EULER CHARACTERISTIC OF DEGENERACY LOCI

A. Parusiński & P. Pragacz

Institute of Mathematics,
Gdańsk University,
Wita Stwosza 57,
PL-80-952 Gdańsk, Poland

Mathematical Institute
Polish Academy of Sciences,
Chopina 12
PL-87-100 Toruń, Poland

Max-Planck-Institut für Mathematik
Gottfried-Claren-Strasse 26
D-5300 Bonn 3,
West Germany

Department of Mathematics, U.B.C.,
121-1984 Mathematics Road,
Vancouver B.C. Canada V6T 1Y4,

MPI/90-68

On representation of large integers by integral ternary positive definite quadratic forms

B.Z. MOROZ

A few years after the famous work of C.L. Siegel's, [14], on representation of integers by a genus of quadratic forms had appeared Yu. V. Linnik, [7], initiated a study of representation of integers by an individual ternary quadratic form. Due to the efforts of many authors (cf., for instance, [8], [9], [1], [12], [16], [6], [3] and references therein), we may now claim a success. Let $f(x) = \frac{1}{2} \sum_{1 \leq i,j \leq 3} a_{ij} x_i x_j$ be a positive definite quadratic form with integral rational coefficients, so that $a_{ij} = a_{ji}$, $a_{ij} \in \mathbb{Z}$, $2 | a_{ii}$ for $1 \leq i, j \leq 3$, and let $r_f(n) = \operatorname{card} \{u | u \in \mathbb{Z}^3,\ f(u) = n\}$ be the representation number of n by f; let $D = \det(a_{ij})$.

<u>Theorem 1</u>. Suppose that $n \in \mathbb{Z}$, $n \geq 1$ and g.c.d.$(n,2D) = 1$. Then $r_f(n) = r(n, \text{gen } f) + O(n^{1/2-\gamma})$ for $\gamma > 1/28$, where $r(n,\text{gen } f)$ denotes the number of representations of n by the genus of f averaged in accordance with Siegel's prescription, [14]. Moreover, if n is primitively represented by f over the ring of p-adic integers for each rational prime p then $r(n,\text{gen } f) \gg_{f,\epsilon} n^{1/2-\epsilon}$ for $\epsilon > 0$.

<u>Proof</u>. Let N be a positive integer such that $2D|N$ and $8|N$, and let $\varphi \in S_0(3/2,N,\chi)$ with $\chi(d) = \left(\frac{2D}{d}\right)$, suppose furthermore that $\varphi \in \mathfrak{T}^{\perp}$, in notations of [12]. Thus φ is a "good" cusp-form of weight 3/2 (and character χ) which does not come from a θ-series. Therefore an argument due to H. Iwaniec, [6], and W. Duke, [3], supplemented by the considerations going back to G. Shimura, [13], and B.A. Cipra, [2], leads to an estimate for the Fourier coefficients of φ (cf. also [4]), and on writing $\varphi(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ we obtain: $a(n) \ll_{\varphi,\gamma} n^{1/2-\gamma}$ as soon as $(n,2D) = 1$ and $\gamma < \frac{1}{28}$. By [12, Korollar 3], it follows then that $r_f(n) = r(n,\text{spn } f) + O(n^{1/2-\gamma})$ for $(n,2D) = 1$ and $\gamma < \frac{1}{28}$, where $r(n,\text{spn } f)$ denotes the representation

EULER CHARACTERISTIC OF DEGENERACY LOCI

A. Parusiński[1] & **P. Pragacz**[2]

CONTENTS

[1] Research carried out during the author stay at the Max-Planck-Institut für Mathematik as a fellow of the Alexander von Humboldt Stiftung.

[2] This research was partially carried out when the author visited the University of British Columbia in Vancouver and was supported by a grant of the National Sciences and Engineering Council of Canada. He is grateful to J.B.Carrell, D.Peterson and S.Kumar for making this visit valuable and enjoyable one.

Introduction.

The aim of this paper is to study the topological Euler-Poincaré characteristic (or, Euler characteristic, for short) of degeneracy loci associated with various bundle homomorphisms. Recall that for a given morphism $\varphi : F \to E$ of vector bundles on a variety X the *r-th degeneracy locus* is the set

$$D_r(\varphi) = \{x \in X |\ rank\ \varphi(x) \leq r\},$$

$r = 0, 1, \dots, min(rankF, rankE)-1$. This concept overlaps a large family of interesting varieties (the set of zeros of a section of a bundle being a very particular case).

Several authors have worked on explicit formulas for the Euler characteristic of $D_r(\varphi)$ in terms of different cohomological and numerical invariants (see **[Pa-Pr]** for a survey concerning this subject).

In the present paper, in Section 1, we give an explicit formula for the Euler characteristic $\chi(D_r(\varphi))$ in terms of the Chern classes of E, F and X, under the assumption that φ is a general holomorphic morphism of vector bundles.

In Section 2 we extend the quoted formula to singular varieties using Chern-MacPherson classes (still for a general morphism φ).

The formula does not hold for a generic (i.e. of the expected codimension of a degeneracy locus) but nongeneral morphism, even in the case of a generic set of zeros of a holomorphic section of a line bundle. We investigate this case in details in Section 3. The difference between the Euler characteristic of a nongeneral hypersurface and the expected polynomial in Chern classes is measured with the help of topological invariants of singularities including some generalizations of the Milnor number and Chern-MacPherson classes.

The present paper gives an answer to the problem posed in **[Pr1]** (8.6).

Notation and conventions

For a complex variety X by $\chi(X)$ we denote its (topological) Euler characteristic, by $H_*(X;\mathbf{Z})$ (resp. $H^*(X;\mathbf{Z})$)-its singular homology (resp. cohomology) groups, by $A_k(X)$- the Chow group of k-dimensional cycles modulo rational equivalence and $A_*(X) = \oplus_k A_k(X)$.

For a given element $z \in H_*(X;\mathbf{Z})$, X compact (resp. $z \in A_*(X)$, X complete) by $\int_X z$ we denote the degree of the 0-th component of z.

By $dimX$ we mean always the complex dimension of X. If X is a nonsingular complex manifold, then the canonical orientation allows one to identify (by Poincaré Duality) the elements of $H_i(X;\mathbf{Z})$ with elements of $H^{2dimX-i}(X;\mathbf{Z})$.

If E is a vector bundle on X and $f : Y \to X$ is a morphism of varieties, then E_Y denotes the pull-back bundle f^*E.

For a given vector bundle E on X by $c_i(E)$, $i = 1, \ldots, rankE$, we denote the i-th Chern class of E. The top Chern class of E will be denoted by $c_{top}(E)$. By $s_k(E)$ we denote the k-th Segre class of E i.e. the k-th complete symetric polynomial in Chern roots of E satysfying $s_i(E) = (-1)^i c_i(-E)$ (Note that this convention differs from that used in [F], where $s_i(E) = c_i(-E)$). We assume also $s_i(E) = c_i(E) = 0$ if $i < 0$.

By $c(E) = 1 + c_1(E) + \ldots + c_{top}(E)$ we denote the total Chern class of E.

For a manifold X, we denote by TX the tangent bundle of X and by $c_k(X)$ the k-th Chern class of TX. We put $c_k(X) = 0$ if $k < 0$.

In topology, the Chern classes $c_i(E)$ of a vector bundle E on a variety X are located in $H^{2i}(X;\mathbf{Z})$, $i = 1, \ldots, rankE$. In algebraic geometry, the Chern classes are operators $c_i(E) \cap - : A_k(X) \to A_{k-i}(X)$ (see [F]). However, if X is smooth, then one usually identifies $c_i(E)$ with $c_i(E) \cap [X] \in A_{dimX-i}(X) \simeq A^i(X)$- the i-th graded component of the Chow ring of X. In Section 1, we use frequently this identification.

The notation for different morphisms between Chow groups is borrowed from [F].

In Section 3 we will use some notation from differential geometry. Let X be a complex manifold and let L be a holomorphic line bundle on X. The norm of a vector in L will be denoted by $\|v\|$. By $\mathring{\mathbf{B}}_\varepsilon \subset \mathbf{C}^n$ (resp. $\mathbf{S}_\varepsilon^{2n-1} \subset \mathbf{C}^n$) we denote the open ball (resp. the sphere) with center at the origin and the radius ε, and by $|c|$ the absolute value of a complex number c.

1. The Euler characteristic of a degeneracy locus of a general holomorphic vector bundle morphism.

Let X be a complex compact manifold and let $\varphi : F \to E$ be a holomorphic morphism of complex holomorphic vector bundles on X. Let $\overline{D_r} \subset \underline{Hom}(F,E)$ stand for the universal (tautological) degeneracy locus (the fibre of $\overline{D_r}$ over $x \in X$ is equal to $\{f \in \underline{Hom}(F(x),E(x));\ rank\, f \leq r\}$). A morphism $\varphi : F \to E$ induces the section $s_\varphi : X \to \underline{Hom}(F,E)$. We say that φ is <u>general</u> if s_φ is transverse to all $\overline{D}_k \subset \underline{Hom}(F,E)$ $k = 0,1,\ldots,min(rankF, rankE) - 1$. Through replacing φ by $\varphi^{\vee} : E^{\vee} \to F^{\vee}$, if necessary, we can assume for all in this (and the next) section, that

$$m = rankF \geq n = rankE\,.$$

To state the main result of this section we need some definitions.

By a <u>partition</u> we mean a sequence of integers $I = (i_1, \ldots, i_k)$, where $i_1 \geq i_2 \geq \ldots \geq i_k \geq 0$. We write $l(I)$ for $card\{p\,;\, i_p \neq 0\}$, $|I|$ for $\sum i_p$, $I^{\sim} = (j_1, j_2, \ldots)$ for the conjugate partition with $j_p = card\{h\,;\, i_h \geq p\}$ and $(i)^k$ for $(i,\ldots,i)$ (k-times). For two partitions I, J we write $I \supset J$ if $i_k \geq j_k$ for each k. Moreover, given two finite sequences $I = (i_1, i_2, \ldots), J = (j_1, j_2, \ldots)$ we write $I + J$ for $(i_1 + j_1, i_2 + j_2, \ldots)$ and I, J for $(i_1, i_2, \ldots, j_1, j_2, \ldots)$.

Fix k, $0 \leq k < n$. For two partitions I, J such that $l(I) \leq n-k$, $l(J) \leq n-k$ define

$$D^k_{I,J} = Det\left[\binom{i_p + j_q + m + n - 2k - p - q}{i_p + n - k - p}\right],\ \ 1 \leq p,q \leq n-k.$$

Finally, given two vector bundles E, F on X and a partition $I = (i_1, \ldots, i_k)$ we define in $H^{2|I|}(X;\mathbf{Z})$ the class

$$s_I(E - F) := Det\left[s_{i_p - p + q}(E - F)\right],\ \ 1 \leq p,q \leq k,$$

where

$$s_i(E-F) := \sum_{p=0}^{i}(-1)^{i-p}s_p(E)c_{i-p}(F).$$

In particular, if $F=0$ then $s_I(E) = Det\left[s_{i_p-p+q}(E)\right]$, $1 \le p,q \le k$; if $E=0$ then $s_I(-F) = (-1)^{|I|}s_{I^\sim}(F)$.

The following formula describes the topological Euler-Poincaré characteristic of the degeneracy locus $D_r(\varphi) = \{x \in X \,|\, rank\,\varphi(x) \le r\}$.

Theorem 1.1. *If φ is general, then*

$$\chi(D_r(\varphi)) = \int_X \sum_{k=0}^{r}(-1)^k\binom{n-r+k-1}{k}g(r-k).$$

Here,

$$g(k) = \sum(-1)^{|I|+|J|}\, D^k_{I,J}\, s_{(m-k)^{n-k}+I,J^\sim}\,(E-F)c_{d-|I|-|J|}(X),$$

where $d = dimD_r(\varphi)$ $(= dimX - (m-r)(n-r))$ and the sum is over all partitions I,J such that $l(I) \le n-k$, $l(J) \le n-k$.

Remark 1.2. Under the assumption $D_{r-1}(\varphi) = \emptyset$, the above formula reads $\chi(D_r(\varphi) = g(r)$. This result was established in **[Pr1]** as a particular case of an algorithm for computation of the Chern numbers of smooth degeneracy loci.

The proof of Theorem 1.1 requires several preliminary definitions and results. Let $i : D_r(\varphi) \to X$ be the inclusion and let $i_* : A_*(D_r(\varphi)) \to A_*(X)$ be the induced map of the Chow groups. Following **[Pr1]** we say that a polynomial $P(c_1,\dots,c_n,c'_1,\dots,c'_m)$, where $\{c_i\}$, $\{c'_j\}$ are independent variables, *describes in a universal way a cycle supported in $D_r(\varphi)$* if $P(c_1(E),\dots,c_n(E),c_1(F),\dots,c_m(F)) \in Im(i_*)$ for every variety X and every morphism $\varphi : F \to E$ of vector bundles on X, such that $rank\, F = m$, $rank\, E = n$, we have

$$P(c_1(E), \ldots, c_n(E), c_1(F), \ldots, c_m(F)) \in Im(i_*)\,.$$

The following fact is a consequence of Lemma 2.5 in **[Pr2]** (see also Theorem 5.3 (i) in **[Pr3]**) and the main Theorem 3.4 of **[Pr1]**.

Proposition 1.3. *No nonzero* $\mathbf{Z}[c.(E)] = \mathbf{Z}[c_1(E), \ldots, c_n(E)]$*-combination of the* $s_I(E - F)$ *with* $I \not\supset (m-r)^{n-r}$ *describes in a universal way a cycle supported in* $D_r(\varphi)$.

We need the following property of $g(r)$ in the sequel. Let $k : D_r(\varphi) \setminus D_{r-1}(\varphi) \to D_r(\varphi)$ be the open imbeding and let K (resp. C) be the kernel (resp. cokernel) bundle of φ restricted to $D_r(\varphi) \setminus D_{r-1}(\varphi)$.

Lemma 1.4. *There exists an element* $a \in A_*(D_r(\varphi))$ *such that* $i_*(a) = g(r)$ *and* $k^*(a) = c_d\Big((ik)^*TX - K^\vee \otimes C\Big)$.

Proof. The proof is a combination of several facts proved in **[Pr1]**. Let $\pi_E : G_r(E) \to X$ (resp. $\pi_F : G^r(F) \to X$) be the Grassmannian bundle parametrizing r-subbundles of E (resp. r-quotients of F). Moreover, let

$$0 \to R_E^{(r)} \to E_{G_r(E)} \to Q_E^{(n-r)} \to 0$$

$$0 \to R_F^{(m-r)} \to E_{G_r(F)} \to Q_F^{(r)} \to 0$$

be the tautological sequences on $G_r(E)$ and $G^r(F)$. Consider the following fibre product of Grassmannian bundles

$$\pi : G = G^r(F) \times_X G_r(E) \xrightarrow{\pi_F \times 1} G_r(E) \xrightarrow{\pi_E} X\,.$$

The morphism φ induces the section s_φ of $\underline{Hom}(F;E)$ and thus the section $\overline{s}_\varphi$ of $\underline{Hom}(F,E)_G/\underline{Hom}(Q_F,R_F)$. Let Z be the subscheme of zeros of $\overline{s}_\varphi$. The restriction ρ of π to Z factorizes through $D_r(\varphi)$. We put

$$a = \rho_* c_d\Big((\pi j)^* TX - (R_F^\vee \otimes Q_E)|_Z\Big).$$

Now, the first assertion follows from a calculation analogous to the one in **[Pr1]**, Proposition 5.7; the only difference being the use of Proposition 3.2 instead of Lemma 5.1 from loc.cit.. The second assertion is immediate as $R_F|_Z$ restricts (via k) to K and $Q_E|_Z$ to C.

□

At the end of the list of preliminary results we record the following consequence of the Littlewood-Richardson rule.

Lemma 1.5. *Let I, J be two partitions such that $l(I^\sim) \leq k$, $l(J) \leq l$. Then the nonzero coefficients β_K occuring in*

$$s_I(F)\cdot s_J(F) = \sum \beta_K \, s_K(F)$$

are indexed by partitions $K \not\supset (l+1)^{k+1}$.

Proof. We use the terminology and formulation of the quoted rule as in **[M]** (1.9). Recall that the diagrams of K for which $\beta_K \neq 0$ are obtained by adding to the diagram of I boxes coming from the diagram of J according to certain rules. One of these rules implies that the number of new boxes added in a single column cannot be greater than $l(J)$. Our assertion now follows from the observation that the (k+1)-th column of the diagram of K, for which $\beta_K \neq 0$, cannot contain $(l+1)$ boxes because $i_1 < k+1$ and $l(J) \leq l$.

□

Consider now the following geometric construction. Fix a general morphism φ and write $D_r = D_r(\varphi)$, for short. Consider the variety Z_r

$$
\begin{array}{lcccc}
 & Z_r = Zeros(F_G & \xrightarrow{\varphi_G} & E_G \to Q) \overset{j}{\hookrightarrow} G = G_r(E) \\
(1.1) & \Big\downarrow \eta & & \Big\downarrow \pi \\
 & D_r & \xrightarrow{i} & X
\end{array}
$$

where $G_r(E)$ is the Grassmannian bundle of r-subbundles associated to $E \to X$, and Q is $(n-r)$-bundle given by the exact (tautological) sequence $0 \to R \to E_G \to Q \to 0$ on G. The key information for the purposes of this section is contained in:

Proposition 1.6.

$$\chi(Z_r) = \int_X g(r)\,.$$

Proof. Our proof is rather conceptual than computational, and will use the main theorem of [**Pr1**]. The proof will be divided into several steps.

Step 1 We claim that the following identity holds:

$$(1.2) \qquad \chi(Z_r) = \int_X \pi_* \left\{ s_{(m)^{n-r}}(Q - F_G) c_d(\pi^* TX + R^\vee \otimes Q - F_G^\vee \otimes Q) \right\} .$$

First, we know arguing as in [**Po**] that for general φ, the variety Z_r is smooth of pure dimension d and its fundamental class is evaluated to be $c_{top}(F_G^\vee \otimes Q)$.

Then

$$
\begin{aligned}
\chi(Z_r) &= \int_X \pi_* j_*(c_{top}(Z_r)) \quad \text{(see [H] 1.4 for example)} \\
&= \int_X \pi_* \left\{ c_{top}(F_G^\vee \otimes Q) c_d(Z_r) \right\} \\
&= \int_X \pi_* \left\{ s_{(m)^{n-r}}(Q - F_G) c_d(Z_r) \right\}
\end{aligned}
$$

(by, for instance, [**Pr1**] Lemma 1.3).

Using the exact sequence

$$(1.3) \qquad 0 \to TZ_r \to TG_{|Z_r} \to F_G^{\vee} \otimes Q_{|Z_r} \to 0$$

and the well-known presentation of the class of TG in the Grothendieck group $K(G)$: $[TG] = [\pi^* TX + R^{\vee} \otimes Q]$ (see e.g. [F] B.5.8) the latter expresion can be rewritten in the form stated in (1.2).

It follows from the formula for Gysin push-forward in Grassmannian bundle (see [**Pr1**] Proposition 2.2, for example) that $\chi(Z_r)$ has the form

$$(1.4) \qquad \chi(Z_r) = \int_X \sum_{i=0}^{d} P_i(c.(E), c.(F)) c_i(X)\,,$$

where $P_i(c.(E), c.(F))$ are elements of $H^*(X; \mathbf{Z})$, given by certain polynomial expressions in the Chern classes of F and E.

<u>Step 2</u> We claim that in order to compute the polynomials $P_i(c.(E), c.(F))$ in (1.4), one can use the Chow groups set-up instead of the (co)homology groups one, and restrict to the case of algebraic varieties.

The first claim follows from the fact that the formulas for Gysin push-forward in Grassmannian bundles are the same for (co)homology $H^*(-; \mathbf{Z})$ and Chow groups $A_*(-)$.

To see that we can assume X to be algebraic consider the generic degeneracy locus $D_r(u, v) \subset X_{v,w}$ described in construction (13) of [**Pr1**]. We claim that if for fixed i we write $P_i(c.(E), c.(F)) = \sum \alpha_{I,J} s_I(E) s_J(F)$, then for $v, w \gg 0$ the coefficients $\alpha_{I,J}$ computed in the above situation for $E = E_{v,w}$, $F = F_{v,w}$ are the same as the universal ones in question. This follows from the property that the vector bundles $E = E_{v,w}$, $F = F_{v,w}$ have generic Chern roots if $v, w \to \infty$ (i.e. every finite set $\{s_{I_1}(E)s_{J_1}(F), \ldots, s_{I_k}(E)s_{J_k}(F)\}$ $l(I_p) \le n$, $l(J_p) \le m$, $(I_p, J_p) \ne (I_q, J_q)$ if $p \ne q$, becomes a family of $\mathbf{Z}$-linearly independent elements for $v, w \to \infty$).

Observe that since $\pi_* j_* = i_* \eta_*$ (see [F] 1.4), each $P_i(c.(E), c.(F))$ describes a class of a cycle supported in D_r . Fix i and let $P = P_i$. Express P as

$$(1.5) \qquad P(c.(E), c.(F)) = \sum_I \alpha_I(c.(E)) s_I(E - F),$$

where the sum is taken over partitions, and α_I depend only on $c.(E)$ and do not depend on $c.(F)$ (This is possible by the linearity formula, see e.g. **[Pr1]** Formula 4).

<u>Step 3</u> We claim that $I \not\supset (m - r + 1)^{n-r+1}$ if $\alpha_I \neq 0$. To prove it, let us look at (1.2) and analyse for which partitions L the following property holds: if

$$P(c.(E), c.(F)) = \sum \beta_L(c.(E)) s_L(F),$$

where $\beta_L \in \mathbf{Z}[c.(E)]$, then $\beta_L \neq 0$.

Note first that every $s_I(F_G)$ appearing in the decomposition of $s_{(m)^{n-r}}(Q - F_G)$ as a $\mathbf{Z}$-combination of $s_K(Q) \cdot s_I(F_G)$ satisfies $l(I^{\sim}) \leq rk\, Q = n - r$ (see e.g. **[Pr1]** Formula 2). Moreover, every $s_J(F_G)$ appearing in the decomposition of $c_i(-F_G^{\vee} \otimes Q) = (-1)^i s_i(F_G \otimes Q)$ as a $\mathbf{Z}$-combination of the $s_L(Q) \cdot s_J(F_G)$ satisfies $l(J) \leq n - r$ (see e.g. **[Pr1]** Lemma 5.6). But, by Lemma 1.5, if $l(I^{\sim}) \leq n - r$ and $l(J) \leq n - r \leq m - r$, then the nonzero β_K occuring in

$$s_I(F) \cdot s_J(F) = \sum_K \beta_K s_K(F) \quad (\beta_K \in \mathbf{Z})$$

are indexed by partitions $K \not\supset (m - r + 1)^{n-r+1}$. Consequently, using the property that $s_K(-F) = \pm s_{K^{\sim}}(F)$ and the linearity formula decomposing $s_I(E - F)$ as a $\mathbf{Z}$-combination of the $s_M(E) s_N(F)$, we easily obtain that if in (1.5) $\alpha_I \neq 0$ then $I \not\supset (m - r + 1)^{n-r+1}$, as claimed.

Step 4 We claim that $\pi_* j_*(c_d(Z_r)) = g(r)$.

We have the following commutative diagram

$$\begin{array}{ccc} A_*(Z_r) & \xrightarrow{k_3^*} & A_*(Z_r \setminus \pi^{-1}(D_{r-1})) \\ \downarrow{\scriptstyle \eta_*} & & {\scriptstyle \eta'_*}\downarrow{\scriptstyle \simeq} \\ A_*(D_r) & \xrightarrow{k_2^*} & A_*(D_r \setminus D_{r-1}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i'_*} \\ A_*(X) & \xrightarrow{k_1^*} & A_*(X \setminus D_{r-1}) \end{array}$$

where i', k_1, k_2, k_3 are the inclusions and η' is the restriction of η (the commutativity of this diagram follows from [F] Proposition 1.7). Let K (resp. C) be the kernel (resp. cokernel) bundle of φ restricted to $D_r \setminus D_{r-1}$. Then, by the pull-back property of Chern classes

$$k_2^* \eta_*(c_d(Z_r)) = \eta'_* k_3^* \Big(c_d(j^*TX + R^\vee \otimes Q - F_G^\vee \otimes Q) \Big) = c_d\big((ik)^*TX - K^\vee \otimes C\big) \ .$$

On the other hand, it follows from Lemma 1.4 that $g(r) \in A_*(X)$ is the image by i_* of an element $a \in A_*(D_r)$ satisfying the property $k_2^*(a) = c_d\big((ik)^*TX - K^\vee \otimes C\big)$. Thus

$$k_1^*[g(r) - \pi_* j_*(c_d(Z_r)] = 0 \ .$$

By the exact sequence (see [F], Proposition 1.8)

$$A_*(D_{r-1}) \xrightarrow{\bar{i}_*} A_*(X) \xrightarrow{k_1^*} A_*(X \setminus D_{r-1}) \to 0 \, ,$$

where $\bar{i} : D_{r-1} \to X$ is the inclusion, we know that $g(r) - \pi_* j_*(c_d(Z_r))$ is contained in $Im(\bar{i}_*)$ and describes in an universal way a cycle supported in D_{r-1}. In fact, the same applies to the coefficients (depending on $c.(E), c.(F)$) of the $c_i(X)$ in this element.

By Step 3 we know that all the coefficients of $c_i(X)$ in $g(r)-\pi_* j_*(c_d(Z_r))$ are $\mathbf{Z}[c.(E)]$-combinations of the $s_I(E-F)$, where $I \not\supset (m-r+1)^{n-r+1}$. In virtue of Proposition 1.3 with r replaced by $r-1$, this forces $g(r) = \pi_* j_*(c_d(Z_r))$, which proves the proposition.

□

Lemma 1.7.

$$\chi(Z_r) = \sum_{k=0}^{r} \binom{n-r+k-1}{k} \chi(D_{r-k}).$$

Proof. Let $Z^k = \pi^{-1}(D_k)$. Then $Z^k \setminus Z^{k-1}$ is a locally trivial fibration over $D_k \setminus D_{k-1}$ with the Grassmannian $G_{r-k}(\mathbf{C}^{n-k})$ as the fibre. Thus

$$\begin{aligned}
\chi(Z_r) &= \sum_{k=0}^{r} \chi(Z^k \setminus Z^{k-1}) = \sum_{k=0}^{r} \binom{n-k}{r-k} \chi(D_k \setminus D_{k-1}) \\
&= \sum_{k=0}^{r} \chi(D_k) \left[\binom{n-k}{r-k} - \binom{n-k-1}{r-k-1} \right] \\
&= \sum_{k=0}^{r} \binom{n-k-1}{r-k} \chi(D_k),
\end{aligned}$$

which is the required result.

□

Lemma 1.8. *For every positive integers a, k, the following equality holds*

$$\binom{a+k}{k} = \sum_{p=1}^{k} (-1)^{p-1} \binom{a+p}{p} \binom{a+k}{k-p}.$$

Proof. The assertion is a consequence of the following two equalities:

$$\binom{a+k}{k} \left[\sum_{p=0}^{k} (-1)^p \binom{k}{p} \right] = 0$$

and

$$\binom{a+k}{k}\binom{k}{p} = \binom{a+p}{p}\binom{a+k}{k-p}$$

($p = 1, \ldots, k$), a verification of the latter being straightforward.

$\square$

Proof of Theorem 1.1. Recall that we want to prove

$$\chi(D_r(\varphi)) = \int_X \sum_{k=0}^{r} (-1)^k \binom{n-r+k-1}{k} g(r-k).$$

We use induction on r. For $r = 0$, the equality $\chi(D_0) = g(0)$ is true by **[Pr1]** Proposition 5.7. Assume that the formula is correct for every $k \leq r-1$. We have

$$\chi(D_r) = \chi(Z_r) - \int_X \sum_{p=0}^{r} \binom{n-r+p-1}{p} \chi(D_{r-p}) \qquad \text{(by Lemma 1.7)}$$

$$= \int_X \left\{ g(r) - \sum_{p=1}^{r} \binom{n-r+p-1}{p} \left[\sum_{q=0}^{r-p} (-1)^q \binom{n-r+p+q-1}{q} g(r-p-q) \right] \right\}$$

(by Proposition 1.6 and the induction assumption)

$$= \int_X \sum_{k=0}^{r} (-1)^k g(r-k) \left[\sum_{p=1}^{k} (-1)^{p-1} \binom{n-r+p-1}{p} \binom{n-r+k-1}{k-p} \right]$$

$$= \int_X \sum_{k=0}^{r} (-1)^k \binom{n-r+k-1}{k} g(r-k)$$

(by Lemma 1.8 with $a = n-r-1$) .

$\square$

2. Generalization to singular varieties.

In this section we allow the ambient space X to have singularities and to be a compact complex pure dimensional analytic space. Let us fix a Whitney stratification $\mathcal{X}$ of X (see, for instance [G-M]). Let $\varphi : F \to E$ be a holomorphic morphism of complex holomorphic vector bundles on X. Assume that φ is general i.e. the induced section $s_\varphi : X \to \underline{Hom}(F, E)$ is transverse (on each stratum of $\mathcal{X}$) to all tautological degeneracy loci $\overline{D}_r \subset \underline{Hom}(F, E)$. We extend the formula of Theorem 1.1 to this case. Since for singular varieties the tangent bundle is not defined we use instead of the Chern classes of TX the Chern-MacPherson classes.

The Chern-MacPherson class $c_*(X) \in H_*(X; \mathbf{Z})$ was introduced (for an algebraic X) in [**McP**]. It equals, via the Alexander isomorphism, to the M.H. Schwartz class [S],[**B-S**] and can be in fact defined for any analytic space.

Let us first recall briefly MacPherson's definition.

Assume first that X is embedded in a smooth variety M. Then the tangent bundle to the nonsingular part X_{reg} of X defines a section over X_{reg} of the Grassmannian bundle $G_n(TM)$ (where $n = dimX$). By *the Nash blowing-up* $\nu : \tilde{X} \to X$ of X we mean the closure $\tilde{X}$ of the image of this section together with a map ν induced by the restriction of the projection of $G_n(TM)$ on M. We denote by $\tilde{T}$ (or $\tilde{T}_X$) the restriction to $\tilde{X}$ of the tautological bundle over $G_n(TM)$. Note that $\tilde{T}|_{\nu^{-1}(X_{reg})}$ is isomorphic to $\nu^* TX_{reg}$. All the above data are analytically independent of the embeding and so defined for all analytic spaces. The Chern-Mather class of X is defined (in $H_*(X; \mathbf{Z})$ or $A_*(X)$) by

$$c_M(X) = \nu_*(c(\tilde{T}) \cap [\tilde{X}]) .$$

We may define c_M for any analytic cycle $\sum n_i V_i$ of X by

$$c_M(\sum n_i V_i) = \sum n_i \, (incl_i)_* c_M(V_i)$$

where $incl_i$ is the inclusion of V_i in X.

In **[McP]** MacPherson defined *the local Euler obstruction* $Eu_X(x)$ of X at $x \in X$. The function Eu_X is constructable with integer values (see e.g.**[L-T]**). Let $\tau : \check{X} \to \tilde{X}$ be a blowing-up of $\nu^{-1}(x)$ and let Y be the exceptional divisor. Then $Eu_X(x)$ can be expressed by the González-Verdier formula (see **[Go]** **[L-T]** or **[F]** Ex. 4.2.9).

$$Eu_X(x) = \int_Y c_{n-1}(\tau^* \tilde{T} - \xi),$$

where ξ is the normal bundle to Y. Here are some properties of the local Euler obstruction which we use in the sequel.

Lemma 2.1.

(1) *$Eu_X(x)$ is constant on the strata of (any) Whitney stratification of X.*

(2) *$Eu_X(x) = 1$ if $x \in X_{reg}$.*

(3) *Assume that X is locally imbeded in $\mathbf{C}^N$ and a nonsingular subspace $W \subset \mathbf{C}^N$ is transverse to a Whitney stratification of X. Then, $Eu_{W \cap X}(x) = Eu_X(x)$ for $x \in W \cap X$.*

(4) *$Eu_{X \times Y}(x, y) = Eu_X(x) \cdot Eu_Y(y)$.*

In **[McP]** MacPherson defined an isomorphism T between the space of analytic cycles on X and the space of constructable functions with integer values on X by: $T(\sum n_i V_i) = \sum n_i Eu_{V_i}$. Let us call $T^{-1}(\mathbf{Id}_X)$ *the Chern-MacPherson cycle of X. The Chern-MacPherson class of X* is defined in $H_*(X; \mathbf{Z})$ (or $A_*(X)$) by

$$c_*(X) = c_M(T^{-1}(\mathbf{Id}_X)),$$

and satisfies good functorial properties (see **[McP]** or **[F]** Ex. 19.1.7.). In particular

$$\chi(X) = \int_X c_*(X) = \sum n_i \int_{V_i} c_M(V_i). \tag{2.1}$$

Theorem 2.2. *If φ is general, then*

$$\chi(D_r(\varphi)) = \int_X \sum_{k=0}^{r} (-1)^k \binom{n-r+k-1}{k} g(r-k).$$

Here,

$$g(k) = \sum (-1)^{|I|+|J|} D^k_{I,J}\, s_{(m-k)^{n-k}+I,J^\sim}\,(E-F) \cap c_*(X)\,,$$

where the sum is over all partitions I, J such that $l(I) \le n-k$, $l(J) \le n-k$.

Proof. We will show how to extend the proof in the nonsingular case to the present situation.

Let us consider first the case of a holomorphic section of a holomorphic vector bundle. So, let E be a holomorphic vector bundle over X and let Z be the zero set of a holomorphic section s of E transverse to the zero section (i.e. transverse on each stratum of $\mathcal{X}$). The image of the fundamental class $[Z]$ of Z by i_* (where $i : Z \hookrightarrow X$ denotes the inclusion) is dual to the top Chern class $c_{top}(E)$ of E in the sense that

$$i_*([Z]) = c_{top}(E) \cap [X]\,.$$

In the nonsingular case the normal bundle to Z is isomorphic to $E|_Z$ and consequently the total Chern class of Z equals $i^* c(TX - E)$. Hence the topological Euler characteristic of Z can be expressed in terms of the Chern classes of E and X as follows

$$\chi(Z) = \int_X c_{top}(E) \cdot c(X) \cdot c(E)^{-1}\,. \tag{2.2}$$

To prove the similar formula for the singular case we note that by the properties of Whitney stratification and Lemma 2.1 we have easily:

Lemma 2.3. *Let E, X and Z be as above. Then:*

(1) *$\mathcal{X}$ induces on Z a stratification $\mathcal{Z}$ which is also Whitney.*

(2) *The Nash blowing-up of Z is induced by ν i.e. equals $\tilde{Z} = \nu^{-1}(Z) \to Z$ and on $\tilde{Z}$ we have an exact sequence of vector bundles*

$$0 \to \tilde{T}_Z \to \tilde{T}_X|_{\tilde{Z}} \to \nu^*(E|_Z) \to 0\,. \tag{2.3}$$

(3) *If $\sum n_i V_i$ is the Chern-MacPherson cycle for X, then $\sum n_i V_i \cap Z$ is the one for Z.*

By (2.3)

$$c(\tilde{T}_Z) = i^*\nu^*c(E)^{-1}\cdot i^*c(\tilde{T}_X)\,,$$

and consequently

$$\begin{aligned} c_M(Z) &= \nu_*(c(\tilde{T}_Z)\cap[\tilde{Z}]) = \nu_*(i^*\nu^*c(E)^{-1}\cdot i^*c(\tilde{T}_X)\cap[\tilde{Z}]) \\ &= \nu_*(\nu^*c(E)^{-1}\cdot c(\tilde{T}_{\tilde{X}})\cdot c_{top}(\nu^*E)\cap[\tilde{X}]) \\ &= c(E)^{-1}\cdot c_{top}(E)\cap c_M(X)\,. \end{aligned}$$

Let Y be a subspace of X given by a union of strata of $\mathcal{X}$. Then, since Z is transverse to $\mathcal{X}$, by the same argument as above

$$c_M(Z\cap Y) = c(E)^{-1}\cdot c_{top}(E)\cap c_M(Y)\,.$$

Therefore, by (3) of Lemma 2.3, we get the following formula.

Proposition 2.4. *Let X, E, s and Z be as above. Then*

$$c_*(Z) = c(E)^{-1}\cdot c_{top}(E)\cap c_*(X)\,.$$

In particular,

$$\chi(Z) = \int_X c(E)^{-1}\cdot c_{top}(E)\cap c_*(X)\,.$$

To prove Theorem 2.2 in the general case we follow completely the proof of Theorem 1.1 and we simply show that all the steps are allowed.

Consider the construction given by (1.1) and note that, as in the nonsingular case, it suffices to prove

(2.4) $$\chi(Z_r) = \int_X g(r)$$

for all r. Fix r and let $Z = Z_r$.

First observe that since $\pi : G \to X$ is a locally trivial bundle with nonsingular fibre we have by Lemma 2.1:

Lemma 2.5.

(1) *A Whitney stratification $\mathcal{X}$ of X induces a Whitney stratification of G by taking strata of the form $\pi^{-1}(S)$ for $S \in \mathcal{X}$.*

(2) *The Nash blowing-up of $\nu_G : \tilde{G} \to G$ of G is canonically isomorphic to the fibre product of ν and π.*

(3) *For any $z \in G$ $Eu_G(z) = Eu_X(\pi(z))$ and if $\sum n_i V_i$ is the Chern-MacPherson cycle for X then $\sum n_i \pi^{-1}(V_i)$ is the one for G.*

Let $\tilde{\pi} : \tilde{G} \to \tilde{X}$ be the induced map (by (2) of Lemma 2.5). By (2.3) we have an exact sequence

$$0 \to \tilde{T}_Z \to \tilde{T}_G|_{\tilde{Z}} \to \nu_G^*(F_G^\vee \otimes Q)_{|\tilde{Z}} \to 0$$

and $[\tilde{T}_G] = [\tilde{\pi}^*\tilde{T}_X + \nu_{\tilde{G}}^*(R^\vee \otimes Q)]$. This implies

$$\int_Z c_M(Z) = \nu_* \tilde{\pi}_* \left(s_{(m)^{n-r}}(Q_{\tilde{G}} - F_{\tilde{G}}) c_d(\tilde{\pi}^*\tilde{T}_X + R_{\tilde{G}}^\vee \otimes Q_{\tilde{G}} - F_{\tilde{G}}^\vee \otimes Q_{\tilde{G}}) \cap [\tilde{G}] \right) ,$$

where $d = dimZ$. By the formula for Gysin push-forward in Grassmannian bundle we have again

$$\begin{aligned}\int_Z c_M(Z) &= \int_{\tilde{X}} \left(\sum_{i=0}^{d} P_i(c.(E_{\tilde{X}}), c.(F_{\tilde{X}})) c_i(\tilde{T}_X) \right) \cap [\tilde{X}] \\ &= \int_X \left(\sum_{i=0}^{d} P_i(c.(E), c.(F)) \cap c_M(X) \right) ,\end{aligned}$$

Repeating the proof of Theorem 1.1 we get

$$\int_Z c_M(Z) = \int_X \left(\sum (-1)^{|I|+|J|} D_{I,J}^k \; s_{(m-k)^{n-k}+I,J^\sim} \left(E - F\right) \cap c_M(X) \right) .$$

As in the proof of Proposition 2.4 we may proceed similarly for any subspace Y of X which is a union of strata of $\mathcal{X}$ in paricular for the cycles V_i occuring in (2.1). By (3) of Lemma 2.5 this gives the required formula $\chi(Z) = \int_Z c_*(Z) = \int_X g(r)$, which ends the proof.

□

3. The Euler characteristic of a nongeneral hypersurface.

Assume now that X is a connected compact n-dimensional manifold and L is a holomorphic line bundle over X. Let $s \in H^0(X,L) \setminus \{0\}$ be a holomorphic section of L and consider the zero set Z of s.

We define the number $\mu(Z,X)$ as follows

$$\mu(Z,X) = (-1)^n(\chi(Z) - \chi(X,L)),$$

where for a vector bundle E, $\chi(X,E) = \int_X \sum_{i=0}^{n}(-1)^{n-i}c_i(X)s_{n-i}(E)$ denotes the Euler characteristic of a zero set of a section transverse to the zero section (for a line bundle L, $s_k(L) = c_1(L)^k$). Usually, we will write $\mu(Z)$ instead of $\mu(Z,X)$.

The aim of this section is to give a formula for $\mu(Z)$ in terms of local invariants of Z and the topology of $Sing(Z)$.

Remark. The methods we use in this section are different to that in previous ones. Our considerations are based on differential geometry (general reference **[G-H]**), differential topology and stratifications of real and complex analytic sets (see e.g. **[G]**).

Example 3.1. Assume that $Z \subset X$ is a hypersurface defined by a section s of a line bundle L and that Z has only isolated singular points. In local coordinates $z = (z_1, \dots, z_n)$ around $x \in Z$ the hypersurface Z is defined by a holomorphic function f. We may assume that x is the origin in $\mathbf{C}^n$ and f is defined in a neighbourhoud of x. For small positive ε and δ (and $0 < \delta << \varepsilon$) the intersection $f^{-1}(\delta) \cap \mathring{\mathbf{B}}_\varepsilon$ has the homotopy type of a bouquet of spheres $\mathbf{S}^{n-1} \vee \dots \vee \mathbf{S}^{n-1}$ of real dimension $n-1$, where the number of spheres μ is called the Milnor number of X at x (see **[Mi]**). Since $Z \cap \mathring{\mathbf{B}}_\varepsilon$ is contractible

$$\chi(Z \cap \mathring{\mathbf{B}}_\varepsilon) - \chi(f^{-1}(\delta) \cap \mathring{\mathbf{B}}_\varepsilon) = (-1)^n \mu\,.$$

It is not difficult to see (cf. **[Pa1]** for instance) that if Z has only isolated singularities, then

$$\mu(Z) = \chi(Z) - \chi(X,L) = (-1)^n \sum_{x \in Sing(Z)} \mu(Z,x)\,,$$

where $\mu(Z,x)$ denotes the Milnor number of Z at x.

The Milnor number of x admits several interpretations. At first

$$\mu = dim_{\mathbf{C}}\, \mathbf{C}\{z\}/(\partial f/\partial z_1, \cdots, \partial f/\partial z_n).$$

Secondly, by a theorem of Mather there is a holomorphic change of coordinates h near the origin such that $f \circ h$ is a polynomial mapping. Thus we may assume that the gradient $\partial f = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ of f has polynomial components. We have $\mu = card\{(\partial f)^{-1}(y)\}$, for y in an open dense subset of a neighbourhood of the origin. The gradient map gives also the following approach. Since ∂f has no zeros in a punctured neighbourhood of the origin we can define (for small $\varepsilon > 0$) $g : \mathbf{S}_\varepsilon^{2n-1} \to \mathbf{S}_1^{2n-1}$ by $g(z) = \partial f(z)/\|\partial f(z)\|$, where $\mathbf{S}_\varepsilon^{2n-1}$. Then, μ equals the topological degree of g.

We refer to [O] for more detailed informations about the Milnor number.

Let x be an arbitrary point of $Sing(Z)$ and assume that in local coordinates around x the hypersuface Z is the zero set of (the germ of) an analytic function $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. Let ε be a small positive number and let $D_\delta = \{z \in \mathbf{C}; |z| \leq \delta\}$. Then, if only $0 < \delta \ll \varepsilon$, f restricted to $\overset{\circ}{\mathbf{B}}_\varepsilon \cap f^{-1}(D_\delta - 0)$ is a locally trivial fibre bundle and the topological type of the fibre does not depend on the choice of ε [Mi]. We call this fibre *the Milnor fibre attached to* x and will denote it by F_x.

DEFINITION. *We define* $\mu(Z,x) = (-1)^{n-1}(\chi(F_x) - 1)$.

Example 3.2. If x is a nonsingular point of Z, then $\mu(Z,x) = 0$. If x is an isolated singularity, then $\mu(Z,x) = (-1)^{n-1}\Big(1 + (-1)^{n-1} dim_{\mathbf{Q}} H^{n-1}(F_x, \mathbf{Q}) - 1\Big) = \mu$, the usual Milnor number of Z at x.

By the existence of "good" stratification of Z (see e.g.[**H-L**] Theorem 1.2.1 or [**H-M-S**] Theorem 7.1.1)) we see that $x \longmapsto \mu(Z,x)$ is a constructable function on Z. Let us recall below briefly the definition of "good" stratification.

"Good" stratification. A stratification $\mathcal{Z}$ of Z is called to be "good" if it satisfies the following local condition (which is independent of the choice of local coordinates). Assume that as above Z is described locally as the zero set of $f : \mathbf{C}^n \to \mathbf{C}$. We say that $\mathcal{Z}$ is a "good" stratification of Z if for each sequence $x_k \in \mathbf{C}^n \setminus Z$ converging to $x \in Z$ and such that the sequence $T_{x_k}(f^{-1}(f(x_k)))$ of the tangent spaces to the fibres of f has a limit T (in $\mathbf{P}^{n-1}$), T contains the tangent space to the stratum containing x. If this stratum is given by $\{z_1 = \ldots = z_j = 0\}$, then the condition above can be described as follows

$$\frac{\|(\partial f/\partial z_{j+1}(z), \ldots, \partial f/\partial z_n(z))\|}{\|(\partial f/\partial z_1(z), \ldots, \partial f/\partial z_j(z))\|} \xrightarrow{z \to 0} 0 . \tag{3.1}$$

Let $\mathcal{Z}$ be a Whitney and "good" stratification of Z. By Thom's First Isotopy Lemma (see e.g. [G] Theorem 5.2) the topological type of the Milnor fibres F_x at x is constant along the strata of $\mathcal{Z}$.

Notation. For $S \in \mathcal{Z}$ we denote by μ_S the value of $x \mapsto \mu(Z, x)$ on S.

The following result, which is a particular case of Theorem A of [N], gives an example of a more elaborate calculation of $\mu(Z, x)$, which will be important in the induction step of the main theorem of this section. We present a proof for the reader's convenience.

Lemma 3.3. *Let Z be a hypersurface in X and let H be a nonsingular hypersurface of X transverse to a "good" and Whitney stratification of Z. Then for each $x \in Z \cap H$*

$$\mu(Z \cup H, x) = (-1)^n$$

(in the other words the Euler characteristic of the Milnor fibre of $Z \cup H$ at x is zero).

Proof. Assume that x is the origin in $\mathbf{C}^n$ and let Z and H be the zero sets of f and z_n respectively, so $Z \cup H$ is the zero set of $g(z) = z_n f(z)$. For $z \in \mathbf{C}^n$ we write $z = (z', z_n)$, where $z' \in \mathbf{C}^{n-1}$. We assume that the line $l = \{(0, z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}\}$ is contained in the stratum of a Whitney and "good" stratification $\mathcal{Z}$ of Z which contains x. Denote by G the Milnor fibre of $Z \cup H$ at the origin. We shall show that, up to homotopy equivalence, G fibres over S^1 and therefore $\chi(G) = 0$, as desired. In the proof we use standard facts of real analytic geometry which can be found in [Ł] and [G].

First, we consider the family of hypersurfaces Z_{z_n} in $\mathbf{C}^{n-1}$ given by the equations $f(*, z_n) = 0$ with fixed z_n. Since $\mathcal{Z}$ is "good", by (3.1), the Milnor fibres of Z_{z_n} at $0 \in \mathbf{C}^{n-1}$ are homeomorphic provided $|z_n|$ is sufficiently small. Moreover, using Łojasiewicz Inequality [Ł], one may find $\varepsilon_0 > 0$ and $m \in \mathbf{N}$ such that for every $0 < \varepsilon \leq \varepsilon_0$, $c \in \mathbf{C}$ such that $0 < |c| \leq \varepsilon^m$, and $|z_n| \leq \varepsilon_0$

$$F(z_n, c) = \{z' \in \mathbf{C}^{n-1}; \|z'\| \leq \varepsilon, f(z', z_n) = c\}$$

is homeomorphic to the Milnor fibre F' of $Z \cap H$ at the origin. Instead of the ordinary representatives of G

$$G_{\varepsilon,c} = g^{-1}(c) \cap \mathring{\mathbf{B}}_\varepsilon \qquad c \in \mathbf{C}\,, 0 < |c| \ll \varepsilon \ll 1$$

we consider

$$\tilde{G}_{\varepsilon,c} = g^{-1}(c) \cap U_\varepsilon \qquad c \in \mathbf{C}\,, 0 < |c| \ll \varepsilon \ll 1\,,$$

where

$$U_\varepsilon = \{(z', z_n) \in \mathbf{C}^n; \|z'\| < \varepsilon, |z_n| < \varepsilon, |f(z', z_n)| < \varepsilon^m\}$$

(we will show below that they are homotopically equivalent). Note that the image of $\tilde{G}_{\varepsilon,c}$ by the standard projection π_n (i.e. $\pi_n(z) = z_n$) is the annulus $\{\frac{c}{\varepsilon^n} < z_n < \varepsilon\}$ and the fibres are homeomorphic to F'. This already gives $\chi(\tilde{G}_{\varepsilon,c}) = 0$ (since π_n restricted to $\tilde{G}_{\varepsilon,c}$ can be stratified). By (3.1) and

$$dg = z_n d'f + (f + z_n \partial f/\partial z_n) dz_n\,,$$

where $d'f = \partial f/\partial z_1 dz_1 + \cdots + \partial f/\partial z_{n-1} dz_{n-1}$, one can prove that in fact $\pi_{n|\tilde{G}_{\varepsilon,c}}$ is a locally trivial fibration.

To complete the proof we will show that $\tilde{G}_{\varepsilon,c}$ are homotopically equivalent to $G_{\varepsilon,c}$ for $0 < |c| \ll \varepsilon \ll 1$. Consider a one-parameter family of neighbourhoods of the origin in $\mathbf{C}^n$

$$V_\varepsilon = \varphi^{-1}([0,\varepsilon)),$$

where $\varphi : \mathbf{C}^n \to \mathbf{R}$ is a semi-analytic function and $\varphi^{-1}(0) = 0$. Fix $c \in \mathbf{C} \setminus \{0\}$ and consider the family of sets

$$G'_{\varepsilon,t} = g^{-1}(tc) \cap V_\varepsilon .$$

By the properties of semi-analytic sets there exist ε_0 and $m \in \mathbf{N}$ such that all $G'_{\varepsilon,t}$ are homeomorphic if only $(\varepsilon,t) \in S = \{(\varepsilon,t);\ 0 < \varepsilon \leq \varepsilon_0, 0 < t \leq \varepsilon^m\}$. Moreover, the homeomorphisms can be obtained by the integration vector fields (First Isotopy Lemma) and therefore for all $(\varepsilon,t),(\varepsilon',t) \in S$, $\varepsilon \leq \varepsilon'$, $G'_{\varepsilon,tc}$ is a deformation retract of $G'_{\varepsilon',tc}$. Take ε_0 good for both families $\mathring{\mathbf{B}}_\varepsilon$ and U_ε and choose ε_i $(i = 1,2,3)$ such that

$$\mathring{\mathbf{B}}_{\varepsilon_0} \supset U_{\varepsilon_1} \supset \mathring{\mathbf{B}}_{\varepsilon_2} \supset U_{\varepsilon_3} .$$

By the above for $t > 0$ and sufficiently small $G_{\varepsilon_0,tc}$ is a deformation retract of $G_{\varepsilon_2,tc}$ and $\tilde{G}_{\varepsilon_1,tc}$ is a deformation retract of $\tilde{G}_{\varepsilon_3,tc}$. Therefore $G_{\varepsilon_2,tc}$ and $\tilde{G}_{\varepsilon_1,tc}$ are homotopically equivalent. This ends the proof.

□

One may ask about a general formula for $\mu(Z)$ in terms of the local invariants of the singularities of Z. The following theorem gives such a formula in the case of a projective variety X.

Theorem 3.4. *Let X be a nonsingular subvariety of $\mathbf{P}^N$ and let Z be the zero set of a holomorphic section of a holomorphic line bundle L over X. Let $\mathcal{Z}$ be a stratification of Z such that $\mu(Z,*)$ is constant on the strata of $\mathcal{Z}$. Then*

$$\mu(Z) = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \int_{\bar{S}} \left(c(L)^{-1} \cap c_*(\bar{S})\right),$$

where $\alpha(S) = \mu_S - \sum_{S' \neq S, \bar{S}' \supset S} \alpha(S')$, *and* $c_*(\bar{S})$ *denotes the Chern-MacPherson class of* $\bar{S}$*(-the closure of stratum* S*).*

Before we start the proof we give some examples illustrating the theorem.

Example 3.5. Assume that $Y = Sing(Z)$ is nonsingular and that the pair $(Z \setminus Y, Y)$ satisfies Whitney Conditions. Then, $(Z \setminus Y, Y)$ is a "good" stratification of Z (**[L-S]**) and for $x \in Y$

$$\mu_Y = \mu(Z, x) = (-1)^m \mu^{n-m}(Z, x),$$

where $m = dim\, Y$ and $\mu^{n-m}(Z, x)$ is the $(n-m)$-th Teissier number of Z at x (see [T]). Now the theorem asserts

$$\mu(Z) = \mu_Y \int_Y c_m(T^{*\prime}Y \otimes L_{|Y}),$$

where $T^{*\prime}Y$ denotes the holomorphic cotangent bundle of Y (see e.g. **[G-H]** Chapter 0 §2.). As it was proved in **[Pa]** the above formula holds without the assumption of projectivity of X. The proof uses the following characterization of $\mu(Z)$ (see **[Pa]** for the details). Let $D = D' + D''$ be the associated metric connection on L compatible with a complex structure. Then $\mu(Z)$ equals the intersection index of $D's$ and the zero section computed near Z.

Example 3.6. Let $Y = Sing(Z)$ be nonsingular and let Z consist of 3 strata: $S_1 = Z - Y, S_2 = Y - S_3$ and S_3. Then the theorem gives

$$\mu(Z) = \mu_{S_2} \int_Y c(Y) \cdot c(L|_Y)^{-1} + (\mu_{S_3} - \mu_{S_2}) \int_{S_3} c(S_3) \cdot c(L|_{S_3})^{-1}.$$

In order to prove Theorem 3.4, let us first prove the following proposition (which holds without the assumption of the projectivity of X).

Proposition 3.7. *Let* L *be a holomorphic line bundle over a connected compact* n*-dimensional manifold* X*. Assume that* Z *is the zero set of* $s \in H^0(X, L) \setminus \{0\}$ *and* *let s' be a holomorphic section of L such that the zero set Z' of s' is nonsingular and transverse to a "good" and Whitney stratification $\mathcal{Z}$ of Z. Then*

$$\mu(Z) = \sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z') = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') .$$

Proof. We will approximate Z by the zero sets of Z_t of the sections $s_t = s - ts'$ ($t \in \mathbf{C}$). First, we note that for small $|t|$ all these sections are nonsingular and transverse to $\mathcal{Z}$. In fact, by Bertini's Theorem (e.g **[G-H]** p.137) the singularities of generic Z_t are contained in $Z \cap Z'$. Fix $x_0 \in Z \cap Z'$ and investigate Z_t around x_0. Denote the sections s and s' by f and g respectively and consider them as functions. By the transversality of Z' to $\mathcal{Z}$ and the fact that $\mathcal{Z}$ is a "good" stratification, the levels of f and g are transverse with the angle bounded from below by nonzero positive constant. In particular, $d(f - tg) = df - t \cdot dg$ nowhere vanishes on Z_t (for $t \neq 0$) and Z_t is transverse to $\mathcal{Z}$.

Let us fix a Hermitian metric on L. For $|t|$ small enough it is easy to see that $Z \cap Z'$ is a strong deformation retract of

$$Z_{t,\epsilon} = \{x \in Z_t;\ \|s'(x)\| \leq \epsilon\}$$

provided ϵ is sufficiently small.

<u>Step 1</u> We claim that for a sufficiently small $|t|$ we can find an universal $\epsilon > 0$ such that $Z \cap Z'$ is a strong deformation retract of $Z_{t,\epsilon}$.

Proof. For $t = 0$ and ϵ small it follows from the transversality. Assume now that $t \neq 0$ and small. We shall show that $Z_{t,\epsilon}$ can be retracted onto $Z \cap Z'$ using the flow generated by the orthogonal projection on $Z_{t,\epsilon}$ of $grad\|s'\|^2$. To prove it, it suffices to show that the projected vector field does not vanish on $Z_{t,\epsilon} \setminus Z \cap Z'$. We proceed locally in a neighbourhood of some $x_0 \in Z \cap Z'$. So assume that x_0 is the origin in $\mathbf{C}^n$ and $s = f \cdot \mathrm{e}$, $s' = g \cdot \mathrm{e}$, where e is a non-vanishing holomorphic section of L defined in a neighbourhood of x_0 and f, g are holomorphic functions. Let D be the associated connection and θ-the connection form with respect to e. Then

$$
\begin{aligned}
d(\|s'\|^2) &= \langle g\cdot \mathrm{e}, Dg\cdot \mathrm{e}\rangle + \langle Dg\cdot \mathrm{e}, g\cdot \mathrm{e}\rangle \\
&= (d\,|g|^2 + |g|^2(\theta+\bar{\theta}))\|\mathrm{e}\|^2 \qquad (3.2)\\
&= ((\bar{g}\,dg + |g|^2\theta) + \overline{(\bar{g}\,dg + |g|^2\theta)})\|\mathrm{e}\|^2\,.
\end{aligned}
$$

By the assumptions x_0 is a regular point of g, so we may choose such coordinates that $g(z) \equiv z_n$, and for $df = d'f + (\partial f/\partial z_n)dz_n$ we have by (3.1)

$$\|d'f\| \geq C\cdot|\partial f/\partial z_n|\,,$$

for some universal $C > 0$.

Take $z \notin Z \cup Z'$ and near x_0 and let $t(z) = f(z)/g(z)$. We show that the levels of $\|s'\|^2$ and f/g are transversal at z. For this purpose we consider the conormal vectors to them. The holomorphic part of the conormal vector to the former, given by (3.2), equals

$$l_1(z) = \bar{g}(z)\,dg(z) + |g(z)|^2\theta(z) = \bar{z}_n dz_n + |z_n|^2\theta(z)\,,$$

and the holomorphic conormal vector to the latter

$$l_2(z) = g(z)^{-2}((df(z) - t\cdot dg(z)) = z_n^{-2}(df(z) - (f(z)/z_n)dz_n)\,.$$

To prove the statement it is enough to show that $l_1(z)$ and $l_2(z)$ are independent.

Assume that this is not the case ; then, for every z such that $l_1(z)$ and $l_2(z)$ are dependent we define $c(z) \in \mathbf{C}$ by $l_1(z) = c(z)l_2(z)$. Then, for such z, $\theta(z) = a(z)dz_n + b(z)df(z) = (a(z)+b(z)\partial f/\partial z_n(z))dz_n + b(z)d'f$. Since $\theta(z)$ is bounded so is $b(z)d'f(z)$ and therefore by (3.3) $b(z)\cdot\partial f/\partial z_n(z)$ is bounded. Hence we conclude that $a(z)$ is bounded. If we write $l_1(z)$ and $l_2(z)$ as combinations of dz_n and $df(z)$ we get

(1) $\bar{z}_n + a(z)\cdot|z_n|^2 + c(z)\cdot z_n f(z) = 0$,

(2) $b(z)\bar{z}_n = c(z)z_n$.

This gives $\bar{z}_n + a(z)|z_n|^2 + b(z)f(z)\cdot\bar{z}_n = 0$ which is possible (near x_0) only if $b(-f(z))$ is close to 1.

Since θ is bounded in a neighbourhood of x_0 this contradicts the following inequality due to Łojasiewicz [Ł] §18 Proposition 1:

$$\|(\partial f/\partial z_1, \dots, \partial f/\partial z_n)\| \geq |f|^\alpha ,$$

for some $0 < \alpha < 1$, which holds in some neighbourhood of x_0. This completes the proof of the assertion.

□

Fix ϵ given by Step 1 and let Y be a manifold with boundary $X \setminus \{x \in X; \|s'(x)\| \leq \epsilon\}$. Note that the stratification $\mathcal{Z}$ is transverse to ∂Y.

One of the main properties of Whitney stratification is a topological equisingularity. It says that if $(Z, \mathcal{Z})$ is a set with Whitney stratification, then the topological type of Z at $x \in S \in \mathcal{Z}$, does not depend on the choice of the point x on a given stratum S. This follows from Thom's First Isotopy Lemma, whose proof is based on the technique of extending vector fields (the reader can consult e.g. [G] Chapter II) and requires a construction of the system of tubular neighbourhoods of the strata (loc.cit. Chapter II §2.).

<u>Step 2</u> (A constuction of a system of tubular neighbourhoods Γ_S of $S \cap Y$ in Y)

For $S \in \mathcal{Z}$ we define Γ_S inductively on $dim S$ as follows:

$$\Gamma_S = \{x \in Y; dist(x, S) \leq \delta_S\} \cup \bigcup_{S' \subset \bar{S} \setminus S} \Gamma_{S'} ,$$

where δ_S is a sufficiently small number such that:

(1) $G_S = \Gamma_S \setminus \bigcup_{S' \subset \bar{S} \setminus S} Int(\Gamma_{S'})$ is a manifold with corners which (as a stratified set) is transverse to $\mathcal{Z}$.

(2) G_S is a locally trivial fibration over $\tilde{S} := S \cap G_S$ (by Thom's First Isotopy Lemma) We denote this fibration by π_S.

(3) $\tilde{S}$ is a manifold with corners with the same homotopy type as $S \cap Y$ (which can be shown by gluing the vector fields given by (2)).

(A more complicated system of tubular neighbourhoods satisfying the above properties was constructed by Dubson [D] Proposition I 1.4.2.B)

Claim: The map $\pi_S|_{Z_t \cap G_S} : Z_t \cap G_S \to \tilde{S}$ is a locally trivial fibration and its fibre $\tilde{F}_x$, $x \in \tilde{S}$, is homotopically equivalent to the Milnor fibre F_x.

Indeed, since $\mathcal{Z}$ is a "good" stratification , Z_t (for $t \neq 0$ and sufficiently small) is transverse to the fibres of π_S. In particular $\pi_S|_{Z_t \cap G_S} : Z_t \cap G_S \to \tilde{S}$ is a locally trivial fibration. Its fibre $\tilde{F}_x$ at $x \in \tilde{S}$ is homotopically equivalent to the Milnor fibre F_x by Thom's First Isotopy Lemma.

Finally we have

$$\begin{aligned}
\chi(Z) - \chi(Z_t) &= \chi(Z \cap Y) - \chi(Z_t \cap Y) \\
&= \sum_{S \in \mathcal{Z}} (\chi(\tilde{S}) - \chi(Z_t \cap G_S)) \\
&= \sum_{S \in \mathcal{Z}} (\chi(\tilde{S}) - \chi(\tilde{S})\chi(\tilde{F}_x)) \qquad \text{(by Claim, here, } x \in \tilde{S}) \\
&= (-1)^n \sum_{S \in \mathcal{Z}} \chi(\tilde{S})\mu_S .
\end{aligned}$$

If ϵ is sufficiently small, then $\chi(\tilde{S}) = \chi(S \cap Y) = \chi(S \setminus Z')$ for every $S \in \mathcal{Z}$. Therefore, by the above we get

$$\begin{aligned}
\chi(Z) - \chi(Z_t) &= (-1)^n \sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z') \\
&= (-1)^n \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') .
\end{aligned}$$

This completes the proof of Proposition 3.7.

□

Proof of Theorem 3.4.

Step 1 We claim that the assertion is true if L is very ample.

Take a stratification $\tilde{\mathcal{Z}}$ of Z which is "good", Whitney and refines $\tilde{\mathcal{Z}}$ (i.e. each $S \in \mathcal{Z}$ is an union of strata of $\tilde{\mathcal{Z}}$). By Bertini's Theorem (**[G-H]** and [V]) there exists a section s' of L whose zero set Z' is nonsingular and transverse to $\tilde{\mathcal{Z}}$. By Proposition 3.5 we have

$$\mu(Z) = \sum_{S \in \tilde{\mathcal{Z}}} \mu_S \cdot \chi(S \setminus Z') = \sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z'),$$

where the last equality follows from the additivity of the Euler characteristic of a complex stratification. So Step 1 follows from Proposition 2.4

$$\sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z') = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \int_{\bar{S}} (c(L)^{-1} \cap c_*(\bar{S})).$$

Consider now the general case. We proceed by induction on $n = dimX$.

Let M be a very ample line bundle on X such that $L \otimes M$ is also very ample (such a bundle exists since X is projective). Let H be the zero set of a section of M such that H is nonsingular and transverse to a good stratification refining $\mathcal{Z}$ (and so also transverse to $\mathcal{Z}$). Let us stratify $Z \cup H$ by taking the following strata: $S \setminus H$ (for $S \in \mathcal{Z}$), $S \cap H$ (for $S \in \mathcal{Z}$) and $H \setminus Z$.

Let T be the zero set of a general section of $L \otimes M$ such that T is nonsingular and transverse to the above stratification of $Z \cup H$.

Step 2 We claim that

$$\begin{aligned}
\mu(Z \cup H) = &\sum_{S \in \mathcal{Z}} \alpha(S) \Big[\chi(\bar{S}) - \chi(\bar{S}, M) - \chi(\bar{S}, L \otimes M)) - \chi(\bar{S}, M \oplus (L \otimes M))\Big] \\
&- \mu(Z \cap H) - \mu(Z \cap H \cap T) \\
&+ (-1)^n (\chi(X, L \oplus M) - \chi(X, L \oplus M \oplus (L \otimes M)),
\end{aligned} \tag{3.4}$$

where $\mu(Z \cap H) = \mu(Z \cap H, H)$ and $\mu(Z \cap H \cap T) = \mu(Z \cap H \cap T, H \cap T)$.

Indeed, by considering the above stratification of $Z \cup H$, we have

$$\begin{aligned}\mu(Z \cup H) &= \sum_{S \in Z} \mu_{S-H}(Z \cup H)\cdot\chi(S - H - T) + \mu_{H-Z}(Z \cup H)\cdot\chi(H - Z) \\ &\quad + \sum_{S \in Z} \mu_{S\cap H}(Z \cup H)\cdot\chi(S \cap H - T) \qquad \text{(by Proposition 3.7)} \\ &= \sum_{S \in Z} \mu_S\cdot\chi(S - H - T) + (-1)^n \sum_{S \in Z} \chi(S \cap H - T)\end{aligned}$$

because obviously $\mu_{S-H}(Z \cup H) = \mu_S(Z) = \mu_S$, $\mu_{H-Z} = 0$, and $\mu_{S\cap H}(Z\cup H) = (-1)^n$ by Lemma 3.3. Thus

$$\mu(Z \cup H) = \sum_{S \in Z} \alpha_S\cdot\chi(\bar{S} - H - T) + (-1)^n[\chi(Z \cap H) - \chi(Z \cap H \cap T)].$$

But we have

$$\chi(\bar{S} - H - T) = \chi(\bar{S}) - \chi(\bar{S}, M) - \chi(\bar{S}, L \otimes M) + \chi(\bar{S}, M \oplus (L \otimes M))$$

and

$$\begin{aligned}(-1)^n\chi(Z \cap H) &= -\mu(Z \cap H) + (-1)^n\chi(H, L) \\ &= -\mu(Z \cap H) + (-1)^n\chi(X, L \oplus M),\end{aligned}$$

$$\begin{aligned}(-1)^{n-1}\chi(Z \cap H \cap T) &= -\mu(Z \cap H \cap T) + (-1)^{n-1}\chi(H \cap T, L) \\ &= -\mu(Z \cap H \cap T) + (-1)^{n-1}\chi(X, L \oplus M \oplus (L \otimes M)).\end{aligned}$$

All these equalities give (3.4).

<u>Step 3</u> We claim that

$$\begin{aligned}\mu(Z \cup H) =&\mu(Z) + \mu(Z \cap H) \\ &+ (-1)^n\left[\chi(X, L) + \chi(X, M) - \chi(X, L \oplus M) - \chi(X, L \otimes M)\right].\end{aligned} \tag{3.5}$$

Indeed, by the definition of $\mu(*)$ and the additivity of Euler characteristic, we have

$$\begin{aligned}\mu(Z \cup H) =&(-1)^n[\chi(Z \cup H) - \chi(X, L \otimes M)] \\ =&(-1)^n[\chi(Z) + \chi(H) - \chi(Z \cap H) - \chi(X, L \otimes M)] \\ =&\mu(Z) + (-1)^n[\chi(H) - \chi(Z \cap H) + \chi(X, L) - \chi(X, L \otimes M)].\end{aligned}$$

But $\chi(H) = \chi(X, M)$ and $(-1)^{n-1}[\chi(Z \cap H) - \chi(X, L \oplus M)] = \mu(Z \cap H)$. Thus the above equation gives (3.5).

Step 4 For arbitrary line bundles L and M on any compact analytic variety Y, the following equality holds

(3.6) $2\chi(Y, L \oplus M) + \chi(Y, L \otimes M) = \chi(Y, L) + \chi(Y, M) + \chi(Y, L \oplus M \oplus (L \otimes M))$.

This equation was proved in a more general framework in [H], Theorem 11.3.1.

We leave to the reader a verification of the following equality

$$\begin{aligned} &2ab(1+a)^{-1}(1+b)^{-1} + (a+b)(1+a+b)^{-1} \\ =&a(1+a)^{-1} + b(1+b)^{-1} + ab(a+b)(1+a)^{-1}(1+b)^{-1}(1+a+b)^{-1}. \end{aligned}$$

This equality (with $a = c_1(L)$ and $b = c_1(M)$) and Proposition 2.4 imply (3.6).

Step 5 (Inductive step) In order to prove the formula we use the induction on $n = dimX$. Assume that the formula holds for $\mu(Z \cap H)$ and $\mu(Z \cap H \cap T)$

$$\mu(Z \cap H) = \sum_{S \in Z} \alpha(S)[\chi(\bar{S}, M) - \chi(\bar{S}, L \oplus M)], \tag{3.7}$$

$$\mu(Z \cap H \cap T) = \sum_{S \in Z} \alpha(S)[\chi(\bar{S}, M \oplus (L \otimes M)) - \chi(\bar{S}, L \oplus M \oplus (L \otimes M))]. \tag{3.8}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} \mu(Z) = &\sum_S \alpha(S) \cdot [\chi(\bar{S}) - \chi(\bar{S}, M) - \chi(\bar{S}, L \otimes M) + \chi(\bar{S}, M \oplus (L \otimes M))] \\ &- 2\mu(Z \cap H) - \mu(Z \cap H \cap T) \\ &+ (-1)^{n-1}\Big[\chi(X, L) + \chi(X, M) - \chi(X, L \otimes M) \\ &\qquad -2\chi(X, L \oplus M) - \chi(X, L \oplus M \oplus (L \otimes M))\Big], \end{aligned}$$

the latter summand being zero by (3.6). Using (3.7) and (3.8) we thus obtain:

$$\begin{aligned}\mu(Z) = \sum_S \alpha(S)\Big[&\chi(\bar{S}) - \chi(\bar{S}, M) - \chi(\bar{S}, L\otimes M) + \chi(\bar{S}, M\oplus(L\otimes M)) \\ &+ 2\chi(\bar{S}, M) - 2\chi(\bar{S}, L\oplus M) \\ &-\chi(\bar{S}, M\oplus(L\otimes M)) - \chi(\bar{S}, L\oplus M\oplus(L\otimes M))\Big] \\ = \sum_S &\alpha(S)[\chi(\bar{S}) - \chi(\bar{S}, L)],\end{aligned}$$

by applying (3.6) once again. This gives the desired assertion.

$\square$

REFERENCES

[B-S] J.P. Brasselet, M.H. Schwartz "Sur les classes de Chern d'un ensemble analytique complexe", *Astérisque*, **82-83** (1981), 93-147

[D] A.S. Dubson "Calcul des invariants numériques des singularités et des applications" , preprint S.F.B. Theoretische Mathematik Universität Bonn, 1981

[F] W. Fulton "Intersection Theory", Springer, 1984

[G] C.G. Gibson et al. "Topological Stability of Smooth Mappings", Lect. Notes in Math. **552**, Springer, 1976

[G-H] P. Griffiths, J. Harris "Principles of Algebraic Geometry", John Wiley & Sons, 1978

[G-M] M.Goresky, R.MacPherson "Stratified Morse Theory", Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **14**, Springer, 1988

[Go] G. González-Springer "L'obstruction locale d'Euler et le théorème de MacPherson", *Astérisque* ,**82-83** (1981), 7-32

[H] F. Hirzebruch, "Topological methods in algebraic geometry", New York, Springer, 1966

[H-L] H.A. Hamm, Lê D.T., "Un théorème de Zariski du type de Lefschetz", *Ann. Sci. Éc. Norm. Sup.* , **6** (1973), 317-366

[H-M-S] J.P.Henry, M.Merle, C.Sabbah, "Sur la condition de Thom stricte pour un morphisme analytique complexe", *Ann. Sci. Éc. Norm. Sup.* , **17** (1984), 227-268

[Ł] Łojasiewicz S. "Ensembles semi-analytiques" I.H.E.S., 1965

[L-S] Lê D.T., K.Saito "La constance du nombre de Milnor donne des bonnes stratifications", *C.R.Acad.Sc. Paris* , **t.277** (1973), 793-795

[L-T] Lê D.T., B. Teissier "Variétés polaires locales et classes de Chern des variétés singulières", *Ann. of Math.* , **114** (1981), 457-491

[M] I.G. Macdonald, "Symmetric Functions and Hall Polynomials", Oxford University Press, 1979

[McP] R.D. MacPherson, "Chern classes for singular algebraic varieties", *Annals of Math.*, **100** (1974), 423-432

[Mi] J. Milnor, "Singular points of complex hypersurfaces", *Ann. of Math. Studies*, **61**, Princeton Univ. Press, 1968

[N] A.Némethi "The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$", to appear in *Compositio Math.*

[O] P. Orlik, "The multiplicity of a holomorphic map at an isolated critical point" in "Real and Complex Singularities", Oslo 1976, 405-474

[Pa] A. Parusiński, "A generalization of the Milnor number", *Math. Ann.*, **281** (1988), 247-254

[Pa-Pr] A. Parusiński, P. Pragacz, "Chern numbers, Euler characteristic of degeneracy loci and Milnor numbers; a review and announcement", Proceedings of "The Zeuthen Symposium", Contemporary Mathematics (to appear)

[Po] I.R. Porteous, "Simple Singularities of Maps", Proc. Liverpool Singularities Symposium I Lecture Notes in Mathematics, **192** (1971), Springer, 286-307

[Pr1] P. Pragacz, "Enumerative geometry of degeneracy loci", *Ann. Sci. Éc. Norm. Sup.* , **21** (1988), 413-454

[Pr2] P. Pragacz, "A note on elimination theory", *Indagationes Math* , **49** (2) (1987), 215-221

[Pr3] P. Pragacz, "Algebro-geometric applications of Schur S- and Q-polynomials", to appear in Séminaire d'Algèbre Dubreil-Malliavin 1989-1990, Springer Lecture Notes in Mathematics

[S] M.H. Schwartz, "Cycles caractéristiques définies par une stratification d'une variété analytique complexe ", *Compt. Rend. Acad. Sc.* , **260** (1965), 3261-3264 and 3535-3537

[T] B. Teissier, "Cycles évanescentes, sections planes et conditions de Whitney", *Astérisque* , **7-8** (1973), 285-362

[V] J.L. Verdier, "Stratifications de Whitney et thèoréme de Bertini-Sard", *Inv. Math.*, **36** (1976), 295-312