(\ \bar\theta([\xi^\infty])\langle\nu\rangle\ ,\ \bar\theta([\xi^0]\langle\mu\rangle\ )_0 = 0.$$

Proof. By (8.2) and (7.2), (7.2)$'$, we see that $(\ \bar\theta([\xi^\infty])\langle\nu\rangle\ ,\ \bar\theta([\xi^0]\langle\mu\rangle\ )_0$ is a multiple of $q^{\nu-\mu}$ by the integer $\langle\ \bar\theta([\xi^\infty])\langle 0\rangle\ ,\ \bar\theta([\xi^0]\langle 0\rangle\ \rangle$. Let $T^0 : CS^{n-1} \to F_u$ and $T^\infty : CS^{n-1} \to F_u$ be continuous maps representing the thimble $\theta([\xi^0])\langle 0\rangle$ over $\xi^0\langle 0\rangle$, and the thimble $\theta([\xi^\infty])\langle 0\rangle$ over $\xi^\infty\langle 0\rangle$, respectively. By the assumptions on the paths ξ^0 and ξ^∞, we have

$$\xi^0\langle 0\rangle(I)\ \cap\ \xi^\infty\langle 0\rangle(I)\ \ =\ \begin{cases} \{p\langle 0\rangle\} & \text{in the case (1), and} \\ \emptyset & \text{in the case (2).} \end{cases}$$

In the case (1), using Lemma 4.1, we can choose the n-chains T^0 and T^∞ in such a way that they intersect only at the critical point of ψ_u over $p\langle 0\rangle$, and that the intersection is transverse. Hence $\langle [T^\infty], [T^0] \rangle = \langle\, \bar{\theta}([\xi^\infty])\langle 0\rangle \,,\, \bar{\theta}([\xi^0]\langle 0\rangle\, \rangle = \pm 1$. In the case (2), the n-cahins T^∞ and T^0 are disjoint. Hence $\langle [T^\infty], [T^0] \rangle$ is zero. □

Now we shall prove the following:

Proposition 8.1. *The intersection forms* $(\ ,\ )_0$ *and* $(\ ,\ )_\infty$ *are non-degenerate.*

Here the non-degeneracy of $(\ ,\ )_0$ means that the map

$$H_n(F_u, \partial_0 F_u) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(\ H_n(F_u, \partial_\infty F_u)\ ,\ \mathbb{Z}[q,q^{-1}]\ )$$

given by $y \mapsto (\ , y)_0$ is a bijection.

Proof. By Remark 8.1, it is enough to prove Proposition 8.1 under the assumption that $u \in \mathcal{U}_N^\sim$. We can take K-regular systems $\{\xi_1^0, \dots, \xi_N^0\}$ and $\{\xi_1^\infty, \dots, \xi_N^\infty\}$ of paths from a_u^0 and from a_u^∞, respectively, which satisfy the following;

$$\xi_i^0(I) \cap \xi_j^\infty(I) = \begin{cases} \emptyset & \text{if } i \neq j, \text{ and} \\ \{p_i\} & \text{if } i = j. \end{cases}$$

By Lemma 8.2, we have

$$(\, \bar{\theta}([\xi_i^\infty])\langle 0\rangle \,,\, \bar{\theta}([\xi_j^0])\langle 0\rangle\,)_0 \;=\; \langle\, \bar{\theta}([\xi_i^\infty])\langle 0\rangle \,,\, \bar{\theta}([\xi_j^0])\langle 0\rangle)\, \rangle \;=\; \pm\, \delta_{ij}.$$

Thus, in terms of the basis $\{\, \bar{\theta}([\xi_i^0])\langle 0\rangle \,;\, i = 1, \dots, N \,\}$ of $H_n(F_u, \partial_0 F_u)$ over $\mathbb{Z}[q, q^{-1}]$ and $\{\, \bar{\theta}([\xi_i^\infty])\langle 0\rangle \,;\, i = 1, \dots, N \,\}$ of $H_n(F_u, \partial_\infty F_u)$ over $\mathbb{Z}[q, q^{-1}]$, the intersection form $(\ ,\ )_0$ is expressed by a diagonal matrix with diagonal coefficients ± 1. □

Definition 8.1. An element $x \in H_n(F_u, \partial_0 F_u)$ (resp. $y \in H_n(F_u, \partial_\infty F_u)$) is called *primitive* if and only if there exists an element $x' \in H_n(F_u, \partial_\infty F_u)$ (resp. $y' \in H_n(F_u, \partial_0 F_u)$) such that $(x', x)_0 = 1$ (resp. $(y', y)_\infty = 1$).

Definition 8.2. Let $U(\mathbb{Z}[q, q^{-1}])$ denote the group of the units $\{\, \pm q^\nu \,;\, \nu \in \mathbb{Z} \,\}$ of the ring $\mathbb{Z}[q, q^{-1}]$. We say that two elements x and x' in a $\mathbb{Z}[q, q^{-1}]$-module is said to be *equal modulo* $U(\mathbb{Z}[q, q^{-1}])$ and write $x \equiv x'$, if there exists a unit $a \in U(\mathbb{Z}[q, q^{-1}])$ such that $x = ax'$.

For example, if x is a primitive element of $H_n(F_u, \partial_\infty F_u)$ and $x \equiv x'$, then x' is also primitive.

§9. Picard-Lefschetz formula for local monodromies around $\mathcal{D}_0$ and $\mathcal{D}_\infty$

§9.1. Definition of simple loops and local monodromies

We fix a base point $b \in \mathcal{U}$.

Definition 9.1.1. A loop $\gamma : I \to \mathcal{U}$ with the base point b is called a *simple loop around* $\mathcal{D}_0$ *(resp.* $\mathcal{D}_\infty$*)* if the following are satisfied; (i) there exist a non-singular point c on

$\mathcal{D}_0 \setminus (\mathcal{D}_0 \cap \mathcal{D}_\infty)$ (resp. $\mathcal{D}_\infty \setminus (\mathcal{D}_0 \cap \mathcal{D}_\infty)$) and a small closed disk Δ in Γ with the center c which intersects $\mathcal{D}_0 \cup \mathcal{D}_\infty$ transversely at only one point c, (ii) there is a path β on $\mathcal{U}$ from b to a point b' on the boundary $\partial\Delta$ of Δ, and (iii) the path γ starts from b, goes to b' along β, draws a circle $\partial\Delta$ in a counter-clockwise direction, and goes back to b along β^{-1}.

Definition 9.1.2. Let $\gamma : I \to \mathcal{U}$ be a simple loop around $\mathcal{D}_0$ (resp. $\mathcal{D}_\infty$). Then the monodromy action $[\gamma]_*$ on various sets or groups are called a *local monodromy around* $\mathcal{D}_0$ *(resp.* $\mathcal{D}_\infty$*).*

Proposition 9.1.1. *Let $\mathcal{D}$ denote $\mathcal{D}_0$ or $\mathcal{D}_\infty$. Let b and b' be two base points on $\mathcal{U}$, and let $\gamma : I \to \mathcal{U}$ and $\gamma' : I \to \mathcal{U}$ be simple loops around $\mathcal{D}$ with the base points b and b', respectively. Then there exists a path $\alpha : I \to \mathcal{U}$ from b to b' such that $[\alpha^{-1}\gamma'\alpha] = [\gamma]$ in $\pi_1(\mathcal{U}, b)$.*

Proof. Since both of the hypersurfaces $\mathcal{D}_0$ and $\mathcal{D}_\infty$ are irreducible, each of the non-singular loci of $\mathcal{D}_0 \setminus (\mathcal{D}_0 \cap \mathcal{D}_\infty)$ and $\mathcal{D}_\infty \setminus (\mathcal{D}_0 \cap \mathcal{D}_\infty)$ is also irreducible. □

§9.2. Picard-Lefschetz formula

Now we shall state our main theorems.

Theorem 9.2.1. *Let $[\gamma_0] \in \pi_1(\mathcal{U}, b)$ be the homotopy class of a simple loop around $\mathcal{D}_0$. There exists a pair*

$$(v[\gamma_0], v\check{}[\gamma_0]) \;\in\; H_n(F_b) \times H_n(F_b, \partial_0 F_b)$$

such that the local monodromy $[\gamma_0]_*$ around $\mathcal{D}_0$ on $H_n(F_b, \partial_\infty F_b)$ is given by

$$x \;\mapsto\; x \;+\; (x, v\check{}[\gamma_0])_0 \cdot v[\gamma_0]. \tag{9.2.1}$$

Moreover, such a pair $(v[\gamma_0], v\check{}[\gamma_0])$ is unique up to $U(\mathbb{Z}[q, q^{-1}])$, and $v\check{}[\gamma_0]$ is primitive. We also have

$$v[\gamma_0] = (-1)^{n(n-1)/2}(q-1) \cdot v\check{}[\gamma_0]. \tag{9.2.2}$$

Theorem 9.2.2. *Let $[\gamma_\infty] \in \pi_1(\mathcal{U}, b)$ be the homotopy class of a simple loop around $\mathcal{D}_\infty$. There exists a pair*

$$(v[\gamma_\infty], v\check{}[\gamma_\infty]) \;\in\; H_n(F_b, \partial_0 F_b) \times H_n(F_b, \partial_\infty F_b)$$

such that the local monodromy $[\gamma_\infty]_*$ around $\mathcal{D}_\infty$ on $H_n(F_b, \partial_0 F_b)$ is given by

$$x \;\mapsto\; x \;+\; (x, v\check{}[\gamma_\infty])_\infty \cdot v[\gamma_\infty]. \tag{9.2.3}$$

Moreover, such a pair $(v[\gamma_\infty], v\check{}[\gamma_\infty])$ is unique up to $U(\mathbb{Z}[q, q^{-1}])$, and $v\check{}[\gamma_\infty]$ is primitive.

Remark 9.2.1. Comparing Theorems 9.2.1 and 9.2.2, we can see that there is a certain kind of duality between "0" and "∞". This duality, however, is not perfect. Contrary to the case in Theorem 9.2.1, the homology class $v[\gamma_\infty] \in H_n(F_b, \partial_0 F_b)$ in Theorem 9.2.2 is not contained in $H_n(F_b)$. This difference comes from the fact that, while the action of

$[\gamma_0]_*$ on $H_{n-1}(X_b^\infty)$ is trivial (cf. Claim 2 in the proof of Proposition 9.4.1 below), the action of $[\gamma_\infty]_*$ on $H_{n-1}(X_b^0)$ is non-trivial (cf. Proposition 9.7.1). Moreover, the relation between $v\check{}[\gamma_\infty]$ and $v[\gamma_\infty]$ is not so simple as (9.2.2). A detailed description of $v[\gamma_\infty]$ is given in Proposition 9.7.2.

Remark 9.2.2. We put $\partial F_b = \partial_0 F_b \cup \partial_\infty F_b$. Using (6.16) and the natural homomorphism $H_n(F_b, \partial_0 F_b) \to H_n(F_b, \partial F_b)$ or $H_n(F_b, \partial_\infty F_b) \to (F_b, \partial F_b)$, we can define the hermitian intersection pairing

$$(\ , \ ) \ : \ H_n(F_b) \times H_n(F_b, \partial F_b) \longrightarrow \mathbb{Z}[q, q^{-1}]$$

from $(\ , \ )_0$ or $(\ , \ )_\infty$. Using (6.16) again, we can derive the statement (2_q) in Introduction from Theorems 9.2.1 and 9.2.2.

Before starting the proof, we make here several remarks (Remarks 9.2.3, 9.2.4, 9.2.5, and §9.3), which make the proof much easier.

Remark 9.2.3. The uniqueness of $(v[\gamma_0], v\check{}[\gamma_0])$ in Theorems 9.2.1 follows easily from the property (9.2.1) and the primitiveness of $v\check{}[\gamma_0]$. Suppose that (9.2.1) holds for all $x \in H_n(F_b, \partial_\infty F_b)$ with some pair $(v[\gamma_0], v\check{}[\gamma_0])$, and that $v\check{}[\gamma_0]$ is primitive. Then the image of the endomorphism $\mathrm{Id} - [\gamma_0]_*$ of $H_n(F_b, \partial_\infty F_b)$ is a free $\mathbb{Z}[q, q^{-1}]$-module of rank 1, and hence its generator $v[\gamma_0]$ is determined uniquely modulo $U(\mathbb{Z}[q, q^{-1}])$. Suppose that a generator $v[\gamma_0]$ is fixed. Then the endomorphism $\mathrm{Id} - [\gamma_0]_*$ is written in the form $x \mapsto l(x) \cdot v[\gamma_0]$ by some $\mathbb{Z}[q, q^{-1}]$-linear form $l : H_n(F_b, \partial_\infty F_b) \to \mathbb{Z}[q, q^{-1}]$. Then $v\check{}[\gamma_0] \in H_n(F_b, \partial_0 F_b)$ is uniquely determined by the non-degeneracy of $(\ , \ )_0$ (Proposition 8.1). The uniqueness of the pair $(v[\gamma_\infty], v\check{}[\gamma_\infty])$ modulo $U(\mathbb{Z}[q, q^{-1}])$ is also derived from (9.2.3) and the primitiveness of $v\check{}[\gamma_\infty]$ in the same way.

Remark 9.2.4. Suppose that Theorem 9.2.1 holds for *one* simple loop γ_0 around $\mathcal{D}_0$ with the base point b. Then it holds for an arbitrary simple loop γ_0' around $\mathcal{D}_0$ with the base point b' arbitrarily chosen. Indeed, by Proposition 9.1.1, there exists a path $\alpha : I \to \mathcal{U}$ from b' to b such that

$$[\alpha^{-1}\gamma_0\alpha] \ = \ [\gamma_0'] \quad \text{in} \quad \pi_1(\mathcal{U}, b').$$

Let

$$\begin{aligned} [\alpha]_* \quad &: \quad H_n(F_{b'}, \partial_\infty F_{b'}) \ \xrightarrow{\sim} \ H_n(F_b, \partial_\infty F_b), \qquad \text{and} \\ [\alpha]_* \quad &: \quad H_n(F_{b'}, \partial_0 F_{b'}) \ \xrightarrow{\sim} \ H_n(F_b, \partial_0 F_b) \end{aligned}$$

be the isomorphisms induced by the path α. Then

$$[\gamma_0']_* \ = \ [\alpha]_*^{-1} \circ [\gamma_0]_* \circ [\alpha]_* \quad \text{on} \quad H_n(F_{b'}, \partial_\infty F_{b'}).$$

By Remark 8.1, we have

$$(\ [\alpha]_*(x), \ \ v\check{}[\gamma_0] \ )_0 = (\ x, \ \ [\alpha]_*^{-1}(v\check{}[\gamma_0]) \ )_0$$

for all $x \in H_n(F_{b'}, \partial_\infty F_{b'})$. Hence the formula (9.2.1) holds for $[\gamma_0']_*$ if we set

$$v\check{}[\gamma_0'] = [\alpha]_*^{-1}(v\check{}[\gamma_0]), \quad \text{and} \quad v[\gamma_0'] = [\alpha]_*^{-1}(v[\gamma_0]). \tag{9.2.4}$$

It is obvious that if $v^{\check{}}[\gamma_0]$ is primitive, then so is $[\alpha]_*^{-1}(v^{\check{}}[\gamma_0])$. The relation (9.2.2) also remains true for the pair $(v[\gamma_0'], v^{\check{}}[\gamma_0'])$ defined by (9.2.4).

Same argument is valid for Theorem 9.2.2.

It is therefore enough to prove each of Theorems 9.2.1 and 9.2.2 only for one suitably chosen simple loop.

Remark 9.2.5. Note that the complement $\mathcal{U} \setminus \mathcal{U}_N$ is of complex codimension 1 in $\mathcal{U}$. Note also that the complement $\mathcal{U} \setminus \mathcal{U}_N^{\sim}$ is of real codimension 1 in $\mathcal{U}$. Combining these with Remark 9.2.4, we may assume that the base point b is contained in $\mathcal{U}_N^{\sim}$, and the simple loops γ_0 and γ_∞ are contained in $\mathcal{U}_N$.

§9.3. Deformation of thimbles

By the definition of $\mathcal{U}_N$, the fundamental group $\pi_1(\mathcal{U}_N, b)$ acts on the sets $\mathrm{Cr}(b) \subset \mathbb{C}$ and $Cr(b) \subset \mathbf{C}$. For an element $[\gamma] \in \pi_1(\mathcal{U}_N, b)$, let $[\gamma]_* : \mathrm{Cr}(b) \to \mathrm{Cr}(b)$ and $[\gamma]_* : Cr(b) \to Cr(b)$ denote its actions. For $u \in \mathcal{U}_N$, we put

$$\mathcal{S}_u \quad := \quad Cr(u) \cup Z_u^0 \cup Z_u^\infty \quad \subset \quad \mathbf{C},$$

where $Z_u^0 = \{\, a_u^0\langle\nu\rangle \,;\, \nu \in \mathbb{Z}\,\}$ and $Z_u^\infty = \{\, a_u^\infty\langle\nu\rangle \,;\, \nu \in \mathbb{Z}\,\}$, and call it the set of *distinguished points*. Let $\gamma : I \to \mathcal{U}_N$ be a loop with the base point b. Then the points of $\mathcal{S}_{\gamma(t)}$ move on $\mathbf{C}$ continuously when t varies, and any two distinct points do not collide during this movement because of the definitions of $\mathcal{U}_N$ and Z_u^0, Z_u^∞. Hence we can denote this movement by the continuous map

$$\mathcal{M}_\gamma \;:\; I \times \mathcal{S}_b \;\longrightarrow\; \mathbf{C}.$$

It is obvious that

(1) $\mathcal{M}_\gamma(0, s) = s$ for all $s \in \mathcal{S}_b$,
(2) $\mathcal{M}_\gamma(1, p) = [\gamma]_*(p)$ for $p \in Cr(b)$,
(3) $\mathcal{M}_\gamma(1, a_b^0\langle\nu\rangle) = a_b^0\langle\nu\rangle$, $\mathcal{M}_\gamma(1, a_b^\infty\langle\nu\rangle) = a_b^\infty\langle\nu\rangle$ for all $\nu \in \mathbb{Z}$, and
(4) $\mathcal{M}_\gamma(t, \cdot) : \mathcal{S}_b \to \mathbf{C}$ is injective for all $t \in I$.

The last property implies that the homotopy type of the pair $(\mathbf{C}, \mathcal{M}_\gamma(t, \mathcal{S}_b))$ does not change when t varies in I. We consequently obtain, for each critical value $p \in Cr(b)$, the bijective maps of the sets of homotopy classes of paths

$$\begin{array}{lllll} [\gamma]_* & : & [\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p)\,] & \longrightarrow & [\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, [\gamma]_*(p))\,] \quad \text{and} \\ [\gamma]_* & : & [\,\mathcal{P}_b^{\sim}(a_b^\infty\langle\nu\rangle, p)\,] & \longrightarrow & [\,\mathcal{P}_b^{\sim}(a_b^\infty\langle\nu\rangle, [\gamma]_*(p))\,] \end{array} \tag{9.3.1}$$

induced by the movement $\mathcal{M}_\gamma$ of the points in $\mathcal{S}_b$.

Now suppose that we are given a path $\omega \in \mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p)$, where $p \in Cr(b)$. We choose a vanishing cycle $\sigma[\omega] \in [S^{n-1}, X_b^0\langle\nu\rangle]$ for $[\omega]$ from among the two possibilities of the signs, and consider the thimble

$$\theta([\omega]; \sigma[\omega]) \;\in\; [(CS^{n-1}, S^{n-1}), (F_b, X_b^0\langle\nu\rangle)]$$

for $[\omega] \in [\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p)\,]$ starting from $\sigma[\omega]$. Note that $[\gamma] \in \pi_1(\mathcal{U}_N, b)$ also acts on the sets $[S^{n-1}, X_b^0\langle\nu\rangle]$ and $[(CS^{n-1}, S^{n-1}), (F_b, X_b^0\langle\nu\rangle)]$ in a natural way, because over the path γ, both of $X_u^0\langle\nu\rangle$ and $(F_u, X_u^0\langle\nu\rangle)$ are locally trivial. By definition, we see that $[\gamma]_*(\sigma[\omega]) \in [S^{n-1}, X_b^0\langle\nu\rangle]$ is one of the vanishing cycles for $[\gamma]_*([\omega]) \in [\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, [\gamma]_*(p))\,]$, and we have a formula:

$$[\gamma]_*(\,\theta([\omega], \sigma[\omega])\,) \;=\; \theta(\,[\gamma]_*([\omega])\,,\,[\gamma]_*(\sigma[\omega])\,). \tag{9.3.2}$$

In particular, we have

$$[\gamma]_*(\,\theta([\omega], \sigma[\omega])\,) \;=\; \theta(\,[\gamma]_*([\omega])\,,\,\sigma[\omega]\,), \;\text{ or }\; -\,\theta(\,[\gamma]_*([\omega])\,,\,\sigma[\omega]\,). \tag{9.3.3}$$

By Proposition 7.1, $H_n(F_b, \partial_0 F_b)$ is generated by the homology classes of thimbles for paths from $a_b^0\langle 0\rangle$. Hence formula (9.3.2) enables us to calculate the action of $[\gamma]_*$ on $H_n(F_b, \partial_0 F_b)$ by looking at the action of $[\gamma]_*$ on $[\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p)\,]$ for $p \in Cr(b)$ and on $[S^{n-1}, X_b^0\langle\nu\rangle]$. Note that the action on $[S^{n-1}, X_b^0\langle\nu\rangle]$ is equivalent to the action on $[S^{n-1}, X_b^0]$ by the isomorphism (6.1).

Same argument holds when 0 is replaced with ∞.

In order to investigate the maps (9.3.1), we introduce the notion of homotopy equivalence of movements of points on $\mathbf{C}$.

Definition 9.3.1. Let $\mathcal{M}_0 : I \times \mathcal{S} \to \mathbf{C}$ and $\mathcal{M}_1 : I \times \mathcal{S} \to \mathbf{C}$ be two movements of a set of points $\mathcal{S}$ on $\mathbf{C}$ such that
(i) $\mathcal{M}_0(0, s) = \mathcal{M}_1(0, s)$ for all $s \in \mathcal{S}$,
(ii) $\mathcal{M}_0(1, s) = \mathcal{M}_1(1, s)$ for all $s \in \mathcal{S}$, and
(iii) for all t, both of the maps from $\mathcal{S}$ to $\mathbf{C}$ given by $s \mapsto \mathcal{M}_0(t, s)$ and by $s \mapsto \mathcal{M}_1(t, s)$ are injective.
These two movements are said to be *homotopically equivalent* if there exists a continuous map $\mathcal{M} : I \times I \times \mathcal{S} \to \mathbf{C}$ such that the movements $\mathcal{M}(\tau) := \mathcal{M}(\tau, \cdot, \cdot) : I \times \mathcal{S} \to \mathbf{C}$ satisfy the following;
(1) $\mathcal{M}(0) = \mathcal{M}_0$, $\mathcal{M}(1) = \mathcal{M}_1$,
(2) $\mathcal{M}(\tau)(0, s) = \mathcal{M}_0(0, s) = \mathcal{M}_1(0, s)$ for all $\tau \in I$ and $s \in \mathcal{S}$,
(3) $\mathcal{M}(\tau)(1, s) = \mathcal{M}_0(1, s) = \mathcal{M}_1(1, s)$ for all $\tau \in I$ and $s \in \mathcal{S}$, and
(4) $\mathcal{M}(\tau)(t, \cdot) : \mathcal{S} \to \mathbf{C}$ is injective for all $(\tau, t) \in I \times I$.

It is obvious that the maps (9.3.1) depend only on the homotopy class of the movement $\mathcal{M}_\gamma$. Therefore, we will find a simpler movement in the homotopy equivalence class containing $\mathcal{M}_\gamma$.

Reduction 1. Note that, for all $p \in Cr(b)$ and for all $t \in I$, the point $\mathcal{M}_\gamma(t, p)$ remains on the right-hand side of the vertical line $R^0_{\gamma(t)}$, which contains the points $a^0_{\gamma(t)}\langle\nu\rangle = \mathcal{M}_\gamma(t, a_b^0\langle\nu\rangle)$, and on the left-hand side of the vertical line $R^\infty_{\gamma(t)}$, which contains the points $a^\infty_{\gamma(t)}\langle\nu\rangle = \mathcal{M}_\gamma(t, a_b^\infty\langle\nu\rangle)$. Hence the movement $\mathcal{M}_\gamma$ is always homotopically equivalent to a movement $\mathcal{M}'_\gamma$ such that

$$\mathcal{M}'_\gamma(t, a_b^0\langle\nu\rangle) \;=\; a_b^0\langle\nu\rangle \quad\text{and}\quad \mathcal{M}'_\gamma(t, a_b^\infty\langle\nu\rangle) \;=\; a_b^\infty\langle\nu\rangle \quad \text{for all } t \in I. \tag{9.3.4}$$

Reduction 2. Suppose that, by the movement along γ, the value $p_i \in \mathrm{Cr}(u) \subset \mathbb{C}^\times$ draws a loop which is homotopically trivial in $\mathbb{C}^\times$. Then each of its lifts to $\mathbf{C}$ is also a loop $\mathcal{M}_\gamma(I \times \{p_i\langle\nu\rangle\})$ on $\mathbf{C}$ for every $\nu \in \mathbb{Z}$. Suppose also that this loop $\mathcal{M}_\gamma(I \times \{p_i\langle\nu\rangle\})$ is disjoint from the trace $\mathcal{M}_\gamma(I \times \{p_j\langle\mu\rangle\})$ if $(i,\nu) \neq (j,\mu)$. Then $\mathcal{M}_\gamma$ is homotopically equivalent to a movement $\mathcal{M}'_\gamma$ which satisfies, in addition to (9.3.4), the property

$$\mathcal{M}_\gamma(t, p_i\langle\nu\rangle) \;=\; p_i\langle\nu\rangle \quad \text{for all} \quad t \in I \;\text{ and }\; \nu \in \mathbb{Z}. \tag{9.3.5}$$

§9.4. Proof of Theorem 9.2.1

Suppose that u is a point of $\mathcal{U}_N$.

Recall that $\mathcal{L}_u \subset \Gamma^\times$ is the affine line $\{\, f_u - t \cdot h^d \,;\, t \in \mathbb{C}\,\}$ with the parameterization $\iota_u : \mathbb{C} \xrightarrow{\sim} \mathcal{L}_u$ given by $t \mapsto f_u - t \cdot h^d$. Let w be an arbitrary point on $\mathcal{L}_u$. By definition, the affine line $\mathcal{L}_w$ is equal with $\mathcal{L}_u$, and we write this affine line simply by $\mathcal{L}$. By Corollary 2.1 and Lemma 2.3, we have

$$\mathcal{L} \setminus \mathcal{D}_0 \;=\; \mathcal{L} \cap \mathcal{U} \;=\; \mathcal{L} \cap \mathcal{U}_N. \tag{9.4.1}$$

Let $c_1, \ldots, c_N$ be the intersection points of $\mathcal{L}$ and $\mathcal{D}_0$. Then, by Corollary 2.1, the critical values of $\hat{\phi}_w$ are accordingly numbered;

$$\mathrm{Cr}(w) = \{p_1(w), \ldots, p_N(w)\}, \quad \text{where} \quad p_i(w) = \iota_w^{-1}(c_i).$$

The point w is on $\mathcal{L} \cap \mathcal{D}_0 = \mathcal{L} \setminus \mathcal{U}_N$ if and only if one of $p_1(w), \ldots, p_N(w)$ is zero.

Lemma 9.4.1. *We have $p_i(w) = p_i(u) + s_w$ for $i = 1, \ldots, N$, where $s_w := \iota_w^{-1}(u) = -\iota_u^{-1}(w)$. In particular, $p_i(u) - p_j(u) = p_i(w) - p_j(w)$ holds for all $u, w \in \mathcal{L}$.*

Proof. The two parameterizations $\iota_u : \mathbb{C} \to \mathcal{L}$ and $\iota_w : \mathbb{C} \to \mathcal{L}$ differ only by translations; and an easy calculation shows that $\iota_w^{-1} \circ \iota_u(s) = s + s_w$. $\square$

This Lemma shows that the set $\mathrm{Cr}(w) \subset \mathbb{C}$ moves by parallel transformation when w moves on $\mathcal{L}$.

Let r be a positive real number which is sufficiently small, and let ρ be a complex number such that $|\rho| = 2r$ and $\rho \notin \mathbb{R}$. We choose the point

$$b \;:=\; \iota_u(p_1(u) - \rho) \;\in\; \mathcal{L} \setminus \mathcal{D}_0 \;\subset\; \mathcal{U}_N \tag{9.4.2}$$

as the base point. Since r is sufficiently small and $\mathrm{Im}\, \rho \neq 0$, we may assume that

$$b \;\in\; \mathcal{U}_N^{\sim}. \tag{9.4.3}$$

In particular, we have $K_b \subset \mathbb{C}^\times$ and the isomorphisms Ψ_b^0, Ψ_b^∞. By Lemma 9.4.1, we have

$$p_i(b) \;=\; p_i(u) - p_1(u) + \rho. \tag{9.4.4}$$

In particular, we have $p_1(b) = \rho$. Moreover, since r is small enough, we may assume that

$$|p_i(b)| > M \cdot r \quad \text{if} \quad i \neq 1, \quad \text{and} \quad |p_i(b) - p_j(b)| > M \cdot r \quad \text{if} \quad i \neq j, \tag{9.4.5}$$

where M is a large natural number, say 10. Again, since r is sufficiently small, we have $\varepsilon(b) = r$ by the definition (3.3). Hence we have $a_b^0 = r$.

Now we consider a closed disk

$$\Delta := \{ \iota_b(p_1(b) - 2rz) \ ; \ |z| \leq 1\} \subset \mathcal{L}$$

on $\mathcal{L}$ with the center $\iota_b(p_1(b)) = c_1$ and of radius $2r$. Since r is small enough, the intersection $\Delta \cap \mathcal{D}_0$ consists of only one point c_1. Moreover, since $u \in \mathcal{U}_N$, $\mathcal{L}$ intersects $\mathcal{D}_0$ transversely by Proposition 2.4. The loop $\gamma : I \to \partial\Delta \subset \mathcal{L}$ given by

$$\gamma(t) := \iota_b \ (\ p_1(b) - \rho e^{2\pi\sqrt{-1}t} \ ) = \iota_b \ (\rho \ (1 - e^{2\pi\sqrt{-1}t} \ ) \ ) \tag{9.4.6}$$

is therefore a simple loop around $\mathcal{D}_0$ with the base point b. By (9.4.1), we have

$$\gamma(I) \subset \mathcal{U}_N. \tag{9.4.7}$$

Let $D_1 \subset \mathbb{C}$ be the closed disk with the center 0 and of radius $2r$. The critical value $p_1(b) = \rho$ is located on the boundary of this disk. We see from (9.4.5) that $D_1 \cap \mathrm{Cr}(b)$ consists of only one point $p_1(b)$. Note also that $K_b \cap D_1$ is simply connected. Therefore, there exists a unique homotopy class $[\xi_1^0] \in [\, \mathcal{P}_b(a_b^0, p_1(b)) \,]$ of paths which is represented by a path ξ_1^0 such that

$$\xi_1^0(I) \subset D_1 \cap K_b. \tag{9.4.8}$$

Now, from Remarks 9.2.3 and 9.2.4, Theorem 9.2.1 follows from the following Proposition.

Proposition 9.4.1. *Let*

$$v\check{} := \bar{\theta}([\xi_1^0])\langle 0\rangle \in H_n(F_b, \partial_0 F_b)$$

be the homology class of the lifted thimble $\theta([\xi_1^0])\langle 0\rangle$, where $[\xi_1^0] \in [\, \mathcal{P}_b(a_b^0, p_1(b)) \,]$ is the unique homotopy class of paths characterized by (9.4.8). We define the element v of $H_n(F_b)$ by

$$v := (-1)^{n(n-1)/2}(q-1)v\check{} \tag{9.4.9}$$

using (6.16). Then, $v\check{}$ is primitive, and the local monodromy action $[\gamma]_*$ on $H_n(F_b, \partial_\infty F_b)$ around $\mathcal{D}_0$ is given by

$$x \mapsto x + (x, v\check{})_0 \cdot v. \tag{9.4.10}$$

Proof. By Lemma 9.4.1 and (9.4.6), we have

$$p_i(\gamma(t)) = p_i(b) - \rho(1 - e^{2\pi\sqrt{-1}t}). \tag{9.4.11}$$

This means that, when t moves from 0 to 1, each $p_i(\gamma(t))$ draws a circle of the radius $2r$ with the center $p_i(b) - \rho$ in the counter-clockwise direction. Let C_i denote this circle, and D_i the disk circumscribed by C_i. Note that D_1 coincides with the D_1 which we have defined just before the statement of Proposition 9.4.1. Since r is sufficiently small, we can see that

$$\varepsilon(\gamma(t)) = |p_1(\gamma(t))|/2 = |\rho|/2 = r \quad \text{for all} \quad t \in I,$$

and thus

$$a^0_{\gamma(t)} = r, \quad \text{and} \quad a^\infty_{\gamma(t)} = 1/r \quad \text{for all} \quad t \in I. \tag{9.4.12}$$

By (9.4.7), $[\gamma]_*$ acts on the set $Cr(b)$. By (9.4.3), each value in $Cr(b)$ is written in the form $p_i(b)\langle\nu\rangle$, where $i = 1, \dots, N$ and $\nu \in \mathbb{Z}$. We see from (9.4.5) that

$$D_i \not\ni 0 \qquad \text{if} \qquad i \neq 1. \tag{9.4.13}$$

On the other hand, $D_1 \ni 0$, and the circle C_1 traverses $\mathbb{R}_{\leq 0}$ in the positive direction. Hence we have

$$[\gamma]_*(p_i(b)\langle\nu\rangle) = \begin{cases} p_i(b)\langle\nu\rangle & \text{if } i \neq 1, \text{ and} \\ p_1(b)\langle\nu + 1\rangle & \text{if } i = 1. \end{cases} \tag{9.4.14}$$

By (9.4.3), we have a K-regular system $\{\xi_1^\infty, \dots, \xi_N^\infty\}$ from a_b^∞. We see from (9.4.5) that

$$D_i \cap D_j = \emptyset \quad \text{if} \quad i \neq j. \tag{9.4.15}$$

It is therefore possible to take the K-regular system in such a way that

$$\xi_i^\infty(I) \cap D_1 = \begin{cases} \emptyset & \text{if } i \neq 1, \text{ and} \\ \{p_1(b)\} & \text{if } i = 1. \end{cases} \tag{9.4.16}$$

Now we choose a vanishing cycle $\sigma_i^\infty := \sigma[\xi_i^\infty] \in [S^{n-1}, X_b^\infty]$ for each $[\xi_i^\infty]$ from among the two possibilities, and consider the lift of the associated thimble

$$\theta_i^\infty\langle 0\rangle := \theta([\xi_i^\infty], \sigma_i^\infty)\langle 0\rangle \quad \in \quad [(CS^{n-1}, S^{n-1}), (F_b, X_b^\infty\langle 0\rangle)],$$

which is the thimble for $[\xi_i^\infty\langle 0\rangle]$ starting from the lifted vanishing cycle $\sigma_i^\infty\langle 0\rangle$ on $X_b^\infty\langle 0\rangle$. Since the homology classes $\bar\theta_1^\infty\langle 0\rangle, \dots, \bar\theta_N^\infty\langle 0\rangle$ of these thimbles form a set of basis of $H_n(F_b, \partial_\infty F_b)$ by Proposition 7.1, it is enough to prove (9.4.10) when x runs through the set of these classes.

The intersection number $(\bar\theta_i^\infty\langle 0\rangle, v^\vee)_0 \in \mathbb{Z}[q, q^{-1}]$ is calculated as follows;

Claim 1

$$(\bar\theta_i^\infty\langle 0\rangle, v^\vee)_0 = (\bar\theta_i^\infty\langle 0\rangle, \bar\theta([\xi_1^0])\langle 0\rangle)_0 = \begin{cases} 0 & \text{if } i \neq 1, \text{ and} \\ \pm 1 & \text{if } i = 1. \end{cases}$$

Proof. Because of (9.4.8) and (9.4.16), we can derive Claim 1 from Lemma 8.2. □

This claim, in particular, shows that $v^\vee$ is primitive.

We choose the sign of the vanishing cycle σ_1^∞ for $[\xi_1^\infty]$ in such a way that

$$(\bar{\theta}_1^\infty\langle 0\rangle, v^\check{})_0 = 1. \tag{9.4.17}$$

Claim 2. *The monodromy action $[\gamma]_*$ on $[S^{n-1}, X_b^\infty]$ is trivial.*

Proof. We see that

$$\begin{aligned} X_{\gamma(t)}^\infty &= \phi_{\gamma(t)}^{-1}(a_{\gamma(t)}^\infty) & \\ &= \phi_{\gamma(t)}^{-1}(1/r) & \text{by (9.4.12)} \\ &= \phi_b^{-1} \circ \iota_b^{-1} \circ \iota_{\gamma(t)}(1/r) & \text{by (2.3)} \\ &= \phi_b^{-1}(1/r + \rho - \rho e^{2\pi\sqrt{-1}t}) & \text{by (9.4.6)}\,; \end{aligned}$$

that is, the family $\{ X_{\gamma(t)}^\infty \,;\, t \in I \}$ over $\partial\Delta$ is isomorphic to the restriction of $\phi_b : E_b \to \mathbb{C}^\times$ to the circle C_∞ of radius $2r$ with the center $1/r + \rho$. Since r can be taken arbitrarily small, this circle can be far away from 0 as much as we want. On the other hand, the critical values $\mathrm{Cr}(b)$ have to remain bounded when r tends to 0 because of (9.4.4). Thus we can conclude that the disk D_∞ circumscribed by C_∞ does not contain any critical values of $\hat{\phi}_b$. Hence Claim 2 follows from Proposition 2.1. $\square$

Claim 3. $[\gamma]_*([\xi_i^\infty\langle 0\rangle]) = [\xi_i^\infty\langle 0\rangle]$ *for* $i = 2, \ldots, N$.

Proof. By (9.4.13), (9.4.15) and Reduction 2 in §9.3, the movement $\mathcal{M}_\gamma$ of the distinguished points $\mathcal{S}_b$ is homotopically equivalent to a movement $\mathcal{M}'_\gamma$ which remains $a_b^0\langle\nu\rangle$, $a_b^\infty\langle\nu\rangle$ fixed for all $\nu \in \mathbb{Z}$, and $p_i\langle\nu\rangle$ also fixed for $i = 2, \ldots, N$ and for all $\nu \in \mathbb{Z}$, while it moves $p_1(b)\langle\nu\rangle$ to $p_1(b)\langle\nu+1\rangle$ along the vertical line $\log|\rho| + \sqrt{-1}\mathbb{R} = e^{-1}(\partial D_1)$. If $i \neq 1$, then the path $\xi_i^\infty\langle\nu\rangle$ is disjoint from this vertical line because of (9.4.16). Therefore we obtain the claim. $\square$

Applying Claims 2 and 3 to the formula (9.3.2), we obtain

$$[\gamma]_*(\theta_i^\infty\langle 0\rangle) = \theta_i^\infty\langle 0\rangle \quad \text{for} \quad i = 2, \ldots, N. \tag{9.4.18}$$

We put

$$v' := [\gamma]_*(\bar{\theta}_1^\infty\langle 0\rangle) - \bar{\theta}_1^\infty\langle 0\rangle \in H_n(F_b, \partial_\infty F_b).$$

By Claim 1, the choice of sign (9.4.17), and (9.4.18), we see that

$$[\gamma]_*(x) = x + (x, v^\check{})_0 \cdot v' \quad \text{for all} \quad x \in H_n(F_b, \partial_\infty F_b).$$

Now we shall prove that v' is equal with $(-1)^{n(n-1)/2}(q-1)v^\check{} \in H_n(F_b)$, and prove (9.4.9). First remark that Claim 2 implies that the thimbles $\theta_1^\infty\langle\nu\rangle$ and $[\gamma]_*(\theta_1^\infty\langle\nu\rangle)$ start with the same vanishing cycle $\sigma_1^\infty\langle\nu\rangle \in [S^{n-1}, X_b^\infty\langle\nu\rangle]$. In particular, we have

$$\partial([\gamma]_*(\bar{\theta}_1^\infty\langle 0\rangle)) = \partial(\bar{\theta}_1^\infty\langle 0\rangle) \quad \text{in} \quad H_{n-1}(X_b^\infty\langle 0\rangle),$$

and hence

$$v' \in \operatorname{Ker}(\partial : H_n(F_b, \partial_\infty F_b) \to H_n(\partial_\infty F_b)) = H_n(F_b). \tag{9.4.19}$$

On the other hand, by the formula (9.3.2) and the remark above, we have,

$$[\gamma]_*(\theta_1^\infty\langle\nu\rangle) = \theta([\gamma]_*([\xi_1^\infty\langle\nu\rangle]), \sigma_1^\infty\langle\nu\rangle). \tag{9.4.20}$$

Note that, by (9.4.14), the homotopy class $[\gamma]_*([\xi_1^\infty\langle\nu\rangle])$ of paths is an element of

$$[\,\mathcal{P}_b^\sim(a_b^\infty\langle\nu\rangle, p_1(b)\langle\nu+1\rangle)\,].$$

Now we shall describe paths which represent this homotopy class.

By the description of the movement $\mathcal{M}'_\gamma$ in the proof of Claim 3, the homotopy class $[\gamma]_*([\xi_1^\infty\langle\nu\rangle])$ is represented by a path $\xi'\langle\nu\rangle$ defined as follows. Note that by (9.4.16), the path $\xi_1^\infty\langle\nu\rangle$ is on the right-hand side of the vertical line

$$\Lambda_0 := \log 2r + \sqrt{-1}\mathbb{R}.$$

Note also that

$$p_1(b)\langle\nu\rangle = \log 2r + \sqrt{-1}(\nu + \arg\rho),$$

where $-\pi < \arg\rho < \pi$. Then $\xi'\langle\nu\rangle$ starts from $a_b^\infty\langle\nu\rangle$, goes to a point $p'_1\langle\nu\rangle := p_1(b)\langle\nu\rangle + \kappa$ along $\xi_1^\infty\langle\nu\rangle$, where κ is a sufficiently small complex number with Re $\kappa > 0$, draws an arc in the counter-clockwise direction to the point $p_1(b)\langle\nu\rangle + \sqrt{-1}|\kappa|$ on the line Λ_0 along the circle of radius $|\kappa|$ with the center $p_1(b)\langle\nu\rangle$, and goes to $p_1(b)\langle\nu+1\rangle$ along Λ_0.

Let $\xi_1^{\infty 0}\langle\nu\rangle$ be the path on $\mathbf{C}\setminus Cr(b)$ from $a_b^\infty\langle\nu\rangle$ to $a_b^0\langle\nu\rangle$ defined as follows. Note that, by (9.4.8), the path $\xi_1^0\langle\nu\rangle$ is on the left-hand side of the vertical line Λ_0. Then the path $\xi_1^{\infty 0}\langle\nu\rangle$ starts from $a_b^\infty\langle\nu\rangle$, goes to $p'_1\langle\nu\rangle$ along $\xi_1^\infty\langle\nu\rangle$, draws an arc on the circle of radius $|\kappa|$ with the center $p_1(b)\langle\nu\rangle$ in the counter-clockwise direction to the point $p_1(b)\langle\nu\rangle - \kappa'$ on $\xi_1^0\langle\nu\rangle(I)$, where κ' is a certain complex number with $|\kappa'| = |\kappa|$, and goes to $a_b^0\langle\nu\rangle$ along $\xi_1^0\langle\nu\rangle^{-1}$. It is easy to see that

$$[\gamma]_*([\xi_1^\infty\langle\nu\rangle]) = [\xi'\langle\nu\rangle] = [\xi_1^0\langle\nu+1\rangle\cdot\delta_b^0\langle\nu+1\rangle\cdot\xi_1^{\infty 0}\langle\nu\rangle] \quad \text{in} \quad [\,\mathcal{P}_b^\sim(a_b^\infty\langle\nu\rangle, p_1(b)\langle\nu+1\rangle)\,].$$

We put

$$\tilde{\xi}\langle 1\rangle := \xi_1^0\langle 1\rangle\cdot\delta_b^0\langle 1\rangle\cdot\xi_1^{\infty 0}\langle 0\rangle.$$

Then, from (9.4.20), we have

$$[\gamma]_*(\theta_1^\infty\langle 0\rangle) = \theta([\tilde{\xi}\langle 1\rangle], \sigma_1^\infty\langle 0\rangle).$$

We decompose the path $\xi_1^\infty\langle 0\rangle$ into two parts at $p'_1\langle 0\rangle$; that is, we write $\xi_1^\infty\langle 0\rangle = \eta_2\cdot\eta_1$, where η_1 is the path from $a_b^\infty\langle 0\rangle$ to $p'_1\langle 0\rangle$ along $\xi_1^\infty\langle 0\rangle$, and η_2 is the remaining part. Then $\xi_1^{\infty 0}\langle 0\rangle$ also decomposes into $\eta_3\cdot\eta_1$.

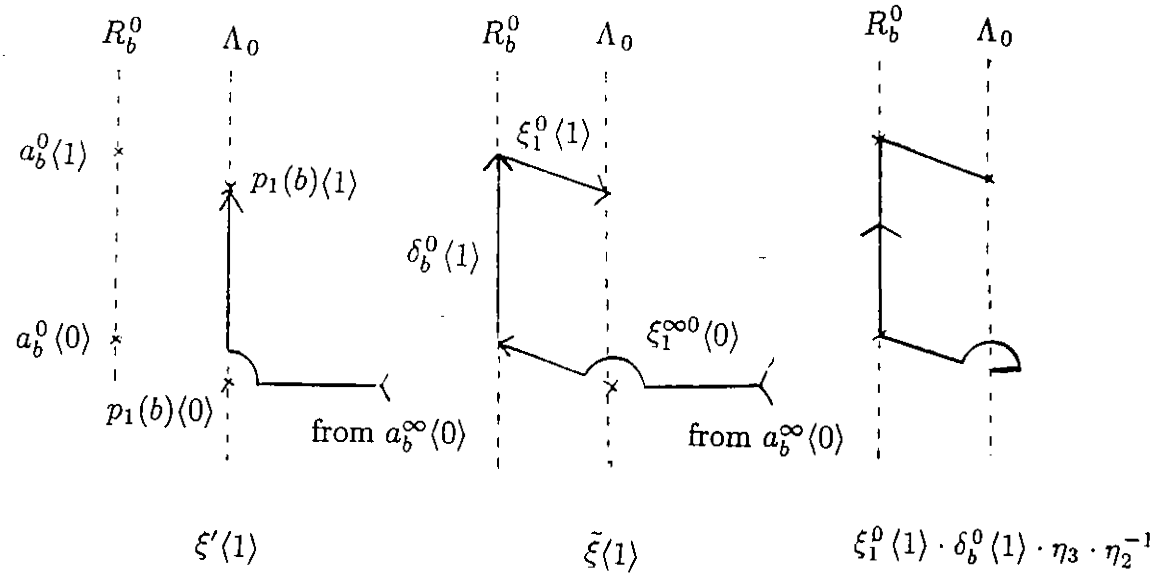

Figure 4

Let

$$T \;:\; CS^{n-1} \longrightarrow F_b, \quad \text{and} \quad T_\gamma \;:\; CS^{n-1} \longrightarrow F_b$$

be continuous maps representing $\theta_1^\infty\langle 0\rangle$ over $\xi_1^\infty\langle 0\rangle$ and $[\gamma]_*(\theta_1^\infty\langle 0\rangle)$ over $\check{\xi}\langle 1\rangle$, respectively. Since $\theta_1^\infty\langle 0\rangle$ and $[\gamma]_*(\theta_1^\infty\langle 0\rangle)$ start with the same vanishing cycle $\sigma_1^\infty\langle 0\rangle$ by Claim 2, we can choose T and T_γ in such a way that their restrictions to the sub-path η_1 coincide;

$$T|_{\eta_1} \;=\; T_\gamma|_{\eta_1}. \tag{9.4.21}$$

(See Definition 4.5 for the definition of the restriction to a sub-path.) Let T' be the restriction of T to the sub-path η_2, and T'_γ the restriction of T_γ to the sub-path $\xi_1^0\langle 1\rangle\delta_b^0\langle 1\rangle\eta_3$. Then we have $\partial T' = \partial T'_\gamma$, and hence we obtain an n-cycle

$$T'' := T'_\gamma - T' \;:\; CS^{n-1} \cup (-CS^{n-1}) \longrightarrow F_b$$

over the path $\xi_1^0\langle 1\rangle\delta_b^0\langle 1\rangle\eta_3\eta_2^{-1}$ from $p_1(b)\langle 0\rangle$ to $p_1(b)\langle 1\rangle$. Its homology class is

$$[T''] \;=\; [T'_\gamma] - [T'] \;=\; [T_\gamma] - [T] \;=\; [\gamma]_*(\bar{\theta}_1^\infty\langle 0\rangle) - \bar{\theta}_1^\infty\langle 0\rangle \;=\; v'.$$

Here we have used (9.4.21). This again shows that $v' \in H_n(F_b)$. The restriction $T_{(q)}$ of this n-cycle T'' to the sub-path $\xi_1^0\langle 1\rangle$ represents a thimble for $[\xi_1^0\langle 1\rangle]$; that is $\theta([\xi_1^0])\langle 1\rangle$ or $-\theta([\xi_1^0])\langle 1\rangle$. Hence its homology class is either $qv^\vee$ or $-qv^\vee$. Let $T_{(1)}$ be the restriction of $-T''$ to the sub-path $\eta_2\eta_3^{-1}$. Since $[\eta_2\eta_3^{-1}] = [\xi_1^0\langle 0\rangle]$ in $[\mathcal{P}_b^\sim(a_b^0\langle 0\rangle, p_1(b)\langle 0\rangle)]$, $T_{(1)}$ represents a thimble for $[\xi_1^0\langle 0\rangle]$. Hence its homology class is either $v^\vee$ or $-v^\vee$. Since the remaining part of T'' after deleting $T_{(q)}$ and $-T_{(1)}$ is contained in $\partial_0 F_b$, we have

$$[T''] \;=\; [T_{(q)}] - [T_{(1)}] \;=\; \pm qv^\vee \pm v^\vee \quad \in \quad H_n(F_b, \partial_0 F_b).$$

Since $v' \in H_n(F_b)$ by (9.4.19), we obtain

$$v' = (q-1)v^{\check{}}, \quad \text{or} \quad v' = -(q-1)v^{\check{}},$$

from (6.16). The sign is determined by the condition (9.4.17) and Lemma 4.1. □

As in Remarks 9.2.3 and 9.2.4, we get the following:

Corollary 9.4.1. *Let γ and γ' be simple loops around $\mathcal{D}_0$ with the base point b and b', respectively. Let α be a path from b to b' in $\mathcal{U}$ such that $[\alpha]^{-1}[\gamma'][\alpha] = [\gamma]$ holds in $\pi_1(\mathcal{U}, b)$. Then we have an equality $(v[\gamma'], v^{\check{}}[\gamma']) \equiv [\alpha]_*(v[\gamma], v^{\check{}}[\gamma])$ modulo $U(\mathbb{Z}[q,q^{-1}])$ in $H_n(F_{b'}) \times H_n(F_{b'}, \partial_0 F_{b'})$.* □

§9.5. A generator of $H_n(F_b)$ as a $\pi_1(\mathcal{U})$-module

Let $\mathbb{Z}[q,q^{-1}][\pi_1(\mathcal{U}, b)]$ be the group ring of $\pi_1(\mathcal{U}, b)$ with coefficients in $\mathbb{Z}[q,q^{-1}]$. We can consider $H_n(F_b)$, $H_n(F_b, \partial_0 F_b)$ and $H_n(F_b, \partial_\infty F_b)$ as modules over this ring in a natural way.

Theorem 9.5.1. *Let $\gamma : I \to \mathcal{U}$ be a simple loop around $\mathcal{D}_0$ with the base point b. Then $v^{\check{}}[\gamma]$ in Theorem 9.2.1 generates the $\mathbb{Z}[q,q^{-1}][\pi_1(\mathcal{U}, b)]$-module $H_n(F_b, \partial_0 F_b)$, and $v[\gamma]$ generates the $\mathbb{Z}[q,q^{-1}][\pi_1(\mathcal{U}, b)]$-module $H_n(F_b)$.*

Before proving Theorem 9.5.1, we need some preparation.

Remark 9.5.1. From now on to the end of this subsection, we use $\varepsilon' = \min\{\varepsilon, r\}$ instead of ε, where r is a sufficiently small positive real number. In particular, we always assume that $a_b^0 = r$ for a base point b given at the outset of each argument.

Definition 9.5.1. We define $\mathcal{U}_N^* \subset \mathcal{U}_N^\sim$ to be the locus of all $u \in \mathcal{U}_N^\sim$ such that, if p_i and p_j are distinct values in $\mathrm{Cr}(u)$, then $|\arg p_i - \arg p_j|$ is not 0 nor π.

It is obvious that $\Gamma \setminus \mathcal{U}_N^*$ is a real semi-algebraic subset of real codimension 1.

Lemma 9.5.1. *Let b be a point of $\mathcal{U}_N^*$, and let $\mathrm{Cr}(b)$ be $\{p_1, \dots, p_N\}$. Let $\lambda_i^0 : I \to \mathbb{C}$ be the path given by $t \mapsto (1-t)r + t \cdot p_i$, where r is the small positive real number in Remark 9.5.1. Then λ_i^0 is an element of $\mathcal{P}_b(a_b^0, p_i)$. Moreover, there exist paths $\xi_i^0 \in \mathcal{P}_b(a_b^0, p_i)$ for $i = 1, \dots, N$ such that $[\xi_i^0] = [\lambda_i^0]$ in $[\,\mathcal{P}_b(a_b^0, p_i)\,]$ for each i, and that $\{\xi_1^0, \dots, \xi_N^0\}$ is a K-regular system of paths from a_b^0.*

Proof. By the definition of $\mathcal{U}_N^*$, the path $t \mapsto t \cdot p_i$ on $\mathbb{C}$ from 0 to p_i does not pass through any critical values of $\hat{\phi}_b$ other than p_i. Since r is small enough, λ_i^0 is also disjoint from $\mathrm{Cr}(b) \setminus \{p_i\}$. Hence $\lambda_i^0 \in \mathcal{P}_b(a_b^0, p_i)$. We put

$$K_b^* := K_b \cup \{\, z \in \mathbb{C} \;;\; |z| \le \varepsilon(b), \text{ and } -\pi + \eta(b)/2 \le \arg(z - r) \le \pi - \eta(b)/2 \},$$

where η is the function defined by (6.2). Then each λ_i^0 is contained in K_b^*. It is easy to see that there is a continuous map $g : K_b^* \to K_b$ which satisfies the following; (i) g is a homeomorphism, (ii) g is a homotopy inverse of the inclusion $K_b \hookrightarrow K_b^*$, and (iii) $g(p_i) = p_i$

for all $p_i \in \mathrm{Cr}(b)$, and $g(a_b^0) = a_b^0$. We put $\xi_i^0 := g \circ \lambda_i^0$. Then we have $[\xi_i^0] = [\lambda_i^0]$ in $[\mathcal{P}_b(a_b^0, p_i)]$. The paths $\lambda_1^0, \dots, \lambda_N^0$ are injective, and they satisfy $\lambda_i^0(I) \cap \lambda_j^0(I) = \{a_b^0\}$ if $i \neq j$. Hence, by (i) and (iii), the system $\{\xi_1^0, \dots, \xi_N^0\}$ of paths from a_b^0 is K-regular. $\square$

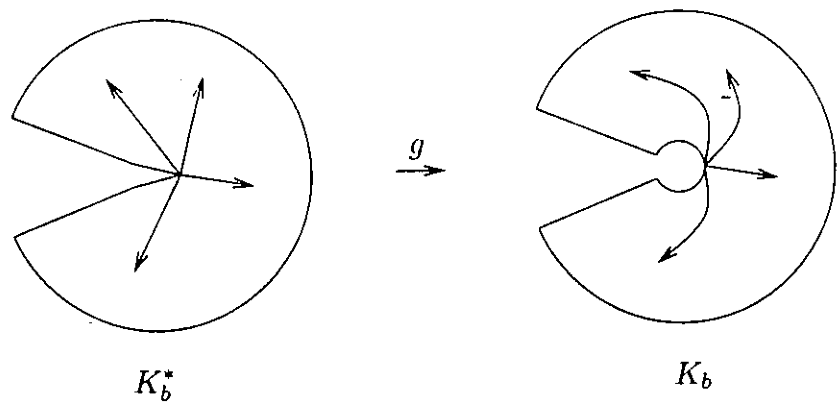

Figure 5

Then Theorem 9.5.1 follows from the following:

Proposition 9.5.1. *Suppose that $b \in \mathcal{U}_N^*$. Let p be a value in $\mathrm{Cr}(b)$, and let $\lambda^0 : I \to \mathbb{C}$ be the path from $a_b^0 := r$ to p given by $t \mapsto (1-t)r + t \cdot p$. Then there exists a simple loop γ_0 in $\mathcal{U}$ with the base point b such that $v\check{}[\gamma_0] \equiv \bar{\theta}([\lambda^0])\langle 0\rangle$ in $H_n(F_b, \partial_0 F_b)$.*

Proof of Theorem 9.5.1. First, we derive Theorem 9.5.1 from Proposition 9.5.1. Since $v[\gamma] = \pm(1-q)v\check{}[\gamma]$ and $H_n(F_b) = (1-q)H_n(F_b, \partial_0 F_b)$, the second assertion follows from the first.

It is enough to prove this theorem under the assumption that $b \in \mathcal{U}_N^*$. We put $\mathrm{Cr}(b) = \{p_1, \dots, p_N\}$, and let $\lambda_i^0 : I \to \mathbb{C}$ denote the path given by $t \mapsto (1-t)r + t \cdot p_i$. By Lemma 9.5.1, there exists a K-regular system $\{\xi_1^0, \dots, \xi_N^0\}$ of paths from $a_b^0 = r$ such that $[\xi_i^0] = [\lambda_i^0]$ in $[\mathcal{P}_b(a_b^0, p_i)]$. In particular, we have $\bar{\theta}([\xi_i^0])\langle 0\rangle = \pm\bar{\theta}([\lambda_i^0])\langle 0\rangle$ in $H_n(F_b, \partial_0 F_b)$ for $i = 1, \dots, N$. By Proposition 7.1, we see that $\bar{\theta}([\xi_1^0])\langle 0\rangle, \dots, \bar{\theta}([\xi_N^0])\langle 0\rangle$ generate $H_n(F_b, \partial_0 F_b)$ as $\mathbb{Z}[q, q^{-1}]$-module. Hence, by Proposition 9.5.1, there exist simple loops $\gamma_1, \dots, \gamma_N$ around $\mathcal{D}_0$ with the base point b such that $v\check{}[\gamma_1], \dots, v\check{}[\gamma_N]$ generate $H_n(F_b, \partial_0 F_b)$ as $\mathbb{Z}[q, q^{-1}]$-module. On the other hand, by Proposition 9.1.1, there exists $[\alpha_i] \in \pi_1(\mathcal{U}, b)$ for each i such that $[\alpha_i]^{-1}[\gamma_i][\alpha_i] = [\gamma]$, where γ is the simple loop given in the statement of Theorem 9.5.1. By Corollary 9.4.1, we have $v\check{}[\gamma_i] \equiv [\alpha_i]_* v\check{}[\gamma]$. Hence $v\check{}[\gamma]$ generates $H_n(F_b, \partial_0 F_b)$ as a $\mathbb{Z}[q, q^{-1}][\pi_1(\mathcal{U}, b)]$-module. $\square$

Proof of Proposition 9.5.1. We use the following notation; for two values $w, z \in \mathbb{C}$, we denote by $\lambda[w, z] : I \to \mathbb{C}$ the path given by $t \mapsto (1-t)w + tz$, and by $\Lambda[w, z]$ its image $\lambda[w, z](I) \subset \mathbb{C}$.

Let $\{c_1, \dots, c_N\}$ be the intersection points of $\mathcal{L}_b$ and $\mathcal{D}_0$. For $u \in \mathcal{L}_b$, we put $p_i(u) := \iota_u^{-1}(c_i)$. We have $\mathrm{Cr}(u) = \{p_1(u), \dots, p_N(u)\}$. By renumbering $c_1, \dots, c_N$, we assume that the point $p \in \mathrm{Cr}(b)$ given in the statement of Proposition 9.5.1 is $p_1(b)$. In particular, we have $\lambda^0 = \lambda[r, p_1(b)]$

Since $b \in \mathcal{U}_N^*$, we have

$$\Lambda[p_1(b), 0] \ \cap \ \mathrm{Cr}(b) \ = \ \{p_1(b)\}. \tag{9.5.1}$$

Let $\Delta_i \subset \mathbb{C}$ denote the closed disk with the center $p_i(b)$ and of radius $2r$ for $i = 1, \ldots, N$. Since r is sufficiently small, there exists a point $p_1(b) - \rho$ on the boundary $\partial\Delta_1$ such that

$$\Lambda[p_1(b) - \rho, 0] \ \cap \ \Delta_i \ = \ \begin{cases} \{p_1(b) - \rho\} & \text{if } i = 1, \text{ and} \\ \emptyset & \text{if } i \neq 1, \end{cases} \tag{9.5.2}$$

where ρ is a complex number with $|\rho| = 2r$. Moreover, we may assume that

$$\begin{array}{ccl} \mathrm{Im}\ \rho > 0 & \text{if} & \mathrm{Im}\ p_1(b) > 0, \\ \mathrm{Im}\ \rho = 0 \ \text{ and } \ \mathrm{Re}\ \rho > 0 & \text{if} & \mathrm{Im}\ p_1(b) = 0, \quad \text{and} \\ \mathrm{Im}\ \rho < 0 & \text{if} & \mathrm{Im}\ p_1(b) < 0. \end{array} \tag{9.5.3}$$

(Note that if $\mathrm{Im}\ p_1(b) = 0$, then $\mathrm{Re}\ p_1(b) > 0$ because of $b \in \mathcal{U}_N^\sim$.) We put

$$b' \ = \ \iota_b\ (p_1(b) - \rho),$$

and let γ_0' be the counter-clockwise loop along $\iota_b\ (\partial\Delta_1)$ with the base point b'. Since r is small enough, and $b \in \mathcal{U}_N^\sim$, (9.5.3) implies that

$$b' \ \in \ \mathcal{U}_N^\sim. \tag{9.5.4}$$

Since $\mathcal{L}_b$ intersects $\mathcal{D}_0$ transversely by Proposition 2.4, and r is sufficiently small, γ_0' is a simple loop around $\mathcal{D}_0$. We have $p_1(b') = \rho$ by Lemma 9.4.1. Let $D_1' \subset \mathbb{C}$ be the closed disk with the center 0 and of radius $2r$. Since r is small enough, we have $D_1' \cap \mathrm{Cr}(b') = \{p_1(b')\}$. We also have $\varepsilon'(b') = \varepsilon(b') = r$. Therefore, there is a path ξ_1^0 from $a_{b'}^0 = r$ to $p_1(b') = \rho$ which represents the homotopy class $[\lambda[r, \rho]]$ in $[\,\mathcal{P}_{b'}(a_{b'}^0, p_1(b'))\,]$ and is contained in $K_{b'} \cap D_1'$. Hence $[\lambda[r, \rho]] \ = \ [\xi_1^0]$ is the unique homotopy class in $[\,\mathcal{P}_{b'}(a_{b'}^0, p_1(b'))\,]$ characterized by (9.4.8). Using Proposition 9.4.1, we have

$$v^\sim[\gamma_0'] \ \equiv \ \bar{\theta}([\lambda[r, \rho]])\langle 0\rangle \quad \text{in} \quad H_n(F_{b'}, \partial_0 F_{b'}). \tag{9.5.5}$$

Let β be a path on $\mathcal{L}_b$ from b' to b given by

$$\beta \ := \ \iota_b \circ \lambda[p_1(b) - \rho, 0].$$

By (9.5.2), this path does not pass through any point of $\mathcal{L}_b \cap \mathcal{D}_0$, and hence it is a path in $\mathcal{U}$. We put

$$\gamma_0 \ := \ \beta\gamma_0'\beta^{-1}.$$

Since γ_0' is a simple loop around $\mathcal{D}_0$, so is the loop γ_0. We shall show that this γ_0 is the hoped-for loop; that is, $v\check{}[\gamma_0]$ is equal with $\bar{\theta}([\lambda^0])\langle 0\rangle$ in $H_n(F_b, \partial_0 F_b)$ modulo $U(\mathbb{Z}[q, q^{-1}])$. By Corollary 9.4.1, we have

$$v\check{}[\gamma_0] \equiv [\beta]_*(v\check{}[\gamma_0']) \quad \text{in} \quad H_n(F_b, \partial_0 F_b). \tag{9.5.6}$$

Combining this with (9.5.5), it is enough to prove

$$\bar{\theta}([\lambda^0])\langle 0\rangle = \pm [\beta]_*(\bar{\theta}([\lambda[r,\rho]])\langle 0\rangle) \quad \text{in} \quad H_n(F_b, \partial_0 F_b). \tag{9.5.7}$$

By Lemma 2.3, we have $\mathcal{L}_b \setminus \mathcal{D}_0 \subset \mathcal{U}_N$. Hence we have a map $[\beta]_* : Cr(b') \to Cr(b)$. The value $p_1(\beta(t))$ draws a straight path $\lambda[\rho, p_1(b)]$ on $\mathbb{C}$ by Lemma 9.4.1. Because of the assumption (9.5.3), this path does not traverse $\mathbb{R}_{\le 0}$. Hence we have

$$[\beta]_*(p_1(b')\langle\nu\rangle) = p_1(b)\langle\nu\rangle \quad \text{for all} \quad \nu \in \mathbb{Z}. \tag{9.5.8}$$

Now consider the bijective map

$$[\beta]_* : [\,\mathcal{P}_{b'}(a^0_{b'}, p_1(b'))\,] \longrightarrow [\,\mathcal{P}_b(a^0_b, p_1(b))\,]. \tag{9.5.9}$$

By (9.5.8), a lifting of (9.5.9) up to $\mathbf{C}$ is given by

$$[\beta]^{\sim}_* : [\,\mathcal{P}^{\sim}_{b'}(a^0_{b'}\langle 0\rangle, p_1(b')\langle 0\rangle)\,] \longrightarrow [\,\mathcal{P}^{\sim}_b(a^0_b\langle 0\rangle, p_1(b)\langle 0\rangle)\,].$$

In order to to prove (9.5.7), it is enough to show that

$$[\beta]_*([\lambda[r,\rho]) = [\lambda^0] \quad \text{in} \quad [\,\mathcal{P}_b(a^0_b, p_1(b))\,]. \tag{9.5.10}$$

In fact, (9.5.10) with (9.5.8) implies that

$$[\beta]^{\sim}_*([\lambda[r,\rho]\langle 0\rangle]) = [\lambda^0\langle 0\rangle] \quad \text{in} \quad [\,\mathcal{P}^{\sim}_b(a^0_b\langle 0\rangle, p_1(b)\langle 0\rangle)\,]. \tag{9.5.11}$$

Since $\theta([\lambda^0])\langle 0\rangle$ is a thimble for $[\lambda^0\langle 0\rangle]$, while $\theta([\lambda[r,\rho]])\langle 0\rangle$ is a thimble for $[\lambda[r,\rho]\langle 0\rangle]$, (9.5.11) and the formula (9.3.3) imply (9.5.7).

In order to prove (9.5.10), we investigate the movements of $a^0_{\beta(t)}$ and $p_i(\beta(t))$ when t varies from 0 to 1. There exists a large closed disk B on $\mathcal{L}_b$ with the following properties;

$$B \supset \iota_b(\Lambda[p_1(b)-\rho, 0]), \quad \text{and} \quad \varepsilon'(u) = r \quad \text{for all} \quad u \in B \setminus \bigcup_{i=1}^{N} \iota_b(\Delta_i). \tag{9.5.12}$$

Indeed, a point u on $\mathcal{L}_b$ such that $\varepsilon'(u) < r$ is either contained in $\iota_b(\Delta_i)$ for some i, or in the union of $A_j := \iota_b(\{\, z \in \mathbb{C}\,;\, |z - p_j(b)| > (2r)^{-1}\,\})$. Since r can be taken arbitrarily small, $\mathcal{L}_b \setminus \cup_{j=1}^{N} A_j$ can be so large that it contains a closed ball B which contains $\iota_b(\Lambda[p_1(b)-\rho, 0])$. By (9.5.2) and (9.5.12), we have $\varepsilon'(\beta(t)) = r$ for all $t \in I$. This implies that

$$a^0_{\beta(t)} = r \quad \text{for all} \quad t \in I. \tag{9.5.13}$$

On the other hand, by Lemma 9.4.1, we see that $p_i(\beta(t))$ draws the path $\lambda[p_i(b'), p_i(b)]$, and that $p_i(b')$ is given by $p_i(b') = p_i(b) - p_1(b) + \rho$. The track of the movement of the ending point $p_1(b') = \rho$ of $\lambda[r, \rho]$ is given by $\Lambda[\rho, p_1(b)]$. We shall see that

$$(\Lambda[r,\rho] \cup \Lambda[\rho, p_1(b)]) \ \cap \ \Lambda[p_i(b'), p_i(b)] = \emptyset \quad \text{if} \quad i \neq 1. \tag{9.5.14}$$

Indeed, the two line segments $\Lambda[0, p_1(b)]$ and $\Lambda[p_i(b) - p_1(b), p_i(b)]$ are parallel, but, if $i \neq 1$, they are not on the same line because of $b \in \mathcal{U}_N^*$. Hence they are disjoint. Since r is small enough and $|\rho| = 2r$, we see that $\Lambda[r,\rho] \cup \Lambda[\rho, p_1(b)]$ and $\Lambda[p_i(b) - p_1(b) + \rho, p_i(b)]$ are still disjoint if $i \neq 1$. Hence (9.5.14) holds. This implies that the path $\lambda[r, \rho]$ stretches to $\lambda[p_1(b), \rho] \cdot \lambda[r, \rho]$ by the movement of the ending point $p_1(\beta(t))$ of the path *without being affected by the movement of any other points* $p_2(\beta(t))$, ..., $p_N(\beta(t))$. Combining this with (9.5.13), we have

$$[\beta]_*([\lambda[r,\rho]]) \ = \ [\,\lambda[p_1(b), \rho] \cdot \lambda[r,\rho]\,]. \tag{9.5.15}$$

It is easy to see from (9.5.3) that the triangle (or the line segment if it degenerates) spanned by the three points $p_1(b)$, ρ and r does not contain 0. Moreover, since r and ρ are sufficiently small, (9.5.1) implies that this triangle does not contain any points of $\mathrm{Cr}(b) \setminus \{p_1(b)\}$. Hence we have

$$[\,\lambda[p_1(b), \rho] \cdot \lambda[r,\rho]\,] \ = \ [\,\lambda[r, p_1(b)]\,] \ = \ [\lambda^0] \quad \text{in} \quad [\,\mathcal{P}_b(a_b^0, p_1(b))\,].$$

Combining this with (9.5.15), we get (9.5.10). □

§9.6. The behavior of $\mathrm{Cr}(u)$ near $\mathcal{D}_\infty$

In this subsection, we shall investigate how the set of the values $\mathrm{Cr}(u)$ of $\hat{\phi}_u$ behaves when u approaches a point of $\mathcal{D}_\infty$. The result will be used in the proof of Theorem 9.2.2.

We choose a general affine line $\mathcal{A}$ in Γ. Let c be an intersection point of $\mathcal{A}$ and $\mathcal{D}_\infty$. Since $\mathcal{A}$ is general, c is a non-singular point of $\mathcal{D}_\infty$, and the intersection of $\mathcal{A}$ and $\mathcal{D}_\infty$ is transverse at c. Let Δ be a sufficiently small closed disk on $\mathcal{A}$ with the center c. We choose a base point b on the boundary $\partial\Delta$, and let $\gamma : I \to \mathcal{A}$ denote the counter-clockwise loop from b to b along $\partial\Delta$. Since Δ is small enough, γ is a simple loop around $\mathcal{D}_\infty$. Since $\mathcal{A}$ is general and Δ is small, we may assume that

$$\Delta \setminus \{c\} \ \subset \ \mathcal{U}_N. \tag{9.6.1}$$

Moreover, by choosing b generally we may also assume that

$$b \in \mathcal{U}_N^{\sim}. \tag{9.6.2}$$

By (9.6.1), $[\gamma]_*$ acts on the set $\mathrm{Cr}(b)$.

Proposition 9.6.1. (1) *The action of* $[\gamma]_*$ *on* $\mathrm{Cr}(b)$ *is trivial.*

This assertion with (9.6.1) enables us to put

$$\mathrm{Cr}(u) = \{p_1(u), \ldots, p_N(u)\} \quad \text{for} \quad u \in \Delta \setminus \{c\},$$

where $p_1(u)$, …, $p_N(u)$ are continuous function defined over $\Delta \setminus \{c\}$. By Proposition 2.3, they are in fact holomorphic functions.

Proposition 9.6.1 *(continued).* (2) *There exists one and only one function among* $\{p_1(u), \ldots, p_N(u)\}$, *say* $p_N(u)$, *which has a pole of order* $d-1$ *at* $u = c$. (3) *The other functions* $p_1(u)$, …, $p_{N-1}(u)$ *can be extended holomorphically over* $u = c$. (4) *The values* $p_1(c)$, …, $p_{N-1}(c)$ *are distinct to each other.* (5) *Moreover* $p_i(c) \neq 0$ *for* $i = 1, \ldots, N-1$.

Proof. Recall that $X_c \subset \mathbb{A}^n$ is the affine hypersurface corresponding to c, and $\overline{X}_c \subset \mathbb{P}^n$ its projective compactification. Since $\mathcal{A}$ is general, we have the following;

(xc-1) $\overline{X}_c$ is non-singular,

(xc-2) $\overline{X}_c$ is tangent to H_∞ at a point P, and

(xc-3) $\overline{X}_c \cap H_\infty$ has an ordinary double point at P as its only singularity.

Since Δ is located on the affine line $\mathcal{A}$, the polynomial $f_u \in \Gamma$ corresponding to $u \in \Delta$ is written in the form

$$f_u = f_c + t(u)g, \tag{9.6.3}$$

where $t : \mathcal{A} \to \mathbb{C}$ is an affine coordinate such that $t(c) = 0$, and g is a certain polynomial in Γ. We consider the punctured affine line $\mathcal{A} \cap \mathcal{U}_N$. The critical values of $\hat{\phi}_u$ and the critical points of $\hat{\phi}_u$ yield *multi-valued algebraic functions on* $\mathcal{A} \cap \mathcal{U}_N$ to $\mathbb{C}$ and to $\mathbb{A}^n$, respectively, when u moves on $\mathcal{A} \cap \mathcal{U}_N$. Let W be a small open neighborhood of b on $\mathcal{A} \cap \mathcal{U}_N$ which is simply-connected. The critical values and the critical points become single valued when they are restricted on W. Let

$$p_i : W \to \mathbb{C}, \quad \text{and} \quad q_i : W \to \mathbb{A}^n \qquad (i = 1, \ldots, N)$$

denote those single valued functions on W such that the critical point $q_i(w) \in \mathbb{A}^n$ of $\hat{\phi}_w$ is mapped to $p_i(w) \in \mathbb{C}$ by $\hat{\phi}_w : \mathbb{A}^n \to \mathbb{C}$ for all $w \in W$. The fundamental group $\pi_1(\mathcal{A} \cap \mathcal{U}_N, b)$ acts on the set $\mathrm{Cr}(b)$, and hence we get a natural homomorphism

$$m \; : \; \pi_1(\mathcal{A} \cap \mathcal{U}_N, b) \longrightarrow \mathfrak{S}(\mathrm{Cr}(b)),$$

where $\mathfrak{S}(\mathrm{Cr}(b))$ is the permutation group of the set $\mathrm{Cr}(b)$. Let

$$\rho' \; : \; (C', b') \longrightarrow (\mathcal{A} \cap \mathcal{U}_N, b)$$

be the finite étale Galois covering corresponding to m. The Galois group is isomorphisc to the image of m. Let $W' \subset C'$ be the unique connected component of $\rho'^{-1}(W)$ which contains the base point b'. Then there exist single valued algebraic functions

$$p'_i \; : \; C' \longrightarrow \mathbb{C} \quad \text{for} \quad i = 1, \ldots, N$$

such that $p'_i(w') = p_i(\rho'(w'))$ for $w' \in W'$, and there also exist algebraic morphisms

$$q'_i \;:\; C' \longrightarrow \mathbb{A}^n \quad \text{for} \quad i = 1, \ldots, N$$

such that $q'_i(w') = q_i(\rho'(w'))$ for $w' \in W'$. These functions p'_i and q'_i are determined uniquely because C' is connected. Then, for all $y \in C'$, the point $q'_i(y) \in \mathbb{A}^n$ is a critical point of $\hat{\phi}_{\rho'(y)}$ lying over the critical value $p'_i(y)$. Let

$$\rho : C \longrightarrow \mathcal{A}$$

be the finite morphism extending the étale covering $\rho' : C' \to \mathcal{A} \cap \mathcal{U}_N$, and let $\tilde{\Delta}$ be the connected component of $\rho^{-1}(\Delta)$ containing b'. There exists a unique point $\tilde{c} \in \tilde{\Delta}$ such that $\rho(\tilde{c}) = c$. The morphisms $q'_i : C' \to \mathbb{A}^n$ naturally extend to

$$\tilde{q}_i \;:\; C \longrightarrow \mathbb{P}^n,$$

and the algebraic functions $p'_i : C' \to \mathbb{C}$ naturally extend to meromorphic functions

$$\tilde{p}_i \;:\; C \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

Claim 1. There is one and only one morphism among $\{\tilde{q}_1, \ldots, \tilde{q}_N\}$, say $\tilde{q}_N$, such that $\tilde{q}_N(\tilde{c})$ is contained in H_∞. Moreover we have $\tilde{q}_N(\tilde{c}) = P$, where P is the point at which $\overline{X}_c$ is tangent to H_∞.

Proof. We choose an affine part $\mathbb{A}^{n\prime}$ of $\mathbb{P}^n$ which contains $\tilde{q}_1(\tilde{c}), \ldots, \tilde{q}_N(\tilde{c})$ and P. Let $(z_1, \ldots, z_n)$ be affine coordinates on $\mathbb{A}^{n\prime}$ such that

$$H_\infty = \{z_n = 0\}, \quad \text{and} \quad P = (0, \ldots, 0).$$

We express the homogeneous polynomials f_c and g as inhomogeneous polynomials in terms of $(z_1, \ldots, z_n)$, and we write them by $f_c(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$, respectively. Note that these inhomogeneous polynomials are determined uniquely only up to multiplications by non-zero constants. By choosing them suitably, we can write the rational function $\hat{\phi}_u = f_u/h^d$ on $\mathbb{A}^{n\prime}$ as follows;

$$\hat{\phi}_u = \frac{f_u(z_1, \ldots, z_n)}{z_n^d} = \frac{f_c(z_1, \ldots, z_n) + t(u) \cdot g(z_1, \ldots, z_n)}{z_n^d}.$$

Let $f_u^{[\nu]}(z_1, \ldots, z_n)$ denote the homogeneous part of degree ν of $f_u(z_1, \ldots, z_n)$. Then the properties (xc-1)-(xc-3) imply that

(fc-1) $f_c^{[0]} = 0$,
(fc-2) $f_c^{[1]} = a z_n$, where a is a non-zero constant, and
(fc-3) $f_c^{[2]}(z_1, \ldots, z_{n-1}, 0)$ is a non-degenerate quadratic form of $z_1, \ldots, z_{n-1}$.

We define polynomials $h_1(u; z_1, \dots, z_n)$, $\dots$, $h_n(u; z_1, \dots, z_n)$ of z_1, $\dots$, z_n as follows;

$$h_i(u; z_1, \dots, z_n) \;:=\; z_n^d \frac{\partial \hat{\phi}_u}{\partial z_i} \;=\; \frac{\partial f_u(z_1, \dots, z_n)}{\partial z_i} \quad \text{for} \quad i = 1, \dots, n-1, \quad \text{and}$$

$$h_n(u; z_1, \dots, z_n) \;:=\; z_n^{d+1} \frac{\partial \hat{\phi}_u}{\partial z_n} \;=\; z_n \frac{\partial f_u(z_1, \dots, z_n)}{\partial z_n} - d \cdot f_u(z_1, \dots, z_n).$$

By (fc-1) and (fc-2), we see that

$$\frac{\partial h_n}{\partial z_i}(c, 0, \dots, 0) = 0 \quad \text{for} \quad i = 1, \dots, n-1 \quad \text{and} \quad \frac{\partial h_n}{\partial z_n}(c, 0, \dots, 0) = (1-d)a \neq 0.$$

Combining these with (fc-3), we obtain the following;

$$\det\left[\ \frac{\partial h_i}{\partial z_j}(c, 0, \dots, 0)\ \right]_{i,j=1,\dots,n} \neq 0. \tag{9.6.4}$$

Let $H_i(u) \subset \mathbb{A}^{n\prime}$ denote the affine hypersurface defined by $h_i(u, z_1, \dots, z_n) = 0$. Because of the properties (fc-1)-(fc-3), none of $H_i(c)$ contains H_∞ as an irreducible component. Since Δ is small, the set of critical points $\{\tilde{q}_1(w), \dots, \tilde{q}_N(w)\}$ of $\hat{\phi}_{\rho(w)}$ remains contained in the affine part $\mathbb{A}^{n\prime}$ for all $w \in \tilde{\Delta}$. Hence the set $\{\tilde{q}_1(w), \dots, \tilde{q}_N(w)\}$ coincides with the intersection

$$I(\rho(w)) := H_1(\rho(w)) \cap \cdots \cap H_n(\rho(w)).$$

We will prove the following two assertions;

Sub-claim 1; $I(c) \cap H_\infty$ consists of only one point P, and

Sub-claim 2; each of $H_1(c)$, $\dots$, $H_n(c)$ is non-singular at P and they intersect transversely at P.

Indeed, the coordinates of a point in $I(c) \cap H_\infty$ are the solution of

$$z_n = f_c(z_1, \dots, z_{n-1}, 0) = 0, \quad \text{and}$$
$$\frac{\partial f_c(z_1, \dots, z_{n-1}, 0)}{\partial z_i} = 0 \quad \text{for} \quad i = 1, \dots, n-1.$$

Since $f_c(z_1, \dots, z_{n-1}, 0) = 0$ defines the hypersurface $\overline{X}_c \cap H_\infty$ on H_∞, the solution must be the coordinates of a singular point of $\overline{X}_c \cap H_\infty$, which must be $P = (0, \dots, 0)$ by the property (xc-3). Sub-claim 2 follows from the non-degeneracy (9.6.4) of the jacobian matrix of the defining equations of the hypersurfaces at P.

Let $V \subset \mathbb{A}^{n\prime}$ be a small open neighborhood of $H_\infty \cap \mathbb{A}^{n\prime}$. Then the above two assertions imply that, if $u \in \Delta$, then $V \cap I(u)$ consists of only one point, say $P(u)$, such that $P(c) = P$, because Δ is small enough. (No new points come into $\mathbb{A}^{n\prime}$ from the infinity because $\tilde{q}_1(c)$, $\dots$, $\tilde{q}_N(c)$ are already contained in $\mathbb{A}^{n\prime}$.) Thus Claim 1 is shown by putting $\tilde{q}_N(w) = P(\rho(w))$. $\square$

Claim 1 implies that the action of $[\gamma]_*$ on $\{q_1(b), \dots, q_N(b)\}$ maps $q_N(b)$ to $q_N(b)$. Hence the corresponding critical value p_N is single-valued on $\Delta \setminus \{c\}$; that is, there exists a single-valued function $p''_N : \Delta \setminus \{c\} \to \mathbb{C}$ such that $\tilde{p}_N|_{\tilde{\Delta}\setminus\{c'\}} = \rho^* p''_N$.

Note also that Claim 1 implies that the values $\tilde{p}_1(w), \dots, \tilde{p}_{N-1}(w)$ are contained in a bounded domain of $\mathbb{C}$ when w moves on $\tilde{\Delta}$. Therefore, the assertion (3) of Proposition follows from the assertion (1).

Claim 2. The function p''_N has a pole of order $d-1$ at $u = c$.

Proof. Let $(\zeta_1(u), \dots, \zeta_n(u))$ denote the coordinates of the point $q_N(u) = P(u)$ in terms of $(z_1, \dots, z_n)$ above. Then

$$\text{each } \zeta_i(u) \text{ has a zero of order 1 at } u = c. \tag{9.6.5}$$

Indeed, when u is close enough to c, the solution $(\zeta_1(u), \dots, \zeta_n(u))$ of $h_1(u; z) = \cdots = h_n(u; z) = 0$ near the origin is obtained approximately by looking at the terms of degree 0 and 1 in $(z_1, \dots, z_n)$. Because $\mathcal{A}$ is chosen generally, we may assume that

$$\frac{\partial g}{\partial z_1}(0, \dots, 0), \dots, \frac{\partial g}{\partial z_n}(0, \dots, 0) \text{ and } g(0, \dots, 0) \text{ are general complex numbers,} \tag{9.6.6}$$

where $g(z_1, \dots, z_n)$ is an inhomogeneous form of the polynomial $g \in \Gamma$ which has appeared in (9.6.3). Using (fc-1) and (fc-2), we can calculate the terms of degree 0 as follows;

$$h_i(u; 0, \dots, 0) = \begin{cases} t(u) \cdot \frac{\partial g}{\partial z_i}(0, \dots, 0) & \text{if } i \neq n, \quad \text{and} \\ -d \cdot t(u) \cdot g(0, \dots, 0) & \text{if } i = n. \end{cases} \tag{9.6.7}$$

Obviously, each of them has a zero of order 1 at $u = c$. On the other hand, the $n \times n$ matrix $(\ \partial h_i / \partial z_j(u; 0, \dots, 0)\ )$ of the coefficients of the degree 1 terms of the polynomials h_i remain non-degenerate even when $u = c$ by (9.6.4). Combining this with (9.6.6) and (9.6.7), we see that (9.6.5) holds. Since

$$p''_N(u) = \hat{\phi}_u(q_N(u)) = \frac{f_u(\zeta_1(u), \dots, \zeta_n(u))}{\zeta_n(u)^d},$$

the property (fc-1) implies that $p''_N(u)$ has a pole of order $\leq d-1$ at $u = c$.

In order to see that the order is $d-1$, we consider the variety $\mathcal{W}$ of all pairs $(f_c(z), g(z))$ of inhomogeneous polynomials of degree $\leq d$ such that $f_c(z)$ has the properties (fc-1)-(fc-3). It is easy to see that $\mathcal{W}$ is irreducible. By looking back at the proof, we can see that sub-claim 2 holds for every pair $(f_c(z), g(z)) \in \mathcal{W}$. Hence the point $P(u) = (\zeta_1(u), \dots, \zeta_n(u))$ of $I(u)$ is well-defined. We have just seen that the locus $\mathcal{W}^1 \subset \mathcal{W}$ of all pairs $(f_c(z), g(z))$ for which (9.6.5) holds is Zariski open dense. Now let $\mathcal{W}^2 \subset \mathcal{W}^1$ be the locus of all $(f_c(z), g(z))$ such that $p''_N(u)$ has a pole of order exactly $d-1$. Under (9.6.5), this condition is obviously open, and hence $\mathcal{W}^2 \subset \mathcal{W}^1$ is Zariski open. We can easily check that if we put $f^0_c(z) = z_n + z_1^2 + \cdots + z_{n-1}^2$ and $g^0(z) = 1 - 2z_1 - \cdots - 2z_{n-1}$, then $(f^0_c(z), g^0(z)) \in \mathcal{W}^2$.

Hence $\mathcal{W}^2 \subset \mathcal{W}$ is a Zariski open and *dense* subset. Now Claim 2 follows from the generality of the affine line $\mathcal{A}$; that is, the generality of the pair $(f_c(z), g(z))$. $\square$

Next we shall prove the assertions (1) and (4). Recall that $L_u^o \subset \mathbb{P}_*(\Gamma)$ is the affine line $L_u \setminus \{\mathfrak{h}_\infty\}$. The property (xc-1) implies that $\overline{X}_u$ is non-singular for all $u \in \Delta$. Hence Proposition 2.3 implies that

$$\mathrm{Cr}(u) = \bar{\iota}_u^{-1}(L_u^o \cap D_0) \quad \text{for all} \quad u \in \Delta.$$

When $u \in \Delta \setminus \{c\}$, L_u^o intersects D_0 at distinct N points transversely by Proposition 2.4 and (9.6.1). Claims 1 and 2 show that, when u approaches c, one of the intersection points tends to the point $\mathfrak{h}_\infty \in L_u$, while the other $N-1$ points remain aloof from $\mathfrak{h}_\infty$. The assertions (1) and (4) are equivalent to the fact that these $N-1$ points remain distinct even when $u = c$. In order to show this fact, it is enough to prove the following:

Claim 3. At every point of $L_c^o \cap D_0$, D_0 is non-singular, and L_c^o intersects D_0 transversely.

Proof. First note that D_∞ has a structure of the cone with the vertex $\mathfrak{h}_\infty \in D_\infty$, and L_c is the line contained in D_∞ passing through this vertex by Proposition 2.2. Hence we have $L_c \cap D_0 \subset D_0 \cap D_\infty$.

Definition 9.6.1. We define the subset $D_{0\infty}$ of $D_0 \cap D_\infty$ to be the locus of all $\overline{X} \in D_0 \cap D_\infty$ such that H_∞ is disjoint from the singular locus of $\overline{X}$. It is obvious that $D_{0\infty}$ is a Zariski open subset of $D_0 \cap D_\infty$.

Remark 9.6.1. If $v \in \mathcal{D}_\infty$ is general, then $\overline{X}_v$ is non-singular, and hence $L_v^o \cap D_0 \subset D_{0\infty}$ because of Lemma 2.1.

Lemma 9.6.1. *The locus $D_{0\infty}$ is irreducible.*

We postpone proving Lemma 9.6.1, and complete the proof of Proposition 9.6.1 first.

Let x be a point of $\mathbb{P}_*(\Gamma)$ and let $\tilde{x}$ be a point of $\Gamma^\times$ such that $pr(\tilde{x}) = x$. Since $L_{\tilde{x}}^o \subset \mathbb{P}_*(\Gamma)$ does not depend on the choice of $\tilde{x}$, we can write L_x^o instead of $L_{\tilde{x}}^o$. Consider the locus $\mathcal{G}$ of all points $x \in D_\infty \setminus \{\mathfrak{h}_\infty\}$ such that, at every point of $L_x^o \cap D_0$, D_0 is non-singular, and L_x^o intersects D_0 transversely. This locus $\mathcal{G}$ is obviously Zariski open in $D_\infty \setminus \{\mathfrak{h}_\infty\}$. By the generality of the position of c in $\mathcal{D}_\infty$, it is enough to show that $\mathcal{G}$ is non-empty. Using Lemma 9.6.1 and Remark 9.6.1, we can reduce the claim $\mathcal{G} \neq \emptyset$ to the following; there exists at least one point $y \in D_{0\infty}$ such that D_0 is non-singular at y and that L_y^o intersects D_0 transversely at y.

Let $(\mathbb{P}^n)\check{}$ denote the dual projective space of $\mathbb{P}^n$, and let $Ver : (\mathbb{P}^n)\check{} \to \mathbb{P}_*(\Gamma)$ be the morphism given by $H \mapsto d \cdot H$. Note that Ver is projectively equivalent to the Veronese embedding of degree d. Let $\overline{X}_w$ be the singular projective hypersurface corresponding to a general point w of D_0, and let $\overline{X}\check{}_w \subset (\mathbb{P}^n)\check{}$ be the dual hypersurface of $\overline{X}_w$; that is, the closure of the locus of all hyperplanes in $\mathbb{P}^n$ which are tangent to $\overline{X}_w$ at its non-singular point. Because $\overline{X}_w$ has one ordinary double point as its only singularity, and because of $d \geq 3$, we see that, for a general point $H \in \overline{X}\check{}_w$, the singular point of $\overline{X}_w$ is disjoint from H. Note also that the degree of $\overline{X}\check{}_w$ is $\geq d+1$ because of $d \geq 3$. Hence $Ver(\overline{X}\check{}_w)$ is not contained in any hyperplane of $\mathbb{P}_*(\Gamma)$. Note that D_0 is non-singular at w

because w is general in D_0. Let $T_w \subset \mathbb{P}_*(\Gamma)$ be the tangent hyperplane of D_0 at w. Then $Ver(\overline{X}_w^{\vee}) \cap T_w$ is of codimension 1 in $Ver(\overline{X}_w^{\vee})$. Hence there exists a hyperplane $H_1 \in \overline{X}_w^{\vee}$ with the following properties;

(h1) $Ver(H_1) \notin T_w$,

(h2) H_1 is tangent to $\overline{X}_w$ at its non-singular point, and

(h3) $\mathrm{Sing}(\overline{X}_w) \cap H_1 = \emptyset$.

Now we consider the automorphism group PGL $(n+1)$ of $\mathbb{P}^n$. This group acts on $(\mathbb{P}^n)^{\vee}$ and $\mathbb{P}_*(\Gamma)$ in such a natural way that Ver is equivariant. Note that $D_0 \subset \mathbb{P}_*(\Gamma)$ is stable under this action. There is an element $g \in$ PGL $(n+1)$ such that $g(H_1) = H_\infty$. Consider the point $g(w) \in D_0$, which corresponds to the singular hypersurface $g(\overline{X}_w) \subset \mathbb{P}^n$. Then D_0 is also non-singular at $g(w)$, and the tangent hyperplane $T_{g(w)} \subset \mathbb{P}_*(\Gamma)$ to D_0 at $g(w)$ is given by $g(T_w)$. By (h2) and (h3), we see that $g(w) \in D_{0\infty}$. Because of (h1), $\mathfrak{h}_\infty = Ver(H_\infty) = g(Ver(H_1))$ is not contained in $T_{g(w)}$. Hence $L^o_{g(w)}$ intersects D_0 transversely. $\square$

The last assertion (5) of Proposition 9.6.1 follows from the generality of the position of c in $\mathcal{D}_\infty$. Indeed, if we replace c with c' such that $f_{c'} = f_c + a \cdot h^d$ for some $a \in \mathbb{C}$, we have $p_i(c') = p_i(c) + a$. $\square$

Proof of Lemma 9.6.1. For a singular projective space $\overline{X} \in D_0$, we put

$$\overline{X}^* := \{\ H \in (\mathbb{P}^n)^{\vee}\ ;\quad H \text{ is tangent to } \overline{X} \text{ at its non-singular point}\}, \quad \text{and}$$
$$\overline{X}^{**} := \{\ H \in \overline{X}^*\ ;\ H \cap \mathrm{Sing}\, \overline{X} = \emptyset\}.$$

The dual hypersurface $\overline{X}^{\vee}$ is the closure of $\overline{X}^*$. If $\overline{X} \in D_0$ is general, then $\overline{X}^{**}$ is an irreducible locally Zariski closed subset of codimension 1 in $(\mathbb{P}^n)^{\vee}$, because of $d \geq 3$. There are no $\overline{X} \in D_0$ such that $\overline{X}^{**}$ is Zariski open dense in $(\mathbb{P}^n)^{\vee}$. We put

$$\mathfrak{X}^{**} \ := \ \{\ (\overline{X}, H)\ ;\ H \in \overline{X}^{**}\} \ \subset \ D_0 \times (\mathbb{P}^n)^{\vee}.$$

Since D_0 is irreducible, the above consideration implies that there exists only one irreducible component $\mathfrak{X}^{**}_{\max}$ of $\mathfrak{X}^{**}$ which is mapped dominantly onto D_0 by the first projection, and moreover, if there exists any other irreducible component $\mathfrak{X}^{**}_1$ of $\mathfrak{X}^{**}$, then we have

$$\dim \mathfrak{X}^{**}_1 \ < \ \dim \mathfrak{X}^{**}_{\max}. \tag{9.6.8}$$

Now consider the second projection

$$pr_2 \ :\ \mathfrak{X}^{**} \ \longrightarrow \ (\mathbb{P}^n)^{\vee}.$$

This projection is a locally trivial fiber space in the sense of complex analytic geometry, because PGL $(n+1)$ acts on both of $\mathfrak{X}^{**}$ and $(\mathbb{P}^n)^{\vee}$ in such a natural way that pr_2 is equivariant, and because this action is transitive on $(\mathbb{P}^n)^{\vee}$. The space $D_{0\infty}$ is nothing but the fiber of pr_2 over $H_\infty \in (\mathbb{P}^n)^{\vee}$. Since $D_{0\infty}$ is Zariski open in $D_0 \cap D_\infty$, every irreducible component of $D_{0\infty}$ is of codimension 2 in $\mathbb{P}_*(\Gamma)$. Hence every irreducible component of $\mathfrak{X}^{**}$

must have a same dimension. Combining this with (9.6.8), we see that $\mathfrak{X}^{**}$ is irreducible. Therefore the fiber $D_{0\infty}$ of pr_2 must be irreducible because $(\mathbb{P}^n)\check{}$ is simply connected. $\square$

Remark 9.6.2. The locus $D_0 \cap D_\infty$ consists of two irreducible components; one is the closure of $D_{0\infty}$ defined above, and the other is the locus of all singular hypersurfaces $\overline{X}$ such that $\operatorname{Sing} \overline{X} \cap H_\infty \neq \emptyset$.

§9.7. Proof of Theorem 9.2.2

We take an affine line $\mathcal{A} \subset \Gamma$, a small closed disk $\Delta \subset \mathcal{A}$ with the center $c \in \mathcal{A} \cap \mathcal{D}_\infty$, and the base point $b \in \partial\Delta$ of the simple loop γ around $\mathcal{D}_\infty$ as in the beginning of §9.6. By (9.6.2), we have $K_b \subset \mathbb{C}^\times$, and the isomorphisms Ψ_b^0 and Ψ_b^∞.

Proposition 9.7.1. *There exist a non-zero element e in the kernel of the natural homomorphism $H_{n-1}(X_b^0) \to H_{n-1}(\overline{X}_b^0)$ and a $\mathbb{Z}$-linear form $l : H_{n-1}(X_b^0) \to \mathbb{Z}$ such that the monodromy action $[\gamma]_*$ on $H_{n-1}(X_b^0)$ is given by*

$$x \mapsto x + l(x) \cdot e. \tag{9.7.1}$$

Moreover the pair (e, l) is unique up to sign.

This proposition will be proved later together with Proposition 9.7.2 below.

From now on, we use ε defined by (3.3) again. Since Δ is small enough, Proposition 9.6.1 implies that

$$\varepsilon(\gamma(t)) = (2|p_N(\gamma(t))|)^{-1}, \quad \text{and} \quad |p_N(\gamma(t))| > |p_i(\gamma(t))| \quad \text{for} \quad i \neq N \tag{9.7.2}$$

hold for all $t \in I$. Consider the domain

$$\Delta_{2\varepsilon(b)}(\infty) \ := \ \{ \ z \in \mathbb{C} \cup \{\infty\} \ ; \ |z| \geq 1/2\varepsilon(b)\}.$$

It is obvious that $K_b \cap \Delta_{2\varepsilon(b)}(\infty)$ is simply connected and its intersection with $\operatorname{Cr}(b)$ consists of only $p_N(b)$ because of (9.7.2). Therefore, there exists a unique homotopy class of paths

$$[\xi_N^\infty] \ \in \ [\,\mathcal{P}_b(a_b^\infty, p_N(b))\,]$$

which is represented by a path ξ_N^∞ such that

$$\xi_N^\infty(I) \ \subset \ K_b \cap \Delta_{2\varepsilon(b)}(\infty). \tag{9.7.3}$$

Now by Remarks 9.2.3 and 9.2.4 in §9.2, Theorem 9.2.2 follows from the following:

Proposition 9.7.2. *Let*

$$v\check{} := \tilde{\theta}([\xi_N^\infty])\langle 0\rangle \ \in \ H_n(F_b, \partial_\infty F_b)$$

be the homology class of the lifted thimble $\theta([\xi_N^\infty])\langle 0\rangle$. Then $v\check{}$ is primitive, and there is an element $v_2 \in H_n(F_b)$ such that the monodromy action $[\gamma]_*$ on $H_n(F_b, \partial_0 F_b)$ is given by

$$x \mapsto x \pm (x, v\check{})_\infty \cdot \{-\Psi_b^0(e \otimes 1) + v_2\}, \tag{9.7.4}$$

where $e \in H_{n-1}(X_b^0)$ is the element in Proposition 9.7.1. Let $\sigma[\xi_N^\infty] \in [S^{n-1}, X_b^\infty]$ be the vanishing cycle from which the thimble $\theta([\xi_N^\infty])$ starts, so that $v^\check{} = -\Psi_b^\infty(\sigma[\xi_N^\infty] \otimes 1)$. Then v_2 is written as follows;

$$\begin{aligned} v_2 &= \Psi_b^\infty(\,\sigma[\xi_N^\infty] \otimes (\pm q^{-d+1} + a_{-d+2}q^{-d+2} + \cdots + a_{-1}q^{-1})\,) + \Psi_b^\infty(w \otimes 1) \\ &= \quad - \,(\pm q^{-d+1} + a_{-d+2}q^{-d+2} + \cdots + a_{-1}q^{-1}) \cdot v^\check{} \quad + \Psi_b^\infty(w \otimes 1), \end{aligned} \tag{9.7.5}$$

where $a_{-d+2}, \ldots, a_{-1}$ are certain integers, and $w \in H_{n-1}(X_b^\infty)$ is a certain homology class.

Remark 9.7.1. The fact that the coefficient of q^{-d+1} in (9.7.5) is 1 or -1 plays an important role in the proof on Irreducibility Theorem in the next section.

Remark 9.7.2. We can determine neither the combination of signs in (9.7.4) and (9.7.5), nor the values of the integers $a_{-d+2} \ldots, a_{-1}$. We would like to fill up this unsatisfactory part of the theory in future.

Proof of Propositions 9.7.1 and 9.7.2. We write the set $\mathrm{Cr}(b)$ simply by $\{p_1, \ldots, p_N\}$ instead of $\{p_1(b), \ldots, p_N(b)\}$. The movement $\mathcal{M}_\gamma$ of the distinguished points $\mathcal{S}_b := Cr(b) \cup Z_b^0 \cup Z_b^\infty$ in $\mathbf{C}$ along the loop γ is homotopically equivalent to the movement $\mathcal{M}'_\gamma : I \times \mathcal{S}_b \to \mathbf{C}$ described as follows; the points $a_b^0\langle\nu\rangle$ and $a_b^\infty\langle\nu\rangle$ remain fixed, the points $p_i\langle\nu\rangle$ also remain fixed if $i \neq N$, and they stay left-hand side of the vertical line

$$\Lambda_\infty \;:=\; \log|p_N| + \sqrt{-1}\mathbb{R} \;=\; e^{-1}(\partial\Delta_{2\varepsilon(b)}(\infty)),$$

while the point $p_N\langle\nu\rangle$ moves down to $p_N\langle\nu - d + 1\rangle$ along the vertical line Λ_∞. This can be seen as follows. Let $p_i(\Delta) \subset \mathbb{C} \cup \{\infty\}$ be the image of the meromorphic function $\Delta \to \mathbb{C}\cup\{\infty\}$ corresponding to the i-th critical value. Since Δ is small enough, Proposition 9.6.1 implies that

$$p_i(\Delta) \;\subset\; \mathbb{C}^\times \quad \text{if} \quad i \neq N, \tag{9.7.6}$$

and

$$p_i(\Delta) \cap p_j(\Delta) \;=\; \emptyset \quad \text{if} \quad i \neq j. \tag{9.7.7}$$

The movements of $a_b^0\langle\nu\rangle$ and $a_b^\infty\langle\nu\rangle$ are homotopically equivalent to the *non*-movement by Reduction 1 in §9.3. By (9.7.6) and (9.7.7), if $i \neq N$, then the movement of $p_i\langle\nu\rangle$ is also homotopically equivalent to the *non*-movement by Reduction 2 in §9.3. On the other hand, Proposition 9.6.1 (2) implies that $p_N \in \mathbb{C}^\times$ makes round trips around ∞ in the clockwise direction $(d-1)$-times. Combining this with (9.7.7), the trace of the movement of $p_N\langle\nu\rangle$ can be deformed to the segment of Λ_∞ between $p_N\langle\nu\rangle$ and $p_N\langle\nu - d + 1\rangle$ without affecting the movements of the other distinguished points.

There exists a K-regular system $\{\xi_1^0, \ldots, \xi_N^0\}$ of paths from a_b^0 which satisfies the following property;

$$\xi_i^0(I) \cap \Delta_{2\varepsilon(b)}(\infty) \;=\; \begin{cases} \emptyset & \text{if } i \neq N, \text{ and} \\ \{p_N\} & \text{if } i = N. \end{cases} \tag{9.7.8}$$

We choose a vanishing cycle

$$\sigma_i^0 := \sigma[\xi_i^0] \in [S^{n-1}, X_b^0]$$

for each $[\xi_i^0]$. By Proposition 5.1, their homology classes $\bar\sigma_1^0, \ldots, \bar\sigma_N^0$ form a set of basis of the free $\mathbb{Z}$-module $H_{n-1}(X_b^0)$. We define a $\mathbb{Z}$-linear form $l : H_{n-1}(X_b^0) \to \mathbb{Z}$ by

$$l(\bar\sigma_i^0) := \begin{cases} 0 & \text{if } i \neq N, \text{ and} \\ 1 & \text{if } i = N. \end{cases} \tag{9.7.9}$$

By Proposition 9.6.1 (1) and (9.6.1), $[\gamma]_*$ acts on the set $[\,\mathcal{P}_b(a_b^0, p_i)\,]$, and if $i \neq N$, it also acts on $[\,\mathcal{P}_b^\sim(a_b^0\langle\nu\rangle, p_i\langle\nu\rangle)\,]$ by the description of $\mathcal{M}'_\gamma$ above. From (9.7.6) and (9.7.7), we can easily see that

$$[\gamma]_*([\xi_i^0]) = [\xi_i^0] \quad \text{in} \quad [\,\mathcal{P}_b(a_b^0, p_i)\,] \quad \text{for} \quad i \neq N, \tag{9.7.10}$$

and, by the description of the movement $\mathcal{M}'_\gamma$, which is homotopically equivalent to $\mathcal{M}_\gamma$, we have

$$[\gamma]_*([\xi_i^0\langle\nu\rangle]) = [\xi_i^0\langle\nu\rangle] \quad \text{in} \quad [\,\mathcal{P}_b^\sim(a_b^0\langle\nu\rangle, p_i\langle\nu\rangle)\,] \quad \text{for} \quad i \neq N. \tag{9.7.11}$$

Since $[\gamma]_*(\sigma_i^0) \in [S^{n-1}, X_b^0]$ is a vanishing cycle for $[\gamma]_*([\xi_i^0])$, (9.7.10) implies that, if $i \neq N$, then $[\gamma]_*(\sigma_i^0)$ is either σ_i^0 or $-\sigma_i^0$. We shall show that

$$[\gamma]_*(\sigma_i^0) = \sigma_i^0 \quad \text{for} \quad i \neq N, \tag{9.7.12}$$

and that

$$e := [\gamma]_*(\bar\sigma_N^0) - \bar\sigma_N^0 \quad \in \quad \mathrm{Ker}\,(H_{n-1}(X_b^0) \to H_{n-1}(\overline{X}_b^0)). \tag{9.7.13}$$

First note that the action of $[\gamma]_*$ on $H_n(\overline{X}_b^0)$ is trivial. Indeed, since Δ is small enough, the property (xc-1) in §9.6 implies that $[\gamma]_*$ acts on $H_{n-1}(\overline{X}_b)$ trivially. By Lemma 5.1, $[\gamma]_*$ acts on $H_{n-1}(\overline{X}_b^0)$ also trivially. This, in particular, implies (9.7.13). Second, note that the image $\bar\sigma_i^{0\,\prime}$ of $\bar\sigma_i^0$ by the natural homomorphism $H_{n-1}(X_b^0) \to H_{n-1}(\overline{X}_b^0)$ is non-zero for $i = 1, \ldots, N$. Indeed, the image of $H_{n-1}(X_b^0) \to H_{n-1}(\overline{X}_b^0)$ is, by definition, $H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0)$ in the exact sequence (5.10). Theorem L3 tells us that, for each i, the element $\bar\sigma_i^{0\prime}$ generates $H^{n-1}_{\mathrm{prim}}(\overline{X}_b^0) \otimes \mathbb{Q}$ as a $\mathbb{Q}[\pi_1(\mathcal{U})]$-module. Therefore, $\bar\sigma_i^{0\prime}$ is not zero for each $i = 1, \ldots, N$. Combining these two facts, we see that $[\gamma]_*(\bar\sigma_i^0)$ cannot be $-\bar\sigma_i^0$. Therefore, $[\gamma]_*(\sigma_i^0)$ cannot be $-\sigma_i^0$. Hence (9.7.12) is proved. We shall show that e defined in (9.7.13) is non-zero. In it were zero, then $[\gamma]_*$ would act on $H_{n-1}(X_b^0)$ trivially because of (9.7.12). However, since $Y_b = \overline{X}_b \cap H_\infty = \overline{X}_b^0 \cap H_\infty$ has an ordinary double point at P, the action of $[\gamma]_*$ on the non-zero sub-module $H^{n-2}_{\mathrm{prim}}(Y_b)$ of $H_{n-1}(X_b^0)$ is non-trivial, because $H^{n-2}_{\mathrm{prim}}(Y_b) \otimes \mathbb{Q}$ coincides with the Poincaré dual of the module of "vanishing cycles" in $H_{n-2}(Y_b) \otimes \mathbb{Q}$ in the sense of [6; §3].

Then, by (9.7.12) and the definitions (9.7.9) and (9.7.13), we obtain

$$[\gamma]_*(x) = x + l(x) \cdot e. \tag{9.7.14}$$

This formula being established, e is characterized as a generater of the image of the endomorphism $\mathrm{Id} - [\gamma]_*$ on $H_{n-1}(X_b^0)$, which is a free $\mathbb{Z}$-module of rank 1, and hence e is uniquely determined up to sign. Therefore the pair (e, l) is also unique up to sign. Thus Proposition 9.7.1 is proved.

Let
$$\theta_i^0 \;:=\; \theta([\xi_i^0], \sigma_i^0) \;\in\; [(CS^{n-1}, S^{n-1}), (E_b, X_b^0)]$$
denote the thimble for $[\xi_i^0]$ starting from σ_i^0, and let
$$\theta_i^0\langle\nu\rangle \;\in\; [(CS^{n-1}, S^{n-1}), (F_b, X_b^0\langle\nu\rangle)]$$
denote its lifting, which is the thimble for $[\xi_i^0\langle\nu\rangle] \in [\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p_i\langle\nu\rangle)\,]$ starting from $\sigma_i^0\langle\nu\rangle$. By Proposition 7.1, the homology classes $\bar\theta_1^0\langle 0\rangle, \ldots, \bar\theta_N^0\langle 0\rangle$ in $H_n(F_b, \partial_0 F_b)$ form a set of basis over $\mathbb{Z}[q, q^{-1}]$. Hence it is enough to prove (9.7.4) when x runs through the set of these classes. By (9.7.8), the paths ξ_i^0 and ξ_N^∞ are disjoint if $i \ne N$, and the paths ξ_N^0 and ξ_N^∞ have a common ending point as their only intersection. Hence, by Lemma 8.2, we have

$$(\bar\theta_i^0\langle 0\rangle, v^{\check{}})_\infty \;=\; (\bar\theta([\xi_i^0])\langle 0\rangle, \bar\theta([\xi_N^\infty])\langle 0\rangle)_\infty \;=\; \begin{cases} 0 & \text{if } i \ne N, \text{ and} \\ \pm 1 & \text{if } i = N. \end{cases} \tag{9.7.15}$$

In particular, this shows that $v^{\check{}}$ is primitive.

We can and will choose the sign of σ_N^0 in such a way that
$$(\bar\theta_N^0\langle 0\rangle, v^{\check{}})_\infty \;=\; 1. \tag{9.7.16}$$

On the other hand, from the formula (9.3.2), the results (9.7.11) and (9.7.12) imply that
$$[\gamma]_*(\theta_i^0\langle 0\rangle) \;=\; \theta_i^0\langle 0\rangle \quad \text{for} \quad i \ne N.$$

Combining this with (9.7.15) and (9.7.16), we see that the action $[\gamma]_*$ on $H_n(F_b, \partial_0 F_b)$ is given by
$$x \;\mapsto\; x + (x, v^{\check{}})_\infty \cdot v',$$
where
$$v' \;:=\; [\gamma]_*(\bar\theta_N^0\langle 0\rangle) - \bar\theta_N^0\langle 0\rangle.$$

Now we shall show that this homology class v' is equal, up to sign, with $-\Psi_b^0(e\otimes 1) + v_2$, where v_2 is an element in $H_n(F_b)$ which can be written in the form (9.7.5) in the statement of Proposition 9.7.2.

From the description of the movement $\mathcal{M}'_\gamma$ of the distinguished points $\mathcal{S}_b$ on $\mathbf{C}$, we see that $[\gamma]_*([\xi_N^0]\langle\nu\rangle)$, which is an element of $[\,\mathcal{P}_b^{\sim}(a_b^0\langle\nu\rangle, p_N\langle\nu - d + 1\rangle)\,]$, is represented by a path $\xi''\langle\nu - d + 1\rangle$ as follows. Note that $\xi_N^0\langle\nu\rangle(I)$ is on the left-hand side of the vertical line $\Lambda_\infty = \log|p_N| + \sqrt{-1}\mathbb{R}$ because of (9.7.8). Then the path $\xi''\langle\nu - d + 1\rangle$ starts from $a_b^0\langle\nu\rangle$, goes to a point $p'_N\langle\nu\rangle := p_N\langle\nu\rangle - \kappa'$ along $\xi_N^0\langle\nu\rangle$ where κ' is a sufficiently small complex number with Re $\kappa' > 0$, goes down to $p'_N\langle\nu - d + 1\rangle = p_N\langle\nu - d + 1\rangle - \kappa'$ along the vertical line parallel to Λ_∞, and then reaches to $p_N\langle\nu - d + 1\rangle$ along $\xi^0_N\langle\nu - d + 1\rangle$.

We define the path $\xi^{0\infty}_N\langle\nu\rangle$ from $a^0_b\langle\nu\rangle$ to $a^\infty_b\langle\nu\rangle$ as follows. Note that $\xi^\infty_N\langle\nu\rangle(I)$ is on the right-hand side of the vertical line Λ_∞ because of (9.7.3). Then $\xi^{0\infty}_N\langle\nu\rangle$ goes from $a^0_b\langle\nu\rangle$ to $p'_N\langle\nu\rangle$ along $\xi^0_N\langle\nu\rangle$, draws an arc on the circle of radius $|\kappa'|$ with the center $p_N\langle\nu\rangle$ in the counter-clockwise direction to a point $p''_N\langle\nu\rangle := p_N\langle\nu\rangle + \kappa''$ on $\xi^\infty_N\langle\nu\rangle(I)$, where κ'' is a complex number such that $|\kappa'| = |\kappa''|$ and Re $\kappa'' > 0$, and then goes to $a^\infty_b\langle\nu\rangle$ along $\xi^\infty_N\langle\nu\rangle^{-1}$. Note that $\xi^{0\infty}_N\langle\nu\rangle$ is a path in $K_b\langle\nu\rangle$.

We put
$$p''_N \ := \ e(p''_N\langle\nu\rangle) \ \in \ \xi^\infty_N(I) \ \subset \ \mathbb{C}^\times,$$
and define a loop τ from a^∞_b to a^∞_b in $\mathbb{C}^\times \setminus \mathrm{Cr}(b)$ as follows; τ goes from a^∞_b to p''_N along ξ^∞_N, draws a circle of radius $|p_N - p''_N|$ with the center p_N in the counter-clockwise direction, and then goes back to a^∞_b along $(\xi^\infty_N)^{-1}$. Note that τ is a path in K_b. Now we are going to be interested exclusively in the case $\nu = 0$. It is easy to see that
$$[\gamma]_*([\xi^0_N\langle 0\rangle]) = [\xi''\langle -d+1\rangle] \ = \ [\zeta] \quad \text{in} \quad [\,\mathcal{P}^\sim_b(a^0_b\langle 0\rangle, p_N\langle -d+1\rangle)\,],$$
where $\zeta := \zeta' \cdot \xi^{0\infty}_N\langle 0\rangle$ and
$$\begin{aligned}\zeta' \ &:= \ \xi^\infty_N\langle -d+1\rangle \cdot (\delta^\infty_b\langle -d+1\rangle)^{-1} \cdot \big(\tau\langle -d+2\rangle \cdot (\delta^\infty_b\langle -d+2\rangle)^{-1}\big) \cdot \cdots \\ &\qquad \cdots \big(\tau\langle -3\rangle \cdot (\delta^\infty_b\langle -3\rangle)^{-1}\big) \cdot \big(\tau\langle -2\rangle \cdot (\delta^\infty_b\langle -2\rangle)^{-1}\big) \cdot \big(\tau\langle -1\rangle \cdot (\delta^\infty_b\langle -1\rangle)^{-1}\big).\end{aligned}$$
(See Figure 6 on the next page.)

We put
$$\sigma^* \ := \ [\gamma]_*(\sigma^0_N\langle 0\rangle)$$
This is a vanishing cycle for $[\zeta]$ in $X^0_b\langle 0\rangle$, and by formula (9.3.2), we have
$$[\gamma]_*(\theta^0_N\langle 0\rangle) \ = \ \theta([\zeta], \sigma^*) \ \in \ [(CS^{n-1}, S^{n-1}), (F_b, X^0_b\langle 0\rangle)].$$

In order to determine its homology class, we choose a continuous map $T : CS^{n-1} \to F_b$ which represents $\theta([\zeta], \sigma^*)$ over the path ζ. Let T_0 and T_1 denote the restrictions of T to the sub-paths $\xi^{0\infty}_N\langle 0\rangle$ and ζ' of ζ, respectively. As n-chains in F_b, we have $T = T_0 + T_1$. Then T_0 is a continuous map from $I \times S^{n-1}$ to $\psi^{-1}_b(K_b\langle 0\rangle)$ because of $\xi^{0\infty}_N\langle 0\rangle(I) \subset K_b\langle 0\rangle$. Its boundary is given by
$$\partial T_0 \ = \ -S^* + S',$$
where $S^* : S^{n-1} \to X^0_b\langle 0\rangle$ represents the vanishing cycle $\sigma^* = [\gamma]_*(\sigma^0_N\langle 0\rangle)$, and $S' : S^{n-1} \to X^\infty_b\langle 0\rangle$ represents a vanishing cycle for $[\zeta']$. Since $\psi^{-1}_b(K_b\langle 0\rangle)$ is contractible (cf. (6.8)), there are n-chains Γ^* and Γ' in $\psi^{-1}_b(K_b\langle 0\rangle)$ such that $\partial\Gamma^* = S^*$ and $\partial\Gamma' = S'$. The sum $T_0 + \Gamma^* - \Gamma'$ is an n-cycle in $\psi^{-1}_b(K_b\langle 0\rangle)$, which is obviously homologous to zero.

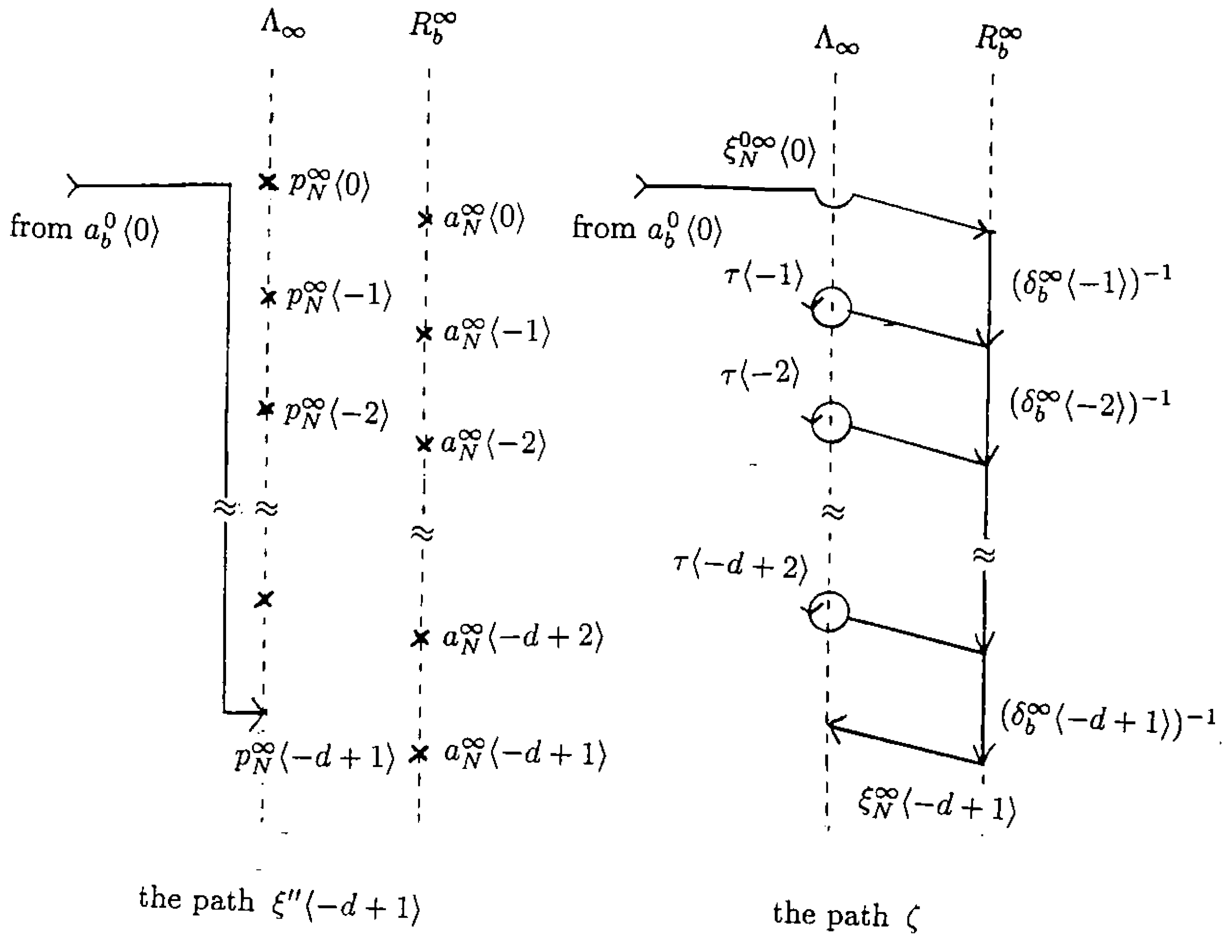

Figure 6

Hence we have

$$[\gamma]_*(\tilde{\theta}_N^0\langle 0\rangle) \;=\; [T] \;=\; [-\Gamma^*] + [\Gamma' + T_1] \quad \text{in} \quad H_n(F_b, \partial_0 F_b).$$

Note that $\Gamma' + T_1$ is an n-cycle in F_b. We put

$$v_2' \;:=\; [\Gamma' + T_1] \;\in\; H_n(F_b).$$

First let us describe the homology class $[-\Gamma^*]$ in $H_n(F_b, \partial_0 F_b)$. Since the homology class of the boundary $\partial\Gamma^* = S^*$ in $X_b^0\langle 0\rangle$ is $\bar{\sigma}^* = [\gamma]_*(\bar{\sigma}_N^0\langle 0\rangle)$, it is mapped to $[\gamma]_*(\bar{\sigma}_N^0)$ by the isomorphism $H_{n-1}(X_b^0\langle 0\rangle) \cong H_{n-1}(X_b^0)$ induced from (6.1). By the definition of e (cf. (9.7.13)), we have $[\gamma]_*(\bar{\sigma}_N^0) = \bar{\sigma}_N^0 + e$. Since Γ^* is contained in $\psi_b^{-1}(K_b\langle 0\rangle)$, we see from Corollary 6.2 that

$$-[\Gamma^*] \;=\; -\,\Psi_b^0(\,[\gamma]_*(\bar{\sigma}_N^0)\otimes 1) \;=\; -\,\Psi_b^0(\,\bar{\sigma}_N^0\otimes 1) - \Psi_b^0(e\otimes 1).$$

On the other hand, we have

$$\bar\theta^0_N\langle 0\rangle \;=\; -\,\Psi^0_b(\,\bar\sigma^0_N\,\otimes 1),$$

because of (7.2). Hence we obtain

$$v' \;=\; [\gamma]_*(\bar\sigma^0_N\langle 0\rangle) - \bar\theta^0_N\langle 0\rangle \;=\; -\,\Psi^0_b(e\otimes 1) + [\Gamma' + T_1] \;=\; -\,\Psi^0_b(e\otimes 1) + v_2'.$$

Next, we express $v_2' := [\Gamma' + T_1] \in H_n(F_b)$ by means of Ψ^∞_b, and show that it is expressed in the form (9.7.5). For $\mu = -1, -2, \dots, -d+2$, let $T_{(\mu)}$ denote the restriction of T_1 to the sub-path $\tau\langle\mu\rangle$ of ζ', and let $T_{(-d+1)}$ denote the restriction of T_1 to the ending piece $\xi^\infty_N\langle -d+1\rangle$ of ζ'. Since the restriction of T_1 to $(\delta^\infty_b\langle\mu\rangle)^{-1}$ is contained in $\partial_\infty F_b$ for all μ, we have

$$v_2' \;=\; [\Gamma' + T_1] \;=\; [\Gamma'] + [T_{(-1)}] + [T_{(-2)}] + \cdots + [T_{(-d+1)}] \quad \text{in} \quad H_n(F_b, \partial_\infty F_b). \qquad (9.7.17)$$

We define $w \in H_{n-1}(X^\infty_b)$ to be the image of the homology class

$$[\partial\Gamma'] \;=\; [S'] \;\in\; H_{n-1}(X^\infty_b\langle 0\rangle)$$

by the isomorphism $H_{n-1}(X^0_b\langle 0\rangle) \cong H_{n-1}(X^0_b)$ induced from (6.1). Since Γ' is contained in $\psi_b^{-1}(K_b\langle 0\rangle)$, we see from Corollary 6.2 that

$$[\Gamma'] \;=\; \Psi^\infty_b(w\otimes 1). \qquad (9.7.18)$$

The continuous map $T_{(-d+1)} : CS^{n-1} \to \psi_b^{-1}(K_b\langle -d+1\rangle)$ represents a thimble for $[\xi^\infty_N\langle -d+1\rangle]$ over the path $\xi^\infty_N\langle -d+1\rangle$, which is either $\theta([\xi^\infty_N])\langle -d+1\rangle$ or $-\theta([\xi^\infty_N])\langle -d+1\rangle$. Therefore we have

$$[T_{(-d+1)}] \;=\; \pm\,\bar\theta([\xi^\infty_N])\langle -d+1\rangle \;=\; \pm\, q^{-d+1}\bar\theta([\xi^\infty_N])\langle 0\rangle \;=\; \pm\, q^{-d+1} v^{\check{}}\;. \qquad (9.7.19)$$

For $\mu = -1, \dots, -d+2$, the boundary of the n-chain $T_{(\mu)} : I\times S^{n-1} \to \psi_b^{-1}(K_b\langle\mu\rangle)$ is of the form $-S_\mu + S'_\mu$, where S_μ and S'_μ are continuous maps from S^{n-1} to $X^\infty_b\langle\mu\rangle$. Their homology classes are related by

$$[S'_\mu] \;=\; [\tau\langle\mu\rangle]_*([S_\mu]) \quad \text{in} \quad H_{n-1}(X^\infty_b\langle\mu\rangle).$$

By Theorem L1 (2), the difference

$$\partial[T_{(\mu)}] \;=\; [S'_\mu] - [S_\mu] = ([\tau\langle\mu\rangle]_* - 1)[S_\mu]$$

is a multiple of the homology class of a vanishing cycle in $X^\infty_b\langle\mu\rangle$ for $[\xi^\infty_b\langle\mu\rangle]$; that is, it is written as $a_\mu\bar\sigma[\xi^\infty_N]\langle\mu\rangle$ by some integer a_μ. The class $\bar\sigma[\xi^\infty_N]\langle\mu\rangle$ is mapped to $\bar\sigma[\xi^\infty_N]$ by the isomorphism $H_{n-1}(X^\infty_b\langle\mu\rangle) \cong H_{n-1}(X^\infty_b)$ induced from (6.1). Since $\tau\langle\mu\rangle(I) \subset K_b\langle\mu\rangle$, $T_{(\mu)}$ is contained in $\psi_b^{-1}(K_b\langle\mu\rangle)$. Therefore, we see from Corollary 6.2 that

$$[T_{(\mu)}] \;=\; a_\mu\cdot\Psi^\infty_b(\,\bar\sigma[\xi^\infty_N]\,\otimes q^\mu) \;=\; -\,a_\mu\cdot q^\mu\cdot\bar\theta([\xi^\infty_N])\langle 0\rangle \;=\; -\,a_\mu\cdot q^\mu\cdot v^{\check{}}\;. \qquad (9.7.20)$$

Combining (9.7.17)-(9.7.20), we get

$$v_2' = \Psi_b^0(w \otimes 1) - (\pm q^{-d+1} + a_{-d+2}q^{-d+2} + \cdots + a_{-2}q^{-2} + a_{-1}q^{-1}) \cdot v^{\check{}},$$

and hence we get (9.7.5). $\square$

Again, by Remarks 9.2.3 and 9.2.4, we get the following:

Corollary 9.7.1. *Let γ and γ' be simple loops around $\mathcal{D}_\infty$ with the base point b and b', respectively. Let α be a path from b to b' in $\mathcal{U}$ such that $[\alpha]^{-1}[\gamma'][\alpha] = [\gamma]$ holds in $\pi_1(\mathcal{U}, b)$. Then we have an equality $(v[\gamma'], v^{\check{}}[\gamma']) \equiv [\alpha]_*(v[\gamma], v^{\check{}}[\gamma])$ modulo $U(\mathbb{Z}[q, q^{-1}])$ in $H_n(F_{b'}, \partial_0 F_{b'}) \times H_n(F_{b'}, \partial_\infty F_{b'})$.* $\square$

§10. Irreducibility of the monodromy representation

Let b be a base point on $\mathcal{U}$. In this section, we deal with the vector space

$$H_n(F_b) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$$

over the quotient field $\mathbb{Q}(q)$ of $\mathbb{Z}[q, q^{-1}]$. For brevity, we denote this space by $H_n(F_b) \otimes \mathbb{Q}(q)$. Let $\overline{\mathbb{Q}(q)}$ be the algebraic closure of $\mathbb{Q}(q)$. A representation on $H_n(F_b) \otimes \mathbb{Q}(q)$ is said to be absolutely irreducible if the induced representation on $H_n(F_b) \otimes \overline{\mathbb{Q}(q)}$ is also irreducible. The purpose of this section is to prove the following:

Irreducibility Theorem. *The monodromy representation of $\pi_1(\mathcal{U}, b)$ on $H_n(F_b) \otimes \mathbb{Q}(q)$ is absolutely irreducible.*

Proof. First remark that the natural injection $H_n(F_b) \hookrightarrow H_n(F_b, \partial_0 F_b)$ induces an isomorphism

$$H_n(F_b) \otimes \mathbb{Q}(q) \cong H_n(F_b, \partial_0 F_b) \otimes \mathbb{Q}(q), \tag{10.1}$$

because $H_n(F_b)$ is identified with $(1-q)H_n(F_b, \partial_0 F_b)$ by (6.16).

Let x be an arbitrary non-zero element of $H_n(F_b) \otimes \overline{\mathbb{Q}(q)}$, and let M be the subspace of $H_n(F_b) \otimes \overline{\mathbb{Q}(q)}$ generated by the set $\{\, [\alpha]_*(x) \,;\, [\alpha] \in \pi_1(\mathcal{U}, b)\,\}$. It is enough to show that M coincides with the total space $H_n(F_b) \otimes \overline{\mathbb{Q}(q)}$, and for this purpose, it suffices to prove that M contains an element $v[\gamma] \in H_n(F_b)$, where γ is a simple loop around $\mathcal{D}_0$, because of Theorem 9.5.1.

We consider the vector space $\Gamma = \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ as an affine part of a projective space $\mathbb{P}^{\dim\Gamma}$, and let $\mathcal{H}$ be the hyperplane $\mathbb{P}^{\dim\Gamma} \setminus \Gamma$. Then $\mathcal{U}$ is the complement to a reducible projective hypersurface $\overline{\mathcal{D}}_0 \cup \overline{\mathcal{D}}_\infty \cup \mathcal{H}$, where $\overline{\mathcal{D}}_0$ and $\overline{\mathcal{D}}_\infty$ denote the closures of $\mathcal{D}_0$ and $\mathcal{D}_\infty$, respectively. Hence, Zariski's hyperplane section theorem [11] implies that $\pi_1(\mathcal{U}, b)$ is generated by the homotopy classes of simple loops around $\mathcal{D}_0$ and $\mathcal{D}_\infty$. In particular, the generater $\mathfrak{c} \in \pi_1(\mathcal{U}, b)$ of the kernel of the natural homomorphism $\pi_1(\mathcal{U}) \to \pi_1(U)$ is written as a product

$$[\gamma_1]^{\delta_1} \cdot [\gamma_2]^{\delta_2} \cdot \cdots \cdot [\gamma_k]^{\delta_k},$$

where each γ_i is a simple loop around $\mathcal{D}_0$ or $\mathcal{D}_\infty$, and δ_i is ± 1. By Proposition 1.1, we have

$$\mathfrak{c}_*(x) = qx \neq x.$$

Hence there exists at least one element among $[\gamma_1], \ldots, [\gamma_k]$, say $[\gamma_l]$, such that $[\gamma_l]_*(x) \neq x$. By Theorems 9.2.1 and 9.2.2, we have

$$[\gamma_l]_*(x) - x = a \cdot v[\gamma_l],$$

where a is a non-zero element of $\overline{\mathbb{Q}(q)}$. Hence M contains $v[\gamma_l]$. Therefore, if γ_l is a simple loop around $\mathcal{D}_0$, then the proof is completed.

Now suppose that γ_l is a simple loop around $\mathcal{D}_\infty$. (Note that in this case, $v[\gamma_l]$ is contained in $H_n(F_b, \partial_0 F_b)$, but not in $H_n(F_b)$. However, by (10.1), we can still say that $v[\gamma_l]$ is contained in M.) The homotopy class $[\gamma_l] \in \pi_1(\mathcal{U}, b)$ is represented by a loop of the form $\alpha^{-1}\gamma_l'\alpha$, where α is a path from b to a point b' which is located in a sufficiently small neighborhood of a non-singular point c of $\mathcal{D}_\infty \setminus (\mathcal{D}_0 \cap \mathcal{D}_\infty)$, and γ_l' is a loop from b' to b' along the boundary $\partial\Delta$ of a small disk Δ with the center c. As is §9.6, we may assume that Δ is situated on an affine line $\mathcal{A}$ intersecting $\mathcal{D}_\infty$ at c transversely. Moreover, perturbing γ_l if necessary, we can assume that

$$b' \in \mathcal{U}_N^* \subset \mathcal{U}_N^\sim. \tag{10.2}$$

In particular, we have $K_{b'} \subset \mathbb{C}^\times$ and the isomorphism $\Psi_{b'}^\infty$. We write $\{p_1, \ldots, p_N\}$ for $\mathrm{Cr}(b')$.

By Proposition 9.6.1, there exists a critical value p_N of $\hat{\phi}_{b'} : \mathbb{A}^n \to \mathbb{C}$ such that

$$\varepsilon(b') = (2|p_N|)^{-1}, \quad \text{and} \quad |p_i| < |p_N| \quad \text{if} \quad i \neq N.$$

Let ξ_N^∞ be an element of $\mathcal{P}_{b'}(a_{b'}^\infty, p_N)$ such that

$$\xi_N^\infty(I) \subset \Delta_{2\varepsilon(b')}^\infty \cap K_{b'}, \tag{10.3}$$

where $\Delta_{2\varepsilon(b')}^\infty := \{\, z \in \mathbb{C} \cup \{\infty\} \,;\, |z| \geq (2\varepsilon(b'))^{-1} \,\}$. By Proposition 9.7.2 and (7.2)$'$, we can put

$$v^\sim[\gamma_l'] = \bar{\theta}([\xi_N^\infty])\langle 0\rangle = -\Psi_{b'}^\infty(\bar{\sigma}[\xi_N^\infty] \otimes 1) \in H_n(F_{b'}, \partial_\infty F_{b'}), \tag{10.4}$$

and

$$\begin{aligned}
v[\gamma_l'] &= \Psi_{b'}^0(e \otimes 1) + \Psi_{b'}^\infty(w \otimes 1) \\
&\quad + \Psi_{b'}^\infty(\,\bar{\sigma}[\xi_N^\infty] \otimes (\pm q^{-d+1} + a_{-d+2}q^{-d+2} + \cdots + a_{-1}q^{-1})\,) \\
&= \Psi_{b'}^0(e \otimes 1) + \Psi_{b'}^\infty(w \otimes 1) \qquad\qquad\qquad\qquad (10.5) \\
&\quad + (\pm q^{-d+1} + a_{-d+2}q^{-d+2} + \cdots + a_{-1}q^{-1}) \cdot v^\sim[\gamma_l'] \\
&\in H_n(F_{b'}, \partial_0 F_{b'})
\end{aligned}$$

by some $e \in H_{n-1}(X^0_{b'})$, $w \in H_{n-1}(X^\infty_{b'})$ and some integers $a_{-1}, \dots, a_{-d+2}$. Consider the element

$$\tilde{v}[\gamma'_l] \;:=\; (1-q)v[\gamma'_l] \quad\in\quad H_n(F_{b'}).$$

By Lemma 6.1, we can write $(1-q)\;\; \Psi^0_{b'}(e \otimes 1) \in H_n(F_{b'})$ as $\Psi^\infty_{b'}(e_1 \otimes q + e_0 \otimes 1)$ by some $e_1, e_0 \in H_n(X^\infty_{b'})$. Putting this into (10.5), we see that $\tilde{v}[\gamma'_l]$ is written in the form

$$\Psi^\infty_{b'}(\alpha_1 \otimes q + \alpha_0 \otimes 1 + \alpha_{-1} \otimes q^{-1} + \cdots + \alpha_{-d+2} \otimes q^{-d+2} \pm \bar{\sigma}[\xi^\infty_N] \otimes q^{-d+1}), \qquad (10.6)$$

where $\alpha_1, \dots, \alpha_{-d+2}$ are elements of $H_{n-1}(X^\infty_{b'})$.

Let λ^0_N be the path from $a^0_{b'}$ to p_N given by

$$\lambda^0_N(t) \;:=\; (1-t)a^0_{b'} + t \cdot p_N. \qquad (10.7)$$

By (10.2) and Lemma 9.5.1, λ^0_N is an element of $\mathcal{P}_{b'}(a^0_{b'}, p_N)$. By Proposition 9.5.1, there is a simple loop β' around $\mathcal{D}_0$ with the base point b' such that

$$v^\vee[\beta'] \;\equiv\; \bar{\theta}([\lambda^0_N])\langle 0\rangle \;=\; -\,\Psi^0_{b'}(\,\bar{\sigma}[\lambda^0_N] \otimes 1) \quad \text{in} \quad H_n(F_{b'}, \partial_0 F_{b'}). \qquad (10.8)$$

Here we used (7.2) again. (Note that $[\lambda^0_N]$ is represented by a path contained in $K_{b'}$ by Lemma 9.5.1.) We shall prove that

$$[\beta']_*(\tilde{v}[\gamma'_l]) \;\neq\; \tilde{v}[\gamma'_l]. \qquad (10.9)$$

Note that since $\tilde{v}[\gamma'_l] \in H_n(H_{b'}) \subset H_n(F_{b'}, \partial_\infty F_{b'})$, we can apply Theorem 9.2.1 to the calculation of $[\beta']_*(\tilde{v}[\gamma'_l])$. By Theorem 9.2.1, in order to prove (10.9), it is enough to show that $(\tilde{v}[\gamma'_l], v^\vee[\beta'])_0$ is not zero. By (10.8), the Laurant polynomial $(\tilde{v}[\gamma'_l], v^\vee[\beta'])_0$ is equal with $(\tilde{v}[\gamma'_l], \bar{\theta}([\lambda^0_N])\langle 0\rangle)_0$ modulo $U(\mathbb{Z}[q, q^{-1}])$. Using the description (10.6) of $\tilde{v}[\gamma'_l]$ and (10.8) of $\bar{\theta}([\lambda^0_N])\langle 0\rangle$, and applying the fomula (8.2), the coefficient of q^{-d+1} in the Laurant polynomial $(\tilde{v}[\gamma'_l], \bar{\theta}([\lambda^0_N])\langle 0\rangle)_0$ is the integer

$$\pm\,(\,\Psi^\infty_{b'}(\,\bar{\sigma}[\xi^\infty_N] \otimes 1)\,,\, \Psi^0_{b'}(\,\bar{\sigma}[\lambda^0_N] \otimes 1)\,)_0 \;=\; \pm\,\langle\, \bar{\theta}([\xi^\infty_N])\langle 0\rangle\,,\, \bar{\theta}([\lambda^0_N])\langle 0\rangle\,\rangle. \qquad (10.10)$$

By (10.3) and (10.7), the paths ξ^∞_N and λ^0_N have a common ending point p_N as their only intersection point. Hence Lemma 8.2 implies that the integer (10.10) is ± 1. Thus (10.9) is proved.

Now we put $\beta := \alpha^{-1}\beta'\alpha$, which is a simple loop around $\mathcal{D}_0$ with the base point b. We also set

$$\tilde{v}[\gamma_l] \;:=\; (1-q)v[\gamma_l] \quad\in\quad H_n(F_b) \cap M.$$

(Recall that $v[\gamma_l]$ is contained in M.) Since $\tilde{v}[\gamma_l] \equiv [\alpha]^{-1}_*(\tilde{v}[\gamma'_l])$ by Corollary 9.7.1, (10.9) implies that

$$[\beta]_*(\tilde{v}[\gamma_l]) \;\neq\; \tilde{v}[\gamma_l].$$

This implies that $v[\beta] \in M$. $\square$

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address

Max-Planck-Institut für Mathematik
Gottfried-Claren Strasse 26
Bonn 53225, Germany
shimada@mpim-bonn.mpg.de

home address
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
shimada@math.hokudai.ac.jp