# The basic structure of polylogarithmic functional equations.

by

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# Non-Archimedean $L$-Functions

# Associated with Siegel and Hilbert Modular Forms

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## 1. Introduction.

The function $\log z$ satisfies the functional equation

$$\log x + \log y = \log (x \cdot y) .$$

The dilogarithm $\mathrm{Li}_2(z) := \int_0^z \frac{-\log (1-z)}{z} dz$ satisfies the following functional equation

$$\mathrm{Li}_2\left[\frac{x}{1-x} \cdot \frac{y}{1-y}\right] = \mathrm{Li}_2\left[\frac{y}{1-x}\right] + \mathrm{Li}_2\left[\frac{x}{1-y}\right] - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \log(1-x)\log(1-y)$$

(see [A]).

Let us set $\mathrm{Li}_0(z) := -\log z$ , $\mathrm{Li}_1(z) := -\log(1-z)$ and $\mathrm{Li}_n(z) := \int_0^z \frac{\mathrm{Li}_{n-1}(z)}{z} dz$ for $n > 1$ . It was expected that functions $\mathrm{Li}_n(z)$ will satisfy functional equations similar to functional equations of $\log z$ and $\mathrm{Li}_2(z)$ . In fact various functional equations of functions $\mathrm{Li}_n(z)$ for small $n$ were found. The basic reference is Lewin's book (see [L]).

Our aim is to find some new functional equations satisfied by these functions and to give some general results about structures of these equations.

Before we shall formulate our results we shall make one observation concering polylogarithms. The functions $Li_n(z)$ are special case of Chen iterated intergrals. We recall their definition. Let $\omega_1,\dots,\omega_n$ be one–forms on a smooth manifold M and let $\gamma$ be a smooth path from x to z . Then we define by a recursive formula:

$$\int_\gamma \omega_1,\dots,\omega_n := \int_\gamma \Big[ \int_{\gamma^t} \omega_1 \Big] \omega_2,\dots,\omega_n ,$$

where $\gamma^t$ denotes the restriction $\gamma |\, [0,t]$ .

(Instead $\int_\gamma$ we shall write also $\int_{x,\gamma}^{z}$ or $\int_x^z$ .)

It is clear that $Li_n(z) = \int_0^z -\frac{dz}{z-1}, \frac{dz}{z}, \dots, \frac{dz}{z}$ .

A rational function $f : P^1(\mathbb{C}) \longrightarrow P^1(\mathbb{C})$ we shall usually write in the form

$$f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} \Big/ \prod_{j=1}^{m} (z-b_j)^{m_j}$$

where $\alpha \in \mathbb{C}$ , $n_i$ and $m_j$ are positive integers and $a_i$ , $b_j$ are complex numbers.

**Definition 1.1.** *We say that f(z) is in an irreducible form if* $a_i \neq b_j$ *for* $i = 1,\dots,n$ *and* $j = 1,\dots,m$ .

**Definition 1.2.** *Let* $a \in \mathbb{C}$ . *If* $f(z) - a = \alpha^* \prod_{k=1}^{r} (z - c_k)^{r_k} / \prod_{j=1}^{p} (z - b_j)^{p_j}$ *is an irreducible form then we define a divisor* $f^{-1}(a)$ *by the formula* $f^{-1}(a) := \sum_{k=1}^{r} r_k \cdot c_k$ .

Now we shall formulate our main results.

**Theorem A.** *Let* $f(z) = \alpha \prod_{i=1}^{n} (z - a_i)^{n_i} / \prod_{j=1}^{m} (z - b_j)^{m_j}$ *be a map from* $P^1(\mathbb{C})$ to $P^1(\mathbb{C})$ *and let* $f^{-1}(1) = \sum_{k=1}^{r} r_k \cdot c_k$ .

*We have the following formula*

*1.3.* $$Li_2(f(z)) - Li_2(f(x)) + \log(1 - f(x)) (\log(f(z)) - \log (f(x)) =$$

$$\sum_{i,k} n_i \cdot r_k \left(Li_2\left[\frac{z - a_i}{c_k - a_i}\right] - Li_2\left[\frac{x - a_i}{c_k - a_i}\right] + \log\frac{x - c_k}{a_i - c_k} \log\frac{z - a_i}{x - a_i}\right) +$$

$$-\sum_{j,k} m_j \cdot r_k \left(Li_2\left[\frac{z - b_j}{c_k - b_j}\right] - Li_2\left[\frac{x - b_j}{c_k - b_j}\right] + \log\frac{x - c_k}{b_j - c_k} \log\frac{z - b_j}{x - b_j}\right) +$$

$$-\sum_{i,j} n_i \cdot m_j \left(Li_2\left[\frac{z - a_i}{b_j - a_i}\right] - Li_2\left[\frac{x - a_i}{b_j - a_i}\right] + \log\frac{x - b_j}{a_i - b_j} \log\frac{z - a_i}{x - a_i}\right) +$$

$$-\sum_{j<j'} m_j \cdot m_{j'} \left[\log\frac{z - b_j}{x - b_j}\right] \left[\log\frac{z - b_{j'}}{x - b_{j'}}\right] - \tfrac{1}{2}\sum m_j^2 \left[\log\frac{z - b_j}{x - b_j}\right]^2 .$$

The following summation convention is used in the formula 1.3 and it will be used through the whole chapter.

$$\sum_{i,k} = \sum_{i=1}^{n} \sum_{k=1}^{r} , \sum_{j<j'} = \sum_{j=1}^{m-1} \sum_{j'=j+1}^{m} , \sum_{j} = \sum_{j=1}^{m} , \sum_{i<i',k} = \sum_{i<i'} \sum_{k=1}^{r}$$

and so on for three or more indices.

One of the difficulties to deal with the formula 1.3 in Theorem A is that the functions $\mathrm{Li}_2(z)$ and $\log z$ are multivalued. For example for some values of $\mathrm{Li}_2(z)$ the formula can be satisfied while for others not. It is not clear at all which values of $\mathrm{Li}_2(z)$ the reader can choose.

However Theorem A is derived from the following formula which has no ambiguity at all.

First we formulate assumptions.

*1.4* *Let* $f(z) = \alpha \prod_{i=1}^{n} (z - a_i)^{n_i} / \prod_{j=1}^{m} (z - b_j)^{m_j}$ *be in an irreducible form and let* $f^{-1}(1) = \sum_{k=1}^{r} r_k \cdot c_k$ . *Let* $X = P^1(\mathbb{C}) \setminus \{a_1,\dots,a_n,b_1,\dots,b_m,c_1\dots,c_r,\infty\}$ *and let* $Y = P^1(\mathbb{C}) \setminus \{0,1,\infty\}$ . *Let* $\gamma$ *be a smooth path in* $X$ *from* $x$ *to* $z$ . *We asume that* $z$ *and* $x$ *are different from* $a_1,\dots,a_n,b_1,\dots,b_m,c_1\dots,c_r,\infty$ .

**Theorem A′.** *(integral form of the functional equation) Let us assume that 1.4 holds. Then we have*

1.3′
$$\int_{f(\gamma)} \frac{-dz}{z-1}, \frac{dz}{z} = \sum_{i,k} n_i \cdot r_k \int_{f_{ik}(\gamma)} \frac{-dz}{z-1}, \frac{dz}{z} - \sum_{j,k} m_j \cdot r_k \int_{h_{jk}(\gamma)} \frac{-dz}{z-1}, \frac{dz}{z} -$$

$$\sum_{i,j} n_i \cdot m_j \int_{g_{ij}(\gamma)} \frac{-dz}{z-1}, \frac{dz}{z} - \sum_{j<j'} m_j \cdot m_j' \Big[\int_\gamma \frac{dz}{z-b_j}\Big]\Big[\int_\gamma \frac{dz}{z-b_{j'}}\Big] -$$

$$\frac{1}{2}\sum_j m_j^2 \int_\gamma \frac{dz}{z-b_j}, \frac{dz}{z-b_j}$$

*where* $f_{ik}(z) = \frac{z-a_i}{c_k-a_i}$, $g_{ij}(z) = \frac{z-a_i}{b_j-a_i}$ *and* $h_{jk}(z) = \frac{z-b_j}{c_k-b_j}$.

To get the expression from Theorem A we must calculate an integral $\int_\varphi \frac{-dz}{z-1}, \frac{dz}{z}$ where $\varphi$ is a path from a to b . We have

$$\int_\varphi \frac{-dz}{z-1}, \frac{dz}{z} = \int_\varphi -(\log(1-z)-\log(1-a))\frac{dz}{z} = \mathrm{Li}_2(b)-\mathrm{Li}_2(a) + \log(1-a)(\log b - \log a)\,.$$

Observe that $\log(1-a)$ we could choose arbitrary, but when we fixed $\log(1-a)$ then $\log(1-z)$ is determined uniquely. $\mathrm{Li}_2(a)$ can be chosen arbitrary, but $\mathrm{Li}_2(b)$ is determined uniquely by $\mathrm{Li}_2(a)$ and $\log(1-z)$ .

Now it is clear which values of $\mathrm{Li}_2$ and log we must choose in the formula 1.1 to have an equality.

Suppose that we have chosen such values of $\mathrm{Li}_2$ and log that we have no equality any more. Then we can always add some expression containing logarithms and constants to the right hand side so that once more we have an equality. This is due to the fact that different branches of $Li_2(z)$ and $\log z$ are given by $Li_2(z) + 2\pi ik \log z$ and $\log z + 2\pi ik$ where $k \in z$ .

This suggests a new formulation of Theorem A.

**Theorem A′′.** *Let $f(z)$ and $f^{-1}(1)$ be as in Theorem A. Then we have*

$$1.3'' \qquad Li_2(f(z)) - Li_2(f(x)) = \sum_{i,j} n_i \cdot r_k \left[Li_2(f_{ik}(z) - Li_2(f_{ik}(x))\right] +$$

$$-\sum_{j,k} m_j \cdot r_k \left[Li_2(h_{jk}(z)) - Li_2(h_{jk}(x))\right] - \sum_{i,j} n_i \cdot m_j \left[Li_2(g_{ij}(z)) - Li_2(g_{ij}(x))\right]$$

$$+ \ l.d.t.(2)$$

*where $l.d.t.(2)$ is a polynomial in logarithms and constants.*

We shall show that from the formula 1.3 one can get all functional equations of the dilogarithm in one variable. Also we shall show that most known functional equations of the dilogarithm one can get from 1.3 choosing suitably the function $f(z)$ .

We have a similar formula for $Li_3(z)$ . In the introduction we state only a special case when the function $f(z)$ is a polynomial function.

**Theorem B.** *Let $f(z) = \alpha \prod_{i=1}^{n} (z - a_i)^{n_i}$ and let $f^{-1}(1) = \sum_{k=1}^{r} r_k \cdot c_k$ . We have the following formula*

$$Li_3\left[\alpha\prod_{i=1}^{n}(z-a_i)^{n_i}\right]-Li_3\left[\alpha\prod_{i=1}^{n}(x-a_i)^{n_i}\right]+$$

$$Cor\left[\alpha\prod_{i=1}^{n}(z-a_i)^{n_i},\alpha\prod_{i=1}^{n}(x-a_i)^{n_i}\right]=$$

$$-\sum_{i<i'}\sum_{k}n_i\cdot n_{i'}\cdot r_k\left[Li_3\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_{i'}}{c_k-a_i}\right]-Li_3\left[\frac{x-a_i}{x-a_{i'}}\cdot\frac{c_k-a_{i'}}{c_k-a_i}\right]\right.$$

$$\left.+Cor\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_{i'}}{c_k-a_i};\frac{x-a_i}{x-a_{i'}}\cdot\frac{c_k-a_{i'}}{c_k-a_i}\right]\right]$$

$$-\sum_{i',i,k}n_i\cdot n_{i'}\cdot r_k\left[Li_3\left[\frac{z-a_i}{c_k-a_i}\right]-\left[Li_3\frac{x-a_i}{c_k-a_i}\right]+Cor\left[\frac{z-a_i}{c_k-a_i};\frac{x-a_i}{c_k-a_i}\right]\right]$$

$$+\sum_{i<i'}\sum_{k}n_i\cdot n_{i'}\cdot r_k\left[Li_3\left[\frac{z-a_i}{z-a_{i'}}\right]-Li_3\left[\frac{x-a_i}{x-a_{i'}}\right]+\right.$$

$$\left.Cor\left[\frac{z-a_i}{z-a_{i'}};\frac{x-a_i}{x-a_{i'}}\right]\right]$$

*where* $Cor(a;b)=-Li_2(b)\log(a/b)-\frac{1}{2}\log(1-b)(\log(a/b))^2$.

We left to the reader the formulation of the integral form of Theorem B. Then one can also fix values of $Li_3$, $Li_2$ and log for which one has an equality.

**Definition 1.5.** *Let* $f(z)=\alpha\prod_{i=1}^{n}(z-a_i)^{n_i}\Big/\prod_{j=1}^{m}(z-b_j)^{m_j}$ *be a rational function in an* *irreducible form. We set* $\deg f := \max(\sum_{i=1}^{n} n_i, \sum_{j=1}^{m} m_j)$ *and we call this number the degree of* $f$.

**Definition 1.6.** *Let* $n$ *be a natural number.*

$$l.d.t.(n)(\text{resp. } \overline{l.d.t.(n)}) := p(c_1,\ldots,c_r, Li_{t_1}(g_1(z)),\ldots,Li_{t_s}(g_s(z)))$$

*where* $p(x_1,\ldots,x_r,y_1,\ldots,y_s)$ *is a polynomial with rational coefficients,* $c_j = 2\pi i$ *or* $Li_k(a_j)$ *where* $a_j \in \mathbb{C}$ *and* $k < n$ *(resp.* $k \leq n$ *) for* $j = 1,\ldots,r$ *;* $g_i(z)$ *are rational functions on* $P^1(\mathbb{C})$ *and* $t_i < n$ *for* $i = 1,\ldots,s$ *.*

Observe that in Theorem A we expressed $\mathrm{Li}_2(\mathrm{f(z)}$ as a sum of $\mathrm{Li}_2(\mathrm{g(z)})$'s where g(z) are rational functions of degree one, of logarithmic terms and constants. The same holds for $\mathrm{Li}_3(\mathrm{f(z)})$ . This is not a general phenomena as we shall see in the next theorem.

**Theorem C.** *Let* $f(z)$ *be a rational function of degree* $k$ *greater than 1. Let us assume that* $f(z)$ *is not a* $k$*–th power. Let* $n$ *be a natural number greater than 3. Then there is no functional equation of the form*

$$Li_n(f(z)) = \sum_{i=1}^{N} n_i \, Li_n(f_i(z)) + \overline{l.d.t.(n)} \ ,$$

*where* $f_i(z)$ *are rational functions of degree 1 and* $n_i$ $(i = 1,\ldots,N)$ *are rational numbers.*

While proving Theorems A and B we met the problem of expressing iterated integrals of the form $\int_x^z \frac{dz}{z-a_1},\ldots,\frac{dz}{z-a_n}$ by classical polylogarithms. The next result related to Theorem C shows that this is usually impossible.

Theorem D. *Let $a_1, a_2, a_3, a_4$ be four different points in $\mathbb{C}$.*

a) *The function $N(z) = \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}$ can be expressed by classical polylogarithms.*

b) *Let $\mathrm{L}(z) = \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}, \frac{dz}{z-a_4}$. There is no polynomial $p(s,t_1,\ldots,t_r)$ such that*

$$p(\mathrm{L}(z), Li_{n_1}(f_1(z)),\ldots,Li_{n_r}(f_r(z))) \equiv 0$$

*where $Li_{n_k}$ are classical polylogarithms (and logarithms) and $f_i(z)$ are rational functions.*

The principal tools in our investigations are two observations.

1. Functions of the type of polylogarithms are horizontal sections of the canonical unipotent connection on $P^1(\mathbb{C})\setminus\{a_1,\ldots,a_n\}$.
2. The functional equations of functions of the type of polylogarithms are consequences of relations between maps induced by regular functions from $P^1(\mathbb{C})\setminus$several points to $P^1(\mathbb{C})\setminus$several points on Lie algebras of fundamental groups.

We illustrate the second principal with few examples.

Example 1. The maps $f(x) = x$ and $g(x) = 1-x$ from $X = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ into itself induce opposite maps on $\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)$, therefore we have a functional equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \overline{\mathrm{l.d.t.}}(2) \ .$$

Example 2. The maps $f(x) = x^2$, $g(x) = x$ and $h(x) = -x$ from $X = P^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ to $P^1(\mathbb{C})\setminus\{0,1,\infty\}$ satisfies

$$f_* - 2g_* - 2h_* = 0$$

on $\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)$, therefore there is a functional equation

$$\mathrm{Li}_2(x^2) - 2\,\mathrm{Li}_2(x) - 2\,\mathrm{Li}_2(-x) = \overline{\mathrm{l.d.t.}}(2).$$

Example 3. Let $f_1(x) = x$, $f_2(x) = \frac{1}{1-x}$, $f_3(x) = \frac{x}{x-1}$, $f_4(x) = \frac{1}{x}$ be maps from $X = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ into itself. In $\mathrm{Hom}\left[\Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x);\Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x) + [V[U,V]]\right]$, where $U$ is a loop around $0$ and $V$ is a loop around $1$ we have

$$f_{1*} = f_{4*} \ \text{and} \ f_{1*} + f_{2*} + f_{3*} = 0 \ .$$

Hence there are functional equations

$$\mathrm{Li}_3(x) = \mathrm{Li}_3(\tfrac{1}{x}) + \overline{\mathrm{l.d.t.}}(3)$$

and

$$\mathrm{Li}_3(x) + \mathrm{Li}_3(\tfrac{1}{1-x}) + \mathrm{Li}_3(\tfrac{x}{x-1}) = \overline{\mathrm{l.d.t.}}(3) \ .$$

Example 4. Let $X = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ , $f(x) = x$ and $g(x) = 1/x$ . Let U be a loop around 0 and let V be a loop around 1. On the quotient $\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x) + L$ , where L is a subgroup of $\Gamma^n\pi_1(X,x)$ generated by all these commutators which contain V at least twice, we have

$$f_* = (-1)^{n-1} g_* \ .$$

Therefore we have a functional equation

$$\mathrm{Li}_n(z) = (-1)^{n-1}\mathrm{Li}_n(1/z) + \overline{\mathrm{l.d.t.}}(n) \ .$$

All these examples follow easily from the following theorem:

**Theorem E.** *Let $X = P^1(\mathbb{C}) \setminus \{a_1,\dots,a_r,\infty\}$ and $Y = P^1(\mathbb{C}) \setminus \{0,1,\infty\}$ . Let $U$ (resp. $V$) be a loop around $0$ (resp. $1$) in $Y$ . Let $f_1,\dots,f_N : X \longrightarrow Y$ be regular maps from $X$ to $Y$ and let $n_1,\dots,n_N$ be integers. There is a functional equation*

$$n_1 Li_n(f_1(z)) + \dots + n_N Li_n(f_N(z)) + \overline{l.d.t.}(n) = 0$$

*if and only if*

$$n_1 f_{1*} + \dots + n_N f_{N*} = 0$$

*in the Z–module $Hom(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)\ ,\ \Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y) + L_n)$ where $L_n$ is a subgroup of $\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)$ generated by all commutators which contain*

*V at least twice and* $f_{i*}$ *is the map induced by* $f_i$ *on fundamental groups.*

Theorem E has the following generalization.

**Theorem F.** *Let* $X$ *be a smooth quasi–projecive algebraic variety over* $\mathbb{C}$ *. Let* $f_1, \dots f_N$ *be regular maps from* $X$ *to* $Y = P^1(\mathbb{C}) \setminus \{0,1,\infty\}$ *and let* $n_1 \dots, n_N$ *be integers. There is a functional equation*

$$n_1 Li_n(f_1(x)) + \dots + n_N Li_n(f_N(x)) + \overline{\text{l.d.t.}}(n) = 0$$

*if and only if*

$$n_1 f_{1*} + \dots + n_N f_{N*} = 0$$

*in the Z–module* $Hom(\Gamma^n \pi_1(X,x)/\Gamma^{n+1} \pi_1(X,x); \Gamma^n \pi_1(Y,y)/\Gamma^{n+1} \pi_1(Y,y) + L_n)$ .

Observe that the defintion of $\overline{\text{l.d.t.}}(n)$ should be modified in Theorem F. One requieries that $g_i(z)$ in Definition 1.5 are regular functions from X to Y . This theorem gives an interpretation of functional equations in several variables as well as functional equations of polylogarithms whose arguments are arbitrary algebraic functions. We shall not prove Theorem E in this chapter. Its proof appears elsewhere.

Theorem E is our pricipal result. From this theorem we derived all our results about functional equations of polylogarithms.

In this moment we should point out that D. Zagier obtained a very short and elegant proof of the related result for higher Bloch–Wigner functions using a version of generalized Bloch homomorphisms (see [Z3] ).

This chapter is based on ideas in our preprint "A note on functional equations of the dilogarithm" (see [W1] and also [W2] ). We would like to thank very much P. Deligne for his comments on our manuscript under the same title, where he reinterpreted our results in terms of Lie algebras of fundamental groups. He also showed us the connection from section 1 in the special case of $\mathbb{C}\setminus\{0,1\}$ . We acknowledge the influence of the lecture of D. Zagier (Bonn, April 1989, see also [Z1] and [Z2] ). We acknowledge the influence of papers of L.J. Rogers (see [Ro] ), H.F. Sandham (see [S] ) and R.F. Coleman (see [C] ). We would like to thank very much J.L. Loday and Ch. Soulé who told us about functional equations of polylogarithms. Whilst writing this chapter we were a visitor at Max–Planck–Institut für Mathematik in Bonn. We would like to thank very much to Prof. F. Hirzebruch for an opportunity to visit Bonn. We would like to thank Y. André and H. Gangl for several useful discussions and L. Lewin for correspondence.

Plan of Chapter:

1. Introduction.
2. Canonical unipotent connection on $P^1(\mathbb{C})\setminus\{a_1,\dots,a_{n+1}\}$.
3. Horizontal sections.
4. Easy lemmas about monodromy.
5. Functional equations.
6. Functional equations of polylogarithms.
7. Functional equations of lower–degree polylogarithms

8. Generalized Bloch groups.

## 2. Canonical unipotent connection on $P^1(\mathbb{C})\setminus\{a_1,\dots,a_{n+1}\}$ .

Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_{n+1}\}$ . Let $A^*(X)$ be a differential, graded subalgebra of $\Omega^*(X)$ generated by linear combinations with complex coefficients of one–forms $\frac{dz}{z-a_i}$ $i = 1,\dots n+1$ . It is a trivial observation that $(A^1(X))^* \approx H_1(X,\mathbb{C})$ . The isomorphism is given by the bilinear form

$$\int : A^1(X) \otimes H_1(X,\mathbb{C}) \longrightarrow \mathbb{C}$$

given by $(\omega,\gamma) \longrightarrow \int_\gamma \omega$ .

Let $L(\pi_1(X,x)) := \varprojlim_N \left[\bigoplus_{n=1}^{N} \Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)) \otimes \mathbb{C}\right]$ be a Lie algebra associated with the lower central series of $\pi_1(X,x)$ . We equipped $L(\pi_1(X,x))$ with a group law given by the Baker–Hausdorff formula and a topology given by the inverse limite of finite dimensional complex vector spaces. This topological group we denote by $\pi(X)$ . The Lie algebra of $\pi(X)$ is $L(\pi_1(X,x))$ .

We shall define a one–form $\omega_X$ on $X$ with values in $L(\pi_1(X,x))$ in the following way. We have natural isomorphisms

2.1 $$A^1(X) \otimes H_1(X,\mathbb{C}) \approx A^1(X) \otimes (A^1(X))^* \approx \mathrm{Hom}(A^1(X)\,,\,A^1(X))$$

**Definition 2.2.** *$\omega_X \in A^1(X) \otimes H_1(X,\mathbb{C})$ is the one–form which corresponds to $2\pi i \cdot id$ under the isomorphisms 2.1. (see also [D] 12.5.5).*

We consider $\omega_X$ as an element of $A^1(X) \otimes L(\pi_1(X,x))$ because of the identification $H_1(X,\mathbb{C}) \approx (\pi_1(X,x)/\Gamma^2\pi_1(X,x)) \otimes \mathbb{C}$.

Let $A_i$ be a loop around $a_i$ in $X$ and let $X_i$ be the image of $A_i$ in $H_1(X,\mathbb{C})$.

Let us assume that $a_{n+1} = \infty$ then

2.2.1. $$\omega_X = \sum_{i=1}^{n} \frac{dz}{z-a_i} \otimes X_i .$$

If $a_i \neq \infty$ for $i = 1,\dots,n+1$ then

2.2.2. $$\omega_X = \sum_{i=1}^{n} \left[ \frac{dz}{z-a_i} - \frac{dz}{z-a_{n+1}} \right] \otimes X_i .$$

Let $\mathbb{C}[[H_1(X,\mathbb{C})]]$ be an algebra of non–commutative, formal power series on $H_1(X,\mathbb{C})$. We shall denote it shortly by $\mathbb{C}[[X]]$. Let $I$ be an augmentation ideal of $\mathbb{C}[[X]]$. Then $C[[X]]/I^n$ is a finite dimensional, complex vector space, $\mathbb{C}[[X]] = \varprojlim C[[X]]/I^n$ and we equipped $C[[X]]$ with a topology of an inverse limite of finite dimensional, complex vector spaces. Let $C[[X]]^*$ be a group of invertible elements in $C[[X]]$. From the discussion given above if follows that $C[[X]]^*$ is a topological group, an inverse limit of finite dimensional, complex Lie groups. We shall denote the group $C[[X]]^*$ by $P(X)$.

The Lie algebra of Lie elements, possibly of infinite length, in $\mathbb{C}[[X]]$ is naturally identified with $L(\pi_1(X,x))$. After this identification the exponential map

$$\exp : \pi(X) \longrightarrow P(X)$$

$$\exp(w) = e^{w} = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \dots$$

is defined. The exponential map is a continous monomorphism of topological groups, whose image is a closed subgroup of $P(X)$. The inverse of $\exp$ is defined on the subgroup $\exp(\pi(X)) \subset P(X)$ and we denote it by $\log$.

Let $\mathrm{Lie}\ P(X)$ be a Lie algebra of $P(X)$. We identify $T \in H_1(X;\mathbb{C}) \subset L(\pi_1(X,X))$ with the tangent vector to $P(X)$ in 1 given by $t \longrightarrow 1 + tT$. After this identification the one–form $\omega_X$ we shall consider as a one–form with values in Lie $P(X)$. We shall denote it by $\overline{\omega}_X$. The homomorphism $\exp$ maps $\omega_X$ into $\overline{\omega}_X$.

Let us consider a principal $\pi(X)$–bundle

$$X \times \pi(X) \longrightarrow X$$

equipped with the integrable connection given by a one–form $\omega_X$, and a principal $P(X)$–bundle.

$$X \times P(X) \longrightarrow X$$

equipped with the integrable connection given by a one–form $\overline{\omega}_X$.

**Lemma 2.3.** *The morphism* $id \times exp : X \times \pi(X) \longrightarrow X \times P(X)$ *over* $id_X$ *maps horizontal section with respect to* $\omega_X$ *into horizontal section with respect to* $\overline{\omega}_X$.

Proof. This is clear from the fact that exp maps $\omega_X$ into $\overline{\omega}_X$.

It is clear that there is no need to distinguish between $\omega_X$ and $\overline{\omega}_X$, hence from now on we shall denote both forms by $\omega_X$.

**3. Horizontal sections.**

Let $X = P^1(\mathbb{C})\setminus\{x_1,\dots x_{n+1}\}$. Let $\gamma$ be a smooth path in $X$ from $x$ to $z$. We shall denote by $(z,l_X(z;x;\gamma))$ (resp. $(z,\lambda_X(z;x,\gamma))$) or shortly by $(z,l_X(z;x))$ (resp. $(z,\lambda_X(z;x))$) the value at $z$ of the horizontal section of the bundle $X \times \pi(X) \longrightarrow X$ (resp. $X \times P(X) \longrightarrow X$) equipped with the connection form $\omega_X$ along the path $\gamma$ with the initial condition $l_X(x;x;\gamma) = 0$ (resp. $\lambda_X(x;x,\gamma) = 1$).

Let us set

$$\omega_i := -\left[\frac{dz}{z-x_i} - \frac{dz}{z-x_{n+1}}\right] \quad i = 1,\dots,n$$

if $x_i \neq \infty$ $i = 1,\dots,n+1$.

If one $x_i = \infty$ then we assume that $x_{n+1} = \infty$ and we set

$$\omega_i := \frac{-dz}{z-x_i} \quad i = 1,\dots,n.$$

Let us define

$$\Lambda_x(\varepsilon_1^{n_1},\dots,\varepsilon_k^{n_k})(z) := \int_{x,\gamma}^{z} \omega_{\varepsilon_k},\dots,\omega_{\varepsilon_k},\dots,\omega_{\varepsilon_1},\dots,\omega_{\varepsilon_1}$$

where $\varepsilon_i \in \{1,\dots,n\}$ and $\omega_{\varepsilon_1}$ repeats $n_1$–times ,..., $\omega_{\varepsilon_k}$ repeats $n_k$–times.

**Lemma 3.1** *The application*

$$X \ni z \longrightarrow (z, 1 + \sum \Lambda_x(\varepsilon_1^{n_1},\dots,\varepsilon_k^{n_k})\,(z)\, X_{\varepsilon_1}^{n_1} \cdot\ldots\cdot X_{\varepsilon_k}^{n_k}) \in X \times P(X)$$

*is horizontal with respect to the connection* $\omega_X = \sum_{i=1}^{n} -\omega_i \otimes X_i$ *and hence it coincides with the map* $z \longrightarrow (z, \lambda_X(z;x))$ . *(The summation is over all noncommutative monomials in variables* $X_1,\dots X_n$ *where* $X_i$ *is the class in* $H_1(X,\mathbb{C})$ *of a loop around* $a_i$ *.)*

Proof. This is a straightforward calculation of horizontal liftings.

Let $X = P^1(\mathbb{C})\setminus\{x_1,\dots,x_{n+1}\}$ and let $Y = P^1(\mathbb{C})\setminus\{y_1,\dots,y_{m+1}\}$ . Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} \,/\, \prod_{j=1}^{m} (z-b_j)^{m_j}$ be a rational function. Let us assume that $f$ restricts to a regular map $f : X \longrightarrow Y$ . The map $f$ induces

$$f^* : A^*(Y) \longrightarrow A^*(X) ,$$

$$H_1(f) : H_1(X) \longrightarrow H_1(Y)$$

and

$$f_{\#} : \pi_1(X,x) \longrightarrow \pi_1(Y,f(x)) .$$

The maps $H_1(f)$ and $f_{\#}$ induce the following three maps

$$f_* : L(\pi_1(X,x)) \longrightarrow L(\pi_1(Y,f(x))) ,$$

$$f_* : \pi(x) \longrightarrow \pi(Y) ,$$

$$f_* : P(X) \longrightarrow P(Y) .$$

In the next proposition $G(X)$ is $\pi(X)$ (resp. $P(X)$) and $G(Y)$ is $\pi(Y)$ (resp. $\pi(Y)$) .

Proposition 3.2. *The map $(f, f \times f_*)$ of principal fibre bundles*

$$\begin{array}{ccc} X \times G(X) & \xrightarrow{f \times f_*} & Y \times G(Y) \\ (1)\downarrow & & (2)\downarrow \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

*satisfies*

$$(\mathrm{id} \otimes f_*)\, \omega_X = (f^* \otimes \mathrm{id})\, \omega_Y .$$

Proof. This is a direct verification for which one can use explicite formulas 2.2.1 and 2.2.2 for $\omega_X$ and $\omega_Y$ .

Corollary 3.3. The map $f \times f_*$ maps horizontal sections of the bundle (1) into horizontal section of the bundle (2). This implies that we have the following equalities

3.3.1. $$f_*(l_X(z;x,\gamma)) = l_Y(f(z);f(x),f(\gamma))$$

and

3.3.2. $$f_*(\lambda_X(z;x,\gamma)) = \lambda_Y(f(z);f(x),f(\gamma))\ .$$

Proof. The corollary is an immediate consequence of Proposition 3.2.

## 4. Easy lemmas about monodromy.

Let $X = P^1(\mathbb{C})\setminus\{x_1,\dots x_{n+1}\}$ . Let $\alpha$ be a loop in $X$ based at $x \in X$ and let $\gamma$ be a path from $x$ to $z$ . The function $l_X(z;x) : P^1(\mathbb{C})\setminus\{x_1,\dots x_{n+1}\} \longrightarrow \pi(X)$ is a multi-valued function. This means that in general $l_X(z;x,\gamma \circ \alpha)$ is different from $l_X(z;x,\gamma)$ . Let us set $l_X^\alpha(z;x,\gamma) := l_X(z;x,\gamma \circ \alpha)$ . The action of $\alpha$ on $l_X(z;x)$ we denote in the following way

$$\alpha : l_X(z;x) \longrightarrow l_X^\alpha(z;x)$$

and we shall call this action of $\alpha$ , the monodromy of the function $l_X(z;x)$ along $\alpha$ .

We recall that $\pi(X) = L(X_1,\dots X_n)$ where $X_k$ is the class of the loop $A_k$ around $a_k$ .

**Lemma 4.1.** *The monodromy of the function $l_X(z;x)$ along the loop $A_k$ is given by the* *following formula*

$$l_X^{A_k}(z;x) = l_X(z;x) \cdot (-2\pi i\, X_k + \text{terms of degree} \geq 2)\ .$$

Proof. The function $l_X(z;x)$ is the horizontal section of the principal $\pi(X)$–bundle. Hence its monodromy along any loop $\alpha \in \pi_1(X,x)$ is given by the following formula

$$l_X^{\alpha}(z;x) = l_X(z;x) \cdot l_X(x;x,\alpha)\ .$$

Observe that

$$l_X(z;x) = \sum_{k=1}^{n} (-\log(z-a_k) + \log(x-a_k))X_k + \text{terms of degree} \geq 2$$

if $x_{n+1} = \infty$ and

$$l_X(z;x) = \sum_{k=1}^{n} (-\log(z-a_k) + \log(z-a_{n+1}) + \log(x-a_k) - \log(x-a_{n+1}))X_k + \text{terms of}$$

degree $\geq 2$ if $x_k \neq \infty$ for $k = 1,2,\dots,n+1$ . This implies that $l_X(x;x,A_k) = -2\pi i X_k$ + terms of degree $\geq 2$ .

Let $L$ be a free Lie algebra on generators $x_1,\dots,x_n$ . Then for any fixed ordering of elements $x_1,\dots,x_n$ there is a base of $L$ consisting of basic Lie elements corresponding to this ordering (see [MCS] ).

Let $B_m = \{e_i\}_{i \in I}$ be a base of $\Gamma^m\pi(X)/\Gamma^{m+1}\pi(X) = (\Gamma^m\pi_1(X,x)/\Gamma^{m+1}\pi_1(X,x)) \otimes \mathbb{C}$

given by basic Lie elements corresponding to the ordering $X_1, X_2, \ldots X_n$ . Let $e_i^*$ be a linear functional dual to $e_i$ with respect to the base $B_m$ . We shall consider $e_i^*$ as a polynomial function on $\pi(X)$ . We are interested in the monodromy of $e_i^*(l_X(z;x))$ .

Corollary 4.2. *Let $e_i$ and $e_j$ belong to $B_m$ . The monodromy of $e_i^*(l_X(z;x))$ is trivial on $\Gamma^d\pi_1(X,x)$ for $d > m$ . The monodromy of $e_i^*(l_X(z;x))$ on $\Gamma^m\pi(X,x)/\Gamma^{m+1}\pi(X,x)$ is given by the following formula*

$$e_j : e_i^*(l_X(z;x)) \longrightarrow e_i^*(l_X(z;x)) + (-2\pi i)^m \delta_i^j .$$

Proof. It follows from Lemma 4.1 that the monodromy of $l_X(z;x)$ on $e_i$ is given by

$e_i : l_X(z;x) \longrightarrow l_X(z;x) + (-2\pi i)^m e_i$ + terms of degree $\geq m$ . This implies the corollary.

Corollary 4.3. *The image of the homomorphism $\pi_1(X,x) \longrightarrow \pi(X)/\Gamma^n\pi(X)$ given by $\pi_1(X,x) \ni \alpha \longrightarrow l_X(x;x,\alpha) \in \pi(X)/\Gamma^n\pi(X)$ is Zariski dense in $\Gamma(X)/\Gamma^n\pi(X)$ for each $n \geq 2$ .*

Proof. Lemma 4.1 implies that the image of the composite homomorphism

$$\pi_1(X,x) \longrightarrow \pi(X) \longrightarrow \pi(X)/\Gamma^2\pi(X)$$

is Zariski dense in $\pi(X)/\Gamma^2\pi(X)$ . Hence it follows that for each n the image of the composite homomorphism

$$\pi_1(X,x) \longrightarrow \pi(X) \longrightarrow \pi(X)/\Gamma^n\pi(X)$$

is Zariski dense in $\pi(X)/\Gamma^n\pi(X)$ .

**5. Functional equations.**

In this section we shall present general results about functional equations. Let $X$ be a complex projective line minus several points. Let $G(X)$ be $\pi(X)$ or $P(X)$ . Observe that $G(X)$ is an affine pro–algebraic group. Let $Alg(G(X))$ be an algebra of polynomial, complex valued functions on $G(X)$ .

Now we set $X = P^1(\mathbb{C})\setminus\{x_1,...x_{n+1}\}$ and $Y = P^1(\mathbb{C})\setminus\{y_1,...y_{m+1}\}$ . Let $f : X \longrightarrow Y$ be a regular map. Let $x \in X$ and $z \in X$ and let $\gamma$ be a path in $X$ from $x$ to $z$ . Our principal tool to derive functional equations are equalities

3.3.1 $$f_*l_X(z;x,\gamma) = l_Y(f(z);f(x),f(y))$$

and

3.3.2 $$f_*\lambda_X(z;x,\gamma) = \lambda_Y(f(z);f(x),f(\gamma)) .$$

In fact these equalities are special cases of functional equations.

**Theorem 5.1.** *Let $f_1,...,f_N : X \longrightarrow Y$ be regular functions. Let $\mathscr{E}_1,...\mathscr{E}_N$ belong to $Alg(G(Y))$ and let $p(t_1,...,t_n)$ be a polynomial in variables $t_1,...,t_n$ .*

i) Let $G(\ ) = \pi(\ )$ . There is a functional equation

(1) $$p(\mathscr{E}_1(l_Y(f_1(z);f_1(x),f_1(\gamma))),\ldots,\mathscr{E}_n(l_Y(f_n(z);f_n(x),f_n(\gamma))) = 0$$

if and only if

(2) $$p(\mathscr{E}_1 \circ f_{1*},\ldots,\mathscr{E}_n \circ f_{n*}) = 0\ .$$

ii) Let $G(\ ) = P(\ )$ . If

$$p(\mathscr{E}_1 \circ f_{1*},\ldots,\mathscr{E}_n \circ f_{n*}) = 0$$

then

$$p(\mathscr{E}_1(\lambda_Y(f_1(z);f_1(x),f_1(\gamma))),\ldots,\mathscr{E}_n(\lambda_Y(f_n(z);f_n(x),f_n(\gamma))) = 0$$

Proof. Let us assume that we have (2). Corollary 3.3 implies that

$$\mathscr{E}_i(f_{i*}(l_X(z;x\gamma))) = \mathscr{E}_i(l_Y(f_i(z);f_i(x),f_i(\gamma)))\ .$$

Replacing $\mathscr{E}_i(f_{i*}(l_X(z;x\gamma)))$ by $\mathscr{E}_i(l_Y(f_i(z);f_i(x),f_i(\gamma)))$ in the formula (2) we get the functional equation (1). The same arguments show also the part ii).

Let us assume that we have a functional equation (1). It follows from Lemma 4.3 that the set of values $l_X(x;x,\gamma)$ for all closed loops $\gamma$ is Zariski dense in $\pi(X)/\Gamma^n\pi(X)$ for all $n$ . Vanishing of a regular function $p(\mathscr{E}_1 \circ f_{1*},\ldots,\mathscr{E}_n \circ f_{n*})$ on a Zariski dense subset implies that this regular function is the zero function.

Now we shall construct some elements of $\mathrm{Alg}(\pi(Y))$ which will be particularly interesting for us. We recall that $\mathrm{Lie}(\pi_1(Y,y))$ is a free Lie algebra on generators $Y_1,\dots Y_m$ where each $Y_i$ is a class in $\pi_1(Y,y)/\Gamma^2\pi_1(Y,y)$ of a loop around $y_i$. Let us choose a base of $\mathrm{Lie}(\pi_1(Y,y))$ given by basic Lie elements corresponding to the ordering $Y_1,\dots,Y_n$. Let $v \in \mathrm{Lie}(\pi_1(Y,y))$ be a basic Lie element and let $v^*$ be a linear functional on $\mathrm{Lie}(\pi_1(Y,y))$ dual to $v$ with respect to the base of basic Lie elements i.e.

$v^* \in \mathrm{Hom}(\mathrm{Lie}(\pi_1(Y,y));Z)$. The linear functional $v^*$ we consider as an element of $\mathrm{Alg}\,\pi(Y)$.

We set

$$\mathscr{L}_v(z;x,y) := v^*(l_Y(z;x,\gamma)) .$$

We shall also write $\mathscr{L}_v(z;x)$ instead of $\mathscr{L}_v(z;x,\gamma)$.

**Corollary 5.2.** *Let $f_1,\dots,f_N : X \longrightarrow Y$ be regular functions, let $n_1,\dots,n_N$ be intergers and let $v_1,\dots,v_N$ in $\mathrm{Lie}(\pi_1(Y,y))$ be basic Lie elements of degree $n$ not necessary different. There is a functional equation*

$$\sum_{i=1}^{N} \mathscr{L}_{v_i}(f_i(z);f_i(x),f_i(\gamma))) = 0$$

*if and only if*

$$\sum_{i=1}^{N} n_i(v_i^* \circ (f_i)_*) = 0$$

*in* $Hom(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);Z)$ *where* $(f_i)_*:\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)\longrightarrow\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)$ *is induced by* $f_i$.

Proof. The corollary follows immediately from Theorem 5.1 if one observes that the condition $\sum_{i=1}^{N} n_i(v_i^* \circ (f_i)_*) = 0$ in $\mathrm{Alg}\pi(X)$ is equivalent to the condition $\sum_{i=1}^{N} n_i(v_i^* \circ (f_i)_*) = 0$ in $\mathrm{Hom}(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);Z)$ because of the identification $(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x))\otimes\mathbb{C}\approx\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X)$.

**Corollary 5.3.** *Let* $b(X)$ *be a base of* $Lie(\pi_1(X,x))$ *given by basic Lie elements. The functions* $\{\mathscr{L}_v(z;x_0)\mid v\in b(X)\}$ *are algebraically independent on* $X$.

Proof. Let $v_1,\dots,v_n$ be different elements of $b(X)$. Let $p(t_1,\dots,t_n)$ be a polynomial with complex coefficients such that

$$p(\mathscr{L}_{v_1}(z;x_0),\dots,\mathscr{L}_{v_n}(z;x_0))\equiv 0 .$$

It follows from Theorem 4.1 that $p(v_1^*,\dots,v_n^*)=0$ in $\mathrm{Alg}(\pi(X))$. The functions $v_1^*,\dots,v_n^*$ are linearly independent generators of the algebra $\mathrm{Alg}(\pi(X))$. Hence the polynomial $p(x_1,\dots,x_n)$ is equal to 0.

**Corollary 5.4.** *The functions* $\{\mathscr{L}_v(z;x_0)\mid v\in b(x)\}$ *are algebraically independent an any open disc around* $x_0$.

Proof. Assume that we have an identity $p(\mathscr{L}_{v_1}(z;x_0),\dots,\mathscr{L}_{v_n}(z;x_0))\equiv 0$ on a small disc around $x_0$ . Then by the analytic continuation we have such an equality along any path. Hence Corollary 5.3 implies that the polynomial $p(x_1,...,x_n)$ is identically equal to zero.

## 6. Functional equations of polylogarithms.

Now we shall restrict our attention to polylogarithms. The following assumptions will be used through the whole section so we extract them at very beginning.

*6.1. Let* $X = P^1(\mathbb{C})\setminus\{x_1,...x_{n+1}\}$ *and let* $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ *. Let* $f_1,...,f_N : X \longrightarrow Y$ *be regular functions and let* $n_1,...,n_N$ *be integers. Let* $x$ *and* $z$ *belong to* $X$ *and let* $\gamma$ *be a smooth path in* $X$ *from* $x$ *to* $z$ *.*

*6.2. Let* $U$ *and* $V$ *be loops in* $Y$ *in a clock–wise direction around points* $0$ *and* $1$ *respectively. We consider* $U$ *and* $V$ *as elements of the Lie algebra* $Lie(\pi_1(Y,y))$ *. Let us set* $e_0 := U$ *,* $e_1 := V$ *,* $e_2 := [V,U]$ *,* $e_n := [e_{n-1},U]$ *for* $n \geq 2$ *. Let* $e_n^*$ *be a linear functional on* $Lie(\pi_1(Y,y))$ *dual to* $e_n$ *with respect to the base of* $Lie(\pi_1(Y,y))$ *given by basic Lie elements corresponding to the ordering* $U$ *,* $V$ *. We consider* $e_n^*$ *as an element of* $Alg\pi(Y)$ *.*

**Definition 6.3.** *Let* $\mathscr{C}_n : P(Y) = C[[U,V]]^* \longrightarrow \mathbb{C}$ *associates to an element of* $P(Y)$ *its coefficients at* $U^nV$ *. We set*

$$Li_n(z;x,\gamma) := (-1)^{n-1}\mathscr{C}_n(\lambda_Y(z;x,\gamma)) .$$

We shall write also $Li_n(z;x)$ when we do not specify the path $\gamma$ .

Observe that $\mathrm{Li}_n(z;x,\gamma) = (-1)^{n-1} \int\limits_{x,\gamma}^{z} \frac{-dz}{z-1}, \frac{-dz}{z}, \ldots, \frac{-dz}{z} = \int\limits_{x,\gamma}^{z} \frac{-dz}{z-1}, \frac{dz}{z}, \ldots, \frac{dz}{z}$, where $\frac{-dz}{z}$ appears $n-1$ times.

Immediate consequence of the results from section 5 is the following theorem.

**Theorem 6.4.** *(functional equation of polylogarithms; integral form and abstract form) Assume 6.1 and 6.2. Then we have:*

*i)* *There is a functional equation*

$$(*_0) \quad \sum_{i=1}^{N} n_i \, \mathscr{L}_{e_n}(f_i(z);f_i(x),f_i(\gamma)) = 0$$

*if and only if one of the following equivalent conditions is satisfied.*

$$(*_1) \quad \sum_{i=1}^{N} n_i \, e_n^{*} \circ (f_i)_* = 0 \ \text{ in the group } \ Hom(\Gamma^n \pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);Z) \, ;$$

$$(*_2) \quad \sum_{i=1}^{N} n_i \, e_n^{*} \circ (f_i)_* = 0 \ \text{ in the group}$$

$$Hom(\Gamma^n \pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x) + [\Gamma^2\pi_1(X,x),\Gamma^2\pi_1(X,x)] \cap \Gamma^n\pi_1(X,x);Z) \, ;$$

$$(*_3) \quad \sum_{i=1}^{N} n_i (f_i)_* = 0 \ \text{ in the group}$$

$$Hom(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)\,;\,\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)+L_n)$$

*where* $L_n$ *is a subgroup of* $\Gamma^n\pi_1(Y,y)$ *generated by all commutators which contain* $V$ *at least twice.*

*ii)* *If* $(*_4)$ $\sum_{i=1}^{N} n_i \mathscr{C}_n \circ (f_i)_* = 0$ *in* $Alg(P(X))$ *then there is a functional equation*

$$(*_5) \qquad \sum_{i=1}^{N} n_i\, Li_n\,(f_i(z)\,;\,f_i(x)\,,\,f_i(\gamma)) = 0\,.$$

The formulas $(*_0)$ and $(*_4)$ are integral forms of functional equations whilst the formulas $(*_1)$, $(*_2)$, $(*_3)$ and $(*_5)$ are abstract forms of functional equations.

Proof. It follows from Corollary 5.2 that $(*_0)$ is equivalent to $(*_1)$. Conditions $(*_1)$, $(*_2)$ and $(*_3)$ are evidently equivalent. Theorem 5.1 implies that the condition $(*_4)$ implies the condition $(*_5)$.

Now we shall show that the function $\mathscr{L}_{e_n}(z;x)$ can be expressed by classical polylogarithms.

**Lemma 6.5.** *We have*

*i)* $Li_n(z;x) = Li_n(z) - Li_n(x) + l.d.t.(n)$.

*ii)* $\mathscr{L}_{e_n}(z;x) - Li_n(z,x) = l.d.t.(n)$.

Proof. The point i) is a direct calculation. Hence it rests to show ii). We recall that a horizontal section of the bundle $Y \times P(Y) \longrightarrow Y$ is $\lambda_Y(z;x)$ while a horizontal section of the bundle $Y \times \pi(Y) \longrightarrow Y$ is $l_Y(z;x)$ . It follows from Lemma 2.6 that $\exp l_Y(z;x) = \lambda_Y(z;x)$ . The coefficient of $\exp l_Y(z;x)$ at $U^nV$ is equal to

$$(-1)^{n+1}\mathscr{L}_{e_{n+1}}(z;x) + \sum_{k=2}^{n} \frac{(-1)^{n+1}}{k!}\left[\int_x^z \frac{dz}{z}\right]^{k-1}\mathscr{L}_{e_{n-k+2}}(z;x) +$$

$$\frac{(-1)^{n+1}}{(n+1)!}\left[\int_x^z \frac{dz}{z}\right]^n \left[\int_x^z \frac{-dz}{z-1}\right]$$

On the other side the coefficient of $\lambda_Y(z;x)$ at $U^nV$ is equal to $(-1)^n \mathrm{Li}_{n+1}(z;x)$ . Comparing these two coefficients it follows by induction and Lemma 6.5 that $\mathscr{L}_{e_n}(z;x) - \mathrm{Li}_n(z,x) = \mathrm{l.d.t.}(n)$ .

Now we can show the following corollary of Theorem 6.4.

Corollary 6.6. *Assume 6.1 and 6.2. Then the following conditions are equivalent:*

*i)* *there is a functional equation*

$$\sum_{i=1}^{N} n_i(Li_n(f_i(z) - Li_n(f_i(x))) + l.d.t.(n) = 0\ ;$$

*ii)* *there is a functional equation*

$$\sum_{i=1}^{N} n_i\, Li_n(f_i(z)) + \overline{l.d.t.}(n) = 0\ ;$$

*iii)* $\sum_{i=1}^{N} n_i (f_i)_* = 0$

*in the group* $Hom(\Gamma^n \pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)\,;\,Hom(\Gamma^n \pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y) + L_n)$ .

Proof. It follows from Lemma 6.5 that $\mathscr{L}_{e_n}(f_i(z);f_i(x),f_i(\gamma)) = Li_n(f_i(z)) - Li_n(f_i(x))$ $+$ l.d.t.(n) . Substituting these expressions for $\mathscr{L}_{e_n}(f_i(z);f_i(x),f_i(\gamma))$ in the formula $(*_0)$ from Theorem 6.4 we get

$\sum_{i=1}^{N} n_i(Li_n(f_i(z) - Li_n(f_i(x))) + \text{l.d.t.}(n) = 0$ . Hence iii) implies ii). Observe that $Li_n(f_i(x)) + \text{l.d.t.}(n) = \overline{\text{l.d.t.}(n)}$ . Hence i) implies ii).

Assume that ii) is satisfied. Then it follows from Lemma 6.5 and Proposition 6.6 that $\sum_{i=1}^{N} n_i(\mathscr{L}_{e_n}(f_i(z);f_i(x),f_i(\gamma)) + \overline{\text{l.d.t.}}(n) = 0$ for some choice of $\overline{\text{l.d.t.}}(n)$ . Let $\gamma \in \Gamma^n \pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)$ . Observe that the monodromy of l.d.t.(n) on $\Gamma^n \pi_1(Y,y)$ is trivial. This follows immediately from Corollary 4.2 and Lemma 6.5. Hence the value $\sum_{i=1}^{N} n_i(\mathscr{L}_{e_n}(f_i(x);f_i(x),f_i(\gamma)) = c$ where $c$ is a constant which does not depend on $\gamma$ . Let $\mathscr{C} = \sum_{i=1}^{N} n_i\, e_n^{\,*} \circ (f_i)_*$ . Then $\mathscr{C}(1_X(x;x,\gamma)) - c = 0$ for each $\gamma \in \Gamma^n \pi_1(x,x)$ . Hence $\mathscr{C} - c$ vanishes on a Zariski dense subste of $\Gamma^n \pi(X)/\Gamma^{n+1}\pi(X)$ . This implies that $\mathscr{C} - c = 0$ . Evaluating $\mathscr{C} - c$ on a constant loop at $x$ we get $c = 0$ . Hence $\mathscr{C} = 0$ .

Observe that we have just proved Theorem E.

Now we shall prove some general results about functional equations of polylogarithms. In functional equations from Theorem 6.4 and Corollary 6.6 coefficients $n_i$ were integers. One can ask whether they cannot be arbitrary complex numbers. We have the following result in this direction.

Corollary 6.7. *If there is a functional equation of the form*

$$(*) \qquad \sum_{i=1}^{N} \alpha_i Li_n(f_i(z)) + \overline{l.d.t.}(n) = 0$$

*then there are rational numbers* $c_1,\dots,c_N$ *not all equal zero such that*

$$\sum_{i=1}^{N} c_i Li_n(f_i(z)) + \overline{l.d.t.}(n) = 0$$

Proof. The equation (*) is equivalent to the relation $\sum_{i=1}^{N} \alpha_i e_n^* \circ (f_i)_* = 0$ in $\mathrm{Hom}(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);\mathbb{C})$. The functionals $e_n^* \circ (f_i)_*$ belong to the $\mathbb{Q}$–vector space $\mathrm{Hom}(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);\mathbb{Q})$. Therefore if there is a non–trivial relation of the form $\sum_{i=1}^{N} \alpha_i e_n^* \circ (f_i)_* = 0$ with $\alpha_i \in \mathbb{C}$, then there is also a non–trivial relation $\sum_{i=1}^{N} c_i e_n^* \circ (f_i)_* = 0$ with $c_i \in \mathbb{Q}$. Hence the corollary follows from Theorem 6.4.

Compare this result with a result in [B]. In our corollary one would like to replace functions $f_1(z),\dots,f_N(z)$ by algebraic numbers $a_1,\dots a_N$ and to take $\alpha_i$ in $\overline{\mathbb{Q}}$.

One would like to get new functional equations from the old one. This is possible as we see from the next result, though unfortunately from functional equations of $\mathrm{Li}_n(z)$ we only get functional equations of $\mathrm{Li}_{n-1}(z)$. We do not know any method which allows to pass from $\mathrm{Li}_n(z)$ to $\mathrm{Li}_{n+1}(z)$.

**Definition 6.8.** *Let $f(z)$ be a rational function. We denote by $\nu_{z-a}(f(z))$ the valuation of $f(z)$ at $(z-a)$.*

Observe that $f(z) = \prod_{a\in\mathbb{C}} (z-a)^{\nu_{z-a}(f(z))}$.

**Lemma 6.9.** *Let $f_1,\dots f_N : X \longrightarrow Y$ be regular functions. Assume that $\sum_{i=1}^{N} n_i\, e_n{}^* \circ (f_i)_* = 0$ in $Hom(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);Z)$. Let $a_1,\dots,a_k$ be complex numbers and let $n-k \geq 2$.*

*Then*

$$\sum_{i=1}^{N} n_i \cdot (\nu_{z-a_1}(f_i(z)) \cdot \nu_{z-a_2}(f_i(z)) \cdot \dots \cdot \nu_{z-a_k}(f_i(z)) \cdot e_{n-k} \circ (f_i)_* = 0$$

*in $Hom(\Gamma^{n-k}\pi_1(X,x)/\Gamma^{n-k+1}\pi_1(X,x);Z)$.*

Proof. This is an easy observation if one writes a map $(f_i)_*$ in terms of a base given by basic Lie elements.

Observe that Lemma 6.9 allows to get functional equations of $\mathrm{Li}_k$ $(2 \leq k < n)$ if we have a functional equation of $Li_n$. This follows from Theorem 6.4 or Corollary 6.7. Observe that the number of functional equations of $Li_k$ grows when $k$ becomes smaller.

Now we shall show that certain functional equations are impossible.

Proof of Theorem $\mathbb{C}$. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}$.

It follows from Example 4 in section 1 that we can assume that $a_1 \neq a_2$. Let $c \in \mathbb{C}$ be such that $f(c) = 1$ with the multiplicity $r$. We consider $f$ as a regular map $f : X = P^1(\mathbb{C}) \setminus \{f^{-1}(0) \cup f^{-1}(1) \cup f^{-1}(\infty) \cup \infty\} \longrightarrow Y = P^1(\mathbb{C}) \setminus \{0,1,\infty\}$. (Warning: here $f^{-1}(*)$ is the inverse image of *.) We choose a base of $H_1(X)$ given by loops around missing points except $\infty$. Let $A_i$ be a loop around $a_i$ and let $\mathbb{C}$ be a loop around $c$. Let us set $\alpha_2 := [C,A_1]$, $\alpha_n := [\alpha_{n-1},A_2]$ and $\beta_3 := [[C,A_1]A_1]$, $\beta_n := [\beta_{n-1},A_2]$. The only maps of degree one which induce something non–trivial on $\alpha_n$ and $\beta_n$ are $g(z) = \frac{z-a_2}{z-a_1} \cdot \frac{c-a_1}{c-a_2}$ and $h(z) = \frac{z-a_1}{z-a_2} \cdot \frac{c-a_2}{c-a_1}$. For these maps we have

$$g_*(\alpha_n) = -e_n \quad , g_*(\beta_n) = e_n$$

and

$$h_*(\alpha_n) = (-1)^{n-2} e_n \quad , h_*(\beta_n) = (-1)^{n-3} e_n$$

in the group

$$(*) \qquad \mathrm{Hom}(\Gamma^n \pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x);Z) \; ; \; \Gamma^n \pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y) + L_n) .$$

Observe that

$$(*) \qquad f_*(\alpha_n) = r \cdot n_1 \cdot n_2^{n-2} e_n \text{ and } f_*(\beta_n) = r \cdot n_1^2 \cdot n_2^{n-3} e_n .$$

Hence the relation of the form

$$f_* = \sum_{i=1}^{N} q_i (f_i)_*$$

where $q_i \in \mathbb{Q}$ and $\deg f_i = 1$ is impossible in the group (*). Therefore Corollary 6.6 implies the theorem.

Closely related to Theorem $\mathbb{C}$ is the following result.

**Theorem 6.10.** *Let $a_1, a_2 \ldots a_n$ be $n$ different points of $\mathbb{C}$. Let $Ł(z) := \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \ldots, \frac{dz}{z-a_n}$. If $n > 3$ then there is no polynomial $p(s, t_1, \ldots, t_r)$ which depends essentially on $s$ such that $p(Ł(z), Li_{n_1}(f_1(z)), \ldots, Li_{n_r}(f_r(z))) \equiv 0$ where $Li_{n_k}$ are classical polylogarithms and logarithms and $f_i(z)$ are rational function.*

Proof. Let $T = \{0,1,\infty\}$ and let $S = \bigcup_{i=1}^{r} f_i^{-1}(T) \cup \{a_1, \ldots, a_n, \infty\}$. Observe that singularities of the functions $p(z) := p(Ł(z), Li_{n_1}(f_1(z)), \ldots, Li_{n_r}(f_r(z)))$, $Ł(z)$ and $Li_{n_i}(f_i(z))$ are contained in the set $S$. On $X = P^1(\mathbb{C}) \setminus S$ these functions are analytic and multivalued. Let $A_i$ be a loop around $a_i$ in $X$. The monodromy of $\int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}, \frac{dz}{z-a_n}$ on the commutator $\alpha = [[A_1, A_2]], [A_3, A_4]]$ is equal to

$(2\pi i)^4$ up to sign. Hence the monodromy of $\mathcal{L}(z)$ on $\alpha$ is also non–trivial.

Now we must calculate the monodromy of $Li_{n_k}(f_k(z))$ on $\alpha$ . We consider the group G of power series $e^{aX} + \sum_{n=0}^{\infty} b_n X^n Y$ with a multiplication given by

$$(e^{aX} + \sum_{n=0}^{\infty} b_n X^n Y)(e^{a'X} + \sum_{n=0}^{\infty} b'_n X^n Y) = e^{(a+a')X} +$$

$$\sum_{n=0}^{\infty} (b_n + b'_n + \left[ \sum_{n=0}^{\infty} \frac{a^k}{k!} b_{n-k} \right] X^n Y .$$

The monodromy of polylogarithms was calculated in [R] and it can be described in the following way.

Let $Li(z) = e^{(-\log z)X} + \sum_{n=0}^{\infty} (-1)^{n-1} Li(z)\, X^n Y$ .

The monodromy of $Li(z)$ along the loop around 0 is given by the multiplication on the right hand side by $e^{(-2\pi i)X}$ and the monodromy along the loop around 1 is given by the multiplication on the right hand side by $1 - 2\pi i Y$ . Observe that for any four elements a,b,c,d in G we have $[[a,b] , [c,d]] = 1$ . Hence the monodromy of $Li_{n_k}(f_k(z))$ on $\alpha$ is trivial. This implies $p(z) \neq 0$ .

Observe that Theorem D point b is a particular case of Theorem 6.10. We left to the reader to show point a of Theorem D.

## 7. Functional equations of lower degree polylogarithms.

In this section we shall prove Theorem A. We shall give also several examples of functional equations of lower–degree polylogarithms.

### 7.1. Functional equations of the dilogarithm.

Proof of Theorem A′. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} \Big/ \prod_{j=1}^{m} (z-b_j)^{m_j}$ and let $f^{-1}(1) = \sum_{k=1}^{r} c_k \cdot r_k$ . Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots a_n, b_1,\dots,b_m, c_1,\dots,c_r,\infty\}$ and let $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ . Let $P(X) = \mathbb{C}[[A_1,\dots,A_n,B_1,\dots,B_m,C_1\dots C_r]]^*$ where $A_i$ (resp. $B_j$ , resp. $C_k$ ) is the class in $H_1(X,\mathbb{C})$ of a loop around $a_i$ (resp. $b_j$ , resp. $c_k$ ). Let $P(Y) = C[[U,V]]^*$ . Let $\gamma$ be a smooth path in $X$ from $x$ to $z$ .

We have

$$f_*(A_i \cdot C_k) = n_i\, r_k\, U{\cdot}V \ \ , f_*(B_j \cdot C_k) = -\, m_j r_k\, U{\cdot}V \ ,$$

$$f_*(A_i \cdot B_j) = n_i\, m_j\, U{\cdot}V \ \ , f_*(B_j \cdot B_j) = -\, m_j m_j\, U{\cdot}V$$

We need maps of degree one from $X$ to $Y$ which induce the same maps on these products. Here there are three families of such maps:

$$f_{ik}(z) = \frac{z-a_i}{c_k-a_i}\,, \qquad\qquad (f_{ik})_*(A_i \cdot C_k) = U \cdot V \ ;$$

$$g_{ij}(z) = \frac{z-a_i}{b_j-a_i}, \qquad (g_{ij})_*(A_i \cdot B_j) = U \cdot V ;$$

$$h_{jk}(z) = \frac{z-b_j}{c_k-b_j}, \qquad (h_{jk})_*(B_j \cdot C_k) = U \cdot V ;$$

Let $\mathscr{C}_2 : P(Y) \longrightarrow \mathbb{C}$ be as in Definition 6.3. Let $\psi_{jj'} : P(X) \longrightarrow \mathbb{C}$ be a coefficient at $B_j \cdot B_j'$ . We have the following identity

$$(*)\ \mathscr{C}_2 \circ f_* = \sum_{i,k} n_i r_k\ \mathscr{C}_2 \circ (f_{ik})_* - \sum_{i,k} n_i m_j\ \mathscr{C}_2 \circ (g_{ij})_* - \sum_{i,k} m_j r_k\ \mathscr{C}_2 \circ (h_{jk})_* + \sum_{j,j'} \psi_{jj'} .$$

We shall calculate the expression $\sum_{j,j'} \psi_{jj'}(\lambda_X(z;x,\gamma))$ .

It follows from the formula

$$\int_\gamma \frac{dz}{z-a}, \frac{dz}{z-b} + \int_\gamma \frac{dz}{z-b}, \frac{dz}{z-a} = \left[\int_\gamma \frac{dz}{z-a}\right]\left[\int_\gamma \frac{dz}{z-b}\right]$$

(see [Ch] 1.5.1) that $\sum_{j,j'} \psi_{jj'}(\lambda_X(z;x,\gamma)) = \frac{1}{2}\sum_{j,j'} m_j \cdot m_{j'} \left[\int_\gamma \frac{dz}{z-b_j}\right]\left[\int_\gamma \frac{dz}{z-b_{j'}}\right]$ . Evaluating the identity (*) on $\lambda_X(z;x,\gamma)$ and applying the equality 3.3.1 we get

$$\int_{f(\gamma)} \omega = \sum_{i,k} n_i r_k \int_{f_{ik}(\gamma)} \omega - \sum_{i,j} n_i m_i \int_{g_{ij}(\gamma)} \omega - \sum_{j,k} m_j r_k \int_{h_{jk}(\gamma)} \omega$$

$$+ \sum_{j,j'} m_j m_{j'} \left[\int_\gamma \frac{dz}{z-b_j}\right]\left[\int_\gamma \frac{dz}{z-b_{j'}}\right]$$

where $\omega = \frac{dz}{z-1}, \frac{dz}{z}$. Theorem A$'$ follows immediately from this equation.

Observe that Theorems A and A$''$ are immediate corollaries of Theorem A$'$. This was already observed in section 1.

Now we shall give an abstract from of the functional equation 1.3. We shall keep the notation from 7.1.

**Theorem 7.1.1.** *We have*

*7.1.2.* $$f_* = \sum_{i,k} n_i r_k (f_{ik})_* - \sum_{i,j} n_i m_j (g_{ij})_* - \sum_{j,k} m_j r_k (h_{jk})_*$$

*in the group* $Hom(\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)\,;\,\Gamma^2\pi_1(Y,y)/\Gamma^3\pi_1(Y,y))$.

Now we shall show that from the functional equation 1.3 choosing suitably a function f(z) and a point x we can get functional equations known before.

Examples: Let $f(z) = z^n$ and $x = 0$. Then we get

7.1.3. $$Li_2(z^n) = \sum_{k=1}^{n} Li_2(\xi^k z)$$

where $\xi = e^{\frac{2\pi i}{n}}$.

Let $f(z) = \frac{yz}{(y-1)(z-1)}$ and $x = 0$. Then we have

7.1.4. $$\mathrm{Li}_2\left[\frac{yz}{(y-1)(z-1)}\right] = \mathrm{Li}_2\left[\frac{z}{1-y}\right] - \mathrm{Li}_2\left[\frac{1-z}{y}\right] + \mathrm{Li}_2\left[\frac{1}{y}\right] - \mathrm{Li}_2(z) -$$

$$\log\left[\frac{y-1}{y}\right]\log(1-z) - \frac{1}{2}\left[\log(1-z)\right]^2 .$$

Let $f(z) = \frac{(1-y)z}{z-1}$ and $x = 0$. Then we have

7.1.5 $$\mathrm{Li}_2\left[\frac{(1-y)z}{z-1}\right] = \mathrm{Li}_2(yz) - \mathrm{Li}_2\left[\frac{y-1}{y}(1-z)\right] + \mathrm{Li}_2\left[\frac{y-1}{y}\right] - \mathrm{Li}_2(z)$$

$$-\log(1-y)\log(z-1) - \frac{1}{2}\log^2(1-z) .$$

Observe that the Abel equation from Section 1 follows from 7.1.4.

Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} \Big/ \prod_{j=1}^{m} (z-b_j)^{m_j}$. Then we have

(*) $$\log f(z) = \log \alpha + \sum_{i=1}^{n} n_i \log(z-a_i) - \sum_{i=1}^{m} m_j \log(z-b_j) .$$

Observe that any functional equation of log of the form $\sum_{i=1}^{N} n_i \log f_i(z) = 0$ where $f_i(z)$ are rational functions is a linear combination with rational coefficients of equations (*). For the dilogarithm we have a similar situation. We shall formulate a theorem only for abstract functional equations.

**Theorem 7.1.6.** *Assume 6.1 and 6.2. Then any relation of the form* $\sum_{i=1}^{N} n_i (f_i)_* = 0$ *in* $Hom(\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)\,;\,\Gamma^2\pi_1(Y,y)/\Gamma^3\pi_1(Y,x))$ *is a linear combination of the re-*

*lations 7.1.2 for functions* $f_i$ .

Proof. The theorem is an easy observation in linear algebra.

7.2. Functional equations of the trilogarithm.

**Theorem 7.2.1.** *(integral form of a functional equation). Let us assume that the condition 1.1 holds. Then we have*

7.2.2.
$$\mathcal{L}_3(f(z;x,\gamma)) = \sum_{1<i',k} n_i n_{i'} r_k \left[ \mathcal{L}_3(d_i^{i'}(z;x,\gamma)) - \mathcal{L}_3(e_{ik}^{i'}(z;x,\gamma)) \right]$$

$$+ \sum_{i,i',k} n_i n_{i'} r_k \left[ \mathcal{L}_3(f_{ik}(z;x,\gamma)) \right] + \sum_{i,j,k} n_i m_j r_k \left[ \mathcal{L}_3(g_{ik}^j(z;x,\gamma)) - \mathcal{L}_3(h_i^j(z;x,\gamma)) + \right.$$

$$\left. - \mathcal{L}_3(l_{ik}(z;x,\gamma)) - \mathcal{L}_3(p_{jk}(z;x,\gamma))\right) + \sum_{j<j',k} m_j m_{j'} r_k \left[ \mathcal{L}_3(q_j^{j'}(z;x,\gamma)) - \mathcal{L}_3(s_{jk}^{j'}(z;x,\gamma)) \right] +$$

$$\sum_{j,j',k} m_j m_{j'} r_k \left[ \mathcal{L}_3(t_{jk}(z;x,\gamma)) \right] + \sum_{i<i',j} n_i n_{i'} m_j \left[ - \mathcal{L}_3(u_i^{i'}(z;x,\gamma)) + \mathcal{L}_3(v_{ij}^{i'}(z;x,\gamma)) \right] +$$

$$\sum_{i,i',j} n_i n_{i'} m_j \left[ - \mathcal{L}_3(w_{ij}(z;x,\gamma)) \right] + \sum_{j<j',i} m_j m_{j'} n_i \left[ \mathcal{L}_3(\varphi_{ji}^{j'}(z;x,\gamma)) - \mathcal{L}_3(\psi_j^{j'}(z;x,\gamma)) \right] -$$

$$\sum_{j,j',i} m_j m_{j'} n_i \left[ \mathcal{L}_3(\chi_{ji}(z;x,\gamma)) \right] .$$

*where*

$$d_i^{i'}(z)=\frac{z-a_i}{z-a_{i'}},\ e_{ik}^{i'}(z)=\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_{i'}}{c_k-a_i},\ f_{ik}(z)=\frac{z-a_i}{c_k-a_i},\ g_{ik}^{j}(z)=\frac{z-a_i}{z-b_j}\cdot\frac{c_k-b_j}{c_k-a_i},$$

$$h_i^j(z)=\frac{z-a_i}{z-b_j},\ l_{ik}(z)=\frac{z-a_i}{c_k-a_i},\ p_{jk}(z)=\frac{z-b_j}{c_k-b_j},\ q_j^{j'}(z)=\frac{z-b_j}{z-b_{j'}},\ s_{jk}^{j'}(z)=\frac{z-b_j}{z-b_{j'}}\cdot\frac{c_k-b_{j'}}{c_k-b_j},$$

$$t_{jk}(z)=\frac{z-b_j}{c_k-b_j},\ u_i^{i'}(z)=\frac{z-a_i}{z-a_{i'}},\ v_{ij}^{i'}(z)=\frac{z-a_i}{z-a_{i'}}\cdot\frac{b_j-a_{i'}}{b_j-a_i},\ w_{ij}(z)=\frac{z-a_i}{b_j-a_i},\ \varphi_{ji}^{j'}(z)$$

$$=\frac{z-b_j}{z-b_{j'}}\cdot\frac{a_i-b_{j'}}{a_i-b_j},\ \psi_j^{j'}(z)=\frac{z-b_j}{z-b_{j'}}\ \text{and}\ \chi_{ji}(z)=\frac{z-b_j}{a_i-b_j}.$$

Proof. One checks that in $\mathrm{Hom}(\Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x)\,;\,\Gamma^3\pi_1(Y,y)/\Gamma^4\pi_1(Y,y)+L_3)$ there is the identity (abstract form of a functional equation)

7.2.3.
$$f_* = \sum_{1<i',k} n_i n_{i'} r_k \left[(d_i^{i'})_* - (e_{ik}^{i'})_*\right] + \sum_{i,i',k} n_i n_{i'} r_k \left[(f_{ik})_*\right] +$$

$$\sum_{i,j,k} n_i m_j r_k \left[(g_{ik}^j)_* - (h_i^j)_* - (l_{ik})_* - (p_{jk})_*\right] + \sum_{j<j',k} m_j m_{j'} r_k \left[(q_j^{j'})_* - (s_{jk}^{j'})_*\right] +$$

$$\sum_{j,j',k} m_j m_{j'} r_k \left[(t_{jk})_*\right] + \sum_{i<i',j} n_i n_{i'} m_j \left[-(u_i^{i'})_* + (v_{ij}^{i'})_*\right] + \sum_{i,i',j} -n_i n_{i'} m_j \left[(w_{ij})_*\right] +$$

$$\sum_{j<j',i} m_j m_{j'} n_i \left[(\varphi_{ji}^{j'})_* - (\psi_j^{j'})_*\right] - \sum_{j,j',i} m_j m_{j'} n_i (\chi_{ji})_* .$$

The theorem follows from Theorem 6.1 i).

We shall not prove Theorem B. We indicate only a general scheme of a proof. First one proves an analog B′ of Theorem B in the same way as we proved Theorem A′. Then one deduces Theorem B from B′ .

Observe that the abstract form of the functional equation from Theorem B is a particular case of 7.2.3.

For the trilogarithm we have an analog of Theorem 7.1.6.

**Theorem 7.2.4.** *Assume 6.1 and 6.2. Then any relation of the form* $\sum_{i=1}^{N} n_i (f_i)_* = 0$ *in* $Hom(\Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x)\,;\Gamma^3\pi_1(Y,y)/\Gamma^4\pi_1(Y,y) + L_3)$ *is a linear combinations of the relations 7.2.3 for functions* $f_i$ .

Proof. The theorem is once more an easy observation in linear algebra.

## 7.3. The fourth–order polylogarithm.

We shall give an example of a functional equation of the fourth–order polylogarithm which seems not to be reported in the literature.

Let $f_1(z) = -\frac{1}{(b-a)^2}(z-a)(z-b)$ and $f_2(z) = \frac{(z^2-a)}{(b-a)(z-b)}$ . Let $c_1$ , $c_2$ be roots of the equation $f_1(z) - 1 = 0$ . Observe that $c_1$ and $c_2$ are also roots of the equation $f_2(z) - 1 = 0$ . Let us set

$$g_1(z) = \frac{z-a}{z-b}\cdot\frac{c_1-b}{c_1-a} \qquad , \qquad g_2(z) = \frac{z-a}{z-b}\cdot\frac{c_2-b}{c_2-a} \qquad ,$$

$$h_1(z) = \frac{z-a}{c_1-a} \quad , \qquad h_2(z) = \frac{z-a}{c_2-a} \quad ,$$

$$k_1(z) = \frac{z-b}{c_1-b} \quad , \qquad k_2(z) = \frac{z-b}{c_2-b} \quad ,$$

$$l_1(z) = \frac{z-a}{z-b} \quad , \qquad l_2(z) = \frac{z-a}{b-a} \quad , \qquad l_3(z) = \frac{z-b}{a-b} .$$

Let $X = P^1(\mathbb{C})\setminus\{a,b,c_1,c_2,\infty\}$ and $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ . Each of the rational functions described above determines a regular map from $X$ to $Y$ .

**Theorem 7.3.1.** *(Abstract form of a functional equation) We have*

$$(f_1)_* + (f_2)_* = 3(g_1)_* + 3(g_2)_* + 6(h_1)_* + 6(h_2)_* + 3(k_1)_* + 3(k_2)_* \\ - 2(l_1)_* - 4(l_2)_* - 2(l_3)_*$$

*in the group* $Hom(\Gamma^4\pi_1(X,x)/\Gamma^5\pi_1(X,x)\,;\,\Gamma^4\pi_1(Y,y)/\Gamma^5\pi_1(Y,y) + L_4)$ .

Notice that this functional equation has less quadric terms then the Kummer functional equation of the fourth–order polylogarithm.

## 8. Generalized Bloch groups.

**Definition 8.1.** Let $K$ be a field. We set

$$B(K) := \coprod_{f \in K\setminus\{0,1\}} Z$$

The group B(K) is by definition a free abelian group on elements of $K\setminus\{0,1\}$ . The generator of B(K) corresponding to $f \in K\setminus\{0,1\}$ , we shall denote by [f] .

We recall the Abel functional equation

$$\mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] - \mathrm{Li}_2\left[\frac{y}{1-x}\right] - \mathrm{Li}_2\left[\frac{x}{1-y}\right] + \mathrm{Li}_2(x) + \mathrm{Li}_2(y) = \log(1-x)\log(1-y) .$$

S. Bloch observed that the element

$$[\tfrac{x}{1-x}\cdot\tfrac{y}{1-y}] - [\tfrac{y}{1-x}] - [\tfrac{x}{1-y}] + [x] + [y] \in B(\mathbb{C}(x,y))$$

belongs to the kernel of the homomorphism

$$\lambda : B(\mathbb{C}(x,y)) \longrightarrow \mathbb{C}^*(x,y) \wedge \mathbb{C}^*(x,y)$$

where $\lambda([f]) = f \wedge (1-f)$ and $\mathbb{C}^*(x,y) \wedge \mathbb{C}^*(x,y)$ is an exterior product of $\mathbb{C}^*(x,y)$ with itself considered as an abelian group (see [DS]).

The aim of this section is to generalize the phenomena observed by S. Bloch and to put in in the picture described in the previous sections.

Let A be an abelian group and let $\mathbb{L}(A)$ be a free Lie algebra on A . Let $\mathbb{L}'(A) = [\mathbb{L}(A),\mathbb{L}(A)]$ and let $\mathbb{L}''(A) = [\mathbb{L}'(A),\mathbb{L}'(A)]$ . We set

$$\mathrm{Li}(A) := \mathbb{L}(A)/\mathbb{L}''(A) .$$

Let K be a function field and let k be its field of constants. Let $K^*$ and $k^*$ be respectively its multiplicative groups. Let $I(K^* : k^*)$ be a Lie ideal in $L(K^*)$ generated by brackets $[\dots[f_1 \dots f_i], [f_{i+1}, \dots] \dots f_n] \dots]$ where at least one $f_i$ is in $k^*$.

Let us set

$$\mathcal{L}i(K^*) := Li(K^*)/I(K^* : k^*) .$$

For any $n \geq 2$ we define a homomorphism

$$B_n : B(K) \longrightarrow \mathcal{L}i(K^*)$$

by the following formula

$$B_n([f]) = [\dots[f-1, f] f] \dots] f] \dots] .$$

The main result of this section is the following theorem.

**Theorem 8.2.** *Let* $X = P^1(\mathbb{C}) \setminus \{a_1, \dots a_m, \infty\}$ *and* $Y = P^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ *. Let* $f_1, \dots, f_N \in C(z)^*$ *be regular functions from* $X$ *to* $Y$ *and let* $k_1, \dots k_N$ *be integers. The following conditions are equivalent:*

*i)* *the element* $\sum_{i=1}^{N} k_i [f_i] \in B(C(z))$ *belongs to the kernel of the map*

$$B_n : B(C(z)) \longrightarrow \mathcal{L}i(C(z)^*) ;$$

*ii)* $\sum_{i=1}^{N} k_i(f_i)_* = 0$ *in the group*

$$Hom(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)\,;\,\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y) + L_n)\,;$$

*iii)* *there is a functional equation*

$$\sum_{i=1}^{N} k_i Li_n(f_i(z)) + \overline{l.d.t.}(n) = 0\,.$$

Proof. Let $f \in \mathbb{C}(z)^*$ and let $f^{-1}(0) = \sum_{i=1}^{n} m_i \cdot \alpha_i$, $f^{-1}(\infty) = \sum_{j=1}^{m} m_{n+j}\,\alpha_{n+j}$ and $f^{-1}(1) = \sum_{k=1}^{r} r_k c_k$. Observe that $f$ defines a regular function from $S = P^1(\mathbb{C})\setminus\{\alpha_1,...\alpha_{n+m},c_1,...c_r,\infty\}$ to $Y$. Let $A_i$ (resp. $C_k$) be a loop in $S$ around $\alpha_i$ (resp. $c_k$). Then $\text{Lie}\ \pi_1(S,s) = L(A_1,...,A_{n+m},c_1,...c_r)$ is a free Lie algebra on $A_1,...,C_r$. We choose a base of $\text{Lie}\ \pi_1(S,s)$ given by basic Lie elements corresponding to the ordering $A_1,...,A_{n+m},C_1,...,C_r$. In the group $\text{Hom}(\Gamma^n\pi_1(S,s)/\Gamma^{n+1}\pi_1(S,s), Z)$ we have $e_n^* \circ f_* = \sum_{i_{n-2}\geq...\geq i_1\geq i} \sum_k r_k m_i m_{i_1} \cdot ... m_{i_{n-2}} (...(C_k,A_i)A_{i_1})...A_{i_{n-2}})...)^* +$

$$\sum_{\substack{i_{n-2}\geq...\geq i_1\geq i\\ i\leq n}} \sum_j m_{j+n} \cdot m_i m_{i_1} \cdots m_{i_{n-2}} (...(A_{j+n},A_i)A_{i_1})...A_{i_{n-2}})...)^*\,.$$

On the other side after the identification $(z-a_i)$ (resp. $(z-c_k)$) with $A_i$ (resp. $C_k$) in the Lie algebra $\mathcal{A}(C(z)^*)$ we have

$$B_n([f]) = \sum_{i_{n-2}\geq\ldots\geq i_1\geq i} \sum_k r_k m_i m_{i_1} \cdot\ldots m_{i_{n-2}} (\gamma(k,i,i_1,\ldots,i_{n-2})$$

$$(\ldots(C_k,A_i)A_{i_1})\ldots A_{i_{n-2}})\ldots) - \delta(k,i,i_1,\ldots,i_{n-2})\,(\ldots(A_{i_1},A_i)A_{i_2})\ldots A_{i_{n-2}})C_k)\ldots)) +$$

$$\sum_{\substack{i_{n-2}\geq\ldots\geq i_1\geq i\\ i\leq n}} \sum_j m_{j+n} m_i m_{i_1} \cdots m_{i_{n-2}}\, \mathscr{C}(j+n,i,i_1\ldots,i_{n-2})(\ldots(A_{j+n},A_i)A_{i_1})\ldots A_{i_{n-2}})\ldots) +$$

$$\sum_{\substack{i_{n-1}\geq\ldots\geq i_1\\ n\geq i\geq i_1}} m_i m_{i_1} \ldots m_{i_{n-1}} \psi(i,i_1,\ldots,i_{n-1})(\ldots(A_i,A_{i_1})A_{i_2})\ldots A_{i_{n-1}})\ldots)\,.$$

This follows from the Jacobi identity and the fact that in the Lie algebra $\mathscr{L}i\,(C(z)^*)$ we have $(\ldots(A,B)A_1)\ldots A_n)\ldots) = (\ldots(A,B)A_{\sigma(1)})\ldots)A_{\sigma(n)})\ldots)$ where $\sigma$ is any permutation of n elements. The coefficients $\gamma(\ldots)$, $\delta(\ldots)$, $\mathscr{C}(\ldots)$ and $\psi(\ldots)$ do not depend on f. For example if $i_{n-2} > \ldots > i_1 > i$ then $\gamma(k,i,i_1,\ldots,i_{n-2}) = (n-1)!$ and $\delta(k,i,i_1,\ldots i_{n-2}) = \frac{1}{2}\cdot(n-1)!$.

Now from the formulas for $e_n^* \circ f_*$ and $B_n([f])$ and from Theorem 6.4 (($*_1$) and ($*_2$) are equivalent) it follows that the conditions i) and ii) are equivalent. By Corollary 6.6 ii) and iii) are also equivalent.

Remark. The groups $Li(K^*)$ and $\mathscr{L}i(K^*)$ are graded. The component in degree n we denote by $Li_n(K^*)$ and $\mathscr{L}i_n(K^*)$ respectively. They are generated additively by brackets of length n. D. Zagier in [Z3] considered the group $(Sym^{n-2}(K^*)\otimes(K^*\wedge K^*))\otimes\mathbb{Q}$. He found a condition to have a functional equation of higher Bloch–Wigner functions in terms of this group.

Observe that there is an epimorphism with non–trivial kernel from $(\mathrm{Sym}^{n-2}(K^*) \otimes (K^* \wedge K^*)) \otimes \mathbb{Q}$ onto $(\mathrm{Li}_n(K^*)) \otimes \mathbb{Q}$ and hence also onto $(\mathscr{L}_n(K^*)) \otimes \mathbb{Q}$. This follows from the fact that in $\mathrm{Li}_n(K^*)$ and $\mathscr{L}_n(K^*)$ we have $(\ldots(A,B)A_1)\ldots A_{n-2})\ldots) = (\ldots A,B)A_{\sigma(1)})\ldots A_{\sigma(n-2)})\ldots)$ for any $\sigma \in \sum_{n-2}$.

Let $Ł_n(z)$ be the higher Bloch–Wigner function considered in [W3] and in [Z2],[Z3]. We would like to show that $B_n\left[\sum_{i=1}^{N} k_i [f_i(z)]\right] = 0$ if and only if $\sum_{i=1}^{N} k_i \left[Ł_n(f_i(z)) - Ł_n(f_i(x))\right] = 0$.

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