ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLE AND LIE ALGEBRAS OF COMPLEX SIZE MATRICES

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Abstract. We uniformly interprete the classical continuous Tchebyshev's and Hahn's orthogonal polynomials of discrete variable in terms of $\mathfrak{gl}(\lambda)$ for $\lambda \in \mathbb{C}$. One can similarly interprete Hahn's q-polynomials and orthogonal polynomials corresponding to Lie superlagebras. We also describe quasifinite modules over $\mathfrak{gl}(\lambda)$ and its real forms.

§1. Introduction

Among Saveliev-Vershik's continum Lie algebras, $\mathfrak{gl}(\lambda)$ is one of the first examples. We observed that the definition and properties of the continuous Tchebyshev's and Hahn's orthogonal polynomials of discrete variable (and their q-analogs) follow from the properties of the Casimir elements for $\mathfrak{gl}(\lambda)$ (and its q-analog). Our starting point was the attempt to calculate explicitely the value of the quadratic Casimir operator on quasifinite modules — a source of orthogonality relations and various identities the polynomials satisfy. (For a more detailed than ours description of quasifinite modules of level 1 see [Sh].)

Summary. In $\mathfrak{sl}(2)$, consider the basis Y, H, X with the commutation relations

$$[X,Y] = H, \quad [H,Y] = -2Y, \quad [H,X] = 2X. \tag{1.1}$$

Then the quadratic Casimir operator

$$\Omega = 2YX + \frac{1}{2}H^2 + H \tag{1.2}$$

lies in the center of $U(\mathfrak{sl}(2))$. Recall the definition of the Lie algebra $\mathfrak{gl}(\lambda)$ introduced by Feigin in [F]. Let I_λ be the two-sided ideal in the associative algebra $U(\mathfrak{sl}(2))$ generated by $\Omega - \frac{1}{2}(\lambda^2 - 1)$. It turns out that the associative algebra $\tilde{\mathfrak{A}}_\lambda = U(\mathfrak{sl}(2))/I_\lambda$ is simple for $\lambda \notin \mathbb{Z} \setminus \{0\}$, otherwise $\tilde{\mathfrak{A}}_\lambda$ contains an ideal such that the quotient is isomorphic to the matrix algebra $\mathrm{Mat}(|\lambda|)$. Set ([Di1])

$$\mathfrak{A}_\lambda = \begin{cases} \tilde{\mathfrak{A}}_\lambda & \text{if } \lambda \notin \mathbb{Z} \setminus \{0\} \\ \mathrm{Mat}(|\lambda|) & \text{otherwise.} \end{cases} \tag{1.3}$$

As associative algebra with unit, $\mathfrak{A}_\lambda$ is generated by X, Y and H subject to the relations (1.1) and

$$XY = \frac{1}{4}(n^2 - (H-1)^2) \text{ and } X^{|\lambda|} = 0 \text{ if } \lambda \in \mathbb{Z} \setminus \{0\}. \tag{1.4}$$

In what follows we set $\mathfrak{gl}(\lambda) := L(\mathfrak{A}_\lambda)$, the Lie algebra associated with the associative algebra $\mathfrak{A}_\lambda$.

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Consider the principal embedding of $\mathfrak{sl}(2)$ into $\mathfrak{G}=\mathfrak{gl}(n)$ (it corresponds to the n-dimensional irreducible $\mathfrak{sl}(2)$-module):

$$Y \mapsto \sum_{i=1}^{n} E_{i+1,i},\ H \mapsto \sum_{i=1}^{n}(n-2i+1)E_{ii},\ X \mapsto \sum_{i=1}^{n} i(n-i)E_{i,i+1}. \tag{1.5}$$

Then $\mathfrak{G}$ considered as an $\mathfrak{sl}(2)$-module with respect to the principal embedding (1.5) is of the form $\mathfrak{G}=\underset{|i|\leq n-1}{\oplus}\mathfrak{G}_i$, where $\mathfrak{G}_i=\{X^i\mathfrak{G}_0\}$ and $\mathfrak{G}_{-i}=\{\mathfrak{G}_0Y^i\}$ for $i>0$ and $\mathfrak{G}_0=\mathbb{C}[H]/(P_n(H))$, where $P_n(H)=\prod_{1\leq i\leq n-1}(H-n+2i-1)$, is the Cartan subalgebra.

The invariant functional, trace, on $\mathfrak{G}$ gives rise to the nondegenerate bilinear form $(A,B)=\operatorname{tr} AB$. The homogeneous components $\mathfrak{G}_i$ — the space of matrices with support on the ith over- (for $i>0$) and under-diagonals (for $i<0$) — are orthogonal with respect to this form and the form is nondegenerate on $\mathfrak{G}_{-i}\oplus\mathfrak{G}_i$ and on $\mathfrak{G}_0$. Reduction of the form to the canonical form on these spaces leads to the classical Hahn and Tchebyshev polynomials of discrete variable, respectively. By expressing the Casimir elements — the generators of the center of $U(\mathfrak{G})$ — in terms of these polynomials in the standard basis one gets all the identities the polynomials satisfy. By applying this approach to $\mathfrak{gl}(\lambda)$ we clarify the known results on the co-called *continuous* versions of the classical Hahn and Tchebyshev polynomials of discrete variable, cf. [NU], [NSU].

By replacing in (1.3) the universal enveloping of $\mathfrak{sl}(2)$ with its q-deformation we get q-versions of the classical polynomials, identities, etc. The details will be given elsewhere. In another paper we also consider superprincipal embeddings of $\mathfrak{osp}(1|2)$ — the closest relative of $\mathfrak{sl}(2)$ among Lie superalgebras — and the corresponding orthogonal polynomials with respect to the nondegenerate indefinite bilinear form, the supertrace. To replace $\mathfrak{sl}(2)$ with ANY Lie (super)algebra or Lie superalgebra $\mathfrak{h}$ in the above scheme is also possible provided $\mathfrak{h}$ possesses Verma modules with finitely many generators and several finite dimensional representations. The theory seems to be reachest if $\mathfrak{h}$ is simple or "close" to simple (nontrivial central extension, the derivation algebra, etc.). Such generalizations of $\mathfrak{gl}(\lambda)$ appear in spin >2 models, see [V]. The description of $\mathfrak{gl}(\lambda)$-modules in §§5–7 is our answer to Vasiliev's question.

§2. The trace formula

2.1. Lemma . *Let A be an associative algebra generated by the Lie algebra $\mathfrak{g}$, conisdered as a subspace. Then $[A,A]=[\mathfrak{g},A]$.*

Proof. It suffices to show that $[x_1\dots x_n,a]\in[\mathfrak{g},A]$ for any $a\in A$ and $x_1,\dots,x_n\in\mathfrak{g}$. The induction on n: for $n=1$ the statement is obvious. Let $n>1$. Let us make use of the identity $[ab,c]=[a,bc]+[b,ca]$ true in any associative algebra, cf. [M]. Then

$$[x_1(x_2\dots x_n),a]=[x_1,x_2\dots x_na]+[x_2\dots x_n,ax_1].$$

By inductive hypothesis, the second summand in the right hand side lies in $[\mathfrak{g},A]$. □

2.1.1. Corollary . ([Di]) *Let $\mathfrak{g}$ be a simple Lie algebra. Then $U(\mathfrak{g})=Z(U(\mathfrak{g}))\oplus[U(\mathfrak{g}),U(\mathfrak{g})]$.*

Proof. Let $U(\mathfrak{g})=Z(U(\mathfrak{g}))\oplus\left(\underset{\lambda\neq0}{\oplus}n_\lambda L^\lambda\right)$ be the decomposition with respect to the adjoint representation. Since the modules L^λ are irreducible, $[\mathfrak{g},U(\mathfrak{g})]=\left(\underset{\lambda\neq0}{\oplus}n_\lambda L^\lambda\right)$. □

2.1.2. Corollary . *Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$, which is either a simple finite dimensional Lie algebra; let $\lambda \in \mathfrak{h}^*$ and $M^{\lambda-\rho}$ the corresponding Verma module; let J_λ be the (left) ideal of $U(\mathfrak{g})$ corresponding to the kernel of the representation of $U(\mathfrak{g})$ in $M^{\lambda-\rho}$. Set $\mathfrak{A}_\lambda = U(\mathfrak{g})/J_\lambda$. Then $\mathfrak{A}_\lambda = \mathbb{C} \oplus [\mathfrak{A}_\lambda, \mathfrak{A}_\lambda]$.*

Proof. By Lemma 2.1 $[\mathfrak{A}_\lambda, \mathfrak{A}_\lambda] = [\mathfrak{g}, \mathfrak{A}_\lambda]$; hence, as in Corollary 2.1.1, $\mathfrak{A}_\lambda = Z(\mathfrak{A}_\lambda) \oplus [\mathfrak{A}_\lambda, \mathfrak{A}_\lambda]$. But $Z(\mathfrak{A}_\lambda)$ is a homomorphic image of $Z(U(\mathfrak{g}))$; hence, is equal to $\mathbb{C}$. □

If $\mathfrak{g}$ is a simple Lie algebra and λ is its highest weight, then $P(\lambda) = \dim L^\lambda$ is a polynomial in $\lambda = (\lambda_1, \dots, \lambda_n)$. Consider $P(\lambda)$ for any $\lambda \in \mathfrak{h}^*$ such that $\prod\limits_{i \neq j} (\lambda_i - \lambda_j) \neq 0$. By Corollary 2.1.2, there is a unique functional tr on $\mathfrak{A}_\lambda$ such that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(1) = P(\lambda)$.

2.2. Lemma . *Let $u \in U(\mathfrak{g})$, $\tilde{u} \in \mathfrak{A}_\lambda$ its image. Then $\mathrm{tr}(\tilde{u})$ is also a polynomial in λ.*

Proof. Let $\sharp : Z(U(\mathfrak{g})) \oplus [U(\mathfrak{g}), U(\mathfrak{g})] \longrightarrow Z(U(\mathfrak{g}))$ be the natural projection and $\tilde{}: U(\mathfrak{g}) \longrightarrow \mathfrak{A}_\lambda$ be given by the formula $u \mapsto \tilde{u}$. By definition of $\sharp$ we have:

$$\mathrm{tr}(\tilde{u}) = \chi_\lambda(u^\sharp)P(\lambda) = \varphi(u^\sharp)(\lambda)P(\lambda), \tag{2.2}$$

where $\varphi : Z(U(\mathfrak{g})) \longrightarrow \mathbb{C}[\mathfrak{h}]$ is the Harish-Chandra homomorphism. □

2.3. Theorem . *Let us identify the space $(\mathbb{C}[\mathfrak{h}])^*$ with the algebra of formal power series on indeterminates $t = (t_1, \dots, t_n)$ by the following formula:*

$$f \mapsto \sum_{\nu=(\nu_1,\dots,\nu_n)} f\left(\frac{h_1^{\nu_1}}{\nu_1!} \dots \frac{h_n^{\nu_n}}{\nu_n!}\right) t_1^{\nu_1} \dots t_n^{\nu_n}, \tag{2.3.1}$$

where $\nu_i \in \mathbb{Z}_+$ for all i and $h_1, \dots, h_n$ is a basis of $\mathfrak{h}$.

Then to the functional tr *there corresponds the series*

$$\psi(\lambda, t) = \frac{\bigoplus\limits_{w\in W} \varepsilon(w)e^{w(\lambda)}}{\bigoplus\limits_{w\in W} \varepsilon(w)e^{w(\rho)}}(t), \tag{2.3.2}$$

where ρ is the half-sum of positive roots and $e^\lambda(t) = e^{\lambda_1 t_1 + \dots + \lambda_n t_n}$. (Recall that the λ_i are supposed to be distinct.)

Proof. By Lemma 1.2 $\mathrm{tr}\left(\frac{h_1^{\nu_1}}{\nu_1!} \dots \frac{h_n^{\nu_n}}{\nu_n!}\right)$ is a polynomial in λ. If $\lambda \in P_{++}$ (the set of highest weights of finite dimensional modules), then

$$\hat{\psi}(\lambda, t) = \sum_{\nu=(\nu_1,\dots,\nu_n)} \mathrm{tr}\left(\frac{h_1^{\nu_1}}{\nu_1!} \dots \frac{h_n^{\nu_n}}{\nu_n!}\right) t_1^{\nu_1} \dots t_n^{\nu_n}$$

coinsides with (2.3.2) thanks to the Weyl's character formula. But $\psi(\lambda, t) = \sum\limits_\nu P_\nu t^\nu$, where P_ν are some polynomials. Indeed, since $P_\nu(\lambda = \mathrm{tr}\left(\frac{h_1^{\nu_1}}{\nu_1!} \dots \frac{h_n^{\nu_n}}{\nu_n!}\right)$ for $\lambda \in P_{++}$, this is true for any $\lambda \in \mathfrak{h}^*$. Hence, $\hat{\psi}(\lambda, t) = \psi(\lambda, t)$. □

§3. Highest weights of quasifinite modules over $\mathfrak{gl}(\lambda)$

Let L^i be the irreducible $\mathfrak{sl}(2)$-module with highest weight i. The Lie algebra $\mathfrak{gl}(\lambda)$ considered as $\mathfrak{sl}(2)$-module with respect to the adjoint action of the latter is of the form

$$\mathfrak{gl}(\lambda) = \begin{cases} L^0 \oplus L^2 \oplus \dots L^{2|\lambda|-2} \text{ for } \lambda \in \mathbb{Z} \setminus \{0\} \\ L^0 \oplus L^2 \oplus \dots \text{ otherwise.} \end{cases} \tag{3.0.1}$$

The Lie algebra $\mathfrak{gl}(\lambda)$ possesses a $\mathbb{Z}$-grading $\mathfrak{gl}(\lambda) = \underset{i\in\mathbb{Z}}{\oplus} \mathfrak{gl}(\lambda)_i$, where

$$\mathfrak{gl}(\lambda)_i = \{z \in \mathfrak{gl}(\lambda) : [H, z] = 2iz\} \tag{3.0.2}$$

and the increasing filtration: $\mathfrak{gl}(\lambda)_{(i)} = \underset{k\leq i}{\oplus} \mathfrak{gl}(\lambda)_k$. Therefore, we can speak about the *triangular decomposition* $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ of $\mathfrak{g} = \mathfrak{gl}(\lambda)$, where $\mathfrak{g}_\pm = \underset{i\geq 0}{\oplus} \mathfrak{gl}(\lambda)_{\pm i}$, *parabolic subalgebras*, *highest weights*, etc., see [KR].

Following [KR] we say that the $\mathfrak{g}$-module V is *quasifinite* if $V = \underset{j\in\mathbb{Z}}{\oplus} V_j$ and $\dim V_j < \infty$ for every j. (We only consider *graded* $\mathfrak{g}$-modules, i.e., such that $\mathfrak{g}_i V_j \subset V_{i+j}$.)

The well-known fact that $\mathfrak{gl}(n)$ has a nontrivial one-dimensional representation ($A \mapsto \operatorname{tr} A$) has its counterpart for $\mathfrak{gl}(\lambda)$.

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called a *parabolic* one if it contains $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ as a proper subalgebra. For example, for any $P \in \mathbb{C}[H]$ set $\mathfrak{p}_{-1} = P\mathbb{C}[H]Y$ and let $\mathfrak{p}$ be generated by $\mathfrak{p}_{-1}$ and $\mathfrak{g}_0 \oplus \mathfrak{g}_+$. This is the *minimal* parabolic subalgebra corresponding to P.

3.1. Lemma . *The minimal parabolic subalgebra $\mathfrak{p}$ (corresponding to a polynomial P) is of the form $\mathfrak{p} = \underset{k\in\mathbb{Z}}{\oplus} \mathfrak{p}_k$, where $\mathfrak{p}_k = \mathfrak{g}_k$ for $k \geq 0$ and $\mathfrak{p}_{-k} = I_k \cdot Y^k$, where I_k is the ideal of $\mathbb{C}[H]$ generated by $P(H)P(H+2)\dots P(H+2k-2)$.*

Proof. See [KR], mutatis mutandis. □

Let $\lambda \in \mathfrak{g}_0^*$, $\mathfrak{g}_+ v_\lambda = 0$ and $M^\lambda = \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_+}(\mathbb{C}v_\lambda)$ the corresponding Verma module; let L^λ be the irreducible $\mathfrak{g}$-module with highest weight λ.

3.2. Lemma . ([KR]) *The following conditions are equivalent:*

i) *M^λ contains a singular vector from $M^\lambda_{(-1)}$, the space of filtration -1;*

ii) *The module L^λ is quasifinite;*

iii) *L^λ is a quotient of the generalized Verma module $M^{\lambda,P} = \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C})$, where $P \in \mathbb{C}[H]$ and $\mathfrak{p}$ is the minimal parabolic subalgebra corresponding to P.*

Shoikhet calls $\deg P$ the *level* of $M^{\lambda,P}$. In [Sh] he described modules of level 1 in more details than we give here.

We have already established that $\mathfrak{g}_0^* \simeq (\mathbb{C}[H])^* \simeq \mathbb{C}[[t]]$ with the algebra isomorphism established by the rule

$$F : (\mathbb{C}[H])^* \longrightarrow \mathbb{C}[[t]], \quad \theta \mapsto F_\theta(t) = \sum_{k=0}^{\infty} \frac{\theta(H^k)}{k!} t^k.$$

The following theorem describes the set of formal power series corresponding to the highest weights of quasifinite modules. Recall that a *quasipolynomial* is an expression of the form $\sum R_i(t)e^{\alpha_i t}$, where the $R_i(t)$ are polynomials.)

Theorem . *The formal power series corresponding to the highest weight of any quasifinite module over $\mathfrak{gl}(\lambda)$ is of the form $\dfrac{R(t)}{1-e^{-2t}}$, where in $R(t)$ is a quasipolynomial such that $R(0)=0$.*

Proof. First, observe the following easy to verify statements: if $\theta \in (\mathbb{C}[H])^*$ and $F_\theta(t) = \sum\limits_{k=0}^{\infty} \frac{\theta(H^k)}{k!} t^k$ the corresponding series, then

$$\sum_{k=0}^{\infty} \frac{\theta((H+a)^k)}{k!} t^k = e^{at} F_\theta(t);$$

and for any $R(H) \in \mathbb{C}[H]$ we have

$$\sum_{k=0}^{\infty} \frac{\theta(R(H)H^k)}{k!} t^k = R(\frac{d}{dt}) F_\theta(t) \text{ and } \sum_{k=0}^{\infty} \frac{\theta(R(H+a)(H+a)^k)}{k!} t^k = R(\frac{d}{dt})(e^{at} F_\theta(t)).$$

If θ is the highest weight of a quasifinite module over $\mathfrak{gl}(\lambda)$, then by Lemma 3.2 there exists a polynomial $P(H) \in \mathbb{C}[H]$ such that θ can be extended to a one-dimensional representation of the minimal parabolic subalgebra corresponding to P. It is not difficult to verify that $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{p}_{-1}]$. Hence, $\theta([\mathfrak{g}_1, \mathfrak{p}_{-1}]) = 0$.

Denote:

$$T(H) = XY = \frac{1}{4}(\lambda^2 - (H+1))^2.$$

Then the condition $\theta([\mathfrak{g}_1, \mathfrak{p}_{-1}]) = 0$ can be expressed as $\theta([X, P(H)H^kY]) = 0$, or, equivalently, as $\theta(XP(H)H^kY - P(H)H^kYX]) = 0$, or, as

$$\theta(T(H-2)P(H-2)(H-2)^k - P(H)H^kT(H)) = 0.$$

Therefore,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\theta(T(H-2)P(H-2)(H-2)^k)}{k!} t^k - \sum_{k=0}^{\infty} \frac{\theta(T(H)P(H)(H)^k)}{k!} t^k = \\
&T(\tfrac{d}{dt})P(\tfrac{d}{dt})(e^{-2t} F_\theta(t)) - T(\tfrac{d}{dt})P(\tfrac{d}{dt})(F_\theta(t)) = \\
&T(\tfrac{d}{dt})P(\tfrac{d}{dt}) \left((e^{-2t} - 1) F_\theta(t)\right) = 0.
\end{aligned}$$

Thus, the function $(e^{-2t}-1)F_\theta(t)$ is a solution of an ordinary differential equation with constant coefficients, hence, is a quasipolynomial. Obviously, $R(0) = 0$.

Conversely, if $R(t)$ is a quasipolynomial, $R(0) = 0$ and $P(\frac{d}{dt})R(t) = 0$, then $T(\frac{d}{dt})P(\frac{d}{dt})R(t) = 0$ and, by setting $F_\theta(t) = \dfrac{R(t)}{e^{-2t}-1}$ (which is well defined since $R(0) = 0$) we satisfy the condition $\theta([\mathfrak{g}_1, \mathfrak{p}_{-1}]) = 0$ for $\mathfrak{p}_{-1} = P(H)\mathfrak{g}_0 Y$. Hence, θ is the highest weight of a quasifinite module. □

3.3. The trace on $\mathfrak{gl}(0)$. If $P = 1$, then the parabolic subalgebra of $\mathfrak{g} = \mathfrak{gl}(0)$ coincides with the whole algebra and, therefore, $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}] \cap \mathfrak{g}_0 \not\ni 1$. This proves that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$ and, therefore, proves also existence of an invariant functional θ on $\mathfrak{g} = \mathfrak{gl}(\lambda)$. Therefore, $R = (1-e^{-2t})F_\theta(t)$ satisfies the equation $T(\frac{d}{dt})R(t) = 0$, more exactly,

$$(\lambda^2 - (\frac{d}{dt} + 1)^2) R(t) = 0.$$

If $\lambda \neq 0$, then solutions are of the form $R(t) = c_1 e^{(\lambda-1)t} + c_2 e^{-(\lambda+1)t}$ and the initial condition $R(0) = 0$ implies that

$$F_\theta(t) = c \frac{e^{(\lambda-1)t} - e^{-(\lambda+1)t}}{1 - e^{-2t}}.$$

If we norm the functional θ (trace) naturally, i.e., by setting $\theta(1) = \lambda$, then $c = 1$.

Clearly, if $\lambda = 0$, then the characteristic equation has multiple roots and

$$F_\theta(t) = c\frac{te^{-t}}{1 - e^{-2t}}.$$

Thus, $\mathfrak{sl}(0)$ is an infinite dimensional Lie algebra with an ∞-dimensional identity module.

§4. The Lie algebras $\mathfrak{gl}(\lambda)$ and $\mathfrak{gl}^-(\infty)$, $\mathfrak{gl}^+(\infty)$, $\mathfrak{gl}(\infty)$

Let V be a vector space with a fixed basis v_i for $i \in \mathbb{Z}$, let V^+ be its subspace spanned by v_i for $i \geq 0$ and V^- its subspace spanned by v_i for $i < 0$. Let $\mathfrak{gl}^-(\infty)$, $\mathfrak{gl}^+(\infty)$, and $\mathfrak{gl}(\infty)$ be the Lie algebras of linear transformations of V^-, V^+ and V, respectively, whose matrices in the fixed bases are supported on a finite number (depending on the matrix) of diagonals parallel to the main one.

On $\mathfrak{gl}(\infty)$, there is a (unique) nontrivial 2-cocycle

$$c(A, B) = \mathrm{tr}\ ([J, A]B) \text{ where } J = \sum_{i\leq 0} E_{ii} - \sum_{i>0} E_{ii}. \tag{4.0}$$

Denote the corresponding central extension by $\widehat{\mathfrak{gl}}(\infty)$; the bracket in $\widehat{\mathfrak{gl}}(\infty)$ is of the form

$$[A, B] = AB - BA + \mathrm{tr}\ ([J, A]B) \cdot z, \text{ where } z \text{ is the new central element.}$$

In particular, for matrix units we have

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} + \delta_{il}\delta_{jk}(\kappa(i) - \kappa(j)) \cdot z,$$

where $\kappa(i) = 1$ if $i \leq 0$ and 0 otherwise.

4.0. Harish-Chandra modules over $\mathfrak{sl}(2)$. Harish-Chandra modules $M^{\lambda,s} = Span(v_i : i \in \mathbb{Z})$ for any $\lambda, s \in \mathbb{C}$ are defined to satisfy the following relations

$$\begin{aligned}
Hv_i &= (s + 2i)v_i\\
Xv_i &= \tfrac{1}{2}\sqrt{(\lambda - s - 2i - 1)(\lambda + s + 2i + 1)}v_{i+1}\\
Yv_i &= \tfrac{1}{2}\sqrt{(\lambda - s - 2i + 1)(\lambda + s + 2i - 1)}v_{i-1}
\end{aligned} \tag{4.0.1}$$

It is not difficult to observe that Ω acts on $M^{\lambda,s}$ as a scalar operator of multiplication by $\frac{1}{2}(\lambda^2 - 1)$ and $M^{\lambda,s} \simeq M^{\lambda,s'}$ if $s - s' \in 2\mathbb{Z}$.

Suppose $\lambda \in \mathbb{Z} \setminus \{0\}$. Then by [Di] if $\lambda - s \notin 2\mathbb{Z} + 1$, then $M^{\lambda,s}$ is irreducible, whereas if $\lambda - s \in 2\mathbb{Z} + 1$, then $M^{\lambda,s} = M_+^{\lambda,s} \oplus M_-^{\lambda,s}$, where $M_+^{\lambda,s}$ is an irreducible $\mathfrak{sl}(2)$-module with the lowest weight $\lambda + 1$ and $M_-^{\lambda,s}$ is an irreducible $\mathfrak{sl}(2)$-module with the highest weight $\lambda - 1$ and these module exhaust all irreducible $\mathfrak{sl}(2)$-modules with a diagonal H-action on which Ω acts as a scalar operator of multiplication by $\frac{1}{2}(\lambda^2 - 1)$.

The module $M^{\lambda,s}$ over $\mathfrak{gl}(\lambda)$ determines an embedding $\phi : \mathfrak{gl}(\lambda) \longrightarrow \mathfrak{gl}(\infty)$. Since $H^2(\mathfrak{gl}(\lambda)) = 0$ (see [F]), there exists a lift of the embedding ϕ to a homomorphism $\hat{\phi} : \mathfrak{gl}(\lambda) \longrightarrow \widehat{\mathfrak{gl}}(\infty)$ which is of the form $\hat{\phi}(u) = \phi(u) + \theta(u) \cdot z$, where θ is a linear functional on $\mathfrak{gl}(\lambda)$ such that $\theta([u, v]) = c(\phi(u), \phi(v))$ for the cocycle on $\mathfrak{gl}(\infty)$ which makes it into $\widehat{\mathfrak{gl}}(\infty)$. Observe that θ is determined uniquely up to a functional proportional to the trace on $\mathfrak{gl}(\lambda)$.

The following theorem describes $\hat{\phi}$ more explicitely.

4.1. Theorem . *Let* $\mathfrak{g} = \mathfrak{gl}(\lambda)$. *Then* $\hat\phi(\mathfrak{g}_i) = \phi(\mathfrak{g}_i)$ *for* $i \neq 0$.

$$\hat\phi(e^{tH}) = \phi(e^{tH}) - \frac{e^{st} - e^{-(\lambda+1)t}}{1 - e^{-2t}} z. \tag{4.1.1}$$

Proof.

$$\begin{aligned}
\phi(X) &= \sum \alpha_i E_{i+1,i}, \text{ where } \alpha_i = \tfrac{1}{2}\sqrt{(\lambda - s - 2i - 1)(\lambda + s + 2i + 1)},\\
\phi(H) &= \sum \gamma_i E_{i,i}, \text{ where } \gamma_i = s + 2i,\\
\phi(Y) &= \sum \beta_i E_{i-1,i}, \text{ where } \beta_i = \tfrac{1}{2}\sqrt{(\lambda - s - 2i + 1)(\lambda + s + 2i - 1)}.
\end{aligned} \tag{4.1.2}$$

Therefore,

$$[J, \phi(X)] = \sum (\kappa(i) - \kappa(i+1))\alpha_i E_{i+1,i}$$

and $\phi(H^k)\phi(Y) = \sum \gamma_i^k \beta_{i+1} E_{i,i+1}$. Hence,

$$[J, \phi(X)]\phi(H^k)\phi(Y) = \sum \gamma_i^k \beta_{i+1}\alpha_i(\kappa(i) - \kappa(i+1))E_{i+1,i+1}$$

and

$$\begin{aligned}
&c(\phi(X), \phi(H^k Y)) = \operatorname{tr}([J, \phi(X)]\phi(H^k Y)) = \textstyle\sum \gamma_i^k \beta_{i+1}\alpha_i(\kappa(i) - \kappa(i+1)) =\\
&\gamma_0^k \beta_1 \alpha_0 = \tfrac{1}{4} s^k(\lambda^2 - (s+1)^2) = T(s)s^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&T(\tfrac{d}{dt})(e^{st}) = T(s)e^{st} = \sum_{k\geq 0} \frac{T(s)s^k}{k!} t^k =\\
&\sum_{k\geq 0} \frac{c(\phi(X), \phi(H^k Y))}{k!} t^k =\\
&\sum_{k\geq 0} \frac{\theta([X, H^k Y])}{k!} t^k = \sum_{k\geq 0} \frac{\theta(XH^kY - H^kYX)}{k!} t^k =\\
&\sum_{k\geq 0} \frac{\theta(T(H-2)(H-2)^k - H^k T(H))}{k!} t^k =\\
&T(\tfrac{d}{dt})\left((e^{-2t} - 1)\theta(e^{tH})\right).
\end{aligned}$$

Therefore, $T(\frac{d}{dt})\left((e^{-2t} - 1)\theta(e^{tH}) - e^{st}\right) = 0$ and

$$(e^{-2t} - 1)\theta(e^{tH}) = \begin{cases} e^{st} + c_1 e^{-(\lambda+1)t} + c_2 e^{(\lambda-1)t} & \text{if } \lambda \neq 0\\ e^{st} + c_1 e^{-t} + c_2 t e^{-t} & \text{if } \lambda = 0. \end{cases}$$

In both cases, subtracting the summand proportional to the trace we get

$$(e^{-2t} - 1)\theta(e^{tH}) = e^{st} - e^{-(\lambda+1)t}.$$

□

4.2. $\mathfrak{gl}(\infty)$ **over the algebra of truncated polynomials.** Let $R_m = \mathbb{C}[\varepsilon]$, where $\varepsilon^{m+1} = 0$. By extending the module $M^{\lambda,s}$ to a module $M^{\lambda,s}_{R_m}$ over R_m, i.e., by assuming that $s \in R_m$ we may define the action of $\mathfrak{sl}(2)$ by the *same* formulas. If the v_i is the initial basis, then for the fixed basis of the extended module $M^{\lambda,s}_{R_m}$ we take $\varepsilon^j v_i$ for all $i \in \mathbb{Z}$ and $j = 1, \dots, m$.

Observe that in this basis of $M^{\lambda,s}_{R_m}$ the action of H is not a diagonal one.

The algebras $\mathfrak{gl}(\infty; R_m)$, its subalgebras $\mathfrak{gl}^{\pm}(\infty; R_m)$ are naturally defined and the central extension $\widehat{\mathfrak{gl}}(\infty; R_m)$ is determined by the same formula.

By considering $s + \varepsilon$ instead of $s \in \mathbb{C}$, we get the following

Corollary . *The homomorphism $\hat{\phi}: \mathfrak{gl}(\lambda) \longrightarrow \widehat{\mathfrak{gl}}(\infty; R_m)$ induced by the embedding $\phi: \mathfrak{gl}(\lambda) \longrightarrow \mathfrak{gl}(\infty; R_m)$ satisfies $\hat{\phi}(\mathfrak{gl}(\lambda)_i) = \phi(\mathfrak{gl}(\lambda)_i)$ for $i \neq 0$ and*

$$\hat{\phi}(e^{tH}) = \phi(e^{tH}) - \left(\frac{e^{st} - e^{-(\lambda+1)t}}{1 - e^{-2t}} + \frac{e^{st}}{1 - e^{-2t}} \sum_{j=1}^{m} \frac{\varepsilon^j t^j}{j!}\right) z. \tag{4.2}$$

4.3. *Remark .* We are "forced" to consider the central extension $\widehat{\mathfrak{gl}}(\infty)$ in order to describe quasifinite modules over $\mathfrak{gl}(\lambda)$ because $\widehat{\mathfrak{gl}}(\infty)$ has more quasifinite representations than $\mathfrak{gl}(\infty)$ and it turns out to be insufficient to consider only $\mathfrak{gl}(\infty)$-modules.

The following statement describes the restrictions onto the higheset weight of an irreducible module necessary and sufficient for the module to be quasifinite.

4.4. The highest weights of quasifinite modules. Denote by $\mathfrak{gl}_f(\infty)$ and $\mathfrak{gl}_f^{\pm}(\infty)$ the subalgebras of $\mathfrak{gl}(\infty)$ and $\mathfrak{gl}^{\pm}(\infty)$, respectively, consisting of matrices with finite support. Let $\mathfrak{h}$ denote the Cartan subalgebra of any of these Lie algebras; consider it spanned by the diagonal elements E_{ii}. Observe that the restriction of the embedding $\hat{\phi}: \mathfrak{gl}_f(\infty) \longrightarrow \widehat{\mathfrak{gl}}(\infty)$ onto $\mathfrak{h}$ is of the form $E_{ii} \mapsto \phi(E_{ii}) + \kappa(i)z$, where ϕ is the natural embedding of $\mathfrak{gl}_f(\infty)$ into $\mathfrak{gl}(\infty)$. If θ is a weight of a (highest weight) module over one of the above Lie algebras, both $\mathfrak{gl}(\infty)$ and its subalgebras and $\mathfrak{gl}_f(\infty)$ and its subalgebras, we may, considering $\mathfrak{h}$ in the above way, determine the coordinates $\theta_i = \theta(E_{ii})$ of θ.

4.4.1. Proposition . *The $\mathfrak{gl}_f(\infty)$-module with highsest weight Λ is quasifinite if and only if there are only finitely many distinct coordinates λ_i, where $\lambda_i = \Lambda(E_{ii})$.*

Proof. If V is quasifinite and generated by the highest weight vector v. Since $\dim V_{-1} < \infty$, it follows that $V_{-1} = Span(E_{i+1,i}v)_{i\in I}$ for a finite set I. Let $k \notin I$ and $k+1 \notin I$. Then $E_{k+1,k}v = \sum\limits_{i\in I} \alpha_i E_{i+1,i}v$, so $E_{k,k+1}(E_{k+1,k}v) = 0$ or, equivalently, $(E_{k,k} - E_{k+1,k+1})v = 0$, i.e., $\lambda_k = \lambda_{k+1}$.

The opposite implication is obvious. □

4.4.2. Proposition . *The $\mathfrak{gl}(\infty)$-module V with highsest weight Λ, where $\lambda_i = \lambda(E_{ii})$, is quasifinite if and only if there are only finitely many nonzero coordinates λ_i, where $\lambda_i = \Lambda(E_{ii})$.*

Proof. By considering V as $\mathfrak{gl}_f(\infty)$-module we deduce that there are only finitely many distinct coordinates λ_i, so we can set $\lambda_k = \lambda_+$ and $\lambda_{-k} = \lambda_-$ for sufficiently large k. Hence, $E_{k,k+1}(E_{k+1,k}v) = 0$ and, thanks to irreducibility of V, $E_{k+1,k}v = 0$ for all k but finitely many.

Let $A = \sum\limits_{i\in\mathbb{Z}} E_{i,i+1}$ and $B_k = \sum\limits_{i\geq k} E_{i+1,i}$. Then

$$[A, B_k] = \sum_{i\geq k} E_{i,i} - \sum_{i\geq k} E_{i+1,i+1} = E_{k,k}.$$

Therefore,

$$\begin{aligned}
&[A, B_k]v = E_{k,k}v = \lambda_k v = AB_k v - B_k A v = AB_k v = \\
&A(\sum_{i\geq k,\, i\in I} E_{i+1,i}v) = \sum_{i\geq k,\, i\in I} [A, E_{i+1,i}]v = \sum_{i\geq k,\, i\in I} (E_{i,i} - E_{i+1,i+1})v = \\
&\left(\sum_{i\geq k,\, i\in I} (\lambda_i - \lambda_{i+1})\right) v = (\lambda_k - \lambda_j)v,
\end{aligned}$$

where j is the largest index in the finite set I. Therefore, $\lambda_k = \lambda_k - \lambda_j$ and $\lambda_j = 0$.

Similarly, if $C_k = \sum\limits_{i\leq k} E_{i+1,i}$, then

$$[A, C_k] = \sum_{i\leq k} E_{i,i} - \sum_{i\leq k} E_{i+1,i+1} = E_{k+1,k+1}.$$

Hence,

$$[A, C_k]v = \lambda_{k+1}v = AC_kv = A(\sum_{i\leq k} E_{i+1,i})v = \sum_{i\leq k}[A, E_{i+1,i}]v$$

$$\sum_{i\leq k}(E_{i,i} - E_{i+1,i+1})v = \sum_{i\leq k}(\lambda_i - \lambda_{i+1})v = (\lambda_j - \lambda_{k+1})v.$$

So, $\lambda_j = 0$ for sufficiently large j. The converse statement follows from Proposition 4.4.1. □

4.4.3. Proposition . *The $\widehat{\mathfrak{gl}}(\infty)$-module V with highest weight (Λ, c), where c is the value of the central charge on z, is quasifinite if and only if there are only finitely many nonzero coordinates λ_i, where $\lambda_i = \Lambda(E_{ii})$.*

Proof. As in the proof of Proposition 4.4.2 we show that there are only finitely many distinct coordinates λ_i, and $E_{k+1,k}v \neq 0$ for finetely many values of k.

Let $A = \sum\limits_{i\in\mathbb{Z}} E_{i,i+1}$ and $J = \sum\limits_{i\leq 0} E_{i,i}$. Then $[A, J] = E_{01}$, so $c(A, B) = \operatorname{tr}\ (E_{01}B)$ for any $B \in \mathfrak{gl}(\infty)$, so $c(A, B_k) = \kappa(k)$. We have

$$[A, B_k]v = E_{k,k}v = (E_{k,k} + \kappa(k)z)v = AB_kv =$$

$$A(\sum_{i\geq k,\, i\in I} E_{i+1,i}v) = \sum_{i\geq k,\, i\in I}[A, E_{i+1,i}]v = \sum_{i\geq k,\, i\in I}(E_{i,i} - E_{i+1,i+1} + \delta_{i0}z)v =$$

$$\kappa(k)z + \left(\sum_{i\geq k,\, i\in I}(\lambda_i - \lambda_{i+1})\right)v = \kappa(k)cv + (\lambda_k - \lambda_+)v = (\kappa(k)c + (\lambda_k)v,$$

so $\lambda_+ = 0$.

For $C_k = \sum\limits_{i\leq k} E_{i+1,i}$ we have $c(A, C_k) = \kappa(-k)$, so

$$[A, C_k]v = (-E_{k+1,k+1} - \kappa(-k))v =$$

$$(-\lambda_{k+1} - \kappa(-k)c)v = AC_kv = A\sum_{i\leq k} E_{i+1,i}v = \sum_{i\leq k}[A, E_{i+1,i}]v =$$

$$\sum_{i\leq k}(E_{i,i} - E_{i+1,i+1} + \delta_{i0}z)v = (\kappa(-k)c + \lambda_- - \lambda_{k+1})v.$$

So, $\lambda_- = 0$. □

4.4.4. Theorem . *Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{gl}(\infty; R_m)$ or $\mathfrak{gl}^{\pm}(\infty; R_m)$ or their hatted versions. The $\mathfrak{g}$-module V with highest weight θ given by its coordinates $\theta_{ij} = (\varepsilon^j E_{ii})$ is quasifinite if and only if there are only finitely many nonzero coordinates θ_{ij}.*

Proof is similar to that of 4.4.1–4.4.3.

Let V be one of the modules $M^{\lambda,s}_{R_m}$ or, if $M^{\lambda,s}_{R_m}$ is reducible, one of its irreducible submodules. We obtain a homomorphism of $\mathfrak{gl}(\lambda)$ into $\mathfrak{g} = \mathfrak{gl}(\infty; R_m)$ or $\mathfrak{gl}^{\pm}(\infty; R_m)$. Let θ be a linear functional satisfying the conditions of Theorem 4.4.4. We may consider θ be a linear functional on the Cartan subalgebra of $\mathfrak{gl}(\lambda)$. Let us calculate the corresponding generating function.

4.5. Theorem . *The generating function $F_\theta(t)$ for the functional θ is as follows:*

i) *For $M^{\lambda,s}_{R_m}$:*

$$F_\theta(t)=\frac{\sum\limits_{i\in\mathbb{Z}}e^{(s-2i)t}\sum\limits_{j=0}^{m}\frac{(\theta_{ij}-\theta_{i-1,j})t^j}{j!}}{1-e^{-2t}}-\left(\frac{e^{st}-e^{-(\lambda+1)t}}{1-e^{-2t}}c+\frac{e^{st}}{1-e^{-2t}}\sum_{j=1}^{m}\frac{c_j}{j!}\right) \qquad (4.5.1)$$

ii) *For the module with highest weight $\lambda-1$:*

$$F_\theta(t)=\frac{\sum\limits_{i\in\mathbb{Z}}e^{(\lambda-2i-1)t}\sum\limits_{j=0}^{m}\frac{(\theta_{ij}-\theta_{i-1,j})t^j}{j!}}{1-e^{-2t}} \qquad (4.5.2)$$

iii) *For the module with highest weight $-\lambda-1$:*

$$F_\theta(t)=\frac{\sum\limits_{i\in\mathbb{Z}}e^{(-\lambda-2i-1)t}\sum\limits_{j=0}^{m}\frac{(\theta_{ij}-\theta_{i-1,j})t^j}{j!}}{1-e^{-2t}} \qquad (4.5.3)$$

iv) *For the module with lowest weight $\lambda+1$:*

$$F_\theta(t)=\frac{\sum\limits_{i\in\mathbb{Z}}e^{(\lambda+2i+1)t}\sum\limits_{j=0}^{m}\frac{(\theta_{ij}-\theta_{i-1,j})t^j}{j!}}{1-e^{-2t}} \qquad (4.5.4)$$

v) *For the module with lowest weight $1-\lambda$:*

$$F_\theta(t)=\frac{\sum\limits_{i\in\mathbb{Z}}e^{(1-\lambda+2i)t}\sum\limits_{j=0}^{m}\frac{(\theta_{ij}-\theta_{i-1,j})t^j}{j!}}{1-e^{-2t}} \qquad (4.5.5)$$

Proof. We will only prove i): the other cases are similar.

Let ϕ be the homomorphism that determines the $\mathfrak{gl}(\lambda)$-action on $M^{\lambda,s}$. Then $\phi(H)=\sum\limits_{i\in\mathbb{Z}}(s+\varepsilon-2i)E_{ii}$, so $\phi(e^{tH})=\sum\limits_{i\in\mathbb{Z}}e^{(s+\varepsilon-2i)t}E_{ii}$ and

$$\phi((1-e^{-2t})e^{tH})=\phi(e^{tH})-\phi(e^{t(H-2)})=\sum_{i\in\mathbb{Z}}e^{(s+\varepsilon-2i)t}(E_{ii}-E_{i-1,i-1})=$$
$$\sum_{i\in\mathbb{Z}}e^{(s-2i)t}\sum_{j=0}^{m}\varepsilon^j(E_{ii}-E_{i-1,i-1})\frac{t^j}{j!}.$$

Therefore,

$$(1-e^{-2t})F_\theta(t)=\theta(\phi(e^{tH})-\phi(e^{t(H-2)}))=$$
$$\sum_{i\in\mathbb{Z}}e^{(s-2i)t}\sum_{j=0}^{m}\frac{\theta_{ij}-\theta_{i-1,j}}{j!}t^j.$$

Since only finitely many coordinates θ_{ij} are nonzero, all the sums are, actually, finite. □

§5. Quasifinite modules over $\mathfrak{gl}(\lambda)$

In what follows we will show that not all quasifinite modules over $\mathfrak{gl}(\lambda)$ can be represented in a canonical form similar to that of $W_{1+\infty}$ from [KR].

Let $\mathfrak{gl}^{hol}(\lambda)$ be the holomorphic completion of $\mathfrak{gl}(\lambda)$, i.e., $\mathfrak{gl}^{hol}_0(\lambda)$ is the algebra of functions in H holomorphic on the whole complex line $\mathbb{C}$ and $\mathfrak{gl}^{hol}_i(\lambda)=\{f(H)X^i\}$ for $i>0$ whereas

$\mathfrak{gl}_i^{hol}(\lambda) = \{f(H)Y^{-i}\}$ for $i < 0$ and $f \in \mathfrak{gl}_0^{hol}(\lambda)$. The relations in the completed algebra follow from (1.1) and (1.4), namely, they are

$$Xf(H) = f(H-2)X,\ Yf(H) = f(H+2)Y \text{ and } XY = \frac{1}{4}(\lambda^2 - (H-1)^2).$$

The following Proposition is proved via the same lines as its counterpart in [KR].

5.1. Proposition . *Let V be quasifinite $\mathfrak{gl}(\lambda)$-module. Then the $\mathfrak{gl}(\lambda)$-action can be naturally extended to a $\mathfrak{gl}_i^{hol}(\lambda)$-action for $i \neq 0$.*

5.2. On $\mathbb{C}$, introduce an equivalence relation by setting

$$[s] = \begin{cases} s + 2\mathbb{Z} & \text{if } s + \lambda \notin 2\mathbb{Z} + 1 \\ s - 2\mathbb{Z}_+ & \text{if } s = \pm\lambda + 1 \\ s + 2\mathbb{Z}_+ & \text{if } s = \pm\lambda - 1 \end{cases}$$

To each class $[s]$ assign an irreducible $\mathfrak{sl}(2)$-module M^s with a diagonal action of H whose set of eigenvalues coinsides with $[s]$; denote by $\mathfrak{gl}_s(\infty)$ the Lie algebra of linear transformations of M^s with finitely many nonzero diagonals in the H-diagonal basis.

The following Theorem is similar and is proved via the same lines as its counterpart in [KR].

Theorem . *Consider $\overset{k}{\underset{i=1}{\oplus}} M^{s_i}_{R_{m_i}}$, where the $[s_i]$ are distinct and the m_i are nonegative integers. Let $\phi : \mathfrak{gl}(\lambda) \longrightarrow \mathfrak{g} = \overset{k}{\underset{i=1}{\oplus}} \mathfrak{gl}_{s_i}(\infty; R_{m_i})$ determined by this module. Then for any quasifinite $\mathfrak{g}$-module V any its $\mathfrak{gl}(\lambda)$-submodule is a $\mathfrak{g}$-submodule. In particular, if V is irreducible as $\mathfrak{g}$-module, it is irreducible as a $\mathfrak{gl}(\lambda)$-module.*

Following [KR], let us describe now the structure of quasifinite $\mathfrak{gl}(\lambda)$-modules. Let θ be the highest weight of a quasifinite $\mathfrak{gl}(\lambda)$-module and $F_\theta(t) = \dfrac{R(t)}{1-e^{-2t}}$, where $R(t) = \sum r_i(t)e^{s_i t}$ the corresponding formal power series, see 3.2. The polynomials $r_i(t)$ will be refered to as *multiplicities.*

5.3. For every s denote by $R_s(t)$ the sum of all quasipolynomials with the exponents from class $[s]$. Then $R(t) = \sum R_s(t)$, where the sum runs over representatives of different equivalency classes. Let $R_s(t) = \sum r_{is}(t)e^{(s-2i)t}$. The following properties of quasipolynomials follow easily from definitions:

i) $R(0) = 0$;

ii) the sum of all multiplicities of $R_{\lambda+1}(t)$ and that of $R_{-\lambda+1}(t)$are equal to 0;

iii) the sum of all multiplicities of $R_{\lambda-1}(t)$ and that of $R_{-\lambda-1}(t)$are equal to a constant.

Denote: $\Lambda = [\lambda-1] \cap [\lambda+1] \cap [-\lambda-1] \cap [-\lambda+1]$. To a quasipolynomial R satisfying i)–iii) assign a $\mathfrak{gl}(\lambda)$-module $V(s)$ as follows:

1) if $s \notin \Lambda$, then $V(s)$ is the irreducible $\widehat{\mathfrak{gl}}(\infty)$-module with central charges $c_j = -\sum\limits_{i\in I} r_i^{(j)}(0)$, where $j = 0, \dots, \max(\deg r_i)$, and the other coordinates of the highest weight are $\theta_{ij} = \sum\limits_{l\leq i}(r_l^{(j)}(0) + \delta_{l0}c_j)$;

2) if $s \notin [\lambda-1] \cap [-\lambda-1]$ and $R_s = \sum\limits_i r_i(y)e^{s-2i}$, then set $c = -\sum\limits_{i\in I} r_i(0)$ and $\theta_{ij} = \sum\limits_{l\leq i}(r_l^{(j)}(0) + \delta_{l0}\delta_{j0}c_j)$ and let $V(s)$ be the corresponding $\mathfrak{gl}^-(\infty)$-module.

3) if $s \notin [-\lambda+1] \cap [\lambda+1]$, then $\theta_{ij} = \sum\limits_{l\leq i} r_l^{(j)}(0)$ and $V(s)$ is the corresponding $\mathfrak{gl}^+(\infty)$-module.

Theorem . *Let the quasipolynomial* $R = \sum_{i=1}^{k} R_{s_i}(t)$ *be decomposed with respect to the distinct equivalence classes of the exponents; let* R *satisfy the conditions* i)–iii) *above. Then the irreducible quasifinite module with highest weight* $R(t)$ *is isomorphic to*

$$V = V(s_1) \otimes \dots\dots V(s_k) \otimes (\alpha\mathrm{tr}),$$

where $\alpha\mathrm{tr}$ *is the irreducible* 1*-dimensional module corresponding to the trace on* $\mathfrak{gl}(\lambda)$ *and the* $V(s_i)$ *are constructed as above.*

Proof. By Theorem 4.5 V is irreducible as $\mathfrak{gl}(\lambda)$-module. Therefore, it suffices to show that the restriction of the highest weight of this module onto $\mathfrak{gl}(\lambda)$ is equal to $\dfrac{R(t)}{1-e^{-2t}}$. Let $s = s_i$ be one of the exponets such that $s \notin \Lambda$. Then Proposition 5.1 implies that there exists a highest weight θ_s for $\widehat{\mathfrak{gl}}(\infty; R_m)$ whose generating function is equal to $\dfrac{R_s(t) + c_s e^{-(\lambda+1)t}}{1-e^{-2t}}$, where $c_s = -R_s(0)$.

If $s \in \Lambda$, then the same Proposition implies that the generating function is equal to $\dfrac{R_s(t)}{1-e^{-2t}}$, where the sum of all the exponents of R_s vanishes.

Finally, the generating function of the trace is equal to $\alpha\dfrac{e^{(\lambda-1)t} - e^{-(\lambda+1)t}}{1-e^{-2t}}$. Therefore, for our V the generating function is of the form $\dfrac{R(t)}{1-e^{-2t}}$, where

$$\begin{aligned}
&R(t) = \sum_{s\notin\Lambda} (R_s(t) + c_s e^{-(\lambda+1)t}) + \\
&R_{\lambda+1}(t) + R_{-\lambda+1}(t) + R'_{-\lambda-1}(t) + R'_{\lambda-1}(t) + \alpha(e^{(\lambda-1)t} - e^{-(\lambda+1)t}) = \\
&(\text{by the choice of } c_s) = \sum_{s\notin\Lambda} R_s(t) + R_{\lambda+1}(t) + R_{-\lambda+1}(t) + \\
&(R'_{-\lambda-1}(t) - \alpha e^{-(\lambda+1)t}) + (R'_{\lambda-1}(t) + \alpha e^{(\lambda-1)t}) = \sum_{s\notin\Lambda} R_s(t).
\end{aligned}$$

□

§6. Unitary modules over $\mathfrak{gl}(\lambda)$

Recall that an *anti-involution* of the Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is a $\mathbb{R}$-linear map $\omega : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$\omega(\alpha x) = \bar{\alpha}\omega(x), \quad \omega([x, y]) = [\omega(x), \omega(y)] \text{ and } \omega^2 = \mathrm{id}$$

for any $\alpha \in \mathbb{C}$ and $x, y \in \mathfrak{g}$.

In presence of an anti-involution ω we can endow V^*, the dual of the $\mathfrak{g}$-module V, with another $\mathfrak{g}$-module structure, namely,

$$(xl)(v) = l(\omega(x)v) \text{ for any } l \in V^*,\ v \in V \text{ and } x \in \mathfrak{g}. \tag{6.0.1}$$

In particular, if $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ and ω interchanges $\mathfrak{g}_-$ with $\mathfrak{g}_+$, then a $\mathfrak{g}$-homomorphism $M \longrightarrow M^*$ is well-defined, where M is a Verma module with highest weight vector v and the $\mathfrak{g}$-module structure on M^* is given by (6.0.1).

Indeed, let $v^* \in M^*$ be such that $v^*(v) = 1$ and $v^*(u) = 0$ for all u of lesser weight than that of v. Hence, if $x \in \mathfrak{g}_+$, then $(xv^*)(u) = v^*(\omega(x)u) = 0$, so v^* is also a highest vector and if $\theta(\omega(H)) = \overline{\theta(H)}$, where θ is the weight of v, then the weight of v^* is also equal to θ. Hence, there exists a $\mathfrak{g}$-isomorphism $M \longrightarrow M^*$ or an Hermitian $\mathfrak{g}$-invariant form $\langle\cdot,\cdot\rangle$ on M. If $\langle\cdot,\cdot\rangle$ is positive definite, M is called *unitary*.

In this section we indicate the conditions on the highest weight of the quasifinite $\mathfrak{gl}(\lambda)$-module to be unitary. First, let us describe automorphisms of $\mathfrak{gl}(\lambda)$.

As is known, the algebra of functions, $\mathcal{F}$, especially polynomial ones, has undescribably many automorphisms. Hence, so does the Lie algebra $\mathfrak{der}(\mathcal{F})$ of the differentiations of $\mathcal{F}$. To diminish the amount of automorphisms to a reasonable number, it is natural to consider only the *outer* automorphisms, i.e., the classes of automorphisms modulo the group of *inner* ones for which automorphisms induced by the automorphisms of $\mathcal{F}$ naturally qualify. Similarly, we say that the automorphism ϕ of $\mathfrak{gl}(\lambda)$ is an *inner* one if it also serves as an automorphism of the associative algebra $\mathfrak{A}_\lambda$.

Recall that a linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an *antiautomorphism* of the Lie algebra $\mathfrak{g}$ if $\phi([x, y]) = [\phi(y), \phi(x)]$. For the associative algebras definition of an antiautomorphism is similar. For example, the map t such that ${}^t|_\mathfrak{g} = -\mathrm{id}$ is called the *principal antiautomorphism* of the Lie algebra $\mathfrak{g}$. The principal antiautomorphism of $\mathfrak{g}$ can be extended to an antiautomorphism of the associative algebra $U(\mathfrak{g})$: ${}^t(x \otimes y) = {}^t(y) \otimes {}^t(x)$. Clearly, t preserves Casimir elements.

6.1. Theorem . *The group of outer automorphisms of $\mathfrak{gl}(\lambda)$ is isomorphic to $\mathbb{Z}/2$ and is generated by the class of $-t$ (where $-t : x \mapsto -{}^t(x)$).*

Proof. Let $\psi \in Aut(\mathfrak{gl}(\lambda))$. Then $\psi(Y)$, $\psi(H)$, and $\psi(X)$ span a Lie algebra isomorphic to $\mathfrak{sl}(2)$. Moreover, they generate $\mathfrak{A}_\lambda$ because the weight of $\psi(X)^n$ is equal to $2n$ with respect to $\psi(H)$. Hence, there exists a surjective homomorphism

$$\tau : U(\mathfrak{sl}(2)) \longrightarrow \mathfrak{A}_\lambda, \quad X \mapsto \psi(X),\ H \mapsto \psi(H),\ Y \mapsto \psi(Y).$$

Since $\tau(\Omega) \in \mathbb{C}$, there exists a $\mu \in \mathbb{C}$ such that $\tau(\Omega) = \frac{1}{2}(\mu^2 - 1)$. Therefore, we get a surjective homomorphism, hence, an isomorphism $\mathfrak{A}_\mu \longrightarrow \mathfrak{A}_\lambda$. In a very difficult technical paper Dixmier proved [Di1] that this may only happen if $\lambda^2 = \mu^2$. So we may assume that $\lambda = \mu$ and τ is an automorphism. Let $\psi_1 = \tau^{-1}\psi$. Then $\psi_1|_{\mathfrak{sl}(2)} = \mathrm{id}$.

Therefore, ψ_1 is an automorphism of $\mathfrak{gl}(\lambda)$, as $\mathfrak{sl}(2)$-module. Therefore, $\psi_1|_{L^{2i}} = c_i \in \mathbb{C}$. Since the Lie algebra $\mathfrak{gl}(\lambda)$ is generated by L^2 and L^4, we deduce that $c_i = c_2^{i-1}$ for every $i \geq 1$. Moreover, since $[L^4, L^4] \supset L^2$, it follows that $c_2^2 = 1$. Hence, $c_2 = \pm 1$; if $c_2 = 1$, then $\psi_1 = \mathrm{id}$ and if $c_2 = -1$, then $\psi_1 = \phi$. □

6.2. Theorem . *If $\lambda^2 \notin \mathbb{R}$, then $\mathfrak{gl}(\lambda)$ has no real forms, i.e., no involutive antilinear automorphisms.*

Proof. Let ω be an involutive antilinear automorphism of $\mathfrak{gl}(\lambda)$. Then $\omega(X)$, $\omega(H)$ and $\omega(Y)$ generate $\mathfrak{A}_\lambda$. Hence, there exists a surjective homomorphism

$$\tau : U(\mathfrak{sl}(2)) \longrightarrow \mathfrak{A}_\lambda, \quad \tau = \mathrm{id} \text{ on } X, H, Y;$$

hence, we obtain an isomorphism $\tau : \mathfrak{A}_\mu \longrightarrow \mathfrak{A}_\lambda$. By [Di1] this may only happen if $\lambda^2 = \mu^2$. Hence, we may assume that $\lambda = \mu$ and τ is an automorphism. Then $\omega_1 = \tau^{-1}\omega$ is an antilinear automorphism of $\mathfrak{gl}(\lambda)$ such that

$$\omega_1(X) = X,\ \omega_1(H) = H,\ \omega_1(Y) = Y.$$

Set $z_i = (\mathrm{ad}\ X)^i(Y^2)$; in particular, $z_0 = z$. Then $Span(z_i : i = 1, \dots, 4)$ form a basis of L^4 and if $\omega_1(Y^2) = cY^2$, then, clearly, $\omega_1(z_i) = cz_i$ for $i = 1, \dots, 4$. According to [GL] in $\mathfrak{gl}(\lambda)$ the following relations hold:

$$3[z_1, z_2] - 2[z_1, z_3] = 24(\lambda^2 - 4)Y$$

$$4[z_3, [z_1, z_1]] - 3[z_2, [z, z_2]] = 576(\lambda^2 - 9)z$$

Having applied ω_1 to both sides of these relations we get

$$c^2(\lambda^2 - 4) = \bar\lambda^2 - 4$$

$$c^3(\lambda^2 - 9) = c(\bar\lambda^2 - 9)$$

Therefore, $c^2 = 1$ and $\lambda^2 = \bar\lambda^2$. □

Corollary . *If $\lambda^2 \neq \bar\lambda^2$, then $\mathfrak{gl}(\lambda)$ has no involutive antilinear automorphisms.*

Proof. Let ω be an involutive antilinear automorphism; clearly $\tau : X \longleftrightarrow Y$ and $\tau(H) = H$ determines an antiautomorphism of $\mathfrak{gl}(\lambda)$. Then $\tau \circ \omega$ is an involutive antilinear automorphism. □

6.3. If $\lambda^2 = \bar\lambda^2$, then $\mathfrak{A}_\lambda$ possesses an involutive antilinear automorphism ω:

$$\omega(X) = Y, \quad \omega(Y) = X, \quad \omega(H) = H \tag{6.3}$$

In what follows, the unitary modules are considered with respect to this automorphism.

Theorem . *Let λ^2 be real and let $F_\theta(t) = \dfrac{R(t)}{1 - e^{-2t}}$, where $R(t)$ is a quasipolynomial none of whose exponents s_i do not belong to $\Lambda = [\lambda - 1] \cap [\lambda + 1] \cap [-\lambda - 1] \cap [-\lambda + 1]$. The $\mathfrak{gl}(\lambda)$-module with character $F_\theta(t)$ is unitary if and only if*

$$F_\theta(t) = \sum n_i \frac{e^{-(\lambda+1)t} - e^{s_i t}}{1 - e^{-2t}}, \text{ where } n_i \in \mathbb{Z}_+.$$

Proof. Let V be the irreducible module with character $F_\theta(t)$ and $P(H)$ the annihilator of Yv, where v is the highest weight vector. Then $P(H)$ is the characteristic polynomial of the operator H in V_{-1}. But since the unitary form is invariant and $\omega(H) = H$, it follows that H is self-adjoint; hence, all the roots of P are real. Further, if α is a multiple root of multiplicity $m > 1$, the polynomial P is of the form $P = (H - \alpha)^m Q(H)$. For $u = (H - \alpha)^{m-1} Q(H)v$ we have

$$\langle u, u\rangle = \langle (H - \alpha)^{m-1} Q(H)v, (H - \alpha)^{m-1} Q(H)v\rangle = \langle Q(H)v, (H - \alpha)^{2m-2} Q(H)v\rangle = 0.$$

Thanks to Hermitean property, $u = 0$. Thus, $m = 1$.

By Theorem 5.3 V is of the form $V(s_1) \otimes \dots \otimes V(s_k)$, where the $V(s_i)$ are irreducible $\widehat{\mathfrak{gl}}(\infty)$-modules. Clearly, V is unitary if and only if each $V(s_i)$ is unitary. Now, thanks to [KR] we know that the $\widehat{\mathfrak{gl}}(\infty)$-module $V(s)$ is unitary if and only if $\theta_i - \theta_{i+1} + \delta_{i0} c \in \mathbb{Z}_+$ for all i, where the θ_i are the coordinates of the highest wieght and c the value of the central charge. □

§7. $\mathfrak{gl}(\lambda)$ AND THE SYMMETRIC GROUP

In this section we deduce an explicit realization of certain irreducible $\mathfrak{gl}(\lambda)$-modules. Namely, we decompose the tensor powers of the Verma module over $\mathfrak{sl}(2)$, indicate the corresponding characteristic polynomials and q-characters.

7.1. Lemma . *Let A be an associative algebra; let $\mathfrak{S}_n$ naturally act on $A^{\otimes n}$. Then the algebra $(A^{\otimes n})^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariants is generated by the elements of the form*

$$a \otimes 1 \otimes 1 \otimes \dots \otimes 1 + \dots 1 \otimes 1 \otimes 1 \otimes \dots \otimes a.$$

Proof. Denote: $s(a_1,\dots,a_n)=\sum\limits_{\sigma\in\mathfrak{S}_n} a_{\sigma(1)}\otimes\dots\otimes a_{\sigma(n)}$. Let B be the algebra generated by the $s(a,1,\dots,1)$ for all $a\in A$. Let $|s(a_1,\dots,a_n)|$ be the number of the a_i distinct from 1. Let us prove that $B\simeq (A^{\otimes n})^{\mathfrak{S}_n}$.

Let us carry on an induction on $|s(a_1,\dots,a_n)|$ in order to prove that $s(a_1,\dots,a_n)\in B$. Indeed, if $|s(a_1,\dots,a_n)|=1$, by definition $s(a_1,\dots,a_n)=s(a,1,\dots,1)\in B$.

Let $|s(a_1,\dots,a_n)|=l>1$. Consider

$$s(a_1,\dots,a_{l-1},1,\dots,1)s(a_l,1,\dots,1)=\alpha s(a_1,\dots,a_l,1,\dots,1)+\dots,$$

where α is a nonzero constant and the dots stand for the linear combination of the terms $s(b_1,\dots,b_m,1,\dots,1)$ with $m<l$. $\square$

7.2. Theorem . *Let $V=M^{\lambda-1}$ be the Verma module with highest weight $\lambda-1$ over $\mathfrak{sl}(2)$ (and $\mathfrak{gl}(\lambda)$). Then*

$$V^{\otimes n}=\bigoplus_{\nu} V^\nu\otimes S^\nu,$$

where V^ν is an irreducible $\mathfrak{gl}(\lambda)$-module and S^ν is an irrecible $\mathfrak{S}_n$-module and ν runs over the partitions of n.

Proof. Clearly, V is irreducible not only as $\mathfrak{gl}(\lambda)$-module but also as $\mathfrak{A}_\lambda$-module. Then, obviously, $W=V^{\otimes n}$ is irreducible as $\mathfrak{A}_\lambda^{\otimes n}$-module.

The image of $U(\mathfrak{gl}(\lambda))$ in $\mathrm{End}(W)$ coinsides with the subalgebra generated by $s(a,1,\dots,1)$ for $a\in\mathfrak{gl}(\lambda)$, so by Lemma it is isomorphic to $(\mathfrak{A}_\lambda^{\otimes n})^{\mathfrak{S}_n}$.

Let us decompose W into isotypical $\mathfrak{S}_n$-modules: $W=\bigoplus\limits_\nu W^\nu$ and represent each W^ν in the form $W^\nu=V^\nu\otimes S^\nu$, where V^ν is an irreducible $\mathfrak{gl}(\lambda)$-module. There is no unique way to do this, for example, set $V^\nu=e_\nu(W)$ for any minimal idempotent in $\mathbb{C}[\mathfrak{S}_n]$ corresponding to the partition ν.

To show that V^ν is irreducible as a $\mathfrak{gl}(\lambda)$-module, consider $V_1^\nu=\mathrm{Hom}_{\mathfrak{S}_n}(S^\nu,W)$. As a $\mathfrak{gl}(\lambda)$-module, V_1^ν is isomorphic to V^ν. Let $\phi,\psi\in V_1^\nu$ and $\phi\neq 0$. Let us show that there exists an $u\in U(\mathfrak{gl}(\lambda))$ such that $u\phi=\psi$. Indeed, since W is irreducible as $\mathfrak{A}_\lambda^{\otimes n}$-module, then the density theorem ([L], Ch. XVII, §3, Th.1) states that there is a $w\in\mathfrak{A}_\lambda^{\otimes n}$ such that $w\phi(v_i)=\psi(v_i)$, where the v_i form a basis of S^ν. (Since S^ν is irreducible and $\phi\neq 0$, then the vectors $\psi(v_i)$ are linealy independent for $i=1,\dots,\dim S^\nu$.)

Let us average the element $w\phi$ with respect to $\mathfrak{S}_n$:

$$(w\phi)^\sharp=\frac{1}{|\mathfrak{S}_n|}\sum\sigma(w\phi)\sigma^{-1}=\frac{1}{|\mathfrak{S}_n|}\sum\sigma(w)\sigma^{-1}(\phi)=w^\sharp\phi.$$

But, on the other hand, since $w\phi=\psi$, we deduce that $(w\phi)^\sharp=(\psi)^\sharp=\psi$, i.e., $w^\sharp\phi=\psi$ and we may assume that $w\in(\mathfrak{A}_\lambda^{\otimes n})^{\mathfrak{S}_n}$. So, there exists a $u\in U(\mathfrak{gl}(\lambda))$ such that $u\phi=\psi$. In other words, V_1^ν is irreducible. $\square$

7.2.1. Corollary . 1) *Let $\nu=(\nu_1,\dots,\nu_n)$. The generating function corresponding to the highest weight ν is of the form*

$$\sum_{i=1}^n \nu_i e^{(\lambda-2i+1)t}. \tag{7.2.1.1}$$

2) *Represent ν in the form $\nu=(\theta_1^{\alpha_1}\dots\theta_m^{\alpha_m})$, where $\theta_1>\dots>\theta_m>0$ and $\alpha_i\neq 0$ for all i and where θ^α denotes the product of α copies: $\theta\dots\theta$. Then the characteristic polynomial of V^ν is equal*

to

$$P(H)=\prod_{i=1}^{m}(H-\lambda-2\alpha_1-2\alpha_2-\dots-2\alpha_i-1).$$

Proof. Heading 1) follows from Theorem 4.5. Multiply (7.2.1.1) by $1-e^{-2t}$; we see that the product is of the form

$$\nu_1 e^{(\lambda-1)t}+\sum_{i=1}^{n-1}(\nu_{i+1}-\nu_i)e^{(\lambda-2i+1)t}-\nu_n e^{(\lambda-2n-1)t}. \tag{7.2.1.2}$$

Therefore, sum (7.2.1.2) without the first summand is a solution of an ordinary differential equation with constant coefficients whose characteristic equation is precisely of the form indicated. □

7.2.2. Corollary . *Set $a=e^{\lambda}$, $q=e^{-\alpha}$, where α is the positive root of $\mathfrak{sl}(2)$. Then the q-character of V^{ν} is equal to*

$$\chi_\nu=\frac{a^{|\nu|}q^{n(\nu)}}{\prod_{x\in\nu}(1-q^{h(x)})}, \tag{7.2.2}$$

where $|\nu|$ is the number of cells in the Young tableau corresponding to ν, $n(\nu)=\sum\limits_{i\geq 1}\nu_i$ and $h(x)$ is the length of the hook corresponding to the cell x.

Proof. By [Mac], Example 2 in §3, for the S-function s_ν we have

$$\chi_\nu=s_\nu(a,aq,aq^2,\dots,)=a^{|\nu|}s_\nu(1,q,q^2,\dots,)=\text{rhs of (7.2.2)}.$$

□

§8. Orthogonal polynomials for $\mathfrak{gl}(n)$

For an overview see [NU], [NSU]. (Setting $\prod\limits_{i=1}^{l}T_1(\alpha_i)\dots T_l(\alpha_i)=1$ for $l=0$ we make formulas (8.1.1) and the like uniform.)

8.1. Theorem . *In $\mathfrak{gl}(n)$:*

i) *Consider the basis $e_{kl}=(\text{ad }Y)^{k-l}(X^k)$ for $0\leq k\leq n-1$ and $-k\leq l\leq k$. We have*

$$\langle e_{kl},e_{k'l'}\rangle=\delta_{k,k'}\delta_{l+l',0}.$$

ii) *Determine the elements f_{kl} from the equations $(0\leq l\leq k)$*

$$(\text{ad }Y)^{k-l}(X^k)=X^l f_{kl}\ \text{and}\ (\text{ad }Y)^{k+l}(X^k)=f_{k,-l}Y^l.$$

Set $T_0(H)=\frac{1}{4}(n^2-(H-1)^2)$ and $T_i(H)=T_0(H+2i)$ and $\alpha_i=n-2i+1$ for $i=1,\dots,n$. For a fixed $l\geq 0$ and any $k\geq l$ the polynomials f_{kl} form an orthonormal basis with respect to the form

$$\langle f,g\rangle=\begin{cases}\sum\limits_{i=l+1}^{n}f(\alpha_i)g(\alpha_i)T_1(\alpha_i)\dots T_l(\alpha_i) & \text{for } l>0\\ \sum\limits_{i=l+1}^{n}f(\alpha_i)g(\alpha_i) & \text{for } l=0.\end{cases} \tag{8.1.1}$$

iii) *Up to a constant factor the polynomials f_{kl} coincide with the Hahn polynomial of one discrete variable:*

$$f_{kl}(H)={}_3F_2\left(\begin{matrix}l-k,l+k+1,\frac{1}{2}(1-n-H)\\ l+1,l+1-n\end{matrix}\ \middle|\ 1\right)\times T_0(\alpha_{l+1})\dots T_0(\alpha_k), \tag{8.1.2}$$

where

$$ {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\mid z\right)=\sum_{i=0}^{\infty}\frac{(\alpha_1)_i(\alpha_2)_i(\alpha_3)_i}{(\beta_1)_i(\beta_2)_i}\frac{z^i}{i!} \tag{8.1.3}$$

is a generalized hypergeometric function and $(\alpha)_0=1$ *whereas* $(\alpha)_i=\alpha(\alpha+1)\dots(\alpha+i-1)$ *for* $i>0$.

Proof. i) First, observe that the subspaces L^{2k} in the decomposition of $\mathfrak{gl}(n)=L^0\oplus L^2\oplus\dots\oplus L^{2n-2}$ are pairwise orthogonal. Indeed, the form $A,B\mapsto \operatorname{tr} AB$ defines an invariant pairing of L^k and L^l, hence, an $\mathfrak{sl}(2)$-homomorphism $L^k\longrightarrow L^l$ which is only possible if $k=l$. Moreover, it is clear that $\langle\mathfrak{gl}(n)_k,\mathfrak{gl}(n)_l\rangle\neq 0$ if and only if $k+l=0$. Now observe that $\mathfrak{gl}(n)_l\cap L^{2k}=Span((\operatorname{ad}Y)^{k-l}(X))$. This proves i).

ii) Let $X^lf(H)\in\mathfrak{gl}(n)_l$ and $g(H)Y^l\in\mathfrak{gl}(n)_{-l}$. Then

$$\operatorname{tr}\,(g(H)Y^lX^lf(H))=\operatorname{tr}\,(f(H)g(H)T_1(H)\dots T_l(H)).$$

As is easy to verify

$$\operatorname{tr}\,f(H)=\sum_{1\le i\le n}f(\alpha_i)\text{ for any }f(H)\in\mathfrak{gl}(n)_0.$$

This implies (8.1.1).

Moreover, i) implies that for any fixed l lying between $-k$ and k the polynomials f_{kl} and $f_{k,-l}$ form two mutually dual bases, i.e., $\langle f_{kl},f_{kl'}\rangle=\delta_{l+l',0}$. But the degrees of polynomials f_{kl} and $f_{k,-l}$ are equal, which means that they coincide up to a constant factor. This proves ii).

iii) follows from the fact that orthogonal polynomials are uniquely determined by the weight function and the interval over which we consider the scalar product. Eq. (8.1.2) follows from comparison of the coefficients of the leading terms. □

Remark . Hahn polynomials are determined for three parameters α, β and N as

$$h_p^{(\alpha,\beta)}(z,N)=\frac{(-1)^p\Gamma(N)(\beta)_p}{p!\Gamma(N-p)}\,{}_3F_2\left(\begin{matrix}-p,\alpha+\beta+p+1,-z\\ \beta+1,l-N\end{matrix}\mid 1\right). \tag{8.1.4}$$

Our f_{kl} coinsides, up to a factor, with $h_p^{(\alpha,\beta)}(z,N)$ at $\alpha=\beta=l$, $p=k-l$, $z=\frac{1}{2}(H+n-1)$ and $N=n-1$.

Let us show now how the main properties of the Hahn polynomials follow from the representation theory of $\mathfrak{sl}(2)$. For any $f(H)\in\mathbb{C}[H]$ set

$$\triangle f(H)=f(H+2)-f(H),\quad \nabla f(H)=f(H)-f(H-2). \tag{8.1.5}$$

8.2. Theorem . *Consider* $\mathfrak{gl}(n)$ *as* $\mathfrak{sl}(2)$*-module with respect to the image of the principal embedding, let* Ω *be the quadratic Casimir operator* (1.2).

i) Ω *is self-adjoint with respect to the form* $\langle\cdot,\cdot\rangle$ *and the polynomials* X^lf_{kl} *are eigenfunctions of* Ω *corresponding to eigenvalue* $2k(k+1)$. *The polynomials* f_{kl} *satisfy the difference equation*

$$T_0(H)\nabla\triangle(f)-(k+1)(k+l)\triangle(f)+(k-l)(k+l+1)f=0. \tag{8.2.i)}$$

ii) $f_{kl}=\begin{cases}\dfrac{\nabla^{k-l}(T_1\dots T_k)}{T_1\dots T_l} & \text{if } l>0\\ \nabla^k(T_1\dots T_k) & \text{if } l=0.\end{cases}$

iii) $\langle f_{kl},f_{kl}T_1\dots T_l\rangle=\dfrac{(k-l)!}{(k+l)!}\dfrac{n(n^2-1^2)\dots(n^2-k^2)}{2k+1}(k!)^2.$

Proof. i) Since the form $\langle\cdot,\cdot\rangle$ is $\mathfrak{sl}(2)$-invariant, $\langle[w,u],v\rangle = -\langle u,[w,v]\rangle$ for any $u,v\in\mathfrak{gl}(n)$ and $w\in\mathfrak{sl}(2)$. Denote the $U(\mathfrak{sl}(2))$-action in $\mathfrak{gl}(n)$ by $*$; by induction we easily deduce that $\langle w*u,v\rangle = -\langle u,w^t*v\rangle$ where now $w\in U(\mathfrak{sl}(2))$ and t is the principal antiinvolution. Since, as is easy to verify, $\Omega^t=\Omega$, we have $\langle\Omega*u,v\rangle = \langle u,\Omega*v\rangle$, i.e., Ω is self-adjoint. Since $\mathfrak{gl}(n)=\oplus L^{2i}$ and $X^lf_{kl}=(\mathrm{ad}Y)^{k-l}(X^k)\in L^{2k}$, it follows that X^lf_{kl} is an eigenfunction of Ω with eigenvalue $2k(k+1)$. Applying Ω to X^lf_{kl} we obtain eq. (8.2.i).

ii) Recall [Di] the identity

$$(\mathrm{ad}y)^p(a)=\sum_{j=0}^{p}(-1)^j\binom{p}{j}y^{p-j}ay^j.$$

Set $a=X^k$, $p=k-l$ and multiply by Y^l from the left. We get

$$Y^lX^lf_{kl}=Y^l(\mathrm{ad}Y)^{k-l}(X^k)=Y^l\left(\sum_{j=0}^{k-l}(-1)^j\tbinom{k-l}{j}Y^{k-l-j}X^kY^j\right)=$$
$$\sum_{j=0}^{k-l}(-1)^j\tbinom{k-l}{j}y^{k-j}X^kY^j=\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}T_{j-1}\dots T_0T_1\dots T_{k-j}.$$

But $\nabla^{k-l}(f)(H)=\sum\limits_{j=0}^{k}(-1)^j\binom{k}{j}f(H-2j)$ for any function f, so

$$Y^lX^lf_{kl}=\nabla^{k-l}(T_1\dots T_k).$$

Since $Y^lX^l=T_1\dots T_l$, we are done.

iii) The module L^{2k} is the linear span of the vectors $v_l=(\mathrm{ad}Y)^{k-l}(X^k)$ for $-k\le l\le k$; hence,

$$\langle v_l,v_m\rangle=-\langle v_{l-1},v_{m+1}\rangle=\dots=(-1)^l\langle v_0,v_{l+m}\rangle$$

by invariance of the form. Hence, $\langle v_l,v_{-l}\rangle=(-1)^l\langle v_0,v_0\rangle$. But $v_l=X^lf_{kl}=(\mathrm{ad}\ Y)^{k-l}(X^k)$ and $v_{-l}=f_{k,-l}Y^l=(\mathrm{ad}\ Y)^{k+l}(X^k)$.

Now recall that f_{kl} and $f_{k,-l}$ are identical up to a constant factor and compare the leading coefficients. We see that

$$f_{k,-l}=(-1)^l\frac{(k+l)!}{(k-l)!}f_{k,l}.$$

Hence,

$$\langle f_{k,l},f_{k,l}T_1\dots T_l\rangle=\mathrm{tr}\ (f_{k,l}Y^l\cdot X^lf_{k,l})=(-1)^l\frac{(k+l)!}{(k-l)!}\langle v_l,v_{-l}\rangle=\frac{(k+l)!}{(k-l)!}\langle v_0,v_0\rangle.$$

It remains to demonstrate that

$$\langle v_0,v_0\rangle=\frac{n(n^2-1^2)\dots(n^2-k^2)}{2k+1}(k!)^2.$$

Obviously, $f_k=\left((\mathrm{ad}\ Y)^k(X^k)\right)^2\in U(\mathfrak{sl}(2))$. So $\langle v_0,v_0\rangle=\mathrm{tr}_n\phi_n(f_k)$, where tr_n is the trace on $\mathfrak{gl}(n)$ and ϕ_n is the homomorphism of $U(\mathfrak{sl}(2))$ into $\mathfrak{gl}(n)$ induced by the principle embedding.

It is clear now that $P_k(n)=\mathrm{tr}_n(\phi_n(f_k))$ is a polynomial of degree $2k+1$; moreover, this polynomial is an odd one. But $\phi_n(X^k)=0$ for $n\le k$ and $\phi_n(f_k)=0$; hence, $P_k(n)=0$ if $n\le k$ and $P_k(-n)=0$ because P_k is odd. Therefore,

$$P_k(n)=c_kn(n^2-1^2)\dots(n^2-k^2).$$

To calculate the constant c_k, it suffices to compute $P_k(k+1) = \mathrm{tr}_{k+1}(\phi_{k+1}(f_k))$. But in this case $X^k = (k!)^2 e_{1,k+1}$ and

$$(\text{ad } Y)^k(X^k) = (k!)^2(\text{ad } Y)^k(e_{1,k+1}) = (k!)^2 \sum_{j=0}^{k}(-1)^j \binom{k}{j} e_{k+1-j,k+1-j}.$$

Thus,

$$\mathrm{tr}_{k+1}(\phi_{k+1}(f_k)) = (k!)^4 \sum_{j=0}^{k} \binom{k}{j}^2 = (k!)^4 \binom{2k}{k}$$

implying $c_k = \dfrac{(k!)^2}{2k+1}$. □

8.3.1. Proposition . *The following orthogonality relations* (i) *and* (ii) *hold:*

(i)
$$\sum_i f_{kl}(\alpha_i) f_{k_1,l}(\alpha_i) T_1(\alpha_i)\dots T_l(\alpha_i) = \delta_{k,k_1} c_{k,l},$$

where $c_{k,l} = \dfrac{(k-l)!}{(k+l)!} \dfrac{n(n^2-1^2)\dots(n^2-k^2)}{2k+1}(k!)^2$.

(ii)
$$\sum_{k=l}^{n-1} \frac{1}{c_{kl}} f_{kl}(\alpha_i) f_{kl}(\alpha_j) T_1(\alpha_i)\dots T_l(\alpha_i) = \delta_{ij} \text{ for any } i,j > l.$$

Proof. (i) follows fromTheorem 8.2, iii). To prove ii), express e_{ij} via $X^l f_{kl}$ and $f_{k,-l}Y^l$. We have, in particular, $e_{i,i+l} = \sum_{k=l}^{n-1} \alpha_{ik} X^l f_{kl}$ for any k between l and $n-1$. But, as is easy to verify,

$$f_{kl}Y^l = (\sum_{i=1}^{n} f_{kl}(\alpha_i)e_{ii})(\sum_{i=1}^{n-1} e_{i+1,i})^l =$$
$$(\sum_{i=1}^{n} f_{kl}(\alpha_i)e_{ii})(\sum_{i=1}^{n-l} e_{i+l,i}) = \sum_{i=1}^{n-l} f_{kl}(\alpha_{i+l})e_{i+l,i}.$$

Hence, by Theorem 8.2 we have

$$f_{kl}(\alpha_{i+l}) = \langle e_{i,i+l}, f_{kl}Y^l\rangle = \alpha_{ik}\langle f_{kl}Y^l, X^l f_{kl}\rangle = \alpha_{ik}c_{kl}.$$

In other words,

$$e_{i,i+l} = \sum_{k=l}^{n-1} \frac{f_{kl}(\alpha_{i+l})}{c_{kl}} X^l f_{kl}.$$

Similarly,

$$e_{i+l,i} = \sum_{k=l}^{n-1} \frac{f_{kl}(\alpha_{i+l})T_1(\alpha_{i+1})\dots T_l(\alpha_{i+l-1})}{c_{kl}} f_{kl}Y^l.$$

Therefore, by setting $i' = i+l$, $j' = j+l$ we deduce that

$$\langle e_{i+l,i}, e_{j,j+l}\rangle = \sum_{i=l}^{n-1} \tfrac{1}{c_{kl}} f_{kl}(\alpha_{i+l}) f_{kl}(\alpha_{j+l}) T_1(\alpha_{i+1})\dots T_l(\alpha_{i+l-1}) =$$
$$\sum_{i=l}^{n-1} \tfrac{1}{c_{kl}} f_{kl}(\alpha_{i'}) f_{kl}(\alpha_{j'}) T_1(\alpha_{i'-1})\dots T_l(\alpha_{i'-l+1}) =$$
$$\sum_{i=l}^{n-1} \tfrac{1}{c_{kl}} f_{kl}(\alpha_{i'}) f_{kl}(\alpha_{j'}) T_1(\alpha_{i'})\dots T_l(\alpha_{i'}).$$

□

Expressing Casimir elements Ω_2, Ω_3, ... , Ω_n for $\mathfrak{gl}(n)$ in terms of the orthogonal polynomials rather than matrix units, we can derive various identities relating the polynomials. Some of these identities might be even new, cf. 9.3. For example, for Ω_2 we have:

8.3.2. Proposition . *In $\mathfrak{gl}(n)$, we have*

$$\sum_{k=0}^{n-1}\left(\frac{1}{c_{k0}}f_{k0}^2+2\sum_{l=1}^{k}\frac{1}{c_{kl}}f_{kl}^2T_1\dots T_l-\frac{2k+1}{n}\right)=-H.$$

Proof. Let us rewrite the Casimir operator for $\mathfrak{gl}(n)$ expressed in terms of matrix units

$$\begin{gathered}\Omega_2=\sum_{k=1}^{n}e_{kk}\otimes e_{kk}+\sum_{i<j}(e_{ij}\otimes e_{ji}+e_{ji}\otimes e_{ij})=\\ \sum_{k=1}^{n}e_{kk}\otimes e_{kk}+\sum_{k=1}^{n}(n-2k+1)e_{kk}2\sum_{i<j}e_{ji}\otimes e_{ij})\end{gathered}$$

via orthogonal polynomials:

$$\begin{gathered}\Omega_2=\sum_{k=0}^{n-1}\tfrac{1}{c_{k0}}f_{k0}\otimes f_{k0}+\sum_{l=1}^{n-1}\sum_{k=l}^{n-1}\tfrac{1}{c_{kl}}(X^lf_{kl}\otimes f_{kl}Y^l+f_{kl}Y^l\otimes X^lf_{kl})=\\ \sum_{k=0}^{n-1}\tfrac{1}{c_{k0}}f_{k0}\otimes f_{k0}+2\sum_{l=1}^{n-1}\sum_{k=l}^{n-1}\tfrac{1}{c_{kl}}f_{kl}Y^l\otimes X^lf_{kl}+\sum_{k>l}\tfrac{1}{c_{kl}}[X^lf_{kl},f_{kl}Y^l].\end{gathered}$$

Let us now transform the last sum. For this, consider a homomorphism $\varphi : U(\mathfrak{gl}(n)) \longrightarrow \mathfrak{gl}(n)$ induced by the identity representation. Since

$$\frac{1}{c_{k0}}f_{k0}\otimes f_{k0}+\sum_{l=1}^{k}\frac{1}{c_{kl}}(X^lf_{kl}\otimes f_{kl}Y^l+f_{kl}Y^l\otimes X^lf_{kl})$$

is $\mathfrak{sl}(2)$-invariant, its image in $\mathfrak{gl}(n)$ is also an $\mathfrak{sl}(2)$-invariant, i.e., is a constant, $D(k)$:

$$\frac{1}{c_{k0}}f_{k0}\cdot f_{k0}+2\sum_{l=1}^{k}\frac{1}{c_{kl}}f_{kl}Y^l\cdot X^lf_{kl}+\sum_{l=1}^{k}\frac{1}{c_{kl}}[X^lf_{kl},f_{kl}Y^l]=D(k).$$

Let us calculate the trace of both sides. We see that

$$nD(k)=\frac{1}{c_{k0}}\mathrm{tr}(f_{k0}^2)+2\sum_{l=1}^{k}\frac{1}{c_{kl}}\mathrm{tr}(f_{kl}^2T_1\dots T_l)=2k+1$$

implying $D(k)=\dfrac{2k+1}{n}$. Therefore,

$$\sum_{l=1}^{k}\frac{1}{c_{kl}}[X^lf_{kl},f_{kl}Y^l]=\frac{2k+1}{n}-\left(\frac{1}{c_{k0}}f_{k0}\cdot f_{k0}+2\sum_{l=1}^{k}\frac{1}{c_{kl}}f_{kl}Y^l\cdot X^lf_{kl}\right).$$

Summing over k we obtain

$$\sum_{k=0}^{n-1}\left(\frac{1}{c_{k0}}f_{k0}\cdot f_{k0}+2\sum_{l=1}^{k}\frac{1}{c_{kl}}f_{kl}Y^l\cdot X^lf_{kl}-\frac{2k+1}{n}\right)=-\sum_{k=0}^{n-1}\sum_{l=1}^{k}\frac{1}{c_{kl}}[X^lf_{kl},f_{kl}Y^l].$$

The linear parts of the Casimir operator coincide, so the last sum is equal to $-\sum\limits_{i=1}^{n}(n-2i+1)e_{ii}=-H$. □

§9. Orthogonal polynomials and $\mathfrak{gl}(\lambda)$

The above results for $\mathfrak{gl}(n)$, $n = 1, 2$, etc. also hold mutatis mutandis for $\mathfrak{gl}(\lambda)$ with any complex $\lambda \notin \mathbb{Z} \setminus \{0\}$. In this section we only consider such values of λ.

Observe that $\mathfrak{gl}(\lambda)$, unlike $\mathfrak{gl}(\infty)$ or $\mathfrak{gl}^{\pm}(\infty)$, has no basis consisting of matrix units. But we can represent it in the form

$$\mathfrak{gl}(\lambda) = \oplus_{k=0}^{\infty} L^{2k} \quad \text{and} \quad \mathfrak{gl}(\lambda) = \oplus_{i=-\infty}^{\infty} \mathfrak{gl}(\lambda)_i,$$

where $\mathfrak{gl}(\lambda)_0 = \mathbb{C}[H]$ while $\mathfrak{gl}(\lambda)_i = X^i \mathfrak{gl}(\lambda)_0$ and $\mathfrak{gl}(\lambda)_{-i} = \mathfrak{gl}(\lambda)_0 Y^i$ for $i > 0$.

As was shown above, there exists a trace on $\mathfrak{gl}(\lambda)$ whose restriction onto $\mathfrak{gl}(\lambda)_0$ has for generating function

$$\frac{e^{\lambda t} - e^{-\lambda t}}{e^t - e^{-t}} \text{ if } \lambda \neq 0 \text{ and } \frac{2t}{e^t - e^{-t}} \text{ if } \lambda = 0.$$

In the first case we have normed the trace so that $\operatorname{tr}(1) = \lambda$ by anology with the finite dimensional case when the scalars are naturally represented by scalar matrices and $\operatorname{tr}(1_n) = n$; in the second case, $\lambda = 0$, we assume that $\operatorname{tr}(1) = 1$.

Observe that for an integer $\lambda = n$ we have

$$\frac{e^{nt} - e^{-nt}}{e^t - e^{-t}} = e^{(1-n)t} + e^{(3-n)t} + \dots + e^{(n-3)t} + e^{(n-1)t}$$

and the respective functional is of the form

$$\operatorname{tr}(f(H)) = \sum_{i=1}^{n} f(\alpha_i), \quad \alpha_i = n - 2i + 1.$$

The functional tr on $\mathfrak{gl}(\lambda)$ gives rise to a nondegenerate symmetric invariant bilinear form $\langle u, v\rangle =$ tr uv.

9.1. Theorem . *In notations of* §8*:* i) $\langle e_{k,l}, e_{k',l'}\rangle = \delta_{k,k'}\delta_{l+l',0}$.

ii) *The polynomials* $f_{k,l}$ *for a fixed* $l \geq 0$ *and* $k \geq l$ *form an orthogonal basis with respect to the scalar product* (8.1.1)*.*

iii) *The polynomials* $f_{k,l}$ *coinside, up to a constant factor, with continuous Hahn polynomials of discrete variable given by* (8.1.2) *for* $n = \lambda$*.*

Proof. The proof of headings i) and ii) is similar to that of respective stements of Theorem 8.1 for integer λ. To prove iii) observe that for a nonnegative integer eq. (9.1) holds thanks to Theorem 8.1. But both sides of (9.1) are polynomials in H whose coefficients are polynomials in λ. Due to continuity of these expressions in Zariski topology we are done. □

9.2. Theorem . *Let* Ω *be the quadratic Casimir operator for* $\mathfrak{sl}(2) \subset \mathfrak{gl}(\lambda)$*.*

i) Ω *is self-adjoint with respect to the form* (8.1.1) *and the polynomials* $X^l f_{kl}$ *for* $k \in \mathbb{Z}_+$ *and* $l \in [-k, k] \cap \mathbb{Z}$ *are eigenfunctions of* Ω *corresponding to eigenvalue* $2k(k + 1)$*. The polynomials* f_{kl} *satisfy the difference equation* (8.2.*i*) *and are of the form* (8.2.*ii*)

$$\text{ii) } \langle f_{kl}, f_{kl}T_1 \dots T_l\rangle = \begin{cases} \dfrac{(k-l)!}{(k+l)!}\dfrac{(k!)^2}{2k+1}\lambda(\lambda^2 - 1^2)\dots(\lambda^2 - k^2) & \text{if } \lambda \neq 0 \\ (-1)^k \dfrac{(k-l)!}{(k+l)!}\dfrac{(k!)^4}{2k+1} & \text{if } \lambda = 0. \end{cases}$$

Proof. Proof of heading i) is the same as that of the corresponding statements of Theorem 8.2. If $\lambda \neq 0$, then both sides of eq. ii) are polynomials in λ which coinside at integer values of λ and, therefore, are equal by continuity in Zariski topology. To embrace $\lambda = 0$, consider $\lim\limits_{\lambda \longrightarrow 0} \frac{\mathrm{tr}}{\lambda}$; as before, both sides become polynomials in λ implying ii). □

It is somewhat unexpected that the dual orthogonality relations hold for sufficiently large $|\lambda|$. Indeed, set

$$c_{kl} = \begin{cases} \frac{(k-l)!}{(k+l)!}\frac{(k!)^2}{2k+1}\lambda(\lambda^2-1^2)\dots(\lambda^2-k^2) & \text{if } \lambda \neq 0 \\ (-1)^k\frac{(k-l)!}{(k+l)!}\frac{(k!)^4}{2k+1} & \text{if } \lambda = 0. \end{cases} \tag{9.2.3}$$

9.3. *Conjecture* . For sufficiently large $|\lambda|$ we have (for $\alpha_i = \lambda - 2i + 1$):

i)
$$\sum_{k=l}^{\infty} \frac{1}{c_{kl}} f_{kl}(\alpha_i) f_{kl}(\alpha_j) T_1(\alpha_i)\dots T_l(\alpha_i) = \delta_{ij} \text{ for } i, j > l. \tag{9.3.1}$$

ii)
$$\sum_{k=0}^{\infty}\left(\frac{1}{c_{k0}}f_{k0}(\alpha_i)^2 + 2\sum_{l=1}^{k}\frac{1}{c_{kl}}f_{kl}(\alpha_i)^2T_1(\alpha_i)\dots T_l(\alpha_i) - \frac{2k+1}{\lambda}\right) = -\alpha_i \text{ for } i = 1, 2, \dots. \tag{9.3.2}$$

A priori, $|\lambda|$ depends on l. We do not know a uniform proof and only checked Conjecture in certain particular cases. We are thankful to V. Gerdt and P. Grozman who helped us with numerical experiments on Maple and MATHEMATICA, respectively. (It seems that Maple is much more convenient and suitable for the task.)

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